Permutations with exactly one copy of a monotone pattern of length k, and a generalization

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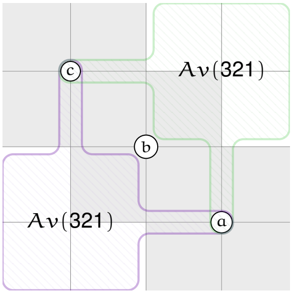
1. One copy of 321
2. Injection for 321
3. Generalization
4. Asymptotics and generating functions

Introduction

- Let $S_n(\sigma;1)$ be the set of permutations containing a single occurrence of pattern σ.
- Knuth (1968,1970): $|S_n(321)| = |S_n(231)| = C_n = \frac{1}{n+1}\binom{2n}{n}$
- Simion, Schmidt (1985): bijection $g : S_n(321) \to S_n(231)$
  - Preserves positions and values of right-to-left minima
- Noonan (1996): $|S_n(321;1)| = \frac{3}{n}\binom{2n}{n-3}$
- Bóna (1998): $|S_n(231;1)| = \binom{2n-3}{n-3}$
- $\lim_{n\to\infty} \frac{|S_n(321;1)|}{|S_n(321)|} = 3 < \infty$ vs. $\lim_{n\to\infty} \frac{|S_n(231;1)|}{|S_n(231)|} = \infty$

Containing a single copy of 321

B. (2011), Zeilberger (2011): Can enumerate $|S_n(321;1)|$ more efficiently by splitting it into the single copy of 321 and two 321-avoiding permutations.



If $\pi \in Av_n(321;1)$ and cba is the single occurrence of 321 in π, then

$$\pi = \pi_1 \; c \; \pi_2 \; b \; \pi_3 \; a \; \pi_4,$$

where

$$\pi_1 \; c \; \pi_2 \; a \in Av(321),$$
$$c \; \pi_3 \; a \; \pi_4 \in Av(321).$$

$S_n(321;1) \hookrightarrow S_{n+2}(231) \cong S_{n+2}(321)$

Define injection $f : S_n(321;1) \to S_{n+2}(231)$ by

$$f : \pi = \pi_1\ c\ \pi_2\ b\ \pi_3\ a\ \pi_4 \mapsto 132[g(\mathrm{red}(\pi_1\ c\ \pi_2\ a)), 1, g(\mathrm{red}(c\ \pi_3\ a\ \pi_4))]$$

Equivalently,

- $c \mapsto b, \qquad b \mapsto a\ (n+2)\ (c+1), \qquad a \mapsto b+1,$
- add 1 to every entry in π_3 and π_4 (to obtain π_3' and π_4'),
- apply g to $\pi_1\ b\ \pi_2\ a$ and to $(c+1)\ \pi_3'\ (b+1)\ \pi_4'$.

Right-to-left minima of π vs. $f(\pi)$ (other than a):

- positions and values preserved to the left of $n+2$;
- positions increased by 2 and values increased by 1 to the right of $n+2$.

Example

Example

Let $\pi = \mathbf{2}5\mathbf{1}4\mathbf{7}3\mathbf{8}6 \in S_8(321;1)$. Then $c = 5$, $b = 4$, $a = 3$, $(\pi_1, \pi_2, \pi_3, \pi_4) = (2, 1, 7, 86)$.

So $\pi_1 b \pi_2 a = \mathbf{2413}$ and $c\pi_3 b\pi_4 = \mathbf{57}4\mathbf{86}$, and hence $(c+1)\pi_3'(b+1)\pi_4' = 68597$.

Therefore, $g(\pi_1 b \pi_2 a) = 4213$ and $g((c+1)\pi_3'(b+1)\pi_4') = 96587$, so

$$f(\pi) = 4\ 2\ 1\ 3\ 10\ 9\ 6\ 5\ 8\ 7 \in S_{10}(231).$$

A generalization of f

The injection f can be generalized as follows:

Theorem (Main Injection)

For any $k \geqslant 3$ and any pattern $\rho \in S_{k-3}$, there is an injection

$$F_k : S_n(321 \ominus \rho; 1) \hookrightarrow S_{n+2}(231 \ominus \rho)$$

Recall also that $S_{n+2}(231 \ominus \rho) \cong S_{n+2}(321 \ominus \rho)$.

Proof of Main Injection

If p is a permutation, we say that entry p_i dominates entry p_j if $i < j$ and $p_i > p_j$. Likewise for p_i dominating a subsequence of entries.

Let $p \in S_n(321 \ominus \rho; 1)$. Let π be the subsequence of all entries of p that dominate an occurrence of ρ in p (call those entries blue). Let τ be the rest of the entries and call those red.

