Increasing subsequences in random separable permutations

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Permutation Patterns 2021 Virtual Workshop
Online, June 15th, 2021.

First main result

Reminder: separable permutations are permutations obtained from 1 by iterating $\oplus$ and $\ominus$ operations. Equivalently they avoid 3142 and 2413.

Theorem

For each $n \geq 1$, let $\boldsymbol{\sigma_n}$ be a uniform random separable permutation of size n. Then, the length of the longest increasing subsequence (LIS) in $\boldsymbol{\sigma}_n$ is sublinear in n, i.e. $\frac{\mathrm{LIS}(\boldsymbol{\sigma_n})}{n}$ converges to 0 in probability.

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Motivations:

- LIS is a standard statistics on uniform random permutations; more recently there has been literature on pattern-avoiding permutations.
- We have an analogue result on cographs (inversion graphs of separable permutations), which answers a question about a probabilistic version of Erdős-Hajnal conjecture.
- The proof is interesting!

The first moment method fails!

Natural approach: let $\boldsymbol{Z}_{n,k}$ be the number of increasing subsequences (not necessarily maximal) of length k in $\boldsymbol{\sigma}_n$.

Hope: if $k = \Theta(n)$, then $\mathbb{E}[\boldsymbol{Z}_{n,k}]$ tends to 0. If this holds, then $\boldsymbol{Z}_{n,k} = 0$ with high probability, i.e. there is no increasing subsequence of length k.

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Theorem (Second main result)

For integers k in $[an, bn]$ (a, b fixed in $(0,1)$), we have

$$\mathbb{E}[\boldsymbol{Z}_{n,k}] \sim D_{k/n} \, n^{-1/2} (E_{k/n})^n, \quad (1)$$

where $E_\beta > 1$ for β sufficiently small ($\beta < 0.58$ numerically).

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Tool: analytic combinatorics. The series

$$S(z,u) = \sum_{\substack{\sigma \text{ separable} \\ J: \sigma/J \text{ increasing}}} z^{|\sigma|} u^{|J|}$$

is the solution of a combinatorial system → can be analyzed.

Instead, we use permutons!

We can define a LIS function on permutons:

permutations σ	permutons μ (=measure on $[0;1]^2$);
subsequence σ/J	submeasure $\nu \leq \mu$;
σ/J increasing	$\nexists \; \bullet^{P} \; \bullet_{Q} \; \in \mathrm{Supp}(\nu)$
normalized length $\|J\|/n$	total mass $\nu([0;1]^2)$

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Definition (Maréchal, '21)

$$\widetilde{\mathrm{LIS}}(\mu) := \sup_{\nu \leq \mu, \nu \text{ ``increasing''}} \nu([0;1]^2).$$

It extends the map $\sigma \mapsto \widetilde{\mathrm{LIS}}(\sigma) := \mathrm{LIS}(\sigma)/n$ to permutons.

Proposition

$\widetilde{\mathrm{LIS}}$ *is lower semi-continuous, i.e. if* $\mu_k \to \mu$*, then* $\limsup \widetilde{\mathrm{LIS}}(\mu_k) \leq \widetilde{\mathrm{LIS}}(\mu)$.

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Our goal is to prove $\widetilde{\mathrm{LIS}}(\boldsymbol{\sigma}_n) \to 0$.

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We do it using a self-similarity property of the Brownian excursion...