Permutations avoiding sets of patterns with long monotone subsequences

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Let A_k be the set of k patterns of length k that start with an increasing subsequence of length $k-1$.

For instance,

$$A_5 = \{12345, 12354, 12453, 13452, 23451\}.$$

Permutation $p = p_1 p_2 \cdots p_n$ avoids A_k if and only if the subsequence $p_1 p_2 \cdots p_{n-1}$ avoids $12 \cdots (k-1)$.

Therefore,

$$\mathrm{Av}_{A_k}(n) = n\mathrm{Av}_{12\cdots(k-1)}(n-1). \qquad (1)$$

This gets more interesting if we remove one element of A_k. Let $A_{k,i} = A_k \setminus \{12 \cdots k\ i\}$, that is, the set A_k with its element ending in i removed.

It is clear that for each $i \leq k$, the chain of inequalities $(k-2)^2 \leq L(A_{k,i}) \leq (k-1)^2$ holds.

The interesting question is *where* in the interval $[(k-2)^2, (k-1)^2]$ are the growth rates $L(A_{i,k})$ located.

When $2 \leq i \leq k - 1$

If a permutation p avoids $A_{k,i}$, but contains an increasing subsequence of length $k - 1$, then the set of entries of p that follow the last entry of that increasing subsequence is very restricted. This leads to the following theorem.

Theorem

For all $k \geq 3$, and all $2 \leq i \leq k - 1$, the equality

$$L(A_{k,i}) = (k - 2)^2$$

holds.

When $i = k$

The case of $i = k$ leads to a different result.

Theorem

For $k \geq 3$, the equality

$$L(A_{k,k}) = (k-2)^2 + 1$$

holds.

The proof uses the Robinson-Schensted correspondence, in a little bit deeper way than in the last case.

When $i = 1$

While we are not able to rigorously compute $L(A_{k,1})$ for $k > 4$, or even just $L(A_{5,1})$, we could to instead rigorously compute the first 642 terms of the counting sequence of $\mathrm{Av}_{5,1}(n)$.

This led to very strong numerical evidence suggesting that $L(A_{5,1}) = 9$.

Data suggest that the numbers $\mathrm{Av}_{5,1}(n)$ grow as $C9^n/n^3$. This would imply that the generating function of the sequence $\mathrm{Av}_{5,1}(n)$ is not algebraic.

Even more strongly, data suggest that the generating function is not even d-finite.

Further directions

We have an injective proof of the following result.

Theorem

For all positive integers n, and all $k \geq 3$, the inequality

$$Av_n(A_{k,k-1}) \leq Av_n(A_{k,k})$$

holds.

Data suggest that

- the sequences $\mathrm{Av}_{5,2}(n)$, $\mathrm{Av}_{5,3}(n)$, and $\mathrm{Av}_{5,4}(n)$ grow as $C9^n/n^3$,
- the sequence $\mathrm{Av}_{5,5}(n)$ grows as $C10^n/n^4$,
- On the other hand, $\mathrm{Av}_{6,1}(n)$ seems to grow as $C16^n/n^6.5$, which could allow for an algebraic generating function.