Sorting with a popqueue

Lapo Cioni

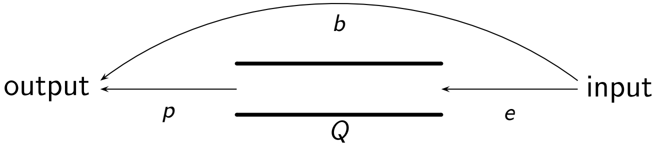
`This talk is based on joint work with Luca Ferrari`

Università degli studi di Firenze

Popqueue

A popqueue is a sorting device in which we can insert and extract elements, following some restrictions. Namely, these are the allowed operations:

- e: *enqueue*, insert the current element into the popqueue, in the rightmost position;
- b: *bypass*, send the current element into the output;
- p: *pop*, remove all the elements currently in the popqueue, from left to right, sending them into the output.



Popqueue

Definition

A permutation is sortable if there exists a sequence of operations that output the identity permutation.

Proposition

If a permutation $\pi \in S_n$ contains an occurrence of the patterns 321 *or* 2413*, then it is not sortable using a popqueue.*

Algorithm

Min

input: a permutation $\pi = \pi_1 \cdots \pi_n$

output: a permutation $Min(\pi)$

for $i = 1, \dots, n$ *do:*

- *if* $Front(Q)$ *is the minimal element not yet in the output (i.e., if* $Front(Q)$ *is smaller than all the unprocessed elements* $\pi_i, \dots, \pi_n$*), then pop and enqueue;*
- *else compare* π_i, $Back(Q)$ *and* $Front(Q)$*:*
  - *if* $Back(Q) < \pi_i$*, enqueue;*
  - *otherwise, if* $Front(Q) > \pi_i$*, then bypass;*
  - *else, pop end enqueue.*

Finally, pop.

The name *Min* comes from the first instruction, which empties the popqueue when $Front(Q)$ is the first element to be output.

Algorithm

Cons
input: a permutation $\pi = \pi_1 \cdots \pi_n$
output: a permutation $Cons(\pi)$
for $i = 1, \ldots, n$ *do:*

- *if* $\pi_i = Back(Q) + 1$*, then enqueue;*
- *else, compare* π_i *and* $Front(Q)$*:*
    - *if* $Front(Q) > \pi_i$*, then bypass;*
    - *else, pop and enqueue.*

Finally, pop.

The name *Cons* comes from the first instruction, that only allows consecutive elements to be in the popqueue, and thus forces the content of the popqueue to be consecutive at all times.

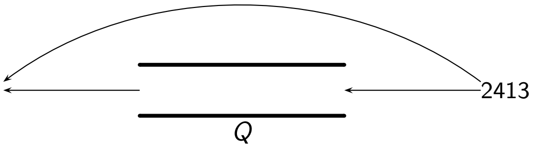
Min and *Cons* are optimal sorting algorithms.

Proposition

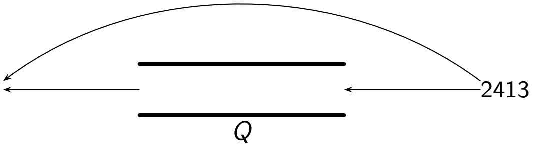
Let $\pi \in S_n$. Then $Min(\pi) \neq id_n$ if and only if $321 \leq \pi$ or $2413 \leq \pi$, and $Cons(\pi) \neq id_n$ if and only if $321 \leq \pi$ or $2413 \leq \pi$.

An example

Min:

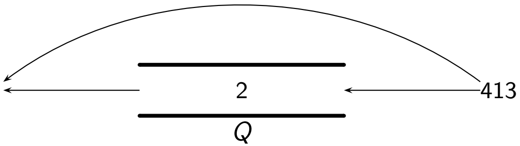


Cons:

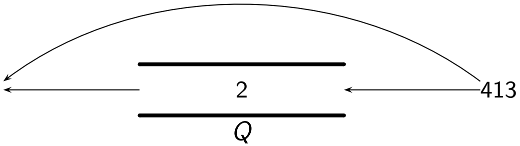


An example

Min:

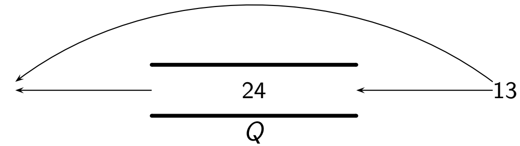


Cons:

