Permutation Patterns 2021

15th & 16th June 2021

A very warm welcome to the **Permutation Patterns 2021 Virtual Workshop**, hosted by the Combinatorics Group at the University of Strathclyde.

Programme

All workshop activities will take place online. Registered participants will receive a Zoom link for the sessions, along with links to the pre-recorded keynote addresses so they can be watched asynchronously.

There will be four sessions on both days. During Sessions A, B and D (one hour), six speakers will each give a five minute presentation and then take questions. During Session C (thirty minutes), one of the keynote speakers will give a 10–15 minute summary of their address, followed by an opportunity for discussion. The full pre-recorded keynote address will be streamed online during the break immediately prior to Session C for the benefit of those who weren't able to watch it earlier.

Schedule

The synchronous sessions will take place during the late afternoon and evening in Europe. The full schedule is on the following two pages, with links to all the abstracts. Session times use British Summer Time (UTC+1). Here are the times in some other locations.

	Session A	Session B	Session C	Session D
US Pacific	08.00	09.30	11.45	12.30
US Central	10.00	11.30	13.45	14.30
US Eastern	11.00	12.30	14.45	15.30
UK & Ireland	16.00	17.30	19.45	20.30
Central Europe	17.00	18.30	20.45	21.30
Israel	18.00	19.30	21.45	22.30
India	20.30	22.00	00.15	01.00
China	23.00	00.30	02.45	03.30
Eastern Australia	01.00	02.30	04.45	05.30
New Zealand	03.00	04.30	06.45	07.30

Proceedings

A special issue of *Enumerative Combinatorics and Applications* (ECA) will be published to mark the conference. Full papers on any topic related to permutation patterns, broadly interpreted (and not restricted to results presented at the conference), are welcome to be submitted for consideration. Papers will be refereed in accordance with the usual standards expected of ECA. Submissions should be emailed to sergey.kitaev@strath.ac.uk by 1st December 2021.

Tuesday 15th June 2021

UK time (UTC+1)

Welcome 16.00–16.05

Session A: Algorithms and computational complexity 16.05–17.05

Chair: Torsten Mütze

An automatic direct enumeration of $\mathrm{Av}(1342)$ — Émile Nadeau (p.89)

Automated bijections with Combinatorial Exploration — Jon Steinn Eliasson (p.27)

Pattern-avoiding rectangulations and permutations — Arturo Merino (p.87)

Counting small patterns and testing for independence — Chaim Even-Zohar (p.34)

Hardness of $\mathcal{C}$-permutation pattern matching — Michal Opler (p.90)

Sorting time of permutation classes — Vít Jelínek (p.56)

Session B: Classical avoidance and pattern densities 17.30–18.30

Chair: Jeff Liese

Permutations with exactly one copy of a monotone pattern of length k, and a generalization — Alex Burstein (p.17)

Permutations avoiding sets of patterns with long monotone subsequences — Miklós Bóna (p.13)

Universal 321-avoiding permutations — Bogdan Alecu (p.10)

Layered permutations and their density maximisers — Adam Kabela (p.63)

Feasible regions and permutation patterns — Raul Penaguiao (p.93)

Permutation limits at infinitely many scales — David Bevan (p.11)

Conference photo

Break 18.45–19.45

Lucas Gerin's pre-recorded keynote address will be streamed during the break.

Session C: Keynote address 19.45–20.15

Chair: Lara Pudwell

Patterns in substitution-closed permutations: a probabilistic approach — Lucas Gerin (p.45)

Session D: Probability 20.30–21.30

Chair: Tony Mendez

Increasing subsequences in random separable permutations — Valentin Feray (p.40)

Fixed points of permutations avoiding increasing patterns — Erik Slivken (p.100)

A simple proof of a CLT for vincular permutation patterns for conjugation invariant permutations — Mohamed Slim Kammoun (p.69)

Two equators of the permutohedron — Joshua Cooper (p.23)

A probabilistic approach to generating trees — Jacopo Borga (p.14)

The density of Costas arrays decays exponentially — Lutz Warnke

Wednesday 16th June 2021

UK time (UTC+1)

Session A: Algebra, permutations and words 16.00–17.00

Chair: Alex Burstein

Rowmotion on 321-avoiding permutations — Ben Adenbaum (p.4)

Permutation groups and permutation patterns — Erkko Lehtonen (p.75)

Spherical Schubert varieties and pattern avoidance — Christian Gaetz (p.44)

Some combinatorial results on smooth permutations — Shoni Gilboa (p.48)

On the existence of bicrucial permutations — Tom Johnston (p.62)

Qubonacci words — Sergey Kirgizov (p.74)

Session B: Permutation statistics 17.30–18.30

Chair: Bruce Sagan

Pattern avoidance in cyclic permutations — Jinting Liang (p.80)

The bivariate generating function on the statistics Peak and Des for cyclic permutations on $[n+2]$ which avoid the patterns $[1324]$ and $[1423]$ — James Schmidt (p.98)

Admissible pinnacle sets and ballot numbers — Rachel Domagalski (p.24)

A formula for counting the number of permutations with a fixed pinnacle set — Quinn Minnich (p.88)

A new algorithm for counting the admissible orderings of a pinnacle set — Alexander Sietsema (p.99)

New refinements of a classical formula in consecutive pattern avoidance — Yan Zhuang (p.110)

Break 18.45–19.45

Luca Ferrari's pre-recorded keynote address will be streamed during the break.

Session C: Keynote address 19.45–20.15

Chair: Rebecca Smith

Sorting with stacks and queues: some recent developments — Luca Ferrari (p.42)

Session D: Permutations and patterns 20.30–21.30

Chair: Vince Vatter

Sorting with a popqueue — Lapo Cioni (p.18)

Triangular permutation matrices — Vadim Lozin (p.81)

On SIF permutations avoiding a pattern — Michael D. Weiner (p.108)

On pattern avoidance in matchings and involutions — Justin M. Troyka (p.101)

Bijections for derangements and pattern-avoiding inversion sequences — Sergi Elizalde (p.29)

Classical pattern-avoiding permutations of length 5 — Anthony Guttmann (p.54)

Closing 21.30–21.35

Rowmotion on 321-avoiding Permutations

Ben Adenbaum

benjamin.m.adenbaum.gr@dartmouth.edu

Dartmouth College

(This talk is based on joint work with Sergi Elizalde.)

We give a natural definition of *rowmotion* for 321-avoiding permutations, by translating, through bijections involving Dyck paths and the Lalanne–Kreweras involution, the analogous notion for antichains of the positive root poset of the $\mathbf{A}^{n-1}$ root system. We prove that some permutation statistics, such as the number of fixed points, are homomesic under rowmotion, meaning that they have a constant average over its orbits. Our proofs use a combination of tools from Dynamical Algebraic Combinatorics and structural properties of pattern-avoiding permutations.

1 Definitions

1.1 Permutations and lattice paths

Let $\mathcal{S}_n(321)$ denote the set of 321-avoding permutations of length n. We can represent $\pi \in \mathcal{S}_n(321)$ as an $n \times n$ array with crosses in squares $(i, \pi(i))$ for $1 \leq i \leq n$. We say that $(i, \pi(i))$ is an *excedance* (respectively weak excendance, deficiency, weak deficiency) if $\pi(i) > i$ (respectively $\pi(i) \geq i$, $\pi(i) < i$, $\pi(i) \leq i$).

It will be convenient to consider two different ways to draw Dyck paths. Let $\mathcal{D}_n^{\ulcorner}$ (resp. $\mathcal{D}_n^{\lrcorner}$) be the set of paths from $(0,0)$ to (n,n) with steps $N = (1,0)$ and $E = (0,1)$ that stay weakly above (resp. weakly below) the diagonal $y = x$.

There are several known bijections between 321-avoiding permutations and Dyck paths. For $\pi \in \mathcal{S}_n(321)$, we define $E_p(\pi) \in \mathcal{D}_n^{\ulcorner}$ to be the path whose peaks occur at the weak excedances of π, $E_v(\pi) \in \mathcal{D}_n^{\ulcorner}$ to be the path whose valleys occur at the excedances of π, and $D_v(\pi) \in \mathcal{D}_n^{\lrcorner}$ to be the path whose valleys occur at the weak deficiencies of π. These are all described in more detail in [2]. See the left of Figure 2 for an example of $E_p(\pi)$ and $D_v(\pi)$ for $\pi = 241358967$. The bijection that maps $E_p(\pi)$ to $D_v(\pi)$ is known as the *Lalanne–Kreweras involution* on Dyck paths [5, 6], which we denote by LK.

1.2 Rowmotion and rowvacuation

Suppose that P is a finite poset. Define $\mathcal{A}(P)$ to be the set of antichains of P. *Antichain rowmotion* is the map $\rho_{\mathcal{A}} : \mathcal{A}(P) \to \mathcal{A}(P)$ defined as follows. For an antichain A, let $\rho_{\mathcal{A}}(A)$ be the minimal elements of the complement of the order ideal generated by A.

There is an equivalent way of defining rowmotion as a composition of antichain toggles, as first studied in [1] and later expanded upon in [8, 9]. Specifically, for $p \in P$, the antichain toggle associated to p is the map $\tau_p : \mathcal{A}(P) \to \mathcal{A}(P)$ defined by

$$\tau_p(A) = \begin{cases} A \setminus \{p\} & \text{if } p \in A, \\ A \cup \{p\} & \text{if } p \notin A \text{ and } A \cup \{p\} \in \mathcal{A}(P), \\ A & \text{otherwise.} \end{cases}$$

In other words, τ_p removes or adds the element p, with the caveat that it can only be added if the resulting set is an antichain. Then, for any linear extension $(p_1, \dots, p_k)$ of P, one can show that $\rho_{\mathcal{A}} = \tau_{p_k}\tau_{p_{k-1}} \dots \tau_{p_1}$.

Next we define *rowvacuation*, which was first studied in [3]. When P is graded of rank r, let $\tau_i = \prod_{p \in P_i} \tau_p$ where P_i is the set of elements of P of rank i. This product is well defined, as two antichain toggles commute if and only if the associated elements are incomparable [9, Lemma 3.12]. The rowvacuation map is then defined as $\tau_r(\tau_r\tau_{r-1}) \dots (\tau_r\tau_{r-1} \dots \tau_1\tau_0)$.

When studying rowmotion, it is common to look for statistics that exhibit a property called *homomesy* [7]. Given a set S and a bijection $\tau : S \to S$ so that each orbit of the action of τ on S has finite order, we say that a statistic on S is *homomesic* under this action if its average on each orbit is constant. More specifically, the statistic is said to be *c-mesic* if its average over each orbit is c.

We will restrict our attention to the positive root poset for the $\mathbf{A}^{n-1}$ root system, which can be described as the set of intervals $\{[i,j] | i,j \in [n], i \le j\}$ ordered by inclusion. This poset is graded with rank function given by $\mathrm{rk}([i,j]) = j - i$. The set $\mathcal{A}(\mathbf{A}^{n-1})$ of antichains of this poset is in bijection with $\mathcal{D}_n$. Let us describe two such bijections. The first one, which we call Path, maps the antichain $\{[i_1, j_1], \dots, [i_k, j_k]\}$ to the path in $\mathcal{D}_n^{\ulcorner}$ with high peaks (namely, peaks whose distance from the diagonal is greater than 1) at the coordinates $\{(i_1 - 1, j_1 + 1), \dots, (i_k - 1, j_k + 1)\}$. The second bijection, denoted by $\mathrm{ant} : \mathcal{D}_n^{\ulcorner} \to \mathcal{A}(\mathbf{A}^{n-1})$, takes a Dyck path to the antichain whose elements correspond to the valleys of the Dyck path in the same way. See Figure 1 for an example of these bijections. It has recently been shown by Hopkins and Joseph [4] that, via the Path correspondence between antichains and Dyck paths, rowvacuation on $\mathcal{A}(\mathbf{A}^{n-1})$ coincides with the Lalanne–Kreweras involution on $\mathcal{D}_n$.

Putting this all together we can define rowmotion on 321-avoiding permutations as follows. First consider the map $\mathrm{Exc} : \mathcal{S}_n(321) \to \mathcal{A}(\mathbf{A}^{n-1})$ where, for $\pi \in \mathcal{S}_n(321)$, we define $\mathrm{Exc}(\pi)$ to be the antichain $A = \{[i, \pi(i) - 1] \,|\, i \text{ is an excedance of } \pi\}$. It can be shown that this map is a bijection, because a 321-avoiding permutation is uniquely determined by the positions and the values of its excedances, which always form an increasing sequence.

Definition 1. *Rowmotion on 321-avoiding permutations is the map* $\rho_{\mathcal{S}} : \mathcal{S}_n(321) \to \mathcal{S}_n(321)$ *defined by* $\rho_{\mathcal{S}} = \mathrm{Exc}^{-1} \circ \rho_{\mathcal{A}} \circ \mathrm{Exc}$.

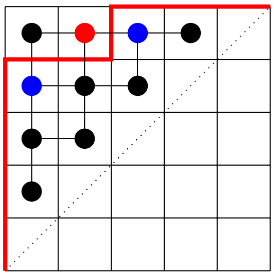


Figure 1: An example of the two bijections from antichains of $\mathbf{A}^4$ to Dyck paths in $\mathcal{D}_5$. The bijection Path maps the antichain $\{[1,3],[3,4]\}$, colored in red, to the path in the figure, which in turn is mapped by ant to the antichain $\{[2,4]\}$, colored in red.

It can be shown that, equivalently, rowmotion can be described in terms of the bijections to Dyck paths as $\rho_{\mathcal{S}} = E_v^{-1} \circ E_p$. See Figure 2 for examples of this map.

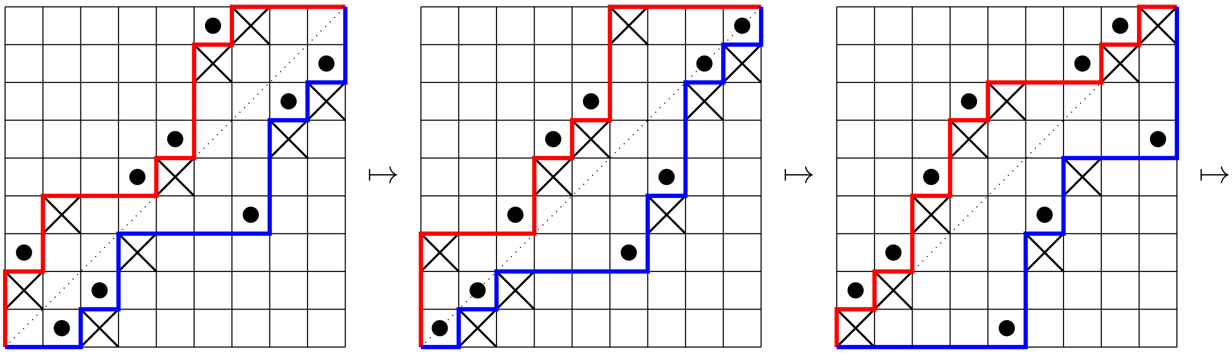


Figure 2: Rowmotion starting at the 321-avoiding permutation $\pi = 241358967$. In the diagram on the left, the crosses represent the elements of π, the red path is $U = E_p(\pi)$, the blue path is $L = D_v(\pi) = \mathrm{LK}(U)$, the dots represent the elements of $\sigma = \rho_{\mathcal{S}}(\pi) = 312569478$, the dots strictly above the diagonal correspond to the elements of the antichain $A = \mathrm{ant}(U) = \mathrm{Exc}(\sigma)$, and the crosses strictly below the diagonal correspond to the elements of $\mathrm{LK}(A)$.

2 Homomesies under $\rho_{\mathcal{S}}$

In this section we will show that certain statistics on 321-avoiding permutations exhibit homomesy under the action of $\rho_{\mathcal{S}}$. The first such statistic is the number of fixed points of a permutation π, denoted by $\mathrm{fp}(\pi) = |\{i : \pi(i) = i\}|$. It is interesting to note that, despite being a natural statistic on permutations, it does not translate to a natural statistic on antichains via the bijection between $\mathcal{S}_n(321)$ and $\mathcal{A}(\mathbf{A}^{n-1})$ described above, which is why it has not been studied in the literature on antichain rowmotion.

The first tool we will use to prove homomesy of fp is the statistic h_i introduced by Hopkins and Joseph [4]. It is defined on antichains $A \in \mathcal{A}(\mathsf{A}^{n-1})$ as

$$h_i(A) = \sum_{j=1}^{i} 1_{[j,i]}(A) + \sum_{j=i}^{n-1} 1_{[i,j]}(A),$$

for $1 \le i \le n-1$, where $1_{[j,i]}(A)$ is the indicator function that equals 1 if $[j,i] \in A$ and 0 otherwise. For $\pi \in \mathcal{S}_n(321)$, define $h_i(\pi)$ by $h_i(\mathrm{Exc}(\pi))$. In terms of the permutation $\pi \in \mathcal{S}_n(321)$, the statistic h_i can be expressed as

$$h_i(\pi) = |\{j \in [i] : \pi(j) = i+1\}| + \chi_{\pi(i)>i} = \chi_{\pi^{-1}(i+1)<i+1} + \chi_{\pi(i)>i}.$$

where χ_B is defined to be 1 if the statement B is true and 0 otherwise. See the left of Figure 3 for a visualization of this statistic.

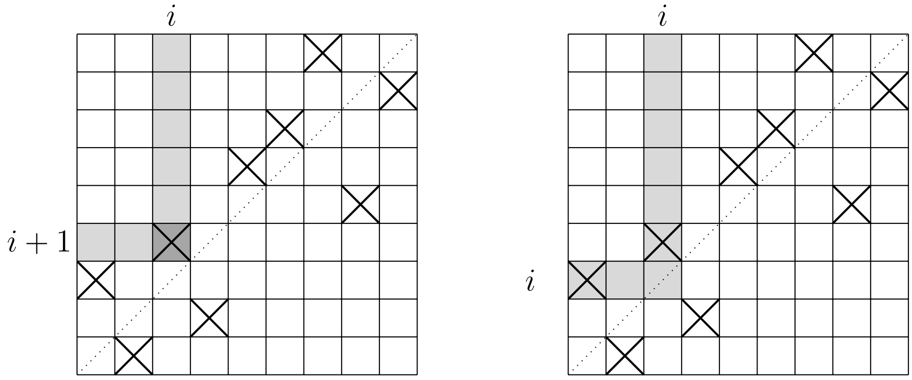


Figure 3: Visualization of the statistics h_i (left) and ℓ_i (right) on the permutation $\pi = 314267958$, as the number of crosses in the shaded squares of the permutation array, for $i = 3$. The darker square at the corner of the diagram on the left is counted twice. In this example, $h_3(\pi) = \ell_3(\pi) = 2$.

Hopkins and Joseph prove the following result for antichains, which we translate here in terms of 321-avoiding permutations.

Theorem 2 ([4])**.** *For $1 \le i \le n-1$, the statistic h_i is 1-mesic under the action of $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

See Figure 4 for a computation of $h_2(\pi)$ over an entire orbit for $n = 4$ and one for $n = 5$. As Theorem 2 asserts, the average of this statistic is 1 over each orbit.

Next we define the statistics ℓ_i, where $1 \le i \le n$ and $\pi \in \mathcal{S}_n(321)$, by letting

$$\ell_i(\pi) = |\{j \in [i] : \pi(j) = i\}| + \chi_{\pi(i)>i} = \chi_{\pi^{-1}(i)\le i} + \chi_{\pi(i)>i}.$$

See the right of Figure 3 for a visualization of this statistic. One of our main results is the following.

Theorem 3. *For $1 \le i \le n$ the statistic ℓ_i is 1-mesic under the action of $\rho_{\mathcal{S}}$ on $\mathcal{S}_n(321)$.*

As an example, we see that the average of the statistic ℓ_2 is 1 in the rowmotion orbits shown in Figure 4.

An important ingredient of our proof of Theorem 3 is the observation that

$$(\rho_{\mathcal{S}} \circ E_p^{-1} \circ \mathrm{LK} \circ E_p)(\pi) = \pi^{-1} \tag{1}$$

for every $\pi \in \mathcal{S}_n(321)$. The following result about fixed points in permutations is a consequence of Theorems 2 and 3.

