PERMUTATIONS IN SUBSTITUTION-CLOSED CLASSES : A PROBABILISTIC APPROACH

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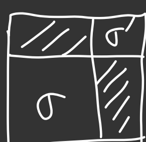
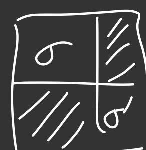
PERMUTATION PATTERNS 2021

(Based on works with
F Bassino, M Bouvel, V Féray, M Maazoun, A Pierrot)
Paris-Nord, Nancy CNRS, Nancy CNRS, Oxford, Paris-Saclay

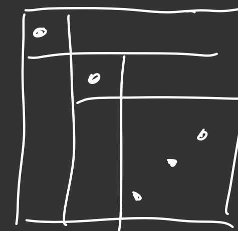
I Separable permutations & Trees

$\sigma \oplus \sigma' =$  $\sigma \ominus \sigma' =$ 

Def σ is separable iff σ can be written only with $1, \oplus, \ominus$

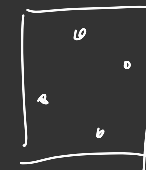
example

$1 \ominus (1 \ominus (1 \oplus (1 \oplus 1)))$

$=$ 

$= 54123$

counter-ex

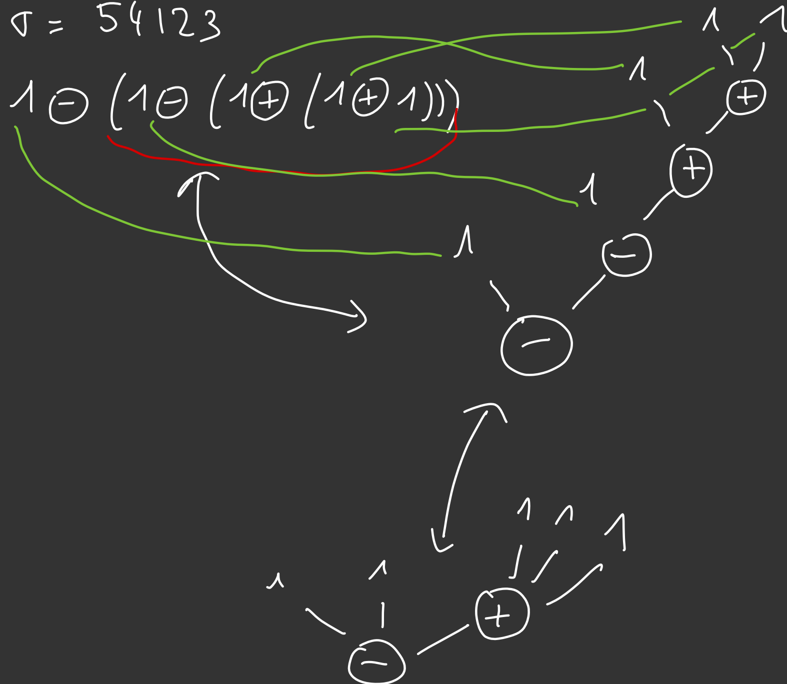
 $= 2413$

σ is separable iff σ avoids both 2413 and 3142

Question : Fix a (small) pattern π, how many times does π occur in a (typical) large separable permutation ?

$\sigma = 54123$

$1 \ominus (1 \ominus (1 \oplus (1 \oplus 1)))$



Theorem (Bose '98)

There is a bijection.

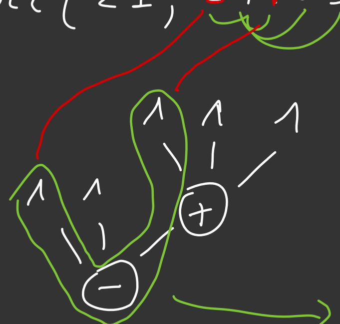
separable permutations of size n $\longleftrightarrow$ plane trees with n leaves such that
- internal vertices are decorated with $\oplus$ - $\ominus$
- $\oplus$ - $\ominus$ alternate
- out-degree $\in \{0, 2, 3, 4, \dots\}$

Schröder Trees

$\text{Schrö}(\sigma)$ = unique tree associated to σ

What about patterns?

$$occ(21, 54123) = 7$$



induced subtree

$\rightsquigarrow$ 21

Proposition.

$$occ(\pi, \sigma) = \#\left\{ \begin{array}{l} \text{Subtrees of } Schrö(\sigma) \\ \text{induced by } k \text{ leaves} \\ \text{which encode} \\ \text{for } \pi \end{array} \right\}$$

