Classical pattern-avoiding permutations of length 5.

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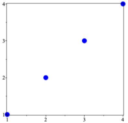
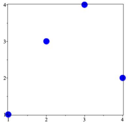
Permutation Patterns, June 2021

Three Wilf classes of length 4

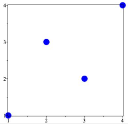
- $Av(1234)$ has a D-finite generating function. The number of length n PAPS $\sim C \cdot \frac{9^n}{n^4}$.



- $Av(1342)$ has an algebraic generating function. The number of length n PAPS $\sim C \cdot \frac{8^n}{n^{5/2}}$.



- $Av(1324)$ is conjectured to have a stretched-exponential generating function. The number of length n PAPS $\sim C \cdot \frac{\mu^n \cdot \mu_1^{\sqrt{n}}}{n^g}$. $\mu \approx 11.598, \; g \approx -1.1$



SIXTEEN WILF CLASSES OF LENGTH 5.

- $Av(12345)$ is solved, and is D-finite. $s_n \sim C \cdot \frac{16^n}{n^{15/2}}$.
- For $Av(31245)$ the growth constant is known: $\mu = 9 + 4\sqrt{2}$.
- For $Av(53421)$, the growth constant $\mu = (1 + \sqrt{\mu(1324)})^2$, perhaps $\mu = (1 + \sqrt{9 + 3\sqrt{3}/2})^2$.
- The OEIS gives 300 terms for $Av(12345)$, 37 terms for $Av(31245)$, 16 terms for two more, 15 terms for one, and 13 terms for the remaining eleven Wilf classes.
- We have extended the 14 shorter series to 20 terms, and expect to go further.
- These are long enough sequences to manifest the asymptotics.

Extracting the asymptotics I.

- For a pure power law, the o.g.f. is

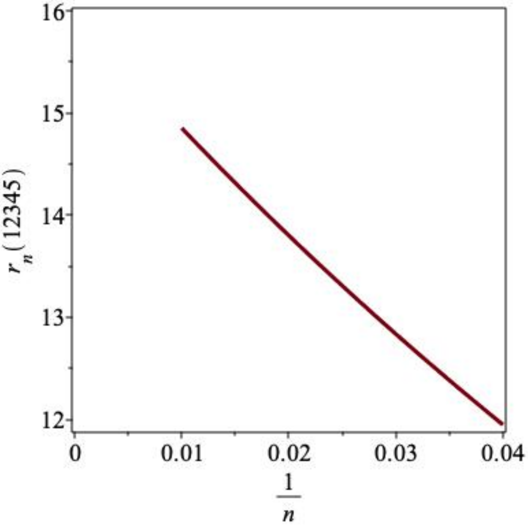
$$S(x) \sim A(1-\mu \cdot x)^{\alpha}, \ \text{so} \ s_n = [x^n]S(x) \sim \frac{A}{\Gamma(-\alpha)} \frac{\mu^n}{n^{\alpha+1}}.$$

- The ratios

$$r_n = \frac{s_n}{s_{n-1}} \sim \mu \left(1 - \frac{\alpha+1}{n} + O\left(\frac{1}{n^2}\right)\right),$$

- So a plot of r_n against $1/n$ will be linear for sufficiently large n, will extrapolate to μ and will have gradient $-\mu(\alpha+1)$.
- Note the r_n converges more rapidly $s_n^{1/n}$.

EXTRACTING THE ASYMPTOTICS I.



EXTRACTING THE ASYMPTOTICS II.

- An alternative method is the *method of differential approximants*.
- Fit available coefficients to an ODE. E.g.

$$Q_2(z)F''(z) + Q_1(z)F'(z) + F(z) = P(z),$$

where $Q_k(z)$ and $P(z)$ are polynomials. Vary their degree until all known coefficients are used.

EXTENDING THE KNOWN SEQUENCES **approximately**.

