Permutation limits at infinitely many scales

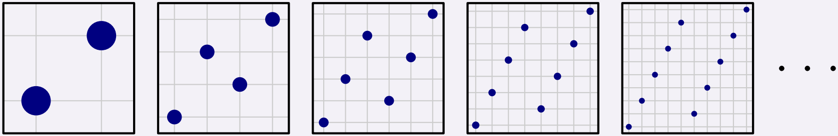
David Bevan
University of Strathclyde

Permutation Patterns 2021

15th June 2021

Global limits: permutons

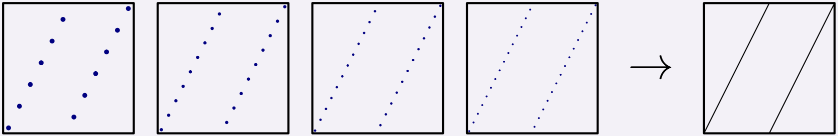
Example



$$\sigma_j = 13579\ldots(2j-1)2468\ldots(2j)$$

Global limits: permutons

Example



$$\sigma_j = 13579\ldots(2j-1)2468\ldots(2j)$$

Definition (permuton)

Probability measure μ **on** the σ-algebra of Borel sets of **the unit square** $[0,1]^2$ such that μ has uniform marginals:

$$\mu([a,b]\times[0,1]) = \mu([0,1]\times[a,b]) = b-a \text{ for every } 0 \leqslant a \leqslant b \leqslant 1$$

Global convergence

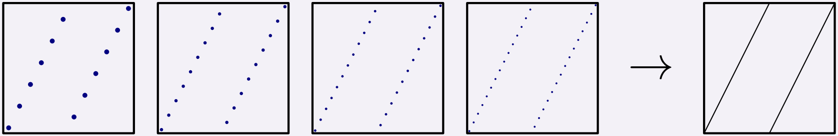
Definition (pattern density)

If $\sigma \in S_n$ and $\pi \in S_k$, and $\nu(\pi, \sigma)$ is the number of occurrences of π in σ, then $\rho(\pi, \sigma) = \nu(\pi, \sigma)/\binom{n}{k}$.

Definition (convergence)

If $|\sigma_j| \to \infty$, then $(\sigma_j)_{j\in\mathbb{N}}$ is convergent if $\rho(\pi, \sigma_j)$ converges *for every pattern* π.

Example

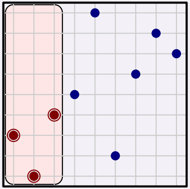
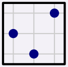


$\rho(12, \sigma_j) = \frac{3j-1}{4j-2} \to \frac{3}{4}$ $\qquad$ $\rho(12\ldots k, \sigma_j) \to (k+1)/2^k$

Patterns at a given scale

Looking through a window:

Example (window of width 3)

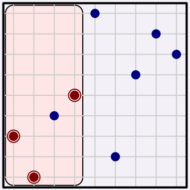
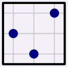
$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3

Patterns at a given scale

Looking through a window:

Example (window of width 4)

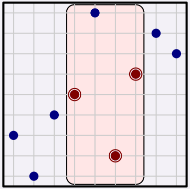
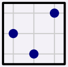
$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3; 2 have width 4

Patterns at a given scale

Looking through a window:

Example (window of width 4)

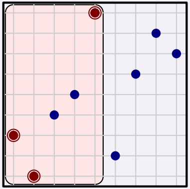
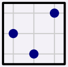
σ =  contains 15 occurrences of π = 

- 1 has width 3; 2 have width 4

Patterns at a given scale

Looking through a window:

Example (window of width 5)

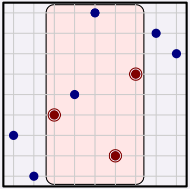
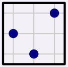
$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3; 2 have width 4; 3 have width 5

Patterns at a given scale

Looking through a window:

Example (window of width 5)

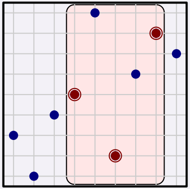
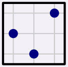
$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3; 2 have width 4; 3 have width 5

Patterns at a given scale

Looking through a window:

Example (window of width 5)

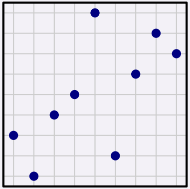
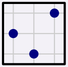
$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3; 2 have width 4; 3 have width 5

Patterns at a given scale

Looking through a window:

Example (window of width 5)

$\sigma =$  contains 15 occurrences of $\pi =$ 

- 1 has width 3; 2 have width 4; 3 have width 5
- 6 have width at most 5

Pattern density at scale 5

- 34 possible choices of three points with width at most 5
- Density of π in σ at scale 5: $\rho_5(\pi, \sigma) = 6/34 = 3/17$.

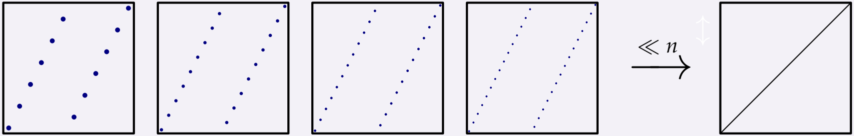
Convergence at a given scale

Typically, the scale (width of window) $f = f(n)$ depends on $n = |\sigma_j|$.

