Some combinatorial results on smooth permutations

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Smooth permutations

A permutation $\sigma \in S_n$ is called smooth if the Schubert variety pertaining to σ is smooth.

Lakshmibai and Sandhya (1990): A permutation is smooth if and only if it is 4231 and 3412 avoiding.

Decorated Dyck paths

We define a decorated Dyck path of semilength n to be a function $f : \{1, \ldots, n\} \to \{1, \ldots, n\}$ together with a function $g : \{1, \ldots, n\} \to \{0, 1\}$ such that

- f is non-decreasing and $f(i) \geq i$ for all i.
- $g(i) = 0$ whenever $f(f(i)) = f(i)$.
- $g(i) = g(i+1)$ whenever $i < n$ and $f(i+1) < f(f(i))$.

For every $1 \leq i < j \leq n$ let $R_{[i,j]} \in S_n$ be the cycle permutation $i \to i+1 \to \cdots \to j \to i$.

Theorem: The following map is a bijection between the decorated Dyck paths of semilength n and the smooth permutations in S_n:

$$(f, g) \to (R_{[j_\ell, f(j_\ell)]} \cdots R_{[j_1, f(j_1)]})^{-1} R_{[i_k, f(i_k)]} \cdots R_{[i_1, f(i_1)]},$$

where

- $g^{-1}(0) = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$.
- $g^{-1}(1) = \{j_1, \ldots, j_\ell\}$, $j_1 < \cdots < j_\ell$.

Enumerative consequences

The bijection between decorated Dyck paths and smooth permutations gives a new approach for proving some known enumerative results.

- **Knuth (1973):** The number of 231 avoiding permutations in S_n (which are automatically smooth) is the Catalan number C_n.
- **Haiman (1992):** The generating function of the number of smooth permutations in S_n is

$$1 + \left(\frac{1}{x} - \frac{1}{\sqrt{1-4x}} - 1 \right)^{-1}.$$

- **West (1996):** The number of 321 avoiding smooth permutations in S_n is the Fibonacci number F_{2n-1}.

2-3-tables

The 2-3-table $\mathrm{C}(\sigma)$ of a permutation $\sigma \in S_n$ is defined to be the set of transpositions and the 3-cycles in S_n that are $\leq \sigma$ with respect to the Bruhat order.

Theorem: The map $\sigma \mapsto \mathrm{C}(\sigma)$ is injective for smooth permutations in S_n.

Moreover, a smooth permutation σ may be recovered from its 2-3-table by taking the product of the transposition in $\mathrm{C}(\sigma)$ (each appearing exactly once) according to a suitable order (governed by the additional data in the 2-3-table).

Relation to Covexillary permutations

A permutation is called covexillary if it is 3412 avoiding.

Theorem: For any covexillary $\tau \in S_n$ there is a (unique) smooth permutation $\sigma \in S_n$ such that $\mathrm{C}(\sigma) = \mathrm{C}(\tau)$.

Moreover, there are covexillary permutations $\tau = \tau_0 < \tau_1 < \cdots < \tau_r = \sigma$ in S_n such that $\tau_{i-1}^{-1}\tau_i$ is a transposition for every $1 \leq i \leq r$.

THANK YOU
FOR YOUR ATTENTION