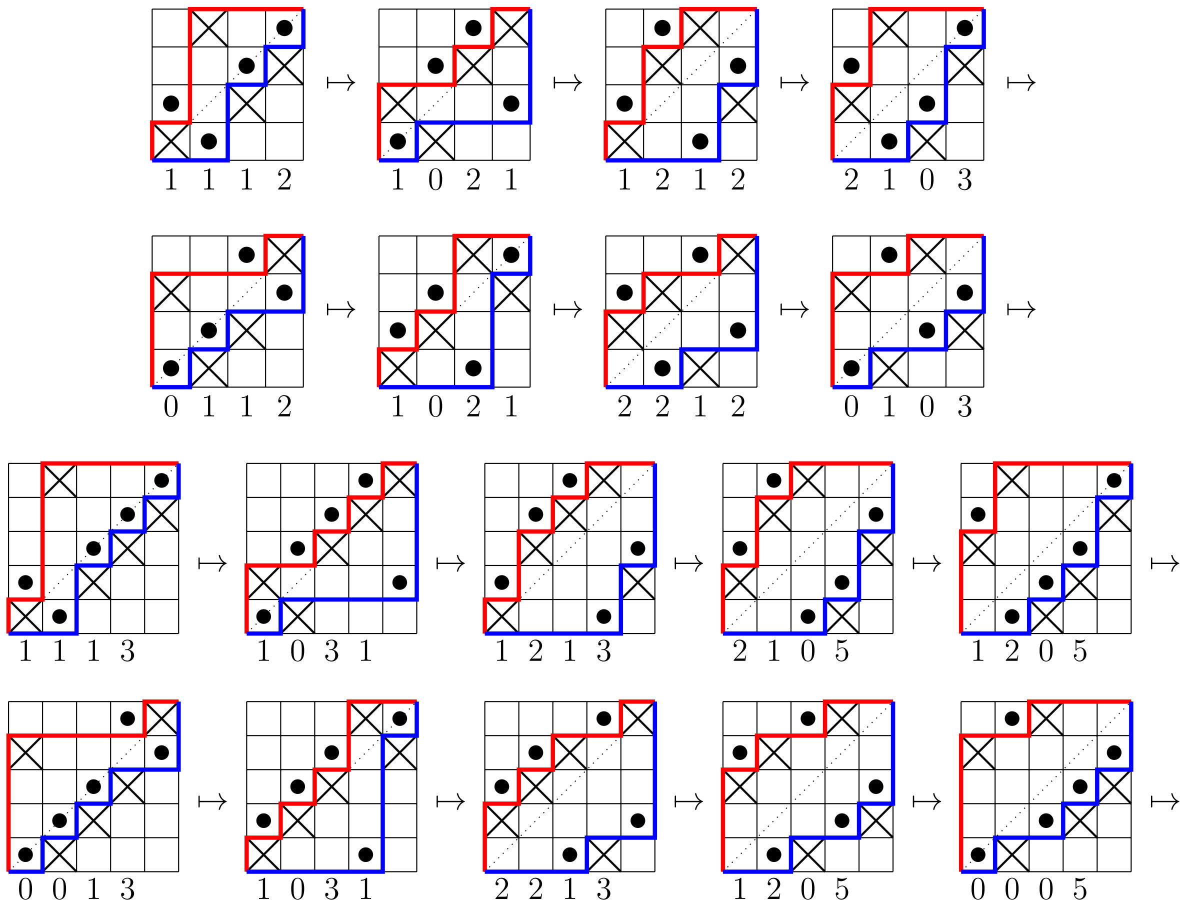


Figure 4: The rowmotion orbits containing 1423 (above) and 15234 (below). The numbers below each diagram are the values $\ell_2(\pi)$, $h_2(\pi)$, $\mathrm{fp}(\pi)$ and $\mathrm{inv}(\pi)$, from left to right.

Corollary 4. *The statistic* fp *is 1-mesic under the action of* $\rho_{\mathcal{S}}$ *on* $\mathcal{S}_n(321)$*.*

Proof. First, observe that we can express fp in terms of the statistics h_i and ℓ_i as $\mathrm{fp} = (\sum_{i=1}^{n} \ell_i) - (\sum_{i=1}^{n-1} h_i)$. Now we use Theorems 2 and 3, and the fact that linear combinations of homomesic statistics are homomesic. $\square$

As an example, the average of the statistic fp is 1 in the rowmotion orbits in Figure 4.

3 Another property of $\rho_{\mathcal{S}}$

Our other main result describes how $\rho_{\mathcal{S}}$ changes the sign of a 321-avoiding permutation.

Theorem 5. *For all* $\pi \in \mathcal{S}_n(321)$*,*

$$\mathrm{sgn}(\rho_{\mathcal{S}}(\pi)) = \begin{cases} -\,\mathrm{sgn}(\pi) & \text{if } n \text{ is even,} \\ \mathrm{sgn}(\pi) & \text{if } n \text{ is odd.} \end{cases}$$

Recall that if $\mathrm{inv}(\pi)$ denotes the number of inversions of π, then $\mathrm{sgn}(\pi) = (-1)^{\mathrm{inv}(\pi)}$. In the orbits in Figure 4, we see that the parity of $\mathrm{inv}(\pi)$, and thus the value of $\mathrm{sgn}(\pi)$, is alternating for $n = 4$ and constant for $n = 5$.

Our proof of Theorem 5 uses the fact that we can express the number of inversions of $\pi \in \mathcal{S}_n(321)$ directly in terms of the antichain $A = \mathrm{Exc}(\pi)$. Specifically, if $(i, \pi(i))$ is an excedance of $\pi \in \mathcal{S}_n(321)$, then the number of inversions of the form (i, j) where $i < j$ and $\pi(i) > \pi(j)$ is equal to $\pi(i) - i$. To see this, suppose that $\pi(i) = i + k$ for some $k > 0$. Then there are k values of j for which $i < j$ and $i + k > \pi(j)$, as otherwise π would not be 321-avoiding. In this case, the element in $A = \mathrm{Exc}(\pi)$ corresponding to the excedance at i is $[i, i + k - 1]$, and its rank in the poset $\mathbf{A}^{n-1}$ is $\mathrm{rk}([i, i + k - 1]) = k - 1$. Putting this together, we have

$$\mathrm{inv}(\pi) = \sum_{[j,k] \in \mathbf{A}^{n-1}} (\mathrm{rk}([j, k]) + 1) 1_{[j,k]}(A),$$

where $A = \mathrm{Exc}(\pi)$.

The other ingredients in our proof of Theorem 5 are [4, Lemma 3.1], which provides a recurrence relating LK and $\rho_{\mathcal{A}}$, and equation (1).

[1] P.J. Cameron and D.G. Fon-Der-Flaass, Orbits of antichains revisited, European J. Combin. 16 (1995), no. 6, 545–554.

[2] S. Elizalde, Fixed points and excedances in restricted permutations, Electron. J. Combin. 18, #P29, 2012.

[3] S. Hopkins, Order polynomial product formulas and poset dynamics, preprint available as 2006.01568 on arXiv, 2020.

[4] S. Hopkins and M. Joseph, The birational Lalanne–Kreweras involution, preprint available as 2012.15795 on arXiv, 2020.

[5] G. Kreweras. Sur les éventails de segments. Cahiers du Bureau universitaire de recherche opérationnelle Série Recherche 15:3–41, 1970.

[6] J.-C. Lalanne. Une involution sur les chemins de Dyck. European J. Combin., 13(6):477–487, 1992.

[7] J. Propp and T. Roby, Homomesy in products of two chains, Electron. J. Combin. 22 (2015), no. 3, Paper 3.4, 29 pp.

[8] J. Striker and N. Williams, Promotion and rowmotion, European J. Combin. 33 (2012), no. 8, 1919–1942.

[9] J. Striker, Rowmotion and Generalized Toggle Groups, Discrete Math. Theor. Comput. Sci. 20 (2018), no. 1, Paper No. 17, 26 pp.

Universal 321-avoiding permutations

Bogdan Alecu

B.Alecu@warwick.ac.uk

University of Warwick

(This talk is based on joint work with Vadim Lozin and Dmitriy Malyshev.)

How big does a 321-avoiding permutation need to be in order to contain all 321-avoiding permutations on n elements as subpatterns? Variations on this question have been around for some time. Arratia [2] asks this question about general (that is, without the 321-avoiding condition) permutations and general "superpatterns", and remarks that a simple counting argument implies an asymptotic lower bound of $(n/e)^2$. If we look instead for *general* permutations containing all 321-avoiding permutations, examples of size $\Theta(n^{3/2})$ have been constructed [4].

The variant of the question that we will discuss (in which we require the superpattern to also avoid 321) was asked in [3], in which an upper bound of n^2 appears. However, as far as we are aware, this bound is not known to be tight. In this talk, we will present an almost-quadratic lower bound proved in [1]. More specifically, we will sketch a proof of the fact that, for any $\alpha < 2$, the size of a 321-avoiding superpattern must grow asymptotically as $\Omega(n^\alpha)$.

[1] B. Alecu, V. Lozin and D. Malyshev, Critical properties of bipartite permutation graphs. *European J. Combin.*, Submitted, Preprint available at `https://arxiv.org/abs/2010.14467`.

[2] R. Arratia, On the Stanley-Wilf Conjecture for the Number of Permutations Avoiding a Given Pattern. *Electron. J. Combin.* **6** (1999), Article N1.

[3] A. Atminas, V. Lozin, S. Kitaev, A. Valyuzhenich, Universal graphs and universal permutations. *Disc. Math., Algorithms and Applications* (4) **5** (2013).

[4] M. J. Bannister, W. E. Devanny, D. Eppstein, Small Superpatterns for Dominance Drawing, *2014 Proceedings of the Meeting on Analytic Algorithmics and Combinatorics (ANALCO)* (2014), pp. 92–103.

Permutation limits at infinitely many scales

David Bevan

david.bevan@strath.ac.uk

University of Strathclyde

We introduce a natural notion of convergence for permutations at any specified scale, in terms of the density of patterns of restricted width. In this setting we prove that limits may be chosen independently at a countably infinite number of scales.

Let $\nu(\pi, \sigma)$ be the number occurrences of π in σ. Then the (*global*) *density* of π in σ, which we denote $\rho(\pi, \sigma)$, is $\nu(\pi, \sigma) / \binom{n}{k}$. Observe that $\rho(\pi, \sigma) = \mathbb{P}\big[\sigma(K) = \pi\big]$, where K is drawn uniformly from the k-element subsets of $[n]$, and $\sigma(K)$ denotes the permutation order-isomorphic to the image of K under σ.

We say that an occurrence $i_1 \leqslant \ldots \leqslant i_k$ of π in σ, has *width* $i_k - i_1 + 1$. Given a real number $f \in [k, n]$, let $\nu_f(\pi, \sigma)$ be the number of occurrences having width no greater than f. Then the density of π in σ *at scale* f, denoted $\rho_f(\pi, \sigma)$, is $\nu_f(\pi, \sigma) / \binom{n}{k}_f$, where

$$\binom{n}{k}_f = \sum_{w=k}^{\lfloor f \rfloor} (n - w + 1) \binom{w-2}{k-2}$$

is the number of k-element subsets of $[n]$ of width no greater than f. Thus, $\rho_f(\pi, \sigma) = \mathbb{P}\big[\sigma(K) = \pi\big]$, where K is drawn uniformly from the k-element subsets of $[n]$ of width no greater than f. Clearly, $\rho_n(\pi, \sigma)$ is the same as $\rho(\pi, \sigma)$.

The scale can be envisaged as specifying a zoom level or magnification, expressed as the horizontal extent of a window through which we inspect a permutation. We typically consider the scale f to be a function $f(n)$ of the length, n, of the host permutation σ, such as $\log n$, $\sqrt{n}$ or $n/\log n$. With a slight abuse of notation, we omit the argument when it is clear from the context.

We say that a function $f : \mathbb{N} \to \mathbb{R}^+$ is a *scaling function* if $1 \ll f \ll n$ and $f(n) \leqslant n$ for all n, where we write $f(n) \ll g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 0$. Note that we require scaling functions to tend to infinity and be sublinear. Our interest is in the behaviour of pattern density at different scales as n tends to infinity.

An infinite sequence $(\sigma_j)_{j\in\mathbb{N}}$ of permutations with $|\sigma_j| \to \infty$ is *globally convergent* if $\rho(\pi, \sigma_j)$ converges for every pattern π. To every convergent sequence of permutations one can associate an analytic limit object known as a *permuton*.

In an analogous manner to the definition of global convergence, we introduce a notion of convergence at a specified scale. Given a scaling function f, an infinite sequence $(\sigma_j)_{j\in\mathbb{N}}$ of permutations with $|\sigma_j| \to \infty$ is *convergent at scale f* if $\rho_f(\pi, \sigma_j)$ converges for every pattern π.

If $(\sigma_j)_{j\in\mathbb{N}}$ is convergent at scale f, then there exists an infinite vector $\Xi \in [0,1]^S$ (where S is the set of all permutations) such that $\rho_f(\pi, \sigma_j) \to \Xi_\pi$ for every $\pi \in S$. Note that, for any $k \geqslant 1$, we have $\sum_{\pi\in S_k} \Xi_\pi = 1$. We consider Ξ itself to be the limit of the sequence at scale f and call such a limit a *scale-specific limit*. Scale-specific limits are independent of the scale in the following sense:

Theorem 1. *Let Ξ be any scale-specific limit and f be any scaling function. Then there exists a sequence of permutations which converges at scale f to Ξ.*

We say that a set $\mathcal{F}$ of functions is *totally ordered by domination* if for every distinct $f, g \in \mathcal{F}$, either $f \ll g$ or $g \ll f$. Our main result is the following:

Theorem 2. *Let $\{f_t : t \in \mathbb{N}\}$ be any countably infinite set of scaling functions, totally ordered by domination, and for each $t \in \mathbb{N}$, let Ξ_t be any scale-specific limit. Then there exists a sequence of permutations which converges to Ξ_t at scale f_t for every $t \in \mathbb{N}$.*

For example, we can construct a sequence of permutations $(\zeta_j)_{j\in\mathbb{N}}$ such that, for each irreducible $p/q \in \mathbb{Q} \cap (0,1)$, a length k subpermutation of ζ_j of width at most $|\zeta_j|^{p/q}$ chosen uniformly at random is asymptotically almost surely the increasing permutation $12\ldots k$ if q is odd, and is asymptotically almost surely the decreasing permutation $k\ldots 21$ if q is even.

Open questions

In general, we believe that certain probability distributions over permutons (that is, certain *random permutons*) should suffice to model scale-specific limits. How can they be characterised?

Question 3. *Which random permutons are scale-specific limits?*

Sometimes, a sequence of permutations converges to the same limit at every scale. We say that an infinite sequence $(\sigma_j)_{j\in\mathbb{N}}$ of permutations with $|\sigma_j| \to \infty$ is *scalably convergent* if, for every pattern π, there exists ρ_π such that $\rho_f(\pi, \sigma_j)$ converges to ρ_π for every scaling function f. We call a limit of a scalably convergent sequence a *scalable limit*.

It seems likely that scalable limits can be characterised as probability distributions over permutons with a specific tiered structure.

Question 4. *Can scalable limits be represented by random tiered permutons?*

[1] David Bevan. Independence of permutation limits at infinitely many scales. arXiv:2005.11568.

Permutations avoiding sets of patterns with long monotone subsequences

Miklós Bóna

bona@ufl.edu

University of Florida

(This talk is based on joint work with Jay Pantone.)

We enumerate permutations that avoid all but one of the k patterns of length k starting with a monotone increasing subsequence of length $k-1$. We compare the size of such permutation classes to the size of the class of permutations avoiding the monotone increasing subsequence of length $k-1$.

A probabilistic approach to generating trees

Jacopo Borga

borgajacopo@gmail.com

University of Zurich

The theory of generating trees has been widely used to enumerate families of combinatorial objects, in particular, families of permutations. We give here an informal definition of generating trees.

A *generating tree* for a combinatorial class $\mathcal{C}$ is an infinite rooted tree whose vertices are the elements of $\mathcal{C}$ (each appearing exactly once in the tree) and such that the objects of size n are at level n (the level of a vertex being the distance from the root plus one). The children of some object $\square \in \mathcal{C}$ correspond to the objects obtained by adding a new *"atom"* to $\square$, i.e. a piece of object which makes the size increase by 1.

See Fig. 1 for an example in the case of $\{1423, 4123\}$-avoiding permutations.

Although generating trees first appeared in the literature in the context of permutations with forbidden patterns [10, 19, 20, 11], they were rigorously defined later, together with the ECO method (Enumerating Combinatorial Objects), in [4]. The latter is a technique that is based on a recursive construction of the objects of a combinatorial class and that provides a useful tool to establish enumerative results.

Generating trees have been used in the last 20 years to establish several enumerative results for various combinatorial classes of partitions, permutations, polyominoes and many other objects (see for instance [7, 17, 18, 14, 16, 5, 6, 12, 13, 1, 2]). We refer to [3] and to the Ph.D. thesis of Guerrini [15, Chapter 1] for two interesting presentations of generating trees and associated enumeration techniques through generating functions.

The goal of this talk is to introduce a new facet of generating trees encoding families of permutations, in order to establish probabilistic results instead of enumerative ones.

Fix a family of permutations $\mathcal{C}$ encoded by a generating tree. We focus on finding a way to construct a uniform permutation in $\mathcal{C}$. This is possible thanks to a new bijection between permutations in $\mathcal{C}$ and a specific family of walks in cones.

This way of constructing uniform permutations in $\mathcal{C}$ can be used as a building-block to determine various probabilistic results such as central limit theorems for consecutive occurrences of patterns in permutations, local limits, permuton limits, ... During the talk we will explore some of these results. For instance, the following results was established in [8]:

Theorem 1. *Let $\mathcal{C}$ be one of the following families of permutations*[1]

$$\begin{gathered}\mathrm{Av}(123), \quad \mathrm{Av}(132), \quad \mathrm{Av}(1423{,}4123), \quad \mathrm{Av}(1234, 2134), \quad \mathrm{Av}(1324, 3124), \\ \mathrm{Av}(2314, 3214), \quad \mathrm{Av}(2413, 4213), \quad \mathrm{Av}(3412, 4312), \quad \mathrm{Av}(213, \bar{2}\,\underline{31}), \quad \mathrm{Av}(213, \bar{2}^{o}\,\underline{31}).\end{gathered}$$

For all $n \in \mathbb{Z}_{>0}$, let $\boldsymbol{\sigma}_n$ be a uniform random permutation in $\mathcal{C}$ of size n. Then, for all patterns $\pi \in \mathcal{S}$, there exist constants μ_π and γ_π^2 such that the following central limit theorem holds

$$\frac{c\text{-}occ(\pi, \boldsymbol{\sigma}_n) - n\mu_\pi}{\sqrt{n}} \xrightarrow{d} \boldsymbol{\mathcal{N}}(0, \gamma_\pi^2),$$

where c-$occ(\pi, \boldsymbol{\sigma}_n)$ denotes the number of consecutive occurrences of the pattern π in $\boldsymbol{\sigma}_n$.

This talk is built on a series of works that include [8, 9] and some other works in preparation.

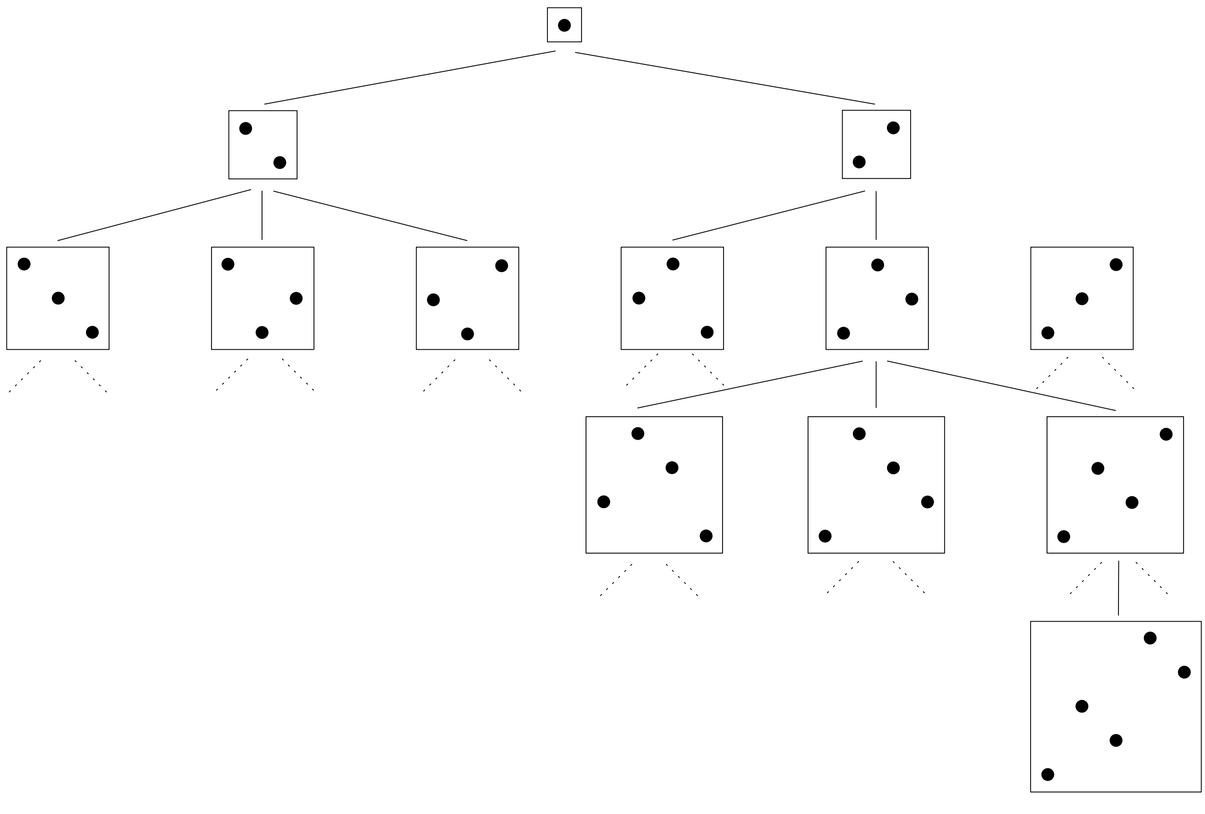


Figure 1: The generating tree for $\{1423, 4123\}$-avoiding permutations. Each permutation diagram is obtained by *appending a new dot on the right* of the diagram of the parent. We completely draw only the first three levels of the tree; for the fourth level, we only draw the children of the permutation 132, and for the fifth level only one of the children of the permutation 1324.

