Universal 321-avoiding permutations

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  Today: at least $\Omega(n^\alpha)$, for any $\alpha < 2$. (A., Lozin, Malyshev, 2020)

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Theorem (Lozin, Rudolf, 2007)

Any bipartite permutation graph on n vertices can be embedded into $H_{n,n}$:

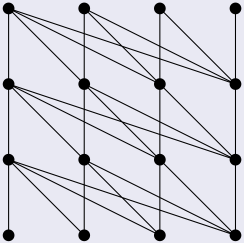

Figure: The graph $H_{4,4}$.

Sketch of the proof

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Find S of size linear in n, and for each $d \in S$, find two linearly sized subsets $X, Y \subseteq V(H)$ with $\text{dist}_H(x, y) = d$ for each $x \in X, y \in Y$.

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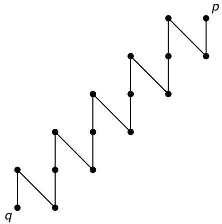

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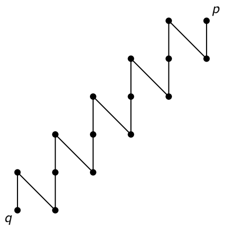

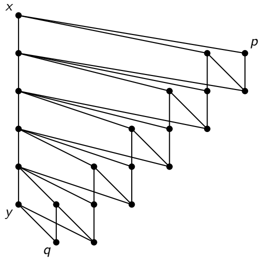

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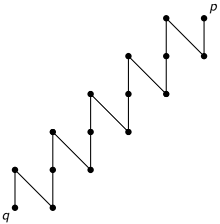

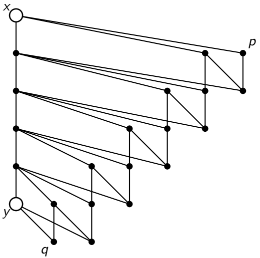

Thank you for your attention!

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