- The differential approximants reproduce all known coefficients, and approximate all subsequent coefficients.
- We average over dozens of DAs and calculate the mean and s.d. of many subsequent coefficients.
- We accept the coefficients as long as the s.d. is $\leq 10^{-6}$ the value of the coefficient. (So we'll have typically 6 sig. digits).
- In this way, we will typically gain an extra 50-100 coefficients estimated with sufficient accuracy to use the ratio method.

EXTENDING THE KNOWN SEQUENCES **exactly.**

- We wrote a general-purpose program to count any classical PAP or combination of PAPs.
- This allowed us to get to 17 terms in a few days.
- Then we learnt of Yuma Inoue's algorithm, based on what he calls a Rot-πDD algorithm, which counts permutations of edges in a given graph.
- This typically gets to order 20 in around 1 minute! So that's what we are using.

THE 120 PERMUTATIONS IN 16 WILF CLASSES

25314	41352								
31524	42513	24153	35142						
35214	41253	23514	41532	25134	43152	25413	31452		
35124	42153	24513	31542	25143	34152	41523	32514		
53124	42135	13542	24531	15243	34251	32415	51423		
42351	15324	14352	25341	24315	51342	41325	52314		
35241	14253	13524	42531	24135	53142	31425	52413		
53241	14235	13425	52431						
43251	15234	13452	25431	23415	51432	41235	53214		
32541	14523	34125	52143						
34215	51243	14532	23541	15423	32451	43125	52134		
31245 23145	54213 54132	12453 23154	35421 45132	12534 31254	43521 45213	21453	35412	21534	43512
42315	51324	15342	24351						
12345 21543	54321 34512	45321 23451	12354 15432	12543 32145	34521 54123	21345 32154	54312 45123	21354 43215	45312 51234
53421	12435	21435	53412	13245	54231	13254	45231		
52341	14325								

The distribution of the 120 possible length-5 pattern-avoiding permutations among the 16 Wilf classes.

Results of numerical experimentation.

- Nine of the sixteen Wilf classes have power-law asymptotics. That is, $s_n \sim C \cdot \mu^n \cdot n^g$. We have estimated μ and g in each case.
- For $Av(41325)$, $Av(14253)$, $Av(14235)$, $Av(51324)$, $Av(12435)$, and $Av(14325)$ we have stretched exponential behaviour. That is, $s_n \sim C \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$. We've estimated μ, σ and in some cases g.
- For three cases, we find $\sigma = 1/3$, and for three cases $\sigma = 1/2$, (just like $Av(1324)$).
- The final case, $Av(31245)$ is not yet fully sorted..

LOWER BOUNDS.

The *Steiltjes moment problem* considers a sequence $\mathbf{a} \equiv \{a_n\}$, $n \geq 0$ such that

$$a_n = \int_\Gamma x^n d\rho(x)$$

for all $n \geq 0$, where $\Gamma \subseteq \mathbb{R}$, and ρ is a measure.
A necessary and suff. condition is that the Hankel matrix $H_n^\infty(\mathbf{a})$ is totally positive.

$$H_n^\infty(\mathbf{a}) = \begin{bmatrix} a_n & a_{n+1} & a_{n+2} & \cdots \\ a_{n+1} & a_{n+2} & a_{n+3} & \cdots \\ a_{n+2} & a_{n+3} & a_{n+4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Lower bounds II.

- Stieltjes sequences are log-convex, so the ratios $\frac{a_n}{a_{n-1}} \leq \mu$.
- For $Av(12345)$ we can prove that the sequence is Stieltjes.
- For the other 15, the Hankel matrices are all positive (increasing with n), so (conjecture) are Stieltjes sequences.
- The ratios then provide quite strong lower bounds.
- For example, for $Av(12345)$ we have $14.8735 \leq \mu = 16$.
- This project is ongoing, with longer series to be produced, and extensive Monte Carlo analyses of the Wilf classes under construction.