[1]We refer the reader to [8, Section 3.4] for the definition of the last two families of permutations in the list, which avoid generalized patterns.

[1] N. R. Beaton, M. Bouvel, V. Guerrini, and S. Rinaldi. Enumerating five families of pattern-avoiding inversion sequences; and introducing the powered Catalan numbers. *Theoret. Comput. Sci.*, 777:69–92, 2019.

[2] N. R. Beaton, M. Bouvel, V. Guerrini, and S. Rinaldi. Slicing of parallelogram polyominoes: Catalan, Schröder, Baxter, and other sequences. *Electron. J. Combin.*, 26(3):Paper 3.13, 36, 2019.

[3] C. Banderier, M. Bousquet-Mélou, A. Denise, P. Flajolet, D. Gardy, and D. Gouyou-Beauchamps. Generating functions for generating trees. *Discrete mathematics*, 246(1-3):29–55, 2002.

[4] E. Barcucci, A. Del Lungo, E. Pergola, and R. Pinzani. ECO: a methodology for the enumeration of combinatorial objects. *J. Differ. Equations Appl.*, 5(4-5):435–490, 1999.

[5] S. Burrill, S. Elizalde, M. Mishna, and L. Yen. A generating tree approach to k-nonnesting partitions and permutations. *Ann. Comb.*, 20(3):453–485, 2016.

[6] M. Bouvel, V. Guerrini, A. Rechnitzer, and S. Rinaldi. Semi-Baxter and strong-Baxter: two relatives of the Baxter sequence. *SIAM J. Discrete Math.*, 32(4):2795–2819, 2018.

[7] M. Bousquet-Mélou. Four classes of pattern-avoiding permutations under one roof: generating trees with two labels. *The electronic journal of combinatorics*, 9(2):19, 2003.

[8] J. Borga. Asymptotic normality of consecutive patterns in permutations encoded by generating trees with one-dimensional labels. *arXiv preprint:2003.08426 (to appear in Random Structures and Algorithms)*, 2020.

[9] J. Borga and M. Maazoun. Scaling and local limits of Baxter permutations and bipolar orientations through coalescent-walk processes. *arXiv preprint:2008.09086*, 2020.

[10] F. R. K. Chung, R. L. Graham, V. E. Hoggatt, Jr., and M. Kleiman. The number of Baxter permutations. *J. Combin. Theory Ser. A*, 24(3):382–394, 1978.

[11] S. Dulucq, S. Gire, and O. Guibert. A combinatorial proof of J. West's conjecture. *Discrete Math.*, 187(1-3):71–96, 1998.

[12] E. Duchi, V. Guerrini, and S. Rinaldi. A generating tree for permutations avoiding the pattern $122^{+}3$. *Fund. Inform.*, 163(1):21–39, 2018.

[13] E. Duchi. A code for square permutations and convex permutominoes. *ArXiv preprint:1904.02691*, 2019.

[14] S. Elizalde. Generating trees for permutations avoiding generalized patterns. *Ann. Comb.*, 11(3-4):435–458, 2007.

[15] V. Guerrini. *On enumeration sequences generalising Catalan and Baxter numbers.* PhD thesis, University of Zurich and Siena, 2017.

[16] J. B. Lewis. Generating trees and pattern avoidance in alternating permutations. *Electron. J. Combin.*, 19(1):Paper 21, 21, 2012.

[17] D. Marinov and R. Rodoicic. Generating trees of (reducible) 1324-avoiding permutations. 2003.

[18] V. Vatter. Finitely labeled generating trees and restricted permutations. *J. Symbolic Comput.*, 41(5):559–572, 2006.

[19] J. West. Generating trees and the Catalan and Schröder numbers. *Discrete Math.*, 146(1-3):247–262, 1995.

[20] J. West. Generating trees and forbidden subsequences. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 363–374, 1996.

Permutations with exactly one copy of a monotone pattern of length k, and a generalization

Alex Burstein

aburstein@howard.edu

Howard University

(This talk is based on joint work with Miklós Bóna.)

We construct an injection from the set of permutations of length n that contain exactly one copy of the decreasing pattern of length k to the set of permutations of length $n+2$ that avoid that pattern. We then prove that the generating function counting the former is not rational, and in the case when k is even and $k \geq 4$, it is not even algebraic. We extend our injection and our nonrationality result to a larger class of patterns.

Sorting with a popqueue

Lapo Cioni

lapo.cioni@unifi.it

Università degli Studi di Firenze

(This talk is based on joint work with Luca Ferrari.)

1 Preliminary notions and results

`Stacksort` is a classical and well-studied algorithm that attempts to sort an input permutation by (suitably) using a stack. It has been introduced and first investigated by Knuth [1], and it is one of the main responsible for the great success of the notion of *pattern* for permutations. Among the many research topics connected with `Stacksort`, one concerns the characterization and enumeration of the permutations sorted by two (or more) iterations of the algorithm. For example, West [3] described the permutations that are sortable by two iterations of `Stacksort` in terms of avoidance of a standard pattern and a barred pattern, and he also formulated a conjecture regarding their enumeration, which was later proved by Zeilberger [4].

We will address the same kind of problems for a different sorting device, namely a popqueue. A popqueue is a sorting device in which we can insert and extract elements, following some restrictions. Namely, these are the allowed operations:

- e: *enqueue*, insert the current element into the popqueue, in the rightmost position;
- p: *pop*, remove all the elements currently in the popqueue, from left to right, sending them into the output;
- b: *bypass*, send the current element into the output.

These operations resemble those of a queue. Indeed, the sole difference is the fact that *pop* removes all the elements instead of removing only the first one, and this is the reason for the name "popqueue".

Starting from an arbitrary input, we can output several different permutations, just by executing the operations in different order. Our interest is to sort the permutation, so we say that a permutation is *sortable* if there exists a sequence of operations that output the identity permutation. In particular, we want to give an optimal sorting algorithm, i.e. we want to describe an algorithm that is able to sort every sortable permutation. We also want our algorithm to give an output when the input permutation is not sortable, so that we can study which permutations are sortable by two iterations of the same algorithm.

We start by giving a necessary condition for a permutation to be sortable.

Proposition 1. *If a permutation $\pi \in S_n$ contains an occurrence of the patterns* 321 *or* 2413*, then it is not sortable using a popqueue.*

2 The algorithms *Min* and *Cons*

We provide two different sorting algorithms which, although similar, will give us remarkably different results when applied twice. The first algorithm is Min.

Algorithm. *Min*
input: a permutation $\pi = \pi_1 \cdots \pi_n$
output: a permutation $Min(\pi)$

for $i = 1, \ldots, n$ do:

- *if $Front(Q)$ is the minimal element not yet in the output (i.e., if $Front(Q)$ is smaller than all the unprocessed elements $\pi_i, \ldots, \pi_n$), then* pop *and* enqueue*;*
- *else compare π_i, $Back(Q)$ and $Front(Q)$;*
  - *if $Back(Q) < \pi_i$,* enqueue*;*
  - *otherwise, if $Front(Q) > \pi_i$, then* bypass*;*
  - *else,* pop *end* enqueue.

Finally, pop.

We will call $Min(\pi)$ the output of the algorithm Min on input π. The algorithm name, Min, comes from the first instruction, which empties the popqueue when the first element of the queue is the first element to be output. Min is an optimal sorting algorithm, as recorded in the next proposition.

Proposition 2. *Let $\pi \in S_n$. Then $Min(\pi) \neq id_n$ if and only if $321 \leq \pi$ or $2413 \leq \pi$.*

The second algorithm is $Cons$.

Algorithm. *Cons*
input: a permutation $\pi = \pi_1 \cdots \pi_n$
output: a permutation $Cons(\pi)$

for $i = 1, \ldots, n$ do:

- *if $\pi_i = Back(Q) + 1$, then* enqueue*;*
- *else, compare π_i and $Front(Q)$;*
  - *if $Front(Q) > \pi_i$, then* bypass*;*
  - *else,* pop *and* enqueue.

Finally, pop.

We will call $Cons(\pi)$ the output of the algorithm $Cons$ on input π. The algorithm name, $Cons$, comes from the first instruction, that only allows consecutive elements to be in the popqueue, and thus forces the content of the popqueue to be consecutive at all times. As for the Min algorithm, it is an optimal sorting algorithm.

Proposition 3. *Let* $\pi \in S_n$*. Then* $Cons(\pi) \neq id_n$ *if and only if* $321 \leq \pi$ *or* $2413 \leq \pi$*.*

We thus have two different optimal sorting algorithms, which follow different heuristics to sort π. It is easy to see that, althought they sort the same permutations, their output is different in general. For example, $Min(2413) = 1243$ but $Cons(2413) = 2134$. Still, it is important to note that both heuristics are reasonable in the context of permutation sorting. Indeed, neither Min nor $Cons$ create new inversions in their output. Also, we can say that if the first element of the popqueue is the minimal element that is not yet in the output, then we may as well empty the popqueue immediately, as Min does, because we would surely empty it before outputting any element from the input. On the other hand, if we were to allow non consecutive elements in the popqueue, then we would be sure that the output would not be sorted, so it is reasonable to maintain consecutivity fot elements inside the popqueue, as $Cons$ does. Both ideas give us optimal algorithms, whose outputs differs only for permutations that are not sortable, moreover the order in which the operations are executed may be different even for sortable permutations. One could be tempted to see what happens when using both heuristics, with an algorithm that only allows consecutive elements in the popqueue, but also empties it whenever the first element is the smallest not-outputted element. That would actually be an algorithm equivalent to $Cons$, because the output would be the same, although the operation would be performed in different orders by prioritizing the outputting of elements over the insertion into the popqueue.

We end this section by proving some properties of $Cons$ that will be useful later. Given a permutation $\pi = \pi_1 \cdots \pi_n$, an element π_i is called a LTR maximum if and only if it is greater than all π_j's for $j < i$. We have the following lemma.

Lemma 4. *Let* $\pi = \pi_1 \cdots \pi_n \in S_n$ *and apply Cons to it. Then an element* π_i *enters the popqueue if and only if it is a LTR maximum. Moreover, the relative order of the non-LTR maxima of* π *is preserved in* $Cons(\pi)$*. That is, if* a *appears before* b *in* π*, and neither are LTR maxima, then* a *appears before* b *even in* $Cons(\pi)$*.*

This lemma does not fully hold for Min, because although every LTR maxima does enter the popqueue during its execution, some non LTR maxima may also enter it (for example, the element 4 in 25143 enters the popqueue). It is interesting to note that the algorithm that sorts using a proper queue (that is, a queue instead of a popqueue) also have the property described in the lemma.

3 Two passes from a popqueue

In this section we will investigate what permutations are sortable when we apply twice each of the previous algorithms. This has been done for Stacksort by West [3], and the resulting set of sortable permutation is not a class, even if it can be expressed by the avoidance of a pattern and a barred pattern.

We start by defining the sets $Sort_M$ and $Sort_C$ of permutations sorted by two applications of Min and $Sort$, respectively, so that $Sort_M = \{\pi \in S \mid Min(Min(\pi)) = id_n, n \in \mathbb{N}\}$ and $Sort_C = \{\pi \in S \mid Cons(Cons(\pi)) = id_n, n \in \mathbb{N}\}$. The following remark shows the relationships between the two sets.

Remark 5. *Consider the permutations* 2431 *and* 35214*. Then* $2431 \in Sort_C \setminus Sort_M$ *and* $35214 \in Sort_M \setminus Sort_C$*. This shows that each of the two algorithms is able to sort some permutations that the other algorithm cannot sort.*

Clearly, there are permutations that both algorithms can (cannot) sort. For example, it is easy to see that none of them can sort permutations containing the pattern 4321; on the other hand both algorithms can sort the permutations 321 and 2413. As we have already remarked, both algorithms follow natural and easy rules, as Min outputs the smaller non outputted element whenever possible, while $Cons$ keeps the popqueue consecutive at all times. Still, the fact that there are two different optimal single-pass sorting algorithm that follow natural rules is by itself interesting. However the two sets $Sort_C$ and $Sort_M$ are different on a deeper level. In fact, we will see that $Sort_C$ is a permutation class, while $Sort_M$ is not.

Proposition 6. *The set* $Sort_M$ *is not a permutation class.*

Proof. Consider the permutation 241653. Then $Min(Min(241653)) = Min(124536) = 123456$, but $Min(Min(2431)) = Min(2413) = 1243$, and $2413 \leq 241653$. □

Conversely, the following proposition shows that $Sort_C$ is a class.

Proposition 7. *Let* $\pi \in S_n$*. Then* $Cons(Cons(\pi)) \neq id_n$ *if and only if* π *contains at least one of the following nine patterns:*

- 4321*;*
- 35241*;*
- 35214*;*
- 52413*;*
- 25413*;*
- 246153*;*
- 246135*;*
- 426153*;*
- 426135*.*

We do not know much about the sequences $|Sort_{M,n}|$ of the number of permutations of length n in $Sort_M$, and $|Sort_{C,n}|$ of the number of permutations of length n in $Sort_C$. Their first terms are 1, 2, 6, 22, 89, 379, 1660, 7380, 33113, 149059 and 1, 2, 6, 23, 99, 445, 2029, 9292, 42608, 195445, respectively, and do not appear in [2]. Still, it seems that $|Sort_{M,n}|$ is smaller than $|Sort_{C,n}|$ for every $n > 3$.

Conjecture 8. *For every $n > 3$, $|Sort_{M,n}| < |Sort_{C,n}|$.*

[1] D. Knuth, The Art of Computer Programming, Volume 1, *Boston: Addison-Wesley*, 1968.

[2] N. J. A. Sloane, editor, The On-Line Encyclopedia of Integer Sequences, published electronically at https://oeis.org.

[3] J. West, Permutations with forbidden subsequences and Stack sortable permutations, *PhD thesis, Massachusetts Institute of Technology*, 1990.

[4] D. Zeilberger, A proof of Julian West conjecture that the number of two-stack-sortable pemutations of length n is $2(3n)!/((n+1)!(2n+1)!)$. *Discrete math.* 102 (1992), no. 1, 85-93.

Two Equators of the Permutohedron

Joshua Cooper

cooper@math.sc.edu

University of South Carolina

(This talk is based on joint work with Rory Mitchell of Nvidia and Eibe Frank and Geoffrey Holmes of University of Waikato.)

Consider permutations $\sigma : [n] \to [n]$ written in one-line notation as a vector $\vec{\sigma} = (\sigma(1), \ldots, \sigma(n))$. The permutohedron $\mathcal{P}_n$ (in one standard presentation) is the convex hull of all such $\vec{\sigma}$, with center $\vec{p} = ((n+1)/2, \ldots, (n+1)/2)$. Then $\mathcal{P}_n$ has a geometric equator: permutations τ so that $(\vec{\tau} - \vec{p}) \cdot (\vec{\mathrm{id}}_n - \vec{p}) = 0$, where id_n is the identity permutation. But $\mathcal{P}_n$ also has a combinatorial equator: permutations τ in the middle level of the weak Bruhat order, i.e., for which $\mathrm{inv}(\tau) = \frac{1}{2}\binom{n}{2}$. We ask: how close are these two equators to each other? This question arose in connection with a problem in machine learning concerned with estimating so-called Shapley values by sampling families of permutations efficiently.

The most interesting special case of our main result is that, for permutations τ close to the geometric equator, i.e., for which $(\vec{\tau} - \vec{p}) \cdot (\vec{\mathrm{id}}_n - \vec{p}) = o(1)$, the number of inversions satisfies

$$\left| \frac{\mathrm{inv}(\tau)}{\binom{n}{2}} - \frac{1}{2} \right| \leq \frac{1}{4} + o(1)$$

and, furthermore, the quantity $1/4$ above cannot be improved beyond $1/2 - 2^{-5/3} \approx 0.185$. The proof is not difficult but uses an amusing application of bubble sorting. We discuss this and a more general and precise version of the result for any ratio $\mathrm{inv}(\tau)/\binom{n}{2}$.

Admissible Pinnacle Sets and Ballot Numbers

Rachel Domagalski

domagal9@msu.edu

Michigan State University

(This talk is based on joint work with Jinting Liang, Quinn Minnich, Bruce Sagan, James Schmidt, and Alexander Sietsema.)

Let $\mathbb{N}$ and $\mathbb{P}$ be the nonnegative and positive integers, respectively. Given $m, n \in \mathbb{P}$ we use the notation $[m, n] = \{m, m+1, \ldots, n\}$ for the interval they define. We abbreviate $[1, n]$ to $[n]$. Let $\mathfrak{S}_n$ be the symmetric group of all permutations $\pi = \pi_1\pi_2 \ldots \pi_n$ of $[n]$ written in one-line notation. An important statistic on $\mathfrak{S}_n$ is the *peak set* of a permutation π which is defined as

$$\operatorname{Pk} \pi = \{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1].$$

For example, if $\pi = 18524376$ then $\operatorname{Pk} \pi = \{2, 5, 7\}$ since $\pi_2 = 8$, $\pi_5 = 4$, and $\pi_7 = 7$ are all bigger than the elements directly to their left and right. Each peak in the peak set is either an occurrence of the consecutive pattern 132 or 231 in the permutation. It is easy to see that $S \subseteq [2, n-1]$ is the peak set of some $\pi \in \mathfrak{S}_n$ if and only if no two elements of S are consecutive. So the number of possible peak sets is a Fibonacci number. One could also ask how many permutations have a given peak set. This question was answered by Billey, Burdzy and Sagan.

Theorem 1 ([BBS13]). *If $n \in \mathbb{P}$ and $S \subseteq [2, n]$ then*

$$\#\{\pi \mid \operatorname{Pk} \pi = S\} = p(S; n) 2^{n - \#S - 1}$$

where $\#$ denotes cardinality and $p(S; n)$ is a polynomial in n depending on S. □

It is natural to study the values at the peak indices. This line of research was initiated by Davis, Nelson, Petersen, and Tenner [DNKPT18] and continued by Rusu [Rus20]; Diaz-Lopez, Harris, Huang, Insko, and Nilsen [DLHH+21]; and Rusu and Tenner [RT]. Define the *pinnacle set* of a permutation $\pi \in \mathfrak{S}_n$ to be

$$\operatorname{Pin} \pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\} \subseteq [2, n-1]$$

Continuing with the example $\pi = 18524376$ we see that $\operatorname{Pin} \pi = \{4, 7, 8\}$. Following Davis et al., call a set S an *admissible pinnacle set* if there is some permutation π with $\operatorname{Pin} \pi = S$. They found a criterion for S to be admissible. This result was stated in recursive fashion, but it is clearly equivalent to the following nonrecursive version.

Theorem 2 ([DNKPT18]). *Let $S = \{s_1 < s_2 < \ldots < s_d\} \subset \mathbb{P}$. The set S is an admissible pinnacle set if and only if we have*

$$s_i > 2i$$

for all $i \in [d]$. $\square$

Davis et al. were able to count the number of admissible pinnacle sets for $\pi \in \mathfrak{S}_n$.

Theorem 3 ([DNKPT18]). *If*

$$\mathcal{A}_n = \{S \mid S = \operatorname{Pin}\pi \text{ for some } \pi \in \mathfrak{S}_n\}$$

then

$$\#\mathcal{A}_n = \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}. \qquad \square$$

They also studied the more refined constants

$$\mathfrak{p}(m,d) = \#\{S \in \mathcal{A}_n \mid \max S = m \text{ and } \#S = d\}$$

where $n \geq m$. Note that if $S = \operatorname{Pin}\pi$ for some $\pi \in \mathfrak{S}_n$ then S is also a pinnacle set of some $\pi' \in \mathfrak{S}_{n'}$ for all $n' \geq n$ since one can just add values larger than n to the beginning of π in decreasing order. It follows that the exact value of n does not play a role in the definition of $\mathfrak{p}(m,d)$.