Then π contains a single occurrence of 321.

Proof of Main Injection (cont'd)

To obtain $F_k(p)$:

- replace the entries c, b, and a, respectively, with the entry b, block $a\,(n+2)\,(c+1)$, and the entry $b+1$, respectively, and color the new entries, except for $n+2$, blue;
- add 1 to every entry in π_3 and π_4 (to obtain π_3' and π_4') and color the new entries blue;
- add 1 to every entry of τ greater than b and color the new entries red.
- apply the map g to the subsequences $\pi_1\,b\,\pi_2\,a$ and $(c+1)\,\pi_3'\,(b+1)\,\pi_4'$. This preserves the right-to-left minima, so these blue entries stay blue.

Note that in $F_k(p)$, as in p, no red entry dominates a blue entry.

Example of F_k

Let $k = 4$, $\rho = 1$, so $321 \ominus \rho = 4321$ and $231 \ominus \rho = 3421$.

Example

Let $p = 481593276 \in S_9(4321; 1)$. Then $\tau = 126$ and $\pi = 485937$ (the unique occurrence of 321 in π is marked in bold), so $p = 481593276$ and $\pi_1 = 4$, $c = 8$, $\pi_2 = \varnothing$, $b = 5$, $\pi_3 = 9$, $a = 3$, and $\pi_4 = 7$. In τ, $1 < b$, $2 < b$, while $6 > b$. Replace entries of π as before and add 1 to the entries of τ greater than $b = 5$ to obtain

$$4\ 5\ 1\ 3\ 11\ 9\ 10\ 6\ 2\ 8\ 7.$$

Now replace subsequences $\pi_1 b \pi_2 a = 453$ with $g(453) = 543$ and $(c+1)\pi_3'(b+1)\pi_4' = 9(10)68$ with $g(9(10)68) = (10)968$, so that $f(\pi) = 543(11)(10)968$ and

$$F_4(p) = 5\ 4\ 1\ 3\ 11\ 10\ 9\ 6\ 2\ 8\ 7 \in S_{11}(3421).$$

Rate of growth of $|S_n(321 \ominus \rho; 1)|$

We also have an injection $S_{n-k}(321 \ominus \rho) \hookrightarrow S_n(321 \ominus \rho; 1)$ by mapping $\sigma \mapsto \sigma \oplus (321 \ominus \rho)$.

So,

$$|S_{n-k}(321 \ominus \rho)| \leqslant |S_n(321 \ominus \rho; 1)| \leqslant |S_{n+2}(321 \ominus \rho)|.$$

Therefore,

$$|S_n(321 \ominus \rho; 1)| \qquad \text{and} \qquad |S_n(321 \ominus \rho)|$$

are of the same exponential order.

E.g. of exp. order $(k-1)^2$ for $321 \ominus \rho = k \cdots 21 = r(\mathrm{id}_k)$.

Generating function for $|S_n(321 \ominus \rho)|$

The above asymptotics let us prove the following result.

Theorem

Let $k \geqslant 3$ and $\rho \in \mathfrak{S}_{k-3}$, then the ordinary generating function for the sequence $|S_n(321 \ominus \rho; 1)|$ is not rational.

Proof Idea: All singularities of rational functions are poles. But $\mathrm{ogf}(|S_n(321 \ominus \rho; 1)|)$ takes a finite value at the dominant singularity (= radius of convergence).

Generating function for $|S_n(k\cdots 21)|$

Regev (1981): For $k \geqslant 2$, there exists a constant γ_k such that

$$|S_n(k\cdots 21)| \simeq \gamma_k \frac{(k-1)^{2n}}{n^{(k^2-2k)/2}}$$

Notice: when $k > 2$ is even, $-\frac{k^2-2k}{2}$ is a negative integer.

This lets us prove the following result.

Theorem

Let $k > 2$ be an even integer. Then the generating function for $|S_n(k\cdots 21;1)|$ is not algebraic.

Note: We can prove non-algebraicity for $|S_n(k\cdots 21;1)|$ because we know precise asymptotics for $|S_n(k\cdots 21)|$.

Further questions

- Conjecture: For any r, there exists $\lim_{n\to\infty} \frac{|S_n(321;r)|}{|S_n(321)|} < \infty$.
  - This would imply that $\mathrm{ogf}(|S_n(321;r|)$ is non-rational.
  - More precise asymptotics $\rightsquigarrow$ non-algebraicity?
- Non-algebraicity for other families of patterns (non-monotone)? Again, need more precise asymptotics.

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