Admissible Pinnacle Sets and Ballot Numbers

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Pinnacle Sets

The *pinnacle set* of a permutation $\pi \in \mathfrak{S}_n$ is

$$\operatorname{Pin}\pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1].$$

If $\pi = 18524376$ then $\operatorname{Pin}\pi = \{4, 7, 8\}$.

A set S is an *admissible pinnacle set* if there is some permutation π with $\operatorname{Pin}\pi = S$.

Definition (Davis et al. [3])

Let $\mathfrak{p}(m; d)$ to be the number of admissible pinnacle sets with maximum m and cardinality d.

We prove these constants are ballot numbers through a bijection between admissible pinnacle sets and ballot sequences.

Ballot Sequence

A (p, q) *ballot sequence* is a permutation $\beta = \beta_1\beta_2 \dots \beta_{p+q}$ of p copies of the letter X and q copies of the letter Y such that in any nonempty prefix $\beta_1\beta_2 \dots \beta_i$ the number of X's is greater than the number of Y's. Let

$$\mathcal{B}_{p,q} = \{\beta \mid \beta \text{ is a } (p, q) \text{ ballot sequence}\}.$$

$\beta = XXXYXXY$ ✓, $\beta = XXYYXXY$ ×

A Bijection to Ballot Sequences

For nonnegative integers $p > q$, we have $\#\mathcal{B}_{p,q} = \frac{p-q}{p+q}\binom{p+q}{q}$ (see [1, 2]).

The Bijection

Let $\mathfrak{P}(m, d) = \{S \mid S \text{ admissible with } \max S = m \text{ and } \#S = d\}$ so that $\#\mathfrak{P}(m, d) = \mathfrak{p}(m, d)$. For $m > 2d$, define a map $\eta : \mathcal{B}_{m-d,d-1} \to \mathfrak{P}(m, d)$ by sending ballot sequence $\beta = \beta_1\beta_2 \ldots \beta_{m-1}$ to

$$\eta(\beta) = \{i \mid \beta_i = Y\} \uplus \{m\}.$$

$$\beta = XXX\mathbf{Y}XX\mathbf{Y} \iff \{4, 7, 8\} \iff \pi = 1\mathbf{8}52\mathbf{4}3\mathbf{7}6.$$

- Well-Defined: Since $\beta \in \mathcal{B}_{m-d,d-1}$, the set $\{i \mid \beta_i = Y\} \subset [m-1]$ with cardinality $d - 1$.
- Admissible: Shown in [3] that $S = \{s_1 < s_2 < \cdots < s_d\}$ is admissible $\Leftrightarrow s_i > 2i$ for all i. Here s_i is the index of the ith Y. Since β a ballot sequence, there are $\geq i + 1$ copies of X.
- Inverse: For $S \in \mathfrak{P}(m, d)$, let $\eta^{-1}(S) = \beta_1\beta_2 \ldots \beta_{m-1}$ where $\beta_i = X$ if $i \notin S$, Y otherwise.

Conclusion

This bijection with ballot sequences gives the exact counts of the number of admissible pinnacle sets S with maximum m and cardinality d.

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Theorem

If $m, d \in \mathbb{P}$ with $m > 2d$, then

$$\mathfrak{p}(m; d) = \frac{m - 2d + 1}{m - 1}\binom{m-1}{d-1} = \#\mathcal{B}_{m-d,d-1}.$$

[1] André, D. (1887). Solution directe du probleme résolu par m. bertrand. *CR Acad. Sci. Paris*, 105(436):7.

[2] Bertrand, J. (1887). Solution d'un problème. *Comptes Rendus de l'Académie des Sciences de Paris*, (105):369.

[3] Davis, R., Nelson, S. A., Kyle Petersen, T., and Tenner, B. E. (2018). The pinnacle set of a permutation. *Discrete Mathematics*, 341(11):3249–3270.