Davis et al. derived a number of properties of the constants $\mathfrak{p}(m,d)$ which count the number of admissible pinnacle sets S with d elements and maximum m. Here, we will prove that these constants are, in fact, ballot numbers. We give two proofs of this result. In the first, we derive a formula for $\mathfrak{p}(m,d)$ using finite differences and then show that it agrees with the well-known expression for ballot numbers. In the second, we give an explicit bijection between these admissible sets and ballot sequences.

Suppose we are given nonnegative integers $p > q$. A (p,q) *ballot sequence* is a permutation $\beta = \beta_1\beta_2\ldots\beta_{p+q}$ of p copies of the letter X and q copies of the letter Y such that in any nonempty prefix $\beta_1\beta_2\ldots\beta_i$ the number of X's is greater than the number of Y's. Let

$$\mathcal{B}_{p,q} = \{\beta \mid \beta \text{ is a } (p,q) \text{ ballot sequence}\}.$$

The following result is well known.

Theorem 4 ([And87],[Ber87]). *For nonnegative integers $p > q$ we have*

$$\#\mathcal{B}_{p,q} = \frac{p-q}{p+q}\binom{p+q}{q}. \qquad \square$$

Note that if we let $p = d+1$ and $q = d$ then the previous result gives get

$$\#\mathcal{B}_{d+1,d} = \frac{1}{2d+1}\binom{2d+1}{d} = C_d$$

where C_d is the dth Catalan number.

Our first proof that the $\mathfrak{p}(m,d)$ are ballot numbers will use the theory of finite differences. If $f(m)$ is a function of a nonnegative integer m then its *forward difference* is the function Δf defined by

$$\Delta f(m) = f(m+1) - f(m).$$

For a fixed $d \in \mathbb{P}$, define the following polynomial in m of degree $d-1$

$$p_d(m) = \frac{m-2d+1}{(d-1)!} \prod_{i=2}^{d-1} (m-i).$$

By showing that $\mathfrak{p}(m,d)$ satisfies the same finite difference equation and initial condition as $p_d(m)$, we obtain the following result.

Theorem 5. *If $m,d \in \mathbb{P}$ with $m > 2d$ then $\mathfrak{p}(m,d) = p_d(m)$. Thus*

$$\mathfrak{p}(m,d) = \frac{m-2d+1}{m-1}\binom{m-1}{d-1} = \#\mathcal{B}_{m-d,d-1}.$$

We would like to give a bijective proof of the relationship between admissible pinnacle sets and ballot sequences from the previous theorem. Let

$$\mathfrak{P}(m,d) = \{S \mid S \text{ admissible with } \max S = m \text{ and } \#S = d\}$$

so that $\#\mathfrak{P}(m,d) = \mathfrak{p}(m,d)$. For $m > 2d$, define a map

$$\eta : \mathcal{B}_{m-d,d-1} \to \mathfrak{P}(m,d)$$

by sending ballot sequence $\beta = \beta_1\beta_2\ldots\beta_{m-1}$ to

$$\eta(\beta) = \{i \mid \beta_i = Y\} \uplus \{m\}.$$

For example, if $m=9$, $d=3$ and $\beta = XXXYXXXYX$ then

$$\eta(\beta) = \{4,7\} \uplus \{9\} = \{4,7,9\}.$$

Theorem 6. *The map η is a well-defined bijection.*

[And87] Désiré André. Solution directe du probleme résolu par m. bertrand. *CR Acad. Sci. Paris*, 105(436):7, 1887.

[BBS13] Sara Billey, Krzysztof Burdzy, and Bruce E. Sagan. Permutations with given peak set. *J. Integer Seq.*, 16(6):Article 13.6.1, 18, 2013.

[Ber87] J. Bertrand. Solution d'un problème. *CR Acad. Sci. Paris*, (105):369, 1887.

[DLHH+21] Alexander Diaz-Lopez, Pamela E. Harris, Isabella Huang, Erik Insko, and Lars Nilsen. A formula for enumerating permutations with a fixed pinnacle set. *Discrete Math.*, 344(6):112375, 2021.

[DNKPT18] Robert Davis, Sarah A. Nelson, T. Kyle Petersen, and Bridget E. Tenner. The pinnacle set of a permutation. *Discrete Math.*, 341(11):3249–3270, 2018.

[Rus20] Irena Rusu. Sorting permutations with fixed Pinnacle set. *Electron. J. Combin.*, 27(3):Paper No. 3.23, 21, 2020.

[RT] Irena Rusu and Bridget E. Tenner. Admissible pinnacle orderings. Preprint `arXiv:2001.08185`.

Automated bijections with Combinatorial Exploration

Jon Steinn Eliasson

jonsteinn@gmail.com

Reykjavik University

(This talk is based on joint work with Christian Bean, Emile Nadeau, Jay Pantone and Henning Ulfarsson.)

Bijections appear in most areas of mathematics. They are of particular importance in the field of combinatorics, where they are often a way of enumerating families of objects. We describe a fully automated and domain agnostic bijection search built on top of Combinatorial Exploration, an existing automated combinatorial specification framework [1, 2, 3]. We show bijections found by the framework between permutation classes, in particular, rediscovering the Simion-Schmidt bijection between $\mathsf{Av}(123)$ and $\mathsf{Av}(132)$ [4] and finding bijections between the three permutation classes $\mathsf{Av}(1234)$, $\mathsf{Av}(1243)$ and $\mathsf{Av}(1432)$. We will also give examples of bijections from simple word classes to permutations.

We say that two specifications for two classes are *parallel* if we can traverse (using rules as edges) both, ignoring equivalence rules, such that at each class pair they are both neutral classes, both atomic classes, have a recursion at an equal distance (ignoring equivalence rules) to an ascendant or the constructor of both rules is the same and there is a bijection between the non-empty right hand side classes of both rules such that we can continue traversing in this fashion. An example of a partial traversal can be seen in Figure 1.

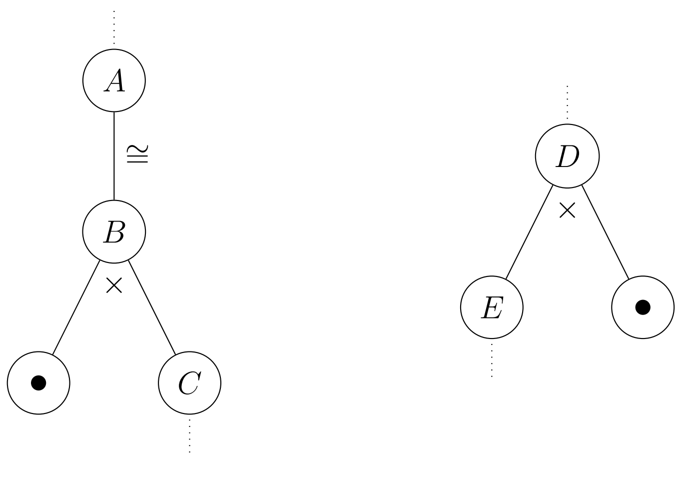


Figure 1: Arriving at classes (A, D) we skip past the equivalence rule from A to B and look at (B, D). They are both left sides of Cartesian products with 2 non-empty children. The bijection between their children will pair the two atoms and then we continue traversing for (C, E).

When searching for specifications, Combinatorial Exploration will expand a universe of rules. Starting with two classes that we suspect are equinumerous, we expand universes for both and then search for specifications that are parallel. This is done by traversing in pairs from the two initial classes using dynamic programming and backtracking. From the resulting specification we can then create a bijection that allows us to map objects from either class to the other. That is done by breaking the object down to atoms in one class and building it up from the matched paths to atoms in the other one. Here we assume every rule in the specifications is bijective and we know how to map objects in both directions.

With the bijection in hand one can experimentally check whether it preserves certain statistics, e.g. the bijection found between $\mathsf{Av}(1234)$ and $\mathsf{Av}(1243)$ seems to preserve the number of peaks and right-to-left maximas under symmetry. A future goal of this work is to explain Wilf-equivalences of many permutations classes avoiding patterns of length 4 with bijections.

[1] C. Bean. Finding structure in permutation patterns. Ph.D. dissertation, Reykjavik University, 2018.

[2] PermutaTriangle GitHub repository. https://github.com/PermutaTriangle/Tilings.

[3] PermutaTriangle GitHub repository. https://github.com/PermutaTriangle/comb_spec_searcher.

[4] R. Simion and F.W. Schmidt. Restricted Permutations. *European Journal of Combinatorics*, vol. 6, pp. 383-406, 1985.

Bijections for derangements and pattern-avoiding inversion sequences

Sergi Elizalde

sergi.elizalde@dartmouth.edu

Dartmouth College

A fixed point of a permutation $\pi \in \mathcal{S}_n$ is an element i such that $\pi(i) = i$. Let $\mathcal{D}_n \subseteq \mathcal{S}_n$ denote the subset of permutations with no fixed points, often called *derangements*, and let $d_n = |\mathcal{D}_n|$. Let $\mathcal{F}_n \subseteq \mathcal{S}_n$ denote the subset of permutations with exactly one fixed point. It is easy to show that $|\mathcal{F}_n| = nd_{n-1}$. Let $\overline{\mathcal{D}}_n \subseteq \mathcal{S}_n$ denote the set of *non-derangements*, that is, permutations with at least one fixed point, and let $\overline{d}_n = |\overline{\mathcal{D}}_n| = n! - d_n$.

It is well known [9, Eq. (2.13)] that the derangement numbers d_n satisfy the recurrence

$$d_n = nd_{n-1} + (-1)^n \tag{1}$$

for $n \geq 1$. This equation states that the number of $\pi \in \mathcal{S}_n$ with no fixed points and the number of $\pi \in \mathcal{S}_n$ with one fixed point differ by one. Stanley [9] acknowledges that proving recurrence (1) combinatorially requires "considerably more work" than proving the other well-known recurrence for derangement numbers,

$$d_n = (n-1)(d_{n-1} + d_{n-2}) \tag{2}$$

for $n \geq 2$. Bijective proofs of (1) have been given by Remmel [8], Wilf [10], Désarménien [5] and, more recently, Benjamin and Ornstein [3]. In Section 1, we present a new bijective proof of Equation (1) that is arguably simpler than these existing proofs.

Section 2 relates derangements to pattern-avoiding inversion sequences. An *inversion sequence* is an integer sequence $e_1e_2\dots e_n$ such that $0 \leq e_i < i$ for each i. Let $\mathbf{I}_n$ denote the set of inversion sequences of length n. Pattern avoidance in inversion sequences, which are often used to encode permutations, has been considered in [4, 6, 7] for classical patterns, and in [1, 2] for consecutive patterns. In [1], the authors enumerate inversion sequences avoiding each consecutive pattern of length 3, and in particular, avoiding the pattern $\underline{000}$. An inversion sequence is said to avoid $\underline{000}$ if it does not contain three consecutive equal entries, that is, there does not exist $i \in [n-2]$ such that $e_i = e_{i+1} = e_{i+2}$. Let

$$\mathbf{I}_n(\underline{000}) = \{e \in \mathbf{I}_n : e \text{ avoids } \underline{000}\}.$$

The following formula from [1, Cor. 3.3] expresses the number of inversion sequences that avoid $\underline{000}$ in terms of the derangement numbers.

Theorem 1 ([1])**.** *For $n \geq 1$,*

$$|\mathbf{I}_n(\underline{000})| = \frac{(n+1)! - d_{n+1}}{n}.$$

The proof in [1] is by induction on n. Finding a bijective proof is left as an open problem in [1]. In Section 2 we provide such a proof.

1 A bijective proof of Equation (1)

We describe a bijection $\psi : \mathcal{D}_n^* \to \mathcal{F}_n^*$, where $\mathcal{D}_n^* = \mathcal{D}_n \setminus \{(1,2)(3,4)\dots(n-1,n)\}$ and $\mathcal{F}_n^* = \mathcal{F}_n$ when n is even, and $\mathcal{D}_n^* = \mathcal{D}_n$ and $\mathcal{F}_n^* = \mathcal{F}_n \setminus \{(1)(2,3)\dots(n-1,n)\}$ when n is odd.

We write derangements in cycle notation so that each cycle begins with its smallest element, and cycles are ordered by increasing first element. We write permutations in $\mathcal{F}_n$ with their fixed point at the beginning.

Let $\pi \in \mathcal{D}_n^*$, and let k be the largest non-negative integer such that the cycle notation of π starts with $(1,2)(3,4)\dots(2k-1,2k)$. Note that $0 \le k < n/2$, since $\pi \neq (1,2)(3,4)\dots(n-1,n)$. To define $\psi(\pi) \in \mathcal{F}_n^*$, consider two cases:

(i) If the cycle containing $2k+1$ has at least 3 elements, change the first $k+1$ cycles of π as follows:

$$\begin{aligned} \pi &= (1,2)(3,4)\dots(2k-1,2k)(2k+1,a_1,a_2,\dots,a_j)\dots \\ \psi(\pi) &= (1)(2,3)(4,5) \quad \dots \quad (2k,a_1)(2k+1,a_2,\dots,a_j)\dots \end{aligned}$$

Note that, if $k=0$, then $\{1,2,\dots,2k\} = \emptyset$ and the fixed point in $\psi(\pi)$ is a_1.

(ii) Otherwise, change the first $k+2$ cycles of π as follows:

$$\begin{aligned} \pi &= (1,2)(3,4)\dots(2k-1,2k)(2k+1,a_1)(2k+2,a_2,\dots,a_j)\dots \\ \psi(\pi) &= (1)(2,3)(4,5) \quad \dots \quad (2k,2k+1)(2k+2,a_1,a_2,\dots,a_j)\dots \end{aligned}$$

The inverse map ψ^{-1} has a similar description. Given $\sigma \in \mathcal{F}_n^*$, let ℓ be the fixed point of σ, and consider two cases. If $\ell \neq 1$, merge the cycles containing ℓ and 1 as follows:

$$\sigma = (\ell)(1,a_2,\dots,a_j)\dots \quad \mapsto \quad \psi^{-1}(\sigma) = (1,\ell,a_2,\dots,a_j)\dots$$

Otherwise, let σ' be the derangement of $\{2,\dots,n\}$ obtained by removing the fixed point 1 from σ; apply ψ to σ' (using the above description, but identifying $\{2,\dots,n\}$ with $\{1,\dots,n-1\}$ in an order-preserving fashion); and replace its fixed point (ℓ) with the 2-cycle $(1,\ell)$ to get $\psi^{-1}(\sigma)$.

As an example, below are the images by ψ of all the derangements in $\mathcal{D}_4$ and some in $\mathcal{D}_5$, with the entry a_1 colored in red in case (i) and in blue in case (ii).

π	(12)(34)	(13)(24)	(14)(23)	(1234)	(1243)	(1324)	(1342)	(1423)	(1432)
$\psi(\pi)$	–	(1)(234)	(1)(243)	(2)(134)	(2)(143)	(3)(124)	(3)(142)	(4)(123)	(4)(132)

π	–	(12)(345)	(12)(354)	(123)(45)	(13)(245)	(14)(235)	(154)(23)	...
$\psi(\pi)$	(1)(23)(45)	(1)(24)(35)	(1)(25)(34)	(2)(13)(45)	(1)(2345)	(1)(2435)	(5)(14)(23)	...

2 A bijective proof of Theorem 1

The right-hand side of Theorem 1 can be interpreted as the cardinality of the set $\overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}$, where $\sqcup$ denotes a disjoint union. Indeed, using Equation (2),

$$\begin{aligned}\frac{(n+1)! - d_{n+1}}{n} &= \frac{(n+1)! - n(d_n + d_{n-1})}{n} = (n+1)(n-1)! - d_n - d_{n-1} \\ &= n! - d_n + (n-1)! - d_{n-1} = \overline{d}_n + \overline{d}_{n-1}.\end{aligned}$$

Our proof of Theorem 1 consists of a bijection

$$\phi : \mathbf{I}_n(\underline{000}) \to \overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}.$$

In order to describe the map ϕ, first we introduce some notation. If $1 \le a, b \le n$, denote by (a, b) the permutation in $\mathcal{S}_n$ that switches a and b; this permutation is a transposition if $a \neq b$, and it is the identity permutation if $a = b$. In particular, if $\sigma = \sigma(1)\sigma(2)\dots\sigma(n) \in \mathcal{S}_n$, then the product $(a, b)\sigma$ denotes the permutation obtained from the one-line notation of σ by switching the entries a and b if they are different. For example, $(3, 5)32514 = 52314$ and $(2, 2)32514 = 32514$. We will also use the fact that $\mathcal{S}_{n-1}$ can be viewed as the subset of $\mathcal{S}_n$ consisting of those permutations where n is a fixed point.

Let $n \ge 1$, and let $e \in \mathbf{I}_n(\underline{000})$. The first step in the construction of $\phi(e)$ is to encode e as a word $w = w_2 \dots w_n$ over the alphabet $[k-1] \cup \{R\}$ as follows. For $2 \le k \le n$, let

$$w_k = \begin{cases} R & \text{if } e_k = e_{k-1}, \\ e_k & \text{if } e_k > e_{k-1}, \\ e_k + 1 & \text{if } e_k < e_{k-1}. \end{cases}$$

This encoding is a bijection between $\mathbf{I}_n(\underline{000})$ and the set of words $w = w_2 \dots w_n$ with $w_k \in [k-1] \cup \{R\}$ (for $2 \le k \le n$) not containing two consecutive Rs.

Next we read w from left to right and build a sequence of permutations $\sigma_1, \sigma_2, \dots, \sigma_n$, where $\sigma_k \in \overline{\mathcal{D}}_k \sqcup \overline{\mathcal{D}}_{k-1}$ for all k; specifically, $\sigma_k \in \overline{\mathcal{D}}_k$ if $w_k \neq R$, and $\sigma_k \in \overline{\mathcal{D}}_{k-1}$ if $w_k = R$. Set $\sigma_1 = 1 \in \overline{\mathcal{D}}_1$. For each k from 2 to n, repeat the following step. If $w_k = R$, let $\sigma_k = \sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$. Otherwise, let

$$\sigma_k = \begin{cases} (w_k, k)\sigma_{k-1} & \text{if } w_{k-1} \neq R \text{ and } \sigma_{k-1} \in \overline{\mathcal{D}}_{k-1} \text{ has fixed points other than } w_k, \\ (w_k, k-1)\sigma_{k-1} & \text{otherwise,} \end{cases}$$

where, in the products $(a, b)\sigma_{k-1}$, we view σ_{k-1} as an element of $\overline{\mathcal{D}}_k$, and thus $\sigma_k \in \overline{\mathcal{D}}_k$.

Finally, we define $\phi(e) = \sigma_n$.

For example, if $e = 001322$, then $w = R133R$. Here is the computation of σ_k for k from 1 to 6:

k	w_k	σ_k in one-line notation	σ_k in cycle notation
1		1	(1)
2	R	1	(1)
3	1	$(1,2)123 = 213$	$(2,1)(3)$
4	3	$(3,3)2134 = 2134$	$(2,1)(3)(4)$
5	3	$(3,5)21345 = 21543$	$(2,1)(5,3)(4)$
6	R	21543	$(2,1)(5,3)(4)$

We get $\phi(e) = \sigma_6 = 21543 \in \overline{\mathcal{D}}_6 \sqcup \overline{\mathcal{D}}_5$.

If $e = 0102230$, then $w = 112R31$, and computing σ_k gives

k	w_k	σ_k in one-line notation	σ_k in cycle notation
1		1	(1)
2	1	$(1,1)12 = 12$	$(1)(2)$
3	1	$(1,3)123 = 321$	$(3,1)(2)$
4	2	$(2,3)3214 = 2314$	$(2,3,1)(4)$
5	R	2314	$(2,3,1)(4)$
6	3	$(3,5)231456 = 251436$	$(2,5,3,1)(4)(6)$
7	1	$(1,7)2514367 = 2574361$	$(2,5,3,7,1)(4)(6)$

We get $\phi(e) = \sigma_7 = 2574361 \in \overline{\mathcal{D}}_7 \sqcup \overline{\mathcal{D}}_6$.

It is sometimes convenient to describe the construction of $\sigma_1, \sigma_2, \dots, \sigma_n$ in cycle notation, where we write permutations as products of disjoint cycles. In this case, $\sigma_1 = (1) \in \overline{\mathcal{D}}_1$, and for each k from 2 to n, we repeat the following step. If $w_k = R$, let $\sigma_k = \sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$. Otherwise, σ_k is the permutation in $\overline{\mathcal{D}}_k$ obtained from the cycle notation of σ_{k-1} as follows:

- if $w_{k-1} = R$, insert $k-1$ right before w_k in the same cycle, and add a new fixed point (k);
- if $w_{k-1} \neq R$ and $\sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$ has fixed points other than w_k, insert k right before w_k in the same cycle,
- otherwise (that is, if $w_{k-1} \neq R$ and w_k is the only fixed point of $\sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$), add a new fixed point (k), and if $w_k \neq k-1$, remove (w_k) and insert w_k right before $k-1$ in the same cycle.