$|\pi| = k$

II Patterns in separable permutations

Schröder tree of size n

$$\xrightarrow{n \to +\infty}$$

continuous & fractal tree

Aldous' Continuum Random Tree 90's

$\hookrightarrow$ One hopes to deduce asymptotics for $occ(\pi, \sigma)$

<u>Theorem</u> (BBFP 2018)

Fix k and a pattern $\pi \in \mathfrak{S}_k$

Let (σ_n) be uniform separable permutations of size n

There exist a random variable $\Lambda_\pi \in [0,1]$ such that:

$$\frac{\mathrm{occ}(\pi, \sigma_n)}{\binom{n}{k}} \xrightarrow{n\to+\infty} \Lambda_\pi$$

(in distribution)

Furthermore, if π separable:

- $0 < \Lambda_\pi < 1$

  $\hookrightarrow$ $\mathrm{occ}(\pi, \sigma_n)$ really scales like $\binom{n}{k} \sim \frac{n^k}{k!}$

- Λ_π is a "true" random variable

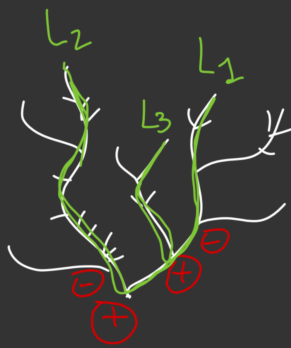
- $$\mathbb{E}[\Lambda_\pi] = \frac{\#\left\{\begin{array}{l}\text{binary trees of size } k \\ \text{which encode } \pi\end{array}\right\}}{\mathrm{Cat}_{k-1} \times 2^{k-1}}$$

- In the case Λ_{12} one can compute $\mathbb{E}[\Lambda_{12}], \mathbb{E}[\Lambda_{12}^2], \dots, \mathbb{E}[\Lambda_{12}^7]$

Probabilistic description of Λ_π

Let T be a sample of the CRT.

We draw fair $\oplus$'s - $\ominus$'s on each internal vertex



Fix k, $\pi \in \mathfrak{S}_k$

We draw $L_1, \dots, L_k$ & uniform leaves in T

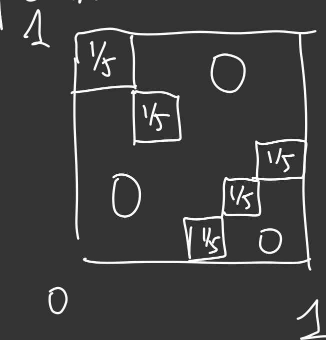
$$\Lambda_\pi = \mathbb{P}\left(\begin{array}{c} L_1, \dots, L_k \\ \text{induce in } T \\ \text{a tree which} \\ \text{encodes } \pi \end{array} \,\middle|\, T, \text{decorations} \right)$$

III Convergence as permutons

"Permuton" (Hoppen et al 2013)

= "Limit of rescaled diagrams of permutations"

$54123 \hookrightarrow$



probability measure on the unit square.

A <u>permuton</u> is a prob measure on $[0,1]^2$ for which both margins are the uniform measure on $[0,1]$

Theorem (Bis)

Let (σ_n) be uniform separable permutation, let P_{σ_n} be the permuton associated to σ_n

Then

$$P_{\sigma_n} \xrightarrow[\text{distrib}]{n \to +\infty} \mu^{(1/2)}$$

(in the space permutons)

where $\mu^{(1/2)}$ is a "continuous and fractal" permuton

informally,

$$\mu^{(1/2)} = \text{Schrö}^{-1}\left(\mathcal{T} \text{ with fair } \oplus\text{'s} - \ominus\text{'s} \right)$$

Aldous' CRT

made rigorous by (Maazoun 2019).

IV What about universality?

- We only consider substitution-closed $\mathcal{C}$

$$\left(\text{If } \theta, \pi_1, \dots, \pi_k \in \mathcal{C} \text{ then } \theta[\pi_1, \dots, \pi_k] \in \mathcal{C} \right)$$

We also can encode permut in $\mathcal{C}$ with (more decorated) trees.

- If simple permutation in $\mathcal{C}$ do not grow too fast then we still can use the CRT

Then under these 2 assumptions:

Theorem (BBFGMP 2020)

Let (σ_n) be uniform in $\mathcal{C}$ of size n, then:

$$\left(\mathcal{P}_{\sigma_n} \right) \xrightarrow{n} \mu^{(p)}$$

$$\downarrow$$

$$\ll \text{Schrö}^{-1}\left(\mathcal{T} \text{ with unfair } \oplus\text{'s} - \ominus\text{'s} \right) \gg$$

(Aldous' $\mathcal{T}$)