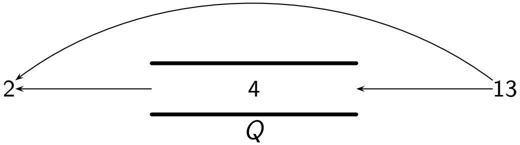


An example

Min:

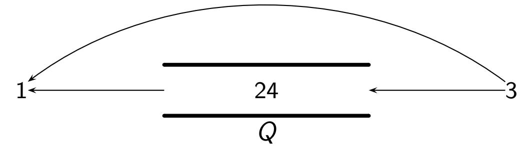


Cons:

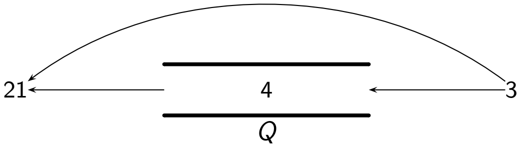


An example

Min:

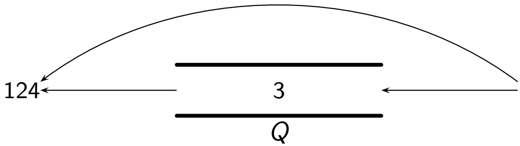


Cons:

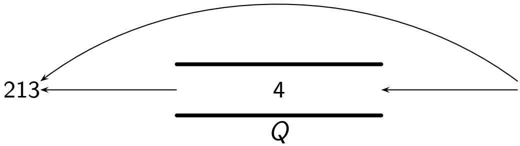


An example

Min:

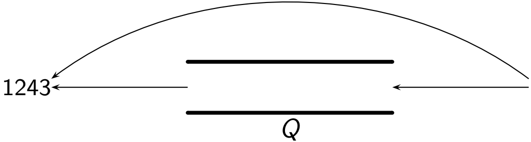


Cons:

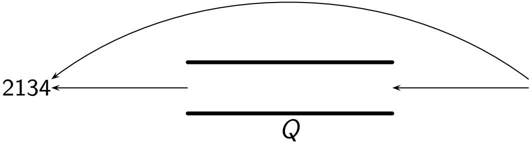


An example

Min:



Cons:



$Sort_M$ and $Sort_C$

$Sort_M$ and $Sort_C$

Define the sets $Sort_M$ and $Sort_C$ of permutations sorted by two applications of Min and $Sort$, respectively, so that
$Sort_M = \{\pi \in S \mid Min(Min(\pi)) = id_n, n \in \mathbb{N}\}$ and
$Sort_C = \{\pi \in S \mid Cons(Cons(\pi)) = id_n, n \in \mathbb{N}\}$.

Remark

Consider the permutations 2431 *and* 35214*. Then* $2431 \in Sort_C \setminus Sort_M$ *and* $35214 \in Sort_M \setminus Sort_C$*. This shows that each of the two algorithms is able to sort some permutations that the other algorithm cannot sort.*

Proposition

The set $Sort_M$ is not a permutation class.

Proof.

Consider the permutation 241653. Then
$Min(Min(241653)) = Min(124536) = 123456$, but
$Min(Min(2431)) = Min(2413) = 1243$, and $2431 \leq 241653$.

□

Differences between $Sort_M$ and $Sort_C$

Proposition

Let $\pi \in S_n$. Then $Cons(Cons(\pi)) \neq id_n$ if and only if π contains at least one of the following nine patterns:

- 4321;
- 35241;
- 35214;
- 52413;
- 25413;
- 246153;
- 246135;
- 426153;
- 426135.

Thus $Sort_C$ is a permutation class.

A conjecture

We do not know much about the sequences $|Sort_{M,n}|$ of the number of permutations of length n in $Sort_M$, and $|Sort_{C,n}|$ of the number of permutations of length n in $Sort_C$. Their first terms are, respectively,
1, 2, 6, 22, 89, 379, 1660, 7380, 33113, 149059 and
1, 2, 6, 23, 99, 445, 2029, 9292, 42608, 195445,
and do not appear in *oeis*. Still, it seems that $|Sort_{M,n}|$ is smaller than $|Sort_{C,n}|$ for every $n > 3$.

Conjecture

For every $n > 3$, $|Sort_{M,n}| < |Sort_{C,n}|$.