To show that ϕ is indeed a bijection, let us describe its inverse $\phi^{-1} : \overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1} \to \mathbf{I}_n(\underline{000})$. Given $\pi \in \overline{\mathcal{D}}_n \sqcup \overline{\mathcal{D}}_{n-1}$, set $\sigma_n = \pi$. We will construct permutations $\sigma_{n-1}, \sigma_{n-2}, \dots, \sigma_1$ in cycle notation, while building a word w from right to left. For k from n to 2, repeat the following step. If $\sigma_k \in \overline{\mathcal{D}}_{k-1}$, let $w_k = R$ and $\sigma_{k-1} = \sigma_k$. Otherwise (that is, if $\sigma_k \in \overline{\mathcal{D}}_k$) proceed as follows.

- If k is not a fixed point of σ_k, let $w_k = \sigma_k(k)$, and let σ_{k-1} be the permutation obtained by removing k from the cycle notation of σ_k.
- Otherwise, remove (k) from the cycle notation of σ_k, and then:
  - if removing $k-1$ from the cycle notation leaves any fixed points, let $\sigma_{k-1} \in \overline{\mathcal{D}}_{k-2}$ be the resulting permutation, and let $w_k = \sigma_k(k-1)$;
  - otherwise, let $w_k = \sigma_k^{-1}(k-1)$, and let $\sigma_{k-1} \in \overline{\mathcal{D}}_{k-1}$ be the permutation obtained by removing w_k from its current cycle and creating a fixed point (w_k). (Note that this produces no change if $w_k = k-1$, since in this case w_k is already a fixed point.)

From w, we obtain the inversion sequence $e = \phi^{-1}(\pi)$ by letting $e_1 = 0$ and, for $2 \leq k \leq n$, letting

$$e_k = \begin{cases} e_{k-1} & \text{if } w_k = R, \\ w_k & \text{if } w_k > e_{k-1}, \\ w_k - 1 & \text{if } w_k \leq e_{k-1}. \end{cases}$$

Finally, we remark that the numbers $\overline{d}_n$ of non-derangements satisfy recurrences that are analogous to Equations (2) and (1), namely

$$\overline{d}_n = (n-1)(\overline{d}_{n-1} + \overline{d}_{n-2})$$

for $n \geq 2$, with initial terms $\overline{d}_0 = 0$, $\overline{d}_1 = 1$ (compare to $d_0 = 1$, $d_1 = 0$); and

$$\overline{d}_n = n\overline{d}_{n-1} - (-1)^n$$

for $n \geq 1$. The tools that we have described in this abstract can be used to prove both of these recurrences bijectively as well.

[1] Juan S. Auli and Sergi Elizalde. Consecutive patterns in inversion sequences. *Discrete Math. Theor. Comput. Sci.*, 21, 2019.

[2] Juan S. Auli and Sergi Elizalde. Consecutive patterns in inversion sequences II: Avoiding patterns of relations. *J. Integer Seq.*, 22:Art. 19.7.5, 2019.

[3] Arthur T. Benjamin and Joel Ornstein. A bijective proof of a derangement recurrence. *Fibonacci Quart.*, 55(5):28–29, 2017.

[4] Sylvie Corteel, Megan A. Martinez, Carla D. Savage, and Michael Weselcouch. Patterns in inversion sequences I. *Discrete Math. Theor. Comput. Sci.*, 18, 2016.

[5] Jacques Désarménien. Une autre interprétation du nombre de dérangements. *Sém. Lothar. Combin.*, B08b, 1982.

[6] Toufik Mansour and Mark Shattuck. Pattern avoidance in inversion sequences. *Pure Math. Appl. (PU.M.A.)*, 25:157–176, 2015.

[7] Megan A. Martinez and Carla D. Savage. Patterns in inversion sequences II: Inversion sequences avoiding triples of relations. *J. Integer Seq.*, 21:Art. 18.2.2, 2018.

[8] Jeffrey B. Remmel. A note on a recursion for the number of derangements. *European J. Combin.*, 4(4):371–374, 1983.

[9] Richard P. Stanley. *Enumerative Combinatorics, Vol. 1.* Cambridge University Press, Second edition, 2011.

[10] Herbert S. Wilf. A bijection in the theory of derangements. *Mathematics Magazine*, 57(1):37–40, 1984.

Counting Small Patterns and Testing for Independence

Chaim Even-Zohar

chaim@ucdavis.edu

The Alan Turing Institute

(This talk is based on joint work with Calvin Leng from USC.)

We discuss the algorithmic problem of counting pattern occurrences in a given permutation. Let $\#\sigma(\pi)$ denote the number of occurrences of σ in π, where as usual, an *occurrence* of $\sigma \in S_k$ in $\pi \in S_n$ is a set of positions $i_1 < i_2 < \cdots < i_k$ such that $\pi_{i_1} \ldots \pi_{i_k}$ are order-isomorphic to $\sigma_1 \ldots \sigma_k$. For example, $\#\mathit{21}\,(1423) = 2$. The *k-profile* of $\pi \in S_n$ consists of the $k!$ numbers $\#\sigma(\pi)$ for all $\sigma \in S_k$. These numbers clearly sum up to $\binom{n}{k}$.

There is considerable combinatorial literature concerning the extremal and probabilistic properties of the numbers $\#\sigma(\pi)$. For example, *permutons* are limits of sequences of permutations, based on these statistics. But pattern counts in permutations are of interest as well in a range of applications, most notably in nonparametric statistics of bivariate data. Consider a sequence of n independent paired samples (X_i, Y_i) drawn from a common continuous distribution (X, Y) on $\mathbb{R}^2$. Various measures have been suggested to detect and quantify a relation between X and Y based on these samples. Often one prefers to rely only on the ranking of X_i and Y_i rather than on their numerical values. In this case it suffices to consider the permutation $\pi \in S_n$, defined by $\operatorname{rank} Y_i = \pi(\operatorname{rank} X_i)$, where the rank of $X_1, \ldots, X_n$ is their order preserving map to $\{1, \ldots, n\}$.

Many classical nonparametric measures depend only on pattern counts in π. Kendall's τ correlation coefficient is based on the 2-profile: $\tau = [\#\mathit{12}\,(\pi) - \#\mathit{21}\,(\pi)] \,/\, \binom{n}{2}$ [1]. Similarly Spearman's ρ is a function of the 3-profile of π [2], and so is a rotation-invariant measure of correlation by Fisher and Lee [3, 4], and other correlation tests [5]. Counting patterns in π can even detect whether there is *any* relationship between X and Y, as demonstrated by Hoeffding's independence test [6]. This classical nonparametric test is based on the 5-profile of π, and so is a refined version with wider consistency [7, 8]. A simpler variant that only relies on the 4-profile has been proposed by Bergsma and Dassios [9, 10], and goes back to a result of Yanagimoto [11]. A general scheme of independence tests expressible by pattern counts is suggested in [12].

Pattern counting also plays a role in *property testing* and *parameter testing* for permutations [13]. In these problems, one estimates features of a large permutation using a randomized algorithm that makes a small number of queries.

Applications to stack-sorting [14] have inspired a significant amount of research around the case $\#\sigma(\pi) = 0$. In particular, there has been interest in the count of such σ-avoiding permutations of order n. The extensive study of σ-avoidance has naturally led to the exploration of its algorithmic aspects. The most obvious question is *permutation pattern* *matching*: deciding whether a given pattern $\sigma \in S_k$ occurs anywhere in a given permutation $\pi \in S_n$. In this generality the problem is NP-complete [15, 16, 17]. Straightforward exhaustive search solves the problem in time $\tilde{O}(\binom{n}{k}\binom{k}{2})$, but faster algorithms have been found, improving the exponent by a constant factor [18, 19, 20, 21]. Numerous works have achieved improved running times for patterns σ that are either short or have some structural properties [22, 23, 24, 15, 25, 26, 27, 28, 29, 19, 30, 31, 32, 33, 34]. Remarkably, this problem is "fixed-parameter tractable". For $\sigma \in S_k$ and large n it can be solved in time $\tilde{O}(f(k) \cdot n)$ [30]. Throughout, the soft O notation suppresses a possible logarithmic factor.

The question of *permutation pattern counting* seeks to determine $\#\sigma(\pi)$, the number of occurrences of a given pattern $\sigma \in S_k$ in a given permutation $\pi \in S_n$. This problem, which is $\#P$-complete in general, has received less attention. It has mostly been considered in papers on the decision problem, as it sometimes happens that a decision algorithm can be adapted to address as well the counting problem [15, 25, 26, 21]. For example, a pattern σ from the special class of *separable* permutations can be counted in time $\tilde{O}(kn^6)$ [15]. However, the general linear time algorithm from [30] does not yield a counting method. A recent paper [21] shows that if $\#\sigma(\pi)$ can be computed for all σ and π in time $f(k)n^{o(k/\log k)}$, then the exponential-time hypothesis fails to hold. For comparison, recall that the run time of the trivial algorithm is $n^{O(k)}$.

While the above hardness results virtually settle the counting problem for general $\sigma \in S_k$ and asymptotically in k, they leave open some questions on specific patterns and particular k. For instance, which patterns can be counted in linear time, and how? Since the decision algorithm of [30] does not help us here, new ideas are clearly needed.

From a practical standpoint, these questions are much relevant to the implementation of rank-based statistical tests. Spearman's ρ is computable in linear time by definition. Kendall's τ naively requires $\tilde{O}(n^2)$ time, but an algorithm by Knight computes it in $\tilde{O}(n)$ [35]. By the same technique, Hoeffding's independence test is readily computable in $\tilde{O}(n)$, despite being stated and sometimes implemented with $\tilde{O}(n^2)$ formulas. Several works have addressed the computation of the Bergsma–Dassios–Yanagimoto test, only bringing it down to $\tilde{O}(n^2)$ time [36, 10, 37, 38, 39]. It seems that these pattern-based statistics have only been treated ad hoc, by dedicated algorithms, without a unified or systematic approach.

We proposes new approaches for permutation pattern counting, and define the key notion of a *corner tree*, a structure that lets us apply dynamic programming to the problem. A corner tree is a rooted tree whose non-root vertices are labeled by `NE`, `NW`, `SE`, and `SW`, as in Figure 1. Informally, an *occurrence* of a corner tree in a permutation is a map from the tree's vertices to the permutation's entries, such that the label of each vertex is

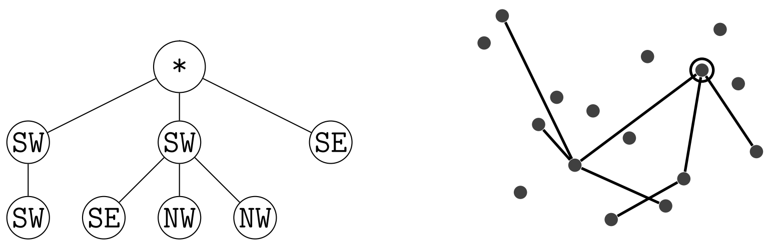


Figure 1: A corner tree and its occurrence in a permutation

compatible with the position of its image relative to its parent. We show that counting these occurrences gives rise to linear combinations of pattern counts. This leads to a class of permutation statistics that have a *corner tree formula.* We present an algorithm that evaluates such statistics efficiently, proving the following theorem.

Theorem 1. *Let $f : S_n \to \mathbb{Z}$ be a permutation statistic, given as a corner tree formula. There exists an algorithm that computes $f(\pi)$ for a given permutation $\pi \in S_n$ in running time $\tilde{O}(n)$.*

Although corner trees can come in any size, they prove particularly useful for counting small patterns. We give corner tree formulas for all 3-patterns and eight 4-patterns.

Corollary 2. *The number of occurrences of any pattern of size* 3 *in a given permutation $\pi \in S_n$ can be computed in $\tilde{O}(n)$ time.*

Corollary 3. *The occurrences of any of the patterns 1234, 1243, 2134, 2143, 3412, 3421, 4312, 4321 in a given permutation $\pi \in S_n$ can be counted in time $\tilde{O}(n)$.*

The total count of these eight patterns is sufficient for computing the Bergsma–Dassios–Yanagimoto statistic. This yields the following practical consequence.

Corollary 4. *The Bergsma–Dassios–Yanagimoto independence test for continuous bivariate data can be computed in $\tilde{O}(n)$ time.*

This is a significant improvement over the best previous method, which requires $\tilde{O}(n^2)$ time and space [38]. Indeed, a near linear time performance is crucial when dealing with large amounts of data. We thus expect our method to be beneficial in real world scenarios. We provide further analysis and optimization of rank-based independence tests in a follow-up work [40], and implementation in `R` and `Python` [41, 42].

Another important ingredient of our approach is treating the k-profile vector as a whole, rather than looking on each pattern separately. The permutation statistics that we study are linear combinations of pattern counts: $F(\pi) = \sum_\sigma f_\sigma \, \#\sigma(\pi)$. Here the sum is over patterns, with finitely many nonzero $f_\sigma \in \mathbb{Q}$. The corner tree formulas that we define span a vector space of such combinations. This viewpoint provides much more information on the k-profile, as demonstrated in the following proposition with $k = 4$.

Proposition 5. *There exist twenty-three linearly independent combinations,*

$$F_i(\pi) \;=\; \sum_{\sigma \in S_4} f_{i\sigma} \, \#\sigma(\pi) \qquad\qquad i \in \{1, 2, \ldots, 23\}$$

that can be evaluated for a given $\pi \in S_n$ in time $\tilde{O}(n)$.

Since the 4-profile is a 24-dimensional vector, only one additional combination is needed to fully reveal it. In other words, one can efficiently compute this vector up to a single linear degree of freedom. All the remaining problems of counting 4-patterns are thus equivalent to each other. Proposition 5 was established with computer assistance [41]. Our code provides general-purpose routines for finding and using corner tree formulas, and it is scalable to larger patterns and combinations. For example, it yields a 100-dimensional space of $\tilde{O}(n)$-time linear functions of the 120-dimensional 5-profile.

Finally, we explore new counting methods beyond corner trees, which naturally focus on the missing piece of the 4-profile. An $\tilde{O}(n^2)$ time computation is straightforward from the existing methods. We present two algorithms that do somewhat better, as follows.

Theorem 6. *The occurrences of any 4-pattern in a given permutation $\pi \in S_n$ can be counted in $\tilde{O}(n^{5/3})$ time and $\tilde{O}(n)$ space.*

Theorem 7. *The occurrences of any 4-pattern in a given permutation $\pi \in S_n$ can be counted in $\tilde{O}(n^{3/2})$ time and space.*

Note. This research was presented in the ACM-SIAM Symposium on Discrete Algorithms (SODA21). Full details of the results described in this abstract are given in the proceedings of that conference [43].

[1] Maurice G Kendall. A new measure of rank correlation. *Biometrika*, 30(1):81–93, 1938.

[2] Charles Spearman. The proof and measurement of association between two things. *The American Journal of Psychology*, 15(1):72–101, 1904.

[3] Nicholas I Fisher and Alan J Lee. Nonparametric measures of angular-angular association. *Biometrika*, pages 315–321, 1982.

[4] Svante Janson. The asymptotic distributions of incomplete U-statistics. *Probability Theory and Related Fields*, 66(4):495–505, 1984.

[5] Casparus F Crouse. Distribution free tests based on the sample distribution function. *Biometrika*, 53(1/2):99–108, 1966.

[6] Wassily Hoeffding. A non-parametric test of independence. *The annals of mathematical statistics*, pages 546–557, 1948.

[7] Wassilij Höffding. Masstabinvariante korrelationstheorie. *Schriften des Mathematischen Instituts und Instituts für Angewandte Mathematik der Universität Berlin*, 5:181–233, 1940.

[8] Julius Robin Blum, Jack Carl Kiefer, and Murray Rosenblatt. Distribution free tests of independence based on the sample distribution function. *The Annals of Mathematical Statistics*, pages 485–498, 1961.

[9] Wicher Bergsma. Nonparametric testing of conditional independence by means of the partial copula. *arXiv preprint, arXiv:1101.4607*, 2010.

[10] Wicher Bergsma and Angelos Dassios. A consistent test of independence based on a sign covariance related to Kendall's tau. *Bernoulli*, 20(2):1006–1028, 2014.

[11] Takemi Yanagimoto. On measures of association and a related problem. *Annals of the Institute of Statistical Mathematics*, 22(1):57–63, 1970.

[12] Ruth Heller, Yair Heller, Shachar Kaufman, Barak Brill, and Malka Gorfine. Consistent distribution-free K-sample and independence tests for univariate random variables. *The Journal of Machine Learning Research*, 17(1):978–1031, 2016.

[13] Carlos Hoppen, Yoshiharu Kohayakawa, Carlos Gustavo Moreira, and Rudini Menezes Sampaio. Testing permutation properties through subpermutations. *Theoretical Computer Science*, 412(29):3555–3567, 2011.

[14] Donald E Knuth. Fundamental algorithms, volume 1 of the art of computer programming. *Addision Wesley, Reading, MA*, 1968.

[15] Prosenjit Bose, Jonathan F Buss, and Anna Lubiw. Pattern matching for permutations. *Information Processing Letters*, 65(5):277–283, 1998.

[16] Ivan Bliznets, Marek Cygan, Paweł Komosa, and Lukáš Mach. Kernelization lower bound for permutation pattern matching. *Information Processing Letters*, 115(5):527–531, 2015.

[17] Joshua Cooper and Anna Kirkpatrick. The complexity of counting poset and permutation patterns. *Australasian Journal of Combinatorics*, 64(1):154–165, 2016.

[18] Shlomo Ahal and Yuri Rabinovich. On complexity of the subpattern problem. *SIAM Journal on Discrete Mathematics*, 22(2):629–649, 2008.

[19] Marie-Louise Bruner and Martin Lackner. A fast algorithm for permutation pattern matching based on alternating runs. *Algorithmica*, 75(1):84–117, 2016.

[20] László Kozma. Faster and simpler algorithms for finding large patterns in permutations. *arXiv preprint arXiv:1902.08809*, 2019.

[21] Benjamin Aram Berendsohn, László Kozma, and Dániel Marx. Finding and counting permutations via CSPs. *arXiv preprint arXiv:1908.04673*, 2019.

[22] Craige Schensted. Longest increasing and decreasing subsequences. *Canadian Journal of Mathematics*, 13:179–191, 1961.

[23] Maw-Shang Chang and Fu-Hsing Wang. Efficient algorithms for the maximum weight clique and maximum weight independent set problems on permutation graphs. *Information Processing Letters*, 43(6):293–295, 1992.

[24] Louis Ibarra. Finding pattern matchings for permutations. *Information Processing Letters*, 61(6):293–295, 1997.

[25] Michael H Albert, Robert EL Aldred, Mike D Atkinson, and Derek A Holton. Algorithms for pattern involvement in permutations. In *International Symposium on Algorithms and Computation*, pages 355–367. Springer, 2001.

[26] V Yugandhar and Sanjeev Saxena. Parallel algorithms for separable permutations. *Discrete Applied Mathematics*, 146(3):343–364, 2005.

[27] Sylvain Guillemot and Stéphane Vialette. Pattern matching for 321-avoiding permutations. In *International Symposium on Algorithms and Computation*, pages 1064–1073. Springer, 2009.

[28] Marie-Louise Bruner and Martin Lackner. A fast algorithm for permutation pattern matching based on alternating runs. In *Scandinavian Workshop on Algorithm Theory*, pages 261–270. Springer, 2012.

[29] Marie-Louise Bruner and Martin Lackner. The computational landscape of permutation patterns. *arXiv preprint arXiv:1301.0340*, 2013.

[30] Sylvain Guillemot and Dániel Marx. Finding small patterns in permutations in linear time. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*, pages 82–101. SIAM, 2014.

[31] Yijie Han and Sanjeev Saxena. Parallel algorithms for testing length four permutations. In *2014 Sixth International Symposium on Parallel Architectures, Algorithms and Programming*, pages 81–86. IEEE, 2014.

[32] Vít Jelínek and Jan Kynčl. Hardness of permutation pattern matching. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 387–396. Society for Industrial and Applied Mathematics, 2017.

[33] Yijie Han and Sanjeev Saxena. Algorithms for testing occurrences of length 4 patterns in permutations. *Journal of Combinatorial Optimization*, 35(1):189–208, 2018.

[34] Benjamin Aram Berendsohn. Complexity of permutation pattern matching. *Master Thesis, FU, Berlin*, 2019.

[35] William R Knight. A computer method for calculating Kendall's tau with ungrouped data. *Journal of the American Statistical Association*, 61(314):436–439, 1966.

[36] Wicher Bergsma and Angelos Dassios. A consistent test of independence based on a sign covariance related to Kendall's tau. *arXiv preprint arXiv:1007.4259*, 2010.

[37] Luca Weihs, Mathias Drton, and Dennis Leung. Efficient computation of the Bergsma–Dassios sign covariance. *Computational Statistics*, 31(1):315–328, 2016.

[38] Yair Heller and Ruth Heller. Computing the Bergsma Dassios sign-covariance. *arXiv preprint arXiv:1605.08732*, 2016.

[39] Luca Weihs, Mathias Drton, and Nicolai Meinshausen. Symmetric rank covariances: a generalized framework for nonparametric measures of dependence. *Biometrika*, 105(3):547–562, 2018.

