Rowmotion on 321-avoiding Permutations

Ben Adenbaum
(joint work with Sergi Elizalde)

Dartmouth College

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Definitions

Definition ($\mathbf{A}^n$ root poset)

Let $\mathbf{A}^n$ denote the positive root poset of the A_n root system. Equivalently this is the set of intervals with endpoints in $\{1, 2, \ldots, n\}$ ordered by inclusion.

Definition (Exc Bijection)

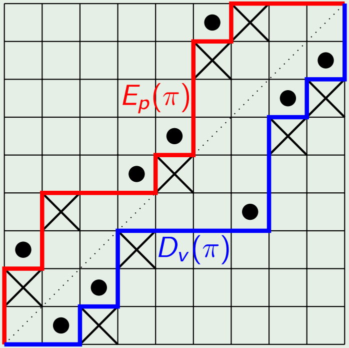
Let $\pi \in \mathcal{S}_n(321)$. Define $Exc(\pi)$ to be the antichain $\{[i, \pi(i) - 1] \mid i \text{ is an excedance of } \pi\}$ of $\mathbf{A}^{n-1}$.

Definition (E_p, E_v, and D_v Bijections)

For $\pi \in \mathcal{S}_n(321)$, define $E_p(\pi)$ to be the upper Dyck path whose peaks occur at the weak excedances of π, $E_v(\pi)$ to be the upper Dyck path whose valleys occur at the excedances of π. Similarly define $D_v(\pi)$ to be the lower Dyck path whose valleys occur at the deficiencies of π.

Example of Maps

Example



The red path is E_p of the permutation $\pi = 241358967$, as well as E_v of the permutation $\sigma = 312569478$, while the blue path is D_v of π. The crosses strictly above the diagonal correspond to $\text{Exc}(\pi)$ and the dots strictly above the diagonal correspond to $\text{Exc}(\sigma)$.

Rowmotion

Definition (Antichain rowmotion)

Let A be an antichain of the poset P. Then $\rho_{\mathcal{A}}(A)$ is defined to be the minimal elements of the complement of the order ideal generated by A.

Definition (Rowmotion on 321-avoiding Permutations)

Let $\pi \in \mathcal{S}_n(321)$. Define

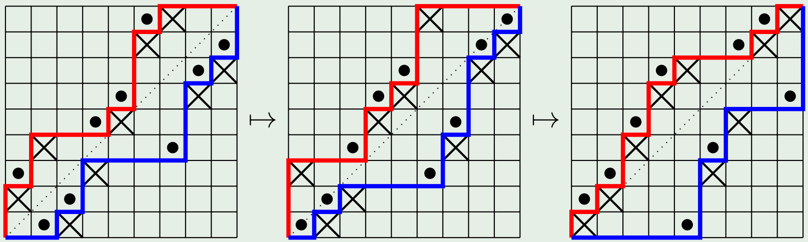
$$\rho_{\mathcal{S}}(\pi) = \mathrm{Exc}^{-1} \circ \rho_{\mathcal{A}} \circ \mathrm{Exc}(\pi) = E_v^{-1} \circ E_p(\pi)$$

Definition (Homomesy)

A statistic on a set S is *homomesic* under a group action if its average on each orbit is constant. More specifically, the statistic is said to be *c-mesic* if its average over each orbit is c.

Example of Rowmotion

Example (Rowmotion for $\pi = 241358967$)



Rowmotion starting at the 321-avoiding permutation $\pi = 241358967$. In the diagram on the left, the crosses represent the elements of π while the dots represent the elements of $\sigma = \rho_{\mathcal{S}}(\pi) = 312569478$.

Some Permutation Statistics

Definition (h_i Statistic)

For $\pi \in \mathcal{S}_n(321)$, define $h_i(\pi)$, for $1 \leq i \leq n-1$ by
$h_i(\pi) = |\{j \in [i] : \pi(j) = i+1\}| + \chi_{\pi(i)>i} = \chi_{\pi^{-1}(i+1)<i+1} + \chi_{\pi(i)>i}$
where χ_B the indicator function for the statement B.