Definition (convergence at scale f)

If $|\sigma_j| \to \infty$, then $(\sigma_j)_{j\in\mathbb{N}}$ is convergent at scale f if $\rho_f(\pi, \sigma_j)$ converges for every pattern π.

Example ($f \ll n$)



$$\rho_f(12\ldots k, \sigma_j) \to 1, \text{ if } f \ll n \quad (n = |\sigma_j| = 2j)$$

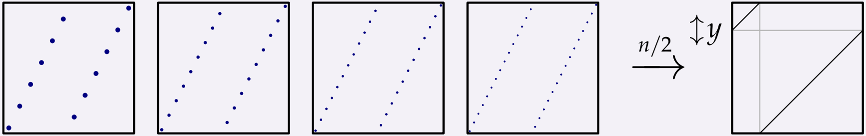
Convergence at a given scale

Typically, the scale (width of window) $f = f(n)$ depends on $n = |\sigma_j|$.

Definition (convergence at scale f)

If $|\sigma_j| \to \infty$, then $(\sigma_j)_{j\in\mathbb{N}}$ is convergent at scale f if $\rho_f(\pi, \sigma_j)$ converges for every pattern π.

Example ($f = n/2$)



$$\rho_{n/2}(12, \sigma_j) = \frac{2}{3} \qquad (n = |\sigma_j| = 2j)$$

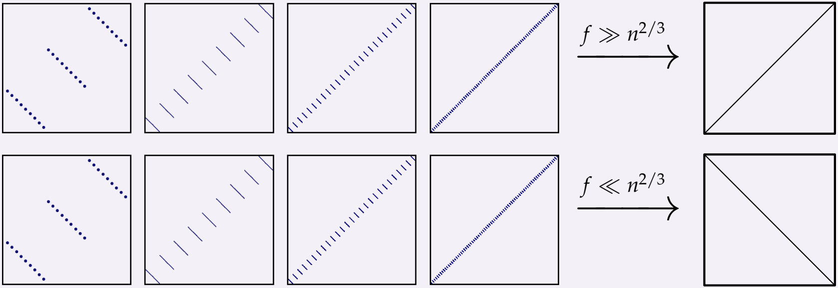
Independence of limits at different scales

We can use *inflation* to give different limits at different scales.

$$\iota_k = 123\ldots k \qquad \delta_k = k\ldots 321$$

Example

$$\sigma_j = \iota_j[\delta_{j^2}] \qquad |\sigma_j| = j^3$$

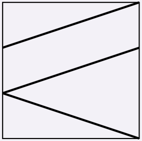


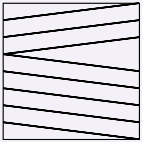
$\sigma_j \xrightarrow{\sqrt{n}}$  $\qquad \sigma_j \xrightarrow{n^{3/4}}$ 

Independence of limits at infinitely many scales

The tiered permutons $\mu_{p,q}$

q tiers; upper p increasing; remainder decreasing

$\mu_{2,3} =$ 

$\mu_{3,8} =$ 

Infinitely many limits

We can construct a sequence of permutations $(\zeta_j)_{j\in\mathbb{N}}$ such that, for each irreducible $p/q \in \mathbb{Q} \cap (0,1]$, we have

$$\zeta_j \xrightarrow{n^{p/q}} \mu_{p,q}.$$

B., Independence of permutation limits at infinitely many scales. arXiv:2005:11568

Main result

We may choose limits independently, in two directions, at a countably infinite number of scales:

Theorem

Let $\{f_t : t \in \mathbb{N}\}$ be any set of scaling functions totally ordered by domination.†

For each $t \in \mathbb{N}$, let Ξ_t and Ξ'_t be any scale-specific limits.

Then there exists a sequence of permutations $(\tau_j)_{j \in \mathbb{N}}$ which converges to Ξ_t at scale f_t for each t, such that $(\tau_j^{-1})_{j \in \mathbb{N}}$ converges to Ξ'_t at scale f_t for each t.

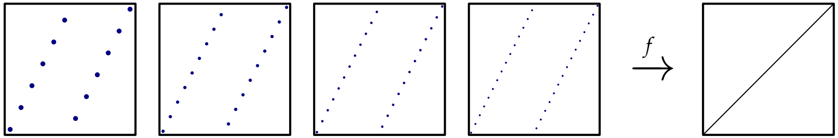
---

†For every $f_i \neq f_j$, either $f_i \ll f_j$ or $f_j \ll f_i$.

Questions

Which random permutons are scale-specific limits?

Sometimes we get the same limit at every scale f such that $1 \ll f \ll n$.



Definition (scalable convergence)

If $|\sigma_j| \to \infty$, then $(\sigma_j)_{j\in\mathbb{N}}$ is scalably convergent if, for every pattern π there exists ρ_π such that $\rho_f(\pi, \sigma_j) \to \rho_\pi$ for every f such that $1 \ll f \ll n$.

Can scalable limits be represented by random *tiered* permutons?