[40] Chaim Even-Zohar. independence: Fast rank tests. *arXiv:2010.09712*, 2020.

[41] Chaim Even-Zohar and Calvin Leng. *GitHub repository:* corners, 2019. http://github.com/chaim-e/corners.

[42] Chaim Even-Zohar. ***independence**: Fast Rank-Based Independence Testing*, 2020. R package version 1.0.

[43] Chaim Even-Zohar and Calvin Leng. Counting small permutation patterns. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 2288–2302. SIAM, 2021.

Increasing subsequences in random separable permutations

Valentin Feray

valentin.feray@univ-lorraine.fr

Université de Lorraine

(This talk is based on joint work with Frédérique Bassino, Mathilde Bouvel, Michael Drmota, Lucas Gerin, Mickaël Maazoun and Adeline Pierrot.)

The asymptotic behavior of the length of the longest increasing subsequence $\mathrm{LIS}(\boldsymbol{s_n})$ in a uniform random permutation $\boldsymbol{s_n}$ of size n is an old and famous problem that led to surprising and deep connections with various areas of pure mathematics (representation theory, combinatorics, linear algebra and operator theory, random matrices,...). In particular, it is well-known that $\mathrm{LIS}(\boldsymbol{s_n})$ is typically close to $2\sqrt{n}$ and has Tracy-Widom fluctuations of order $n^{1/6}$. We refer to [Rom15] for a nice and modern introduction to this topic. Longest increasing subsequences in random permutations in permutation classes are a much newer topic: see, e.g., [MRRY20] and references therein. Here we consider the well-studied class of separable permutations and derive results on increasing subsequences in a uniform element of size n in this class.

We first prove the following result.

Theorem 1. *For each $n \geq 1$, let $\boldsymbol{\sigma_n}$ be a uniform random separable permutation of size n. Then, the maximal length of an increasing subsequence in $\boldsymbol{\sigma}_n$ is sublinear in n, namely $\frac{\mathrm{LIS}(\boldsymbol{\sigma_n})}{n}$ converges to 0 in probability.*

Second, and unexpectedly given the above results, we show that for $\beta > 0$ sufficiently small, the expected number of increasing subsequences of size βn grows exponentially fast with n. More precisely, we prove the following:

Theorem 2. *For each $n \geq 1$, let $\boldsymbol{\sigma_n}$ be a uniform random separable permutation of size n, and let $\boldsymbol{Z}_{n,k}$ be the number of increasing subsequences of length k in $\boldsymbol{\sigma}_n$. Then there exist some computable functions $D_\beta > 0$, $E_\beta > 0$ ($0 < \beta < 1$) with the following property. For every fixed closed interval $[a, b] \subseteq (0, 1)$, we have*

$$\mathbb{E}[\boldsymbol{Z}_{n,k}] \sim D_{k/n}\, n^{-1/2} (E_{k/n})^n, \tag{1}$$

uniformly for $an \leq k \leq bn$. Furthermore,

1. *When $\beta \to 0$, we have $E_\beta = 1 + \beta|\log(\beta)| + \mathrm{o}(\beta \log(\beta))$.*
2. *Consequently, there exists $\beta_1 > 0$ such that $E_\beta > 1$ for every $\beta \in (0, \beta_1)$; numerically, we can estimate*

$$\beta_1 \approx 0.5827\ldots$$

Let us briefly discuss proof methods. A standard strategy to prove a non-existence result in probability as Theorem 1 is the so-called *first moment method.* In the present case, we would need to show that, for any fixed $\beta > 0$, the expectation $\mathbb{E}[\boldsymbol{Z}_{n,\lfloor n\beta \rfloor}]$ tends to 0 as n grows. Theorem 2 shows however that this is not the case, and we need a more sophisticated strategy.

Our proof of Theorem 1 relies on the permuton limit of separable permutations found by some of us in [BBF+18] (see also the invited talk of L. Gerin). Roughly, we use a self-similarity property of the limit to establish an inequation in distribution for the law of the limit of $\frac{\mathrm{LIS}(\boldsymbol{\sigma_n})}{n}$, and we show that the only solution of this inequation is the Dirac measure in 0. The proof extends verbatim to random permutations in all classes with the same permuton limit as separable permutations, i.e. to many substitution-closed classes [BBF+20] and some classes with finitely many simple permutations [BBF+19b].

The proof of Theorem 2 is of completely different nature. We write down equations for the bivariate generating series of separable permutations with a marked increasing subsequence, and perform bivariate singularity analysis.

Details can be found in the preprint [BBD$^+$21]. We remark that this preprint also contains analog results on independent sets in random cographs (which are inversion graphs of separable permutations).

[BBD$^+$21] F. Bassino, M. Bouvel, M. Drmota, V. Féray, L. Gerin, M. Maazoun, and A. Pierrot. Linear-sized independent sets in random cographs and increasing subsequences in separable permutations. preprint arXiv:2104.07444, 2021.

[BBF+18] F. Bassino, M. Bouvel, V. Féray, L. Gerin, A. Pierrot. The Brownian limit of separable permutations. *Ann. Probab.*, vol.46 (4), p.2134-2189, 2018.

[BBF+20] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M.Maazoun, A. Pierrot. Universal limits of substitution-closed permutation classes, *Journal of the European Mathematical Society*, vol. 22 (11), pp. 3565–3639, 2020.

[BBF+19b] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, A. Pierrot. Scaling limits of permutation classes with a finite specification: a dichotomy. Preprint arXiv:1903.07522, 2019.

[MRRY20] T. Mansour, R. Rastegar, A. Roitershtein, G. Yıldırım. The longest increasing subsequence in involutions avoiding 3412 and another pattern. Preprint arXiv:2001:10030, 2020.

[Rom15] D. Romik. *The Surprising Mathematics of Longest Increasing Subsequences.* Cambridge University Press (2015).

KEYNOTE ADDRESS

Sorting with stacks and queues: some recent developments

Luca Ferrari

luca.ferrari@unifi.it

University of Firenze

(This talk is based on joint work with Giulio Cerbai, Lapo Cioni, Anders Claesson, Einar Steingrímsson.)

The problem of sorting permutations using stacks and queues (and other devices) is a very classical one in the realm of Permutation Patterns. Three of the most important research lines in this context are the following:

- characterize and enumerate sortable permutations;
- design optimal sorting algorithms;
- investigate properties of the associated sorting map.

In my talk I will illustrate three results, one for each of the above general questions.

First, I will consider an extension of a device studied by Rebecca Smith, consisting of a decrasing stack followed by an increasing one. Since the analysis of such a DI machine reveals strong analogies with the case of a single stack, I will investigate what happens by adding one more decreasing stack in front. The resulting DDI machine shows in fact some similarities with the general 2-stacksort problem (for instance, in both cases the class of sortable permutations has an infinite basis). However, a major difference is that for the DDI machine I will be able to describe an optimal sorting algorithm (which instead is at present unknown for the case of two stacks in series).

Then I will introduce a new family of sorting devices, consisting of two successive passes through a stack, such that during the first pass the content of the stack is required to avoid a given permutation σ (whereas during the second pass the stack has to be increasing). The resulting machine is called *σ-machine*, and I will give some results concerning the characterization and enumeration of permutations sortable using a σ-machine (called *σ-sortable permutations*). In particular, I will describe complete results for three patterns of length 3, whereas the remaining three patterns are still unsolved. Along the way, I will also show a rather curious appearance of Catalan numbers.

Finally, I will consider the algorithm `Queuesort` (which is obtained from `Stacksort` by replacing the stack with a queue and adding the operation of bypass of the queue), and I will provide results concerning the combinatorics of the preimages of the associated sorting operator. Specifically, after an alternative description of `Queuesort` (directly on permutations, without any reference to the queue), I will describe a recursive procedure to generate all preimages of a given permutations, then I will address (and completely solve) the problem of characerizing the allowed cardinalities for the set of preimages of a given permutations, finally I will give a formula for the number of preimages of permutations having a specific shape.

Scattered through the talk, I will also mention several open problems that may give suggestions for further research.

Spherical Schubert varieties and pattern avoidance

Christian Gaetz

crgaetz@gmail.com

Massachusetts Institute of Technology

Schubert varieties X_w are indexed by permutations w and are of central importance in several areas of algebraic geometry and representation theory. Going back to the groundbreaking work of Lakshmibai–Sandhya [4] characterizing smooth Schubert varieties as those where w avoids 3412 and 4231, many important geometric and combinatorial properties of Schubert varieties have been shown to be determined by pattern avoidance conditions.

A normal variety X is called *H-spherical* for the action of the complex reductive group H if it contains a dense orbit of some Borel subgroup of H. We resolve a conjecture of Hodges–Yong [3] by showing that their *spherical permutations* are characterized by avoiding 21 patterns of length 5. Together with results of Gao–Hodges–Yong [2] this implies that the sphericality of a Schubert variety X_w with respect to the largest possible Levi subgroup is characterized by this same pattern avoidance condition. Spherical permutations are closely related to Tenner's [5] *Boolean permutations* for which we give a new characterization.

[1] Christian Gaetz Spherical Schubert varieties and pattern avoidance. 2021. arXiv:2104.03264 [math.CO]

[2] Yibo Gao, Reuven Hodges, and Alexander Yong. Classification of Levi-spherical Schubert varieties. 2021. arXiv:2104.10101 [math.CO].

[3] Reuven Hodges and Alexander Yong. Coxeter combinatorics and spherical Schubert geometry. 2020. arXiv:2007.09238 [math.RT].

[4] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$. *Proc. Indian Acad. Sci. Math. Sci.*, 100(1):45–52, 1990.

[5] Bridget Eileen Tenner. Pattern avoidance and the Bruhat order. *J. Combin. Theory Ser. A*, 114(5):888–905, 2007.

KEYNOTE ADDRESS

Patterns in substitution-closed permutations: a probabilistic approach

Lucas Gerin

lucas.gerin@polytechnique.edu

École Polytechnique (Palaiseau, France)

(This talk is based on joint work with F.Bassino, M.Bouvel, V.Féray, M.Maazoun, A.Pierrot.)

The aim of the talk is to survey our recent papers [1, 2, 3] regarding the asymptotic behaviour of pattern occurences in a large uniform permutation in $\mathcal{C}$, where $\mathcal{C}$ is a given permutation class.

We developped a probabilistic approach which could *in principle* give interesting results for every pattern in every substitution-closed class. The method is better explained with the iconic class of separable permutations. Some of our results were presented in the previous PERMUTATION PATTERNS conferences so we will try here to focus on the probabilistic flavour of our work.

Patterns in separable permutations: random variables Λ_π

A *separable permutation* is a permutation which avoids both 2413 and 3142. Equivalently, the class $\mathcal{S}$ of separable permutations can be recursively defined as follows:

1. The trivial permutation 1 belongs to $\mathcal{S}$;
2. If $\sigma, \sigma' \in \mathcal{S}$ then $\sigma \oplus \sigma' \in \mathcal{S}$;
3. If $\sigma, \sigma' \in \mathcal{S}$ then $\sigma \ominus \sigma' \in \mathcal{S}$.

The latter definition is more convenient for our purpose as it allows us to encode separable permutations with simple trees called *separation trees*. Let now σ_n be a uniform separable permutation of size n (very soon $n \to +\infty$). Then the separation tree associated to σ_n has a nice combinatorial and probabilistic description which enables us to obtain fairly accurate asymptotic results.

For a pattern π of size k (k is typically fixed and small) we denote by

$$\mathrm{occ}(\pi, \sigma_n) \in \left\{0, 1, 2, \dots, \binom{n}{k}\right\}$$

the number of *occurences* of π in σ_n. When $n \to +\infty$ the random variable $\mathrm{occ}(\pi, \sigma_n)$ grows at most like $\binom{n}{k} \sim \frac{n^k}{k!}$. Here is our main result.

Theorem 1. *For every $k \geq 1$ and every π of size k there exists a random variable $\Lambda_\pi \in [0,1]$ such that*

$$\frac{1}{\binom{n}{k}} \operatorname{occ}(\pi, \sigma_n) \overset{n \to +\infty}{\to} \Lambda_\pi, \tag{1}$$

in distribution. Furthermore if π is separable then

- *With probability one we have $0 < \Lambda_\pi < 1$. In words: $\operatorname{occ}(\pi, \sigma_n)$ really scales like $\binom{n}{k}$.*
- *Λ_π is a* true *random variable (*i.e. *not a constant).*

We will show the results of our best efforts to have a description of the Λ_π's. Even the simple case of Λ_{12} is challenging, so far we are only able to compute its first moments:

$$\mathbb{E}[\Lambda_{12}] = \frac{1}{2}; \ \mathbb{E}[\Lambda_{12}^2] = \frac{17}{60}; \ \mathbb{E}[\Lambda_{12}^3] = \frac{7}{40}; \ \mathbb{E}[\Lambda_{12}^4] = \frac{6361}{55440}; \ \mathbb{E}[\Lambda_{12}^5] = \frac{1741}{22176}.$$

We do not even know if Λ_{12} has a density in $(0,1)$. In the talk we will give a probabilistic representation of Λ_π and explain how to compute the above numbers.

A strategy for proving Theorem 1 is to introduce some generating functions which allow us to compute (almost) explicitly some quantities related to $\operatorname{occ}(\pi, \sigma_n)$. We then can use the robust and powerful framework of *Analytic Combinatorics* [4] to derive asymptotics.

The Brownian limit of separable permutations

We then realized that in order to have a deeper understanding of the convergence in (1) and to obtain similar results for more general permutation classes we need to have a higher-level description of the convergence. Indeed Theorem 1 can be reformulated with the settings of permutons [5] which can be seen as convergence of rescaled permutation diagrams. Our results reveal that separable permutations converge in distribution in the sense of permutons towards a continuous and fractal permuton that we called the *Brownian separable permuton*. See Figure 1 below.

We will try to give some insights provided by such a convergence result. If time permits we will also discuss how to go beyond the case of separable permutations and show *universality* of the Brownian separable permuton.

[1] F. Bassino, M. Bouvel, V. Féray, L. Gerin, A. Pierrot. The Brownian limit of separable permutations. *Ann. Probab.*, vol.46 (4), p.2134-2189, 2018.

[2] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M.Maazoun, A. Pierrot. Universal limits of substitution-closed permutation classes, *J. Eur. Math. Soc.*, vol. 22 (11), pp. 3565–3639, 2020.

[3] F. Bassino, M. Bouvel, V. Féray, L. Gerin, M. Maazoun, A. Pierrot. Scaling limits of permutation classes with a finite specification: a dichotomy. Preprint arXiv:1903.07522, 2019.

[4] Ph. Flajolet, R. Sedgewick. *Analytic combinatorics.* Cambridge University Press, 2009.

[5] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Rath, R. M. Sampaio. Limits of permutation sequences. *Journal of Combinatorial Theory, Series B*, vol. 103 (2013) n.1, p.93–113.

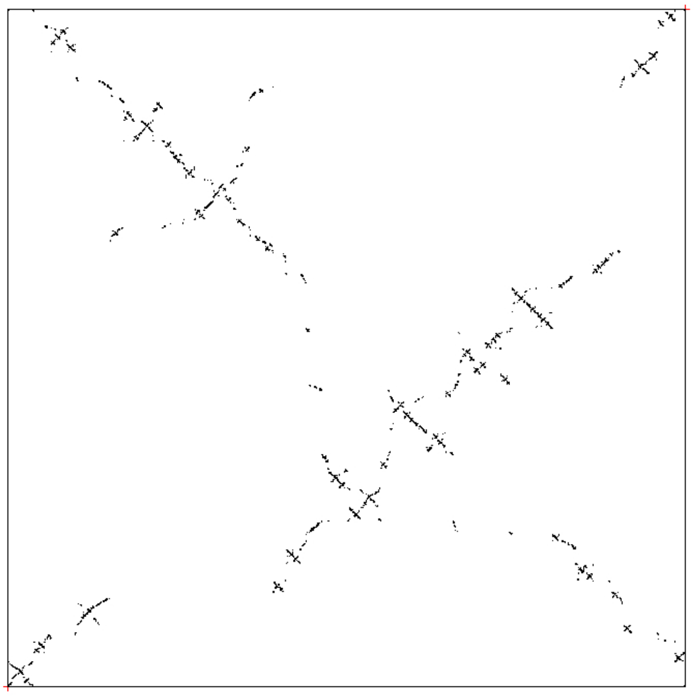


Figure 1: The diagram of a uniform separable permutation of size $n = 17705$ (simulation by M.Maazoun). This can be seen as a realization of the Brownian separable permuton.

Some combinatorial results on smooth permutations

Shoni Gilboa

tipshoni@gmail.com

The Open University of Israel

(This talk is based on joint work with Erez Lapid.)

1 Introduction

This is an extended abstract to the paper [5], which contains all the proofs and additional details.

Fix an integer $n \geq 1$. Consider the symmetric group S_n of all the permutations of the set $[n] = \{1, 2, \ldots, n\}$. For any $1 \leq i < j \leq n$, let $T_{i,j} \in S_n$ be the transposition interchanging i and j, and let $\mathcal{T} = \{T_{i,j} : 1 \leq i < j \leq n\}$ be the set of transpositions in S_n. Recall that the Bruhat order on S_n may be described combinatorially as the partial order generated by the relations $\sigma < \sigma T_{i,j}$ for every $\sigma \in S_n$ and every $1 \leq i < j \leq n$ for which $\sigma(i) < \sigma(j)$. It is also given by

$$\tau \leq \sigma \iff \#(\tau([i]) \cap [j]) \geq \#(\sigma([i]) \cap [j]) \text{ for every } i, j \in [n].$$

For every permutation $\sigma \in S_n$ define the *2-table* of σ to be

$$\mathbf{C}_{\mathcal{T}}(\sigma) = \{\tau \in \mathcal{T} : \tau \leq \sigma\}.$$

For every $\sigma \in S_n$ we have $\#\mathbf{C}_{\mathcal{T}}(\sigma) \geq \mathrm{inv}(\sigma)$ (where $\mathrm{inv}(\sigma) = \#\{i < j : \sigma(i) > \sigma(j)\}$ is the number of inversions of σ) [7]. If $\#\mathbf{C}_{\mathcal{T}}(\sigma) = \mathrm{inv}(\sigma)$, then σ is called *smooth*, a terminology that is justified by the fact that this condition also characterizes the smoothness of the Schubert variety pertaining to σ [ibid.]. Another well-known combinatorial characterization of the smoothness of σ is that σ is 4231 and 3412 avoiding [6].

We give another way of looking at smooth permutations combinatorially. Our main result is the characterization of smooth permutations in terms of their 2-3-*table*. By definition, the 2-3-table of a permutation σ is the set of transpositions and the 3-cycles that are $\leq \sigma$. The 2-3-table of a smooth permutation satisfies some simple combinatorial properties and conversely, any set of transpositions and 3-cycles satisfying these conditions arises from a smooth permutation. Moreover, a smooth permutation σ may be recovered from its 2-3-table by taking the product of the transposition $\leq \sigma$ in a suitable order, governed by the additional data in the 2-3-table (Theorem 1).

This also gives a bijection between smooth permutations and Dyck paths with additional data (Theorem 3), which yields another approach for enumerating smooth permutations and subclasses thereof. Another interesting consequence is yet another combinatorial characterization of smooth permutations: we show that $\sigma \in S_n$ is smooth if and only if the intersection of $\{\tau \in S_n : \tau \leq \sigma\}$ with every conjugate of a parabolic subgroup of S_n admits a maximum (Theorem 9). Finally, we obtain an intriguing relation between *covexillary* (i.e., 3412 avoiding) permutations and smooth ones (Theorem 12).

2 The 2-3-table of a permutation

Distinct smooth permutations may have the same 2-table (for example, for $n = 3$, $\mathbf{C}_{\mathcal{T}}((231)) = \{T_{1,2}, T_{2,3}\} = \mathbf{C}_{\mathcal{T}}((312))$). However, we show that smooth permutations are distinguishable from each other at the 'next level'. More precisely, let $\mathcal{C}^{2,3} \subset S_n$ be the set of permutations consisting of a single cycle of length 2 or 3. Denote the 3-cycle permutation $i \mapsto j \mapsto k \mapsto i$ with $i < j < k$ by $R_{i,j,k}$, so that

$$\mathcal{C}^{2,3} = \mathcal{T} \cup \{R_{i,j,k}, R_{i,j,k}^{-1} : 1 \leq i < j < k \leq n\}$$

We define the 2-3-*table* of a permutation $\sigma \in S_n$ to be

$$\mathbf{C}(\sigma) = \{\tau \in \mathcal{C}^{2,3} : \tau \leq \sigma\}.$$

We say that a downward closed subset A of $\mathcal{C}^{2,3}$ is *admissible* if it satisfies the following two conditions.