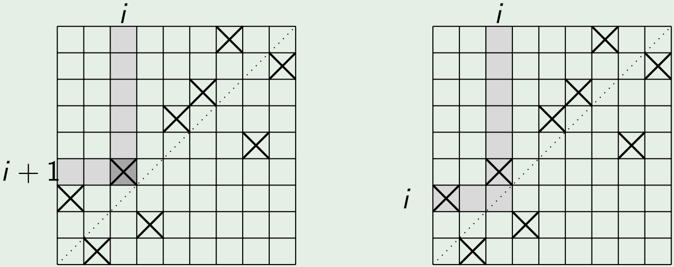
Definition (ℓ_i Statistic)

Define the statistics ℓ_i, where $1 \leq i \leq n$ and $\pi \in \mathcal{S}_n(321)$, by letting

$$\ell_i(\pi) = |\{j \in [i] : \pi(j) = i\}| + \chi_{\pi(i)>i} = \chi_{\pi^{-1}(i)\leq i} + \chi_{\pi(i)>i}.$$

Examples of h_i and ℓ_i Statistics

Example



Visualization of the statistics h_i (left) and ℓ_i (right) on the permutation $\pi = 314267958$, as the number of crosses in the shaded squares of the permutation array, for $i = 3$. The darker square at the corner of the diagram on the left is counted twice. In this example, $h_3(\pi) = \ell_3(\pi) = 2$.

Results

Theorem

The h_i statistic is 1-mesic under $\rho_{\mathcal{S}}$ for $1 \leq i \leq n-1$.

Theorem

The ℓ_i statistic is 1-mesic under $\rho_{\mathcal{S}}$ for $1 \leq i \leq n$.

Corollary

The fixed point statistic, denoted by fp, *is 1-mesic under $\rho_{\mathcal{S}}$.*

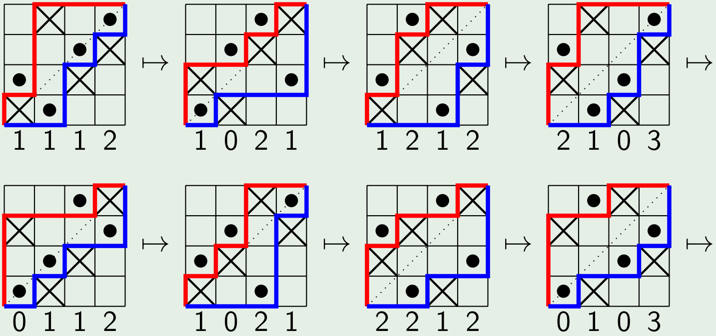
Theorem

For all $\pi \in \mathcal{S}_n(321)$,

$$\operatorname{sgn}(\rho_{\mathcal{S}}(\pi)) = \begin{cases} -\operatorname{sgn}(\pi) & \text{if } n \text{ is even,} \\ \operatorname{sgn}(\pi) & \text{if } n \text{ is odd.} \end{cases}$$

More Examples

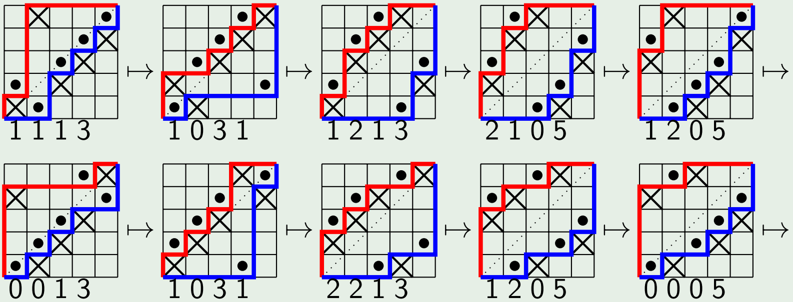
Example



The numbers below each diagram are the values $\ell_2(\pi)$, $h_2(\pi)$, $\mathrm{fp}(\pi)$ and $\mathrm{inv}(\pi)$, from left to right.

More Examples (cont)

Example



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