- If $R_{i,j,l}, R_{i,k,l}^{-1} \in A$ (with $i < j < l$ and $i < k < l$), then $T_{i,l} \in A$.
- Whenever $T_{i,j}, T_{j,k} \in A$ (with $i < j < k$), at least one of $R_{i,j,k}$ and $R_{i,j,k}^{-1}$ is in A.

For an admissible set $A \subseteq \mathcal{C}^{2,3}$, we say that a linear order $\prec$ on $A \cap \mathcal{T}$ is *compatible* (with A) if whenever $T_{i,j}, T_{j,k} \in A$ (with $i < j < k$):

- If $T_{i,k} \in A$, then either $T_{i,j} \prec T_{i,k} \prec T_{j,k}$ or $T_{j,k} \prec T_{i,k} \prec T_{i,j}$.
- If $T_{i,k} \notin A$, then $R_{i,j,k} \in A \iff T_{i,j} \prec T_{j,k}$.

We show that for any admissible $A \subseteq \mathcal{C}^{2,3}$, a compatible order on $A \cap \mathcal{T}$ always exists and moreover, any compatible order may be obtained from any other compatible order by a sequence of the following operations.

- Interchanging the order of two adjacent commuting transpositions.
- Switching the order of consecutive $T_{i,j}$, $T_{i,k}$, $T_{j,k}$ (with $i < j < k$) to $T_{j,k}$, $T_{i,k}$, $T_{i,j}$, or vice versa.

Using the braid relations $T_{i,j}T_{i,k}T_{j,k} = T_{j,k}T_{i,k}T_{i,j}$ it follows that the product $\pi(A)$ of the transpositions in A according to a compatible order is well defined.

Our main result is the following.

Theorem 1. *The map $\sigma \mapsto \mathbf{C}(\sigma)$ defines a bijection between the smooth permutations of S_n and the admissible sets. The inverse bijection is given by $A \mapsto \pi(A)$. In particular, every smooth permutation may be written as the product, in an appropriate order, of the transpositions in its 2-table (each appearing exactly once).*

Moreover, for any admissible $A \subseteq \mathcal{C}^{2,3}$,

$$\pi(A) = \max\{\tau \in S_n : \mathbf{C}(\tau) = A\} = \max\{\tau \in S_n : \mathbf{C}_{\mathcal{T}}(\tau) = A \cap \mathcal{T},\, \mathbf{C}(\tau) \subseteq A\}$$

where the maximum (i.e., the greatest element, which in particular exists) is with respect to the Bruhat order.

The proof of Theorem 1 also yields the following corollary, which may be of independent interest.

Corollary 2. *For any smooth permutation $e \neq \sigma \in S_n$ there is a smooth permutation $\tilde{\sigma} \in S_n$ such that $\mathrm{inv}(\tilde{\sigma}) = \mathrm{inv}(\sigma) - 1$ and at least one of $\sigma^{-1}\tilde{\sigma}$ or $\tilde{\sigma}\sigma^{-1}$ is a simple reflection, i.e., in the set $\{T_{i,i+1} : i \in [n-1]\}$.*

3 Relation to Dyck paths

We may also interpret the bijection of Theorem 1 in terms of more familiar combinatorial objects, namely Dyck paths. For any $n \geq 1$, let

$$\mathcal{F}_n = \{f : [n] \to [n] : f \text{ is weakly increasing, } f(i) \geq i \text{ for all } i \in [n]\}.$$

We can view elements of $\mathcal{F}_n$ as Dyck paths from $(0,0)$ to (n,n) by taking $f(i)$ to be the minimal x such that the lattice point (x,i) lies in the path. We define a *decorated* Dyck path to be a function $f \in \mathcal{F}_n$ together with a function $g : [n] \to \{0,1\}$ such that

- $g(i) = 0$ whenever $f(f(i)) = f(i)$.
- $g(i) = g(i+1)$ whenever $i < n$ and $f(i+1) < f(f(i))$.

In terms of Dyck paths, such a decoration g corresponds to an (unrestricted) 2-coloring of a certain distinguished set of vertices of the path. Denote by $\mathcal{P}_n$ the set of decorated Dyck paths.

For every $1 \leq i < j \leq n$ let $R_{[i,j]} \in S_n$ be the cycle permutation $i \to i+1 \to \cdots \to j \to i$. For consistency, $R_{[i,i]}$ is the identity permutation for all i. The following theorem is in the spirit of Skandera's factorization of smooth permutation [9].

Theorem 3. *The map*

$$(f,g) \to \sigma(f,g) = (R_{[j_l,f(j_l)]} \cdots R_{[j_1,f(j_1)]})^{-1} R_{[i_k,f(i_k)]} \cdots R_{[i_1,f(i_1)]}, \qquad (*)$$

where $g^{-1}(0) = \{i_1, \ldots, i_k\}$, $g^{-1}(1) = \{j_1, \ldots, j_l\}$ with $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$, is a bijection between the decorated Dyck paths and the smooth permutations in S_n. Moreover, the expression on the right-hand side of (*) is reduced. Furthermore, for any $(f,g) \in \mathcal{P}_n$,

$$\mathbf{C}_{\mathcal{T}}(\sigma(f,g)) = \{T_{i,j} : i < j \leq f(i)\}.$$

3.1 Enumerative consequences

Using Theorem 3 we can recover several known enumerative results concerning smooth permutations.

First note that by Theorem 3, the permutation $\sigma(f,g)$, for $(f,g) \in \mathcal{P}_n$, is indecomposable (i.e, there does not exist $1 \leq k < n$ such that $(\sigma(f,g))([k]) = [k]$) if and only if $f(i) > i$ for all $i \in [n-1]$.

Proposition 4. *For every $n \geq 1$, let*

$$\tilde{p}_n = \#\{(f,g) \in \mathcal{P}_n : f(i) > i \text{ for all } i \in [n-1]\}.$$

Then, $\tilde{p}_n = \tilde{p}_{n-1} + 2\sum_{i=1}^{n-2} C_{i-1}\tilde{p}_{n-i}$ for every $n \geq 2$. Consequently,

$$\sum_{n=1}^{\infty} \tilde{p}_n x^n = \left(\frac{1}{x} - \frac{1}{\sqrt{1-4x}}\right)^{-1}, \quad \sum_{n=1}^{\infty} \#\mathcal{P}_n x^n = \left(\frac{1}{x} - \frac{1}{\sqrt{1-4x}} - 1\right)^{-1}.$$

Corollary 5 (cf. [1, 2, 8] and the references therein)**.** *The generating function of smooth indecomposable permutations in S_n (for $n \geq 1$) is $\left(\frac{1}{x} - \frac{1}{\sqrt{1-4x}}\right)^{-1}$ and the generating function of smooth permutations in S_n (for $n \geq 1$) is $\left(\frac{1}{x} - \frac{1}{\sqrt{1-4x}} - 1\right)^{-1}$.*

The following proposition, combined with Theorem 3, enable us to recover additional enumerative results concerning smooth permutations.

Proposition 6. *Let $(f,g) \in \mathcal{P}_n$. Then,*

1. *$\sigma(f,g)$ is 231 avoiding if and only if $g \equiv 0$.*
2. *$\sigma(f,g)$ is 321 avoiding if and only if $f(i) \leq i+1$ for all $i \in [n-1]$.*

Note that Theorem 3 and the second part of Proposition 6 imply that a smooth permutation is 321 avoiding if and only if its 2-table consists of simple reflections.

It is well known that $\#\mathcal{F}_n$, $n \geq 1$ is the n-th Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Since clearly $\#\{(f,g) \in \mathcal{P}_n : g \equiv 0\} = \#\mathcal{F}_n$, combining Theorem 3 and the first part of Proposition 6 we recover the standard fact that the number of 231 avoiding permutations in S_n (which are automatically smooth) is C_n.

Next we use the second part of Proposition 6 in a similar manner; first we establish the following result.

Proposition 7. *For every $n \geq 1$,*

$$\#\{(f,g) \in \mathcal{P}_n : f(i) \leq i+1 \text{ for every } i \in [n-1]\} = F_{2n-1},$$

where $(F_k)_{k \geq 1}$ is the Fibonacci sequence (given by $F_1 = F_2 = 1$ and $F_k = F_{k-1} + F_{k-2}$ for $k > 2$).

Corollary 8 ([10, 3])**.** *The number of 321 avoiding smooth permutations in S_n is F_{2n-1}.*

4 Intersection of Bruhat intervals with conjugates of parabolic subgroups

As an application of Theorem 1 we give another remarkable characterization of smooth permutations. Let $\mathfrak{X}$ be a partition of $[n]$. Consider the subgroup $S_{\mathfrak{X}}$ of S_n preserving every $X \in \mathfrak{X}$. The group $S_{\mathfrak{X}}$ is isomorphic to the direct product of $S_{\#X}$ over $X \in \mathfrak{X}$. The product order on $S_{\mathfrak{X}}$, which we denote by $\leq_{\mathfrak{X}}$, is (in general, strictly) stronger than the one induced from S_n. We say that an element of $S_{\mathfrak{X}}$ is $\mathfrak{X}$-*smooth* if all its coordinates in $S_{\#X}$, $X \in \mathfrak{X}$ are smooth. (This is weaker than smoothness in S_n; for instance, if $\mathfrak{X}$ is the partition $\{\{1,3\},\{2,4\}\}$ then the permutation (3412) is $\mathfrak{X}$-smooth but not smooth.)

Theorem 9. *A permutation $\sigma \in S_n$ is smooth if and only if for every partition $\mathfrak{X}$ of $[n]$, the set $\{\tau \in S_{\mathfrak{X}} : \tau \leq \sigma\}$ admits a maximum $\sigma_{\mathfrak{X}}$ with respect to $\leq_{\mathfrak{X}}$. Moreover, in this case $\sigma_{\mathfrak{X}}$ is $\mathfrak{X}$-smooth, and it is the identity permutation if and only if $\mathbf{C}_{\mathcal{T}}(\sigma) \cap S_{\mathfrak{X}} = \emptyset$.*

In the proof of Theorem 9 we use the following two results, which may be of independent interest. For any $\sigma \in S_n$ define the "maximal function" $\mu_\sigma : [n] \to [n]$ of σ by

$$\mu_\sigma(i) = \max \sigma([i]).$$

Lemma 10. *Let σ be 4231 avoiding but not smooth. Then, there exists an index i such that*

$$\mu_\sigma(\mu_{\sigma^{-1}}(i)) > \mu_\sigma(i) > i \text{ and } \mu_{\sigma^{-1}}(\mu_\sigma(i)) > \mu_{\sigma^{-1}}(i) > i.$$

Clearly, if $\tau \leq \sigma$ then $\mu_\tau \leq \mu_\sigma$ pointwise, although the converse in not true in general. We say that $\sigma \in S_n$ is *defined by inclusions* if for every $\tau \in S_n$ we have

$$\tau \leq \sigma \iff \mu_\tau(i) \leq \mu_\sigma(i) \text{ and } \mu_{\tau^{-1}}(i) \leq \mu_{\sigma^{-1}}(i) \text{ for all } i.$$

By [4], a permutation is defined by inclusions if and only if it is 4231, 35142, 42513 and 351624 avoiding. We give another characterization of permutations defined by inclusions.

Proposition 11. *A permutation $\sigma \in S_n$ is defined by inclusions if and only if for every partition $\mathfrak{X}$ of $[n]$ and $\tau \in S_{\mathfrak{X}}$ we have*

$$\tau \leq \sigma \iff \tau_X \leq \sigma \ \forall X \in \mathfrak{X},$$

where $\tau_X \in S_{\mathfrak{X}}$ is defined by $\tau_X(r) = \tau(r)$ if $r \in X$ and $\tau_X(r) = r$ otherwise.

5 Relation to Covexillary permutations

Using Theorem 1, we can also give a curious relation between smooth permutations and covexillary ones. Recall that a permutation is called covexillary if it is 3412 avoiding.

Theorem 12. *The 2-3-table of any covexillary permutation is admissible. Therefore, the map $\tau \mapsto \pi(\mathbf{C}(\tau))$ is an idempotent function from the set of covexillary permutations onto the subset of smooth permutations. Moreover, this map is order preserving and for any covexillary $\tau \in S_n$,*

$$\pi(\mathbf{C}(\tau)) = \min\{\sigma \in S_n \text{ smooth} : \sigma \geq \tau\}.$$

Furthermore, the following is true.

Proposition 13. *For any covexillary $\tau \in S_n$ there are covexillary permutations $\tau = \tau_0 < \tau_1 < \cdots < \tau_r = \pi(\mathbf{C}(\tau))$ in S_n such that $\tau_{i-1}^{-1}\tau_i$ is a transposition for every $1 \le i \le r$.*

In the proof of Proposition 13 we use the following lemmas which may be of independent interest.

Lemma 14. *Suppose that $\tau \in S_n$ is covexillary but not smooth. Then, there exist $i < j < k < l$ such that $\tau(l) < \tau(j) < \tau(k) < \tau(i)$ and $\tau T_{j,k}$ is covexillary.*

Lemma 15. *A permutation σ is defined by inclusions if and only if for any $\tau \le \sigma$ and $i < j < k < l$ such that $\tau(l) < \tau(j) < \tau(k) < \tau(i)$ we have $\tau T_{j,k} \le \sigma$.*

[1] Miklós Bóna. The permutation classes equinumerous to the smooth class. *Electron. J. Combin.*, 5:Research Paper 31, 12, 1998.

[2] Mireille Bousquet-Mélou and Steve Butler. Forest-like permutations. *Ann. Comb.*, 11(3-4):335–354, 2007.

[3] C. K. Fan. Schubert varieties and short braidedness. *Transform. Groups*, 3(1):51–56, 1998.

[4] V. Gasharov and V. Reiner. Cohomology of smooth Schubert varieties in partial flag manifolds. *J. London Math. Soc. (2)*, 66(3):550–562, 2002.

[5] Shoni Gilboa and Erez Lapid. Some combinatorial results on smooth permutations. To appear in *Journal of Combinatorics*; preprint available at `arXiv:1912.04725`.

[6] V. Lakshmibai and B. Sandhya. Criterion for smoothness of Schubert varieties in $\mathrm{Sl}(n)/B$. *Proc. Indian Acad. Sci. Math. Sci.*, 100(1):45–52, 1990.

[7] V. Lakshmibai and C. S. Seshadri. Singular locus of a Schubert variety. *Bull. Amer. Math. Soc. (N.S.)*, 11(2):363–366, 1984.

[8] Edward Richmond and William Slofstra. Staircase diagrams and enumeration of smooth Schubert varieties. *J. Combin. Theory Ser. A*, 150:328–376, 2017.

[9] Mark Skandera. On the dual canonical and Kazhdan-Lusztig bases and 3412-, 4231-avoiding permutations. *J. Pure Appl. Algebra*, 212(5):1086–1104, 2008.

[10] Julian West. Generating trees and forbidden subsequences. In *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics (New Brunswick, NJ, 1994)*, volume 157, pages 363–374, 1996.

Classical pattern-avoiding permutations of length 5

Anthony Guttmann

guttmann@unimelb.edu.au

The University of Melbourne

(This talk is based on joint work with Nathan Clisby, Andrew Conway.)

We have made a systematic numerical study of the 16 Wilf classes of length-5 classical pattern-avoiding permutations from their generating function coefficients.

For the 16 Wilf classes of length-5 PAPs there is only one, $Av(12345)$, for which the generating function is known [5], (and is D-finite). In one other case, $Av(31245)$, the growth constant is known [3], and for $Av(53421)$ the growth constant can be expressed in terms of that of $Av(1324)$ [4], which has been estimated to some accuracy in [6]. These three known growth constant results can all be obtained from Theorem 4.2 in [4], alternatively proved as Theorem 3.2 in [1].

We wrote a general purpose program to generate the coefficients, which produced the first seventeen coefficients of all 16 Wilf classes. That is, all coefficients up to and including $O(x^{16})$. This extends by three coefficients the available series in 12 of the 16 Wilf classes, and confirms all known coefficients in two further classes. As mentioned, $Av(12345)$ is completely solvable, so an arbitrary number of coefficients is available, and for $Av(31245)$ a special purpose program has been written giving 38 coefficients [2]. Our program is described in the next section.

We have extended the number of known coefficients in twelve of the sixteen classes. Careful analysis, including sequence extension, has allowed us to estimate the growth constant of all classes, and in some cases to estimate the sub-dominant power-law term associated with the exponential growth.

In ten of the sixteen classes cases we find the familiar power-law behaviour, so that the coefficients behave like $s_n \sim C \cdot \mu^n \cdot n^g$, while in the remaining six cases we find a stretched exponential as the sub-dominant term, so that the coefficients behave like $s_n \sim C \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$, where $0 < \sigma < 1$. We have also classified the 120 possible permutations into the 16 distinct classes.

We give compelling numerical evidence, and in one case a proof, that all 16 Wilf-class generating function coefficients can be represented as moments of a nonnegative measure on $[0, \infty)$. Such sequences are known as *Stieltjes moment sequences*. They have a number of nice properties, such as log-convexity, which can be used to provide quite strong rigorous lower bounds.

Stronger bounds still can be established under plausible monotonicity assumptions about the terms in the continued-fraction expansion of the generating functions implied by the Stieltjes property. In this way we provide strong (non-rigorous) lower bounds to the growth constants, which are typically within a few percent of the exact value.

[1] M. Albert, J. Pantone and V. Vatter, *On the growth of merges and staircases of permutation classes*, Rocky Mtn. J. Math. **49**(2) 355–367, (2019).

[2] Y. Biers-Ariel, *Flexible schemes for pattern-avoiding permutations* arXiv:1905.07283v2. Series in OEIS A116485.

[3] M. Bóna *The limit of a Stanley-Wilf sequence is not always rational and layered patterns beat monotone patterns.* J. Combin. Theory Ser. A **110** (2), 223-235, (2005).

[4] M. Bóna *New records in Stanley-Wilf limits.* J. Combin.**28** (1), 75-85, (2007).

[5] A. Bostan, A Elvey Price, A J Guttmann and J-M Maillard, *Stieltjes moment sequences for pattern-avoiding permutations* arXiv:2001.00393, Elec. J. Combin. **27**(4) #P4.20 (2020).

[6] A. R. Conway, A. J. Guttmann and P. Zinn-Justin, *1324-avoiding permutations revisited*, Adv. in Appl. Math., **96**, 312-333, (2018).

Sorting Time of Permutation Classes

Vít Jelínek

jelinek@iuuk.mff.cuni.cz

Computer Science Institute, Charles University, Prague

(This talk is based on joint work with Michal Opler and Jakub Pekárek.)

Abstract

In this talk, we will deal with two well-known algebraic structures defined on the set of (finite) permutations. The first structure of interest is the poset induced by the classical Wilf containment relation. The second relevant structure is the group structure induced on S_n by the composition operation. We will thus view a permutation $\sigma \in S_n$ both as a bijection $[n] \to [n]$ and as a sequence of numbers $(\sigma(1), \sigma(2), \ldots, \sigma(n))$. If $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$ is another permutation of size n, the *composition of* π *and* σ, denoted $\pi \circ \sigma$, is the permutation $(\pi(\sigma(1)), \pi(\sigma(2)), \ldots, \pi(\sigma(n)))$.

Consider a permutation class $\mathcal{C}$ and a permutation $\pi \in S_n$. We say that $\mathcal{C}$ *can sort* π *in* k *steps*, if π can be obtained by composing k (not necessarily distinct) permutations from $\mathcal{C}$. The $\mathcal{C}$*-sorting time of* π, denoted $\mathrm{st}(\mathcal{C}; \pi)$, is the smallest k such that $\mathcal{C}$ can sort π in k steps; if no such k exists, we put $\mathrm{st}(\mathcal{C}; \pi) = +\infty$. For an integer $n \in \mathbb{N}$, the *worst-case* $\mathcal{C}$*-sorting time*, denoted $\mathrm{wst}(\mathcal{C}; n)$, is defined as $\max\{\mathrm{st}(\mathcal{C}; \pi);\ \pi \in S_n\}$.

Our goal is to describe the possible asymptotic behaviors of the function $\mathrm{wst}(\mathcal{C}; n)$. Our main result is that any class $\mathcal{C}$ falls into one of the following five categories:

- $\mathrm{wst}(\mathcal{C}; n) = +\infty$ for every n large enough,
- $\mathrm{wst}(\mathcal{C}; n) = \Theta(n^2)$,
- $\Omega(\sqrt{n}) \leq \mathrm{wst}(\mathcal{C}; n) \leq O(n)$,
- $\Omega(\log n) \leq \mathrm{wst}(\mathcal{C}; n) \leq O(\log^2 n)$, or
- $\mathrm{wst}(\mathcal{C}; n) = 1$ for all n.

In addition, we can characterize the classes in each of the five categories. We now sketch the main ideas in the proof of this result.

Notation

Let S_n be the set of all permutations of size n, and for a permutation class $\mathcal{C}$, let $\mathcal{C}_n$ denote the set $\mathcal{C} \cap S_n$.

We write π^r, π^c and π^{-1} for the reverse, complement and inverse of π, respectively. For a set of permutations $\mathcal{C}$, we write $\mathcal{C}^r = \{\pi^r;\ \pi \in \mathcal{C}\}$, $\mathcal{C}^c = \{\pi^c;\ \pi \in \mathcal{C}\}$ and $\mathcal{C}^{-1} = \{\pi^{-1};\ \pi \in \mathcal{C}\}$. Note that if $\mathcal{C}$ is a class, then so are $\mathcal{C}^r$, $\mathcal{C}^c$ and $\mathcal{C}^{-1}$.

For a pair of permutation classes $\mathcal{A}$ and $\mathcal{B}$, we write $\mathcal{A} \circ \mathcal{B}$ for $\{\alpha \circ \beta;\ \alpha \in \mathcal{A}, \beta \in \mathcal{B}\}$, $\mathcal{A}^{\circ k}$ for the k-fold composition $\mathcal{A} \circ \mathcal{A} \circ \cdots \circ \mathcal{A}$, and $\mathcal{A}^*$ for $\bigcup_{k \geq 1} \mathcal{A}^{\circ k}$. Note that $\mathcal{A} \circ \mathcal{B}$, $\mathcal{A}^{\circ k}$ and $\mathcal{A}^*$ are again permutation classes.

Let ι_k denote the identity permutation $(1, 2, \ldots, k)$, and let δ_k be its reversal. The following classes will be useful in stating our results:

- $\mathcal{L} := \mathrm{Av}(231, 312)$, known as *layered permutations*
- $\mathcal{F} := \mathcal{L} \cap \mathrm{Av}(321)$, known as the *Fibonacci class*, consisting of layered permutations with layers of length 1 or 2
- $\mathcal{PBT} := \mathcal{F}[\mathrm{Av}(21)]$, the *parallel block transposition class*, consisting of permutations that can be obtained from a Fibonacci permutation by inflating each element by an increasing permutation
- $\mathcal{R} := \{\iota_a \ominus \iota_b;\ a, b \in \mathbb{N}_0\}$, the *rotation class* – note that for $\rho \in \mathcal{R}_n$ and $\pi \in S_n$, $\rho \circ \pi$ is a cyclic rotation of the sequence π
- $\mathcal{RR} := \mathcal{R} \cup \mathcal{R}^r$, the class corresponding to rotations and their reversals
- $\mathcal{T} := \{\iota_a \oplus 21 \oplus \iota_b;\ a, b \in \mathbb{N}_0\} \cup \{1 \ominus \iota_c \ominus 1;\ c \in \mathbb{N}_0\} \cup \mathrm{Av}(21)$, the class of (cyclic) adjacent transpositions
- k-*Fringe* $:= \{\pi \oplus \iota_a \oplus \sigma;\ a \in \mathbb{N}_0, |\sigma| \leq k, |\pi| \leq k\}$, the permutations in which only the prefix of length k and the suffix of length k can be permuted.

Classes that cannot sort

Let us say that a class $\mathcal{C}$ *cannot sort*, if $\mathrm{wst}(\mathcal{C}; n) = +\infty$ for some n. Note that this is equivalent to $\mathcal{C}_n^* \neq S_n$. When seeking to characterize the maximal classes that cannot sort, we may restrict our attention to the classes $\mathcal{C}$ that are *composition closed*, that is, those that satisfy $\mathcal{C} = \mathcal{C}^*$. Composition closed permutation classes have been characterized by Atkinson and Beals [2]. Using their results, we obtain the following proposition.

Proposition 1. *A permutation class $\mathcal{C}$ cannot sort if and only if it satisfies one of the following two conditions:*

1. *there is an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, we have $\mathcal{C}_n \subseteq \mathcal{RR}_n$, or*
2. *there is a $k \in \mathbb{N}$ such that $\mathcal{C} \subseteq k$-Fringe $\cup$ (k-Fringe$)^r$.*

From infinity to n^2

Let us sketch the argument showing that any class that can sort has at most quadratic sorting time.

A *concatenation* of permutation classes $\mathcal{A}$ and $\mathcal{B}$, denoted $\mathcal{A}|\mathcal{B}$, is the set of permutations that can be obtained by concatenating a (possibly empty) sequence order-isomorphic to a permutation from $\mathcal{A}$ with a (possibly empty) sequence order-isomorphic to a permutation from $\mathcal{B}$. A *monotone juxtaposition* is a class of the form $\mathcal{A}|\mathcal{B}$ or $(\mathcal{A}|\mathcal{B})^{-1}$ with $\mathcal{A}, \mathcal{B} \in \{\mathrm{Av}(21), \mathrm{Av}(12)\}$ (so there are eight monotone juxtapositions in total).

It is not hard to see that any monotone juxtaposition has worst-case sorting time of order $O(\log n)$. Also, for the Fibonacci class $\mathcal{F}$ we can easily observe that $\mathrm{wst}(\mathcal{F}; n) = \Theta(n)$ and $\mathrm{wst}(\mathcal{F}^r; n) = \Theta(n)$.

Thus, if a class $\mathcal{C}$ has at least quadratic $\mathrm{wst}(\mathcal{C}; n)$, it cannot contain any monotone juxtaposition and any symmetry of $\mathcal{F}$ as a subclass. It follows from the results of Huczynska and Vatter [7] and further refinements by Albert et al. [1] and Homberger and Vatter [6], that such a class $\mathcal{C}$ is contained in a monotone grid class; more precisely [1, 6], such a class $\mathcal{C}$ can be written as a finite union of *peg classes.* A peg class is a monotone grid class in which each row and column of its gridding matrix has exactly one nonempty cell, and each nonempty cell is either the singleton class $\{1\}$ or one of the two monotone classes $\mathrm{Av}(12)$ and $\mathrm{Av}(21)$.

For a peg class $\mathcal{C}$, it is easy to check that if it can sort, then $\mathrm{wst}(\mathcal{C}; n)$ is at most quadratic. Additionally, if $\mathcal{C}$ is a peg class contained in $\mathcal{RR}$, and $\mathcal{C}'$ is a peg class contained in *k-Fringe* for some k, and if neither $\mathcal{C}$ nor $\mathcal{C}'$ are contained in $\mathrm{Av}(21) \cup \mathrm{Av}(12)$, then $\mathcal{C} \cup \mathcal{C}'$ has at most quadratic worst-case sorting time.

This line of reasoning eventually leads to the following proposition.

Proposition 2. *A class $\mathcal{C}$ that can sort satisfies* $\mathrm{wst}(\mathcal{C}; n) = O(n^2)$.

From quadratic to linear

To show there is a gap between quadratic and linear worst-case sorting times, and to describe the classes with quadratic time, we need to introduce another concept: the *reduced inversion number* of a permutation π, denoted $\mathrm{rin}(\pi)$, is the smallest integer $k \in \mathbb{N}_0$ such that π belongs to $\mathcal{T}^{\circ k} \circ \mathcal{RR}$. Intuitively, this means π can be transformed by a rotation and an optional reversal to a permutation with at most k inversions.

One may easily observe that the $\mathrm{rin}(\cdot)$ parameter satisfies

$$\mathrm{rin}(\pi \circ \sigma) \leq \mathrm{rin}(\pi) + \mathrm{rin}(\sigma).$$

In particular, defining $\mathrm{rin}(\mathcal{C}_n) := \max\{\mathrm{rin}(\sigma);\ \sigma \in \mathcal{C}_n\}$ and considering any $\pi \in S_n$, we see that

$$\mathrm{st}(\mathcal{C}; \pi) \geq \frac{\mathrm{rin}(\pi)}{\mathrm{rin}(\mathcal{C}_n)}.$$

Since $\max\{\mathrm{rin}(\pi);\ \pi \in S_n\} = \Theta(n^2)$, this yields a general lower bound

$$\mathrm{wst}(\mathcal{C}; n) \geq \Omega\left(\frac{n^2}{\mathrm{rin}(\mathcal{C}_n)}\right).$$

This simple lower bound is sufficient to characterize the classes with quadratic sorting speed.

Proposition 3. *For a class $\mathcal{C}$, the following are equivalent:*

- $\mathrm{wst}(\mathcal{C}; n) = \Omega(n^2)$, *and*
- $\mathrm{rin}(\mathcal{C}_n)$ *is bounded by a constant as* $n \to \infty$.

Additionally, if $\mathcal{C}$ *is a peg class and* $\mathrm{rin}(\mathcal{C}_n)$ *is not bounded, then* $\mathrm{rin}(\mathcal{C}_n) = \Omega(n)$ *and* $\mathrm{wst}(\mathcal{C}; n) = O(n)$.

Proposition 3 is proven by an analysis of possible sorting times of peg classes. We omit the details here.

From $\sqrt{n}$ to polylog

The most difficult part of our argument deals with the gap between sorting time $O(\log^2 n)$ and $\Omega(\sqrt{n})$. The criterion distinguishing the two types of classes is rather simple to state, though.

Proposition 4. *A permutation class* $\mathcal{C}$ *satisfies* $\mathrm{wst}(\mathcal{C}; n) = O(\log^2 n)$ *if and only if* $\mathcal{C}$ *contains as a subclass a monotone juxtaposition or any of the classes* $\mathcal{L}$, $\mathcal{L}^r$, $\mathcal{PBT}$ *or* $\mathcal{PBT}^r$. *In any other case,* $\mathrm{wst}(\mathcal{C}; n) = \Omega(\sqrt{n})$.

As we pointed out above, a monotone juxtaposition has logarithmic worst-case sorting time. For $\mathcal{L}$, $\mathcal{PBT}$ and their symmetries, we are able to show an upper bound of $O(\log^2 n)$, while the best lower bound we can give is of order $\Omega(\log n)$. Tightening these bounds is an open problem.

Question 5. *Can you get better bounds for* $\mathrm{wst}(\mathcal{L}; n)$, $\mathrm{wst}(\mathcal{PBT}; n)$, *and their symmetries?*

We remark that sorting by layered permutations corresponds to sorting by a sequence of pop-stacks in "genuine series", which has been considered (for two pop-stacks) by Atkinson and Stitt [3].

The hardest part in the proof of Proposition 4 is to prove a lower bound for a class $\mathcal{C}$ not containing a monotone juxtaposition or a symmetry of $\mathcal{L}$ or $\mathcal{PBT}$. Fix a class $\mathcal{C}$ of this form. For this bound, we will work with the concept of *tree-width*, a graph-theoretic parameter whose definition we omit here.

For a permutation π, the *adjacency graph* $G(\pi)$ is a graph whose vertices are the elements of π, and two vertices are connected by an edge if and only if the corresponding two elements have adjacent positions or adjacent values. In particular, the adjacency graph is a union of two paths, one visiting the vertices in left-to-right order, and the other in bottom-to-top order.

The *tree-width* of the permutation π, denoted $\mathrm{tw}(\pi)$, is then defined as the treewidth of $G(\pi)$. Note that there are permutations of size n with tree-width $\Omega(n)$.

Our argument is based on an idea of Berendsohn [4], which implicitly relates sorting time with bounds on the tree-width of a certain staircase-shaped permutation. For an integer k, let the $(k, \mathcal{C})$-*staircase* be the generalized grid class whose gridding matrix has size $k \times k$, whose entries on the main diagonal are equal to Av(21), the entries on the diagonal immediately below are all equal to $\mathcal{C}$, and all the other entries are empty. Building upon the ideas of Berendsohn [4], we can show the following fact.

Proposition 6. *Let $\mathcal{C}$ be a permutation class. If $\pi \in \mathcal{C}^{\circ k}$ for some π and k, then the $(k+1, \mathcal{C})$-staircase contains a permutation τ of size $O(kn)$ whose adjacency graph $G(\tau)$ contains $G(\pi)$ as a minor, and in particular* $\mathrm{tw}(\tau) \geq \mathrm{tw}(\pi)$.

We combine this proposition with the following upper bound on treewidth of graphs in a $(k, \mathcal{C})$-staircase class.

Proposition 7. *Let $\mathcal{C}$ be a class not containing a monotone juxtaposition or a symmetry of $\mathcal{L}$ or $\mathcal{PBT}$. There is then a constant c such that for any k and any permutation τ from the $(k, \mathcal{C})$-staircase class, we have* $\mathrm{tw}(\tau) \leq c\sqrt{k|\tau|}$.

The proof of Proposition 7 uses the results of Dujmović et al. [5] dealing with the treewidth of graph that can be embedded on a surface of bounded genus with a bounded number of crossings per edge. We are able to show that under the assumptions of Proposition 7, the adjacency graph of τ admits an embedding of this form. We again omit the details of the argument.

Notice that Propositions 6 and 7 together establish the required linear lower-bound for $\mathrm{wst}(\mathcal{C}; n)$. Indeed, choose a permutation $\pi \in S_n$ with $\mathrm{tw}(\pi) = \Omega(n)$, and let $k = \mathrm{st}(\mathcal{C}; \pi)$. By Proposition 6, there is a permutation τ of size $O(kn)$ and with $\mathrm{tw}(\tau) \geq \mathrm{tw}(\pi) = \Omega(n)$. Proposition 7 then gives

$$\Omega(n) \leq \mathrm{tw}(\tau) \leq ck\sqrt{n} \leq c\sqrt{n}\,\mathrm{wst}(\mathcal{C}; n),$$

implying $\mathrm{wst}(\mathcal{C}; n) \geq \Omega(\sqrt{n})$.

From $\log n$ to 1

A simple counting argument, combined with the famous Marcus–Tardos theorem [8] shows that any proper permutation class $\mathcal{C}$, i.e., a class $\mathcal{C}$ not containing all permutations, has at least logarithmic worst-case sorting time.

Proposition 8. *If $\mathcal{C}$ is a permutation class that does not contain all permutations, then* $\mathrm{wst}(\mathcal{C}; n) = \Omega(\log n)$.

Proof. By the Marcus–Tardos theorem, there is a constant $c > 0$ such that $|\mathcal{C}_n| \leq c^n$ for all n. Consequently, $|\mathcal{C}_n^{\circ k}| \leq c^{kn}$ for any $k \geq 1$. Taking $k := \mathrm{wst}(\mathcal{C}; n)$ yields

$$n! = |S_n| = \left| \bigcup_{m=1}^{k} \mathcal{C}_n^{\circ m} \right| \leq \sum_{m=1}^{k} |\mathcal{C}_n^{\circ m}| \leq kc^{kn},$$

which implies $k = \Omega(\log n)$. □

Obviously, if $\mathcal{C}$ is the class of all permutations, then $\mathrm{wst}(\mathcal{C}; n) = 1$ for all n. This completes our classification of possible worst-case sorting times of permutation classes.

[1] Michael H. Albert, M. D. Atkinson, Mathilde Bouvel, Nik Ruškuc, and Vincent Vatter. Geometric grid classes of permutations. *Trans. Amer. Math. Soc.*, 365(11):5859–5881, 2013.

[2] M. D. Atkinson and Robert Beals. Permutation involvement and groups. *Q. J. Math.*, 52(4):415–421, 2001.

[3] M. D. Atkinson and T. Stitt. Restricted permutations and the wreath product. *Discrete Math.*, 259(1-3):19–36, 2002.

[4] Benjamin Aram Berendsohn. Complexity of permutation pattern matching. Master's thesis, Freie Universität Berlin, Berlin, 2019.

[5] Vida Dujmović, David Eppstein, and David R. Wood. Structure of graphs with locally restricted crossings. *SIAM J. Discrete Math.*, 31(2):805–824, 2017.

[6] Cheyne Homberger and Vincent Vatter. On the effective and automatic enumeration of polynomial permutation classes. *J. Symbolic Comput.*, 76:84–96, 2016.

[7] Sophie Huczynska and Vincent Vatter. Grid classes and the Fibonacci dichotomy for restricted permutations. *Electron. J. Combin.*, 13(1):Research Paper 54, 14, 2006.

[8] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley-Wilf conjecture. *J. Combin. Theory Ser. A*, 107(1):153–160, 2004.

On the existence of bicrucial permutations

Tom Johnston

thomas.johnston@maths.ox.ac.uk

University of Oxford

(This talk is based on joint work with Carla Groenland.)

A *square* of length 2ℓ is a factor

$$(S_1; S_2) = (\sigma_k, \ldots, \sigma_{k+\ell-1}; \sigma_{k+\ell}, \ldots, \sigma_{k+2\ell-1})$$

where S_1 and S_2 have the same reduced form, and we say a permutation is *square-free* if it contains no squares of length at least 4. We say a permutation π of length $n+1$ is a *right-extension* of σ if π can be formed by appending an entry $x \in \{1, \ldots, n+1\}$ to σ and replacing σ_i by $\sigma_i + 1$ if $\sigma_i \geq x$. We say that the permutation σ is *right-crucial* if it is square-free and every right-extension π contains a square, and we define left-extensions and left-crucial permutations similarly. In this talk we will be interested in permutations which are simultaneously left-crucial and right-crucial, which we call *bicrucial*.

In 2011, Avgustinovich, Kitaev, Pyatkin and Valyuzhenich [1] initiated the study of bicrucial permutations, showing that such permutations exist of odd lengths $8k+1$, $8k+5$ and $8k+7$ for all $k \geq 1$. Using the constraint solver Minion, it was shown by Gent, Kitaev, Konovalov, Linton and Nightingale [2] that bicrucial permutations exist when n is 19, 27 or 32, and they conjectured that larger permutations should exist.

Conjecture 1. *There exist bicrucial permutations of length $8k+3$ for all $k \geq 2$.*

Conjecture 2. *There exist arbitrarily long bicrucial permutations of even length.*

In this talk, we confirm both these conjectures and completely classify the n for which there exist bicrucial permutations of length n.

Theorem 3. *A bicrucial permutation of length n exists if and only if $n = 9$, $n \geq 13$ is odd or $n \geq 32$ is even and not 38.*

[1] S. Avgustinovich, S. Kitaev, A. Pyatkin, and A. Valyuzhenich. On square-free permutations. *Journal of Automata, Languages and Combinatorics*, 16(1):3–10, 2011.

[2] I. Gent, S. Kitaev, A. Konovalov, S. Linton, and P. Nightingale. S-crucial and bicrucial permutations with respect to squares. *Journal of Integer Sequences*, 18(6), 2015.

Layered permutations and their density maximisers

Adam Kabela

kabela@kma.zcu.cz

University of West Bohemia

(This talk is based on joint work with Dan Kráľ, Jon Noel and Théo Pierron.)

We consider a layered permutation π and its layered density maximisers, and we investigate the question whether the number of their layers can be bounded. We show that if the first layer of π is long and the second layer is a singleton, then the answer is negative for each sequence of optimal permutations of increasing lengths. This disproves a conjecture of Albert et al. [1] which suspected a positive answer if the first and the last layer of π are non-singleton and no two singleton layers of π are consecutive. On the other hand, we show that the conjecture is essentially true in a limit setting under an additional condition on the shortness of the first and last layer of π.

We recall that a permutation is *layered* if it can be obtained as a direct sum of decreasing permutations. It is known that for every layered permutation π and every positive integer n, the set of all permutations of length n maximising the density of π contains a layered permutation (for instance, see [7, 1, 2]). Furthermore, if no layer of π is a singleton, then all such permutations are layered and the number of their layers is bounded in a limit setting (see [1]). The results of [6, 1, 3, 8, 10] outline that singleton layers of a permutation pattern play an important role in relation to the layered density maximisers.

We present several results motivated by the conjecture mentioned above. Following Warren [9], we say that a sequence (λ_i) of permutations is *near-optimal* for a given permutation π if the density of π in λ_i goes to the packing density of π as i goes to infinity. We show the following.

Theorem 1. *Let π be a layered permutation with layers of sizes $(n, 1, \ell_1, \dots, \ell_k)$. If n is sufficiently large, then the number of layers goes to infinity for every near-optimal sequence of layered permutations of increasing lengths.*

The considered class of layered patterns contains counterexamples to the conjecture mentioned, for instance, the permutation consisting of three layers of sizes $(13, 1, 2)$; and it shows that the sought sufficient conditions have to also address non-singleton layers of a pattern. In addition, we extend the assumptions on a pattern and show the following.

Theorem 2. *Let π be a layered permutation whose first and last layer are non-singleton, and each pair of consecutive layers contains a non-singleton layer, and furthermore the first and last layer are of the same length and no non-singleton layer is shorter. Then every layered permuton maximising the density of π has only finitely many decreasing layers and no identity section (see the definition below).*

For the sake of simplicity, the main part of the argument showing Theorem 1 is also done in a limit setting. To this end, we introduce the notion of layered permutons. We recall that a *permuton* is a probability measure Π on the σ-algebra of Borel sets from $[0,1]^2$ satisfying $\Pi([a,b] \times [0,1)) = \Pi([0,1] \times [a,b]) = b - a$ for every interval $[a,b]$ within $[0,1]$. We say that a permuton Π is *layered* if there exist a partition of $[0,1]$ into (countably many) non-trivial intervals, and a red-black colouring of the intervals (such that no two red intervals are adjacent), and the support of Π is formed by points (x,y) such that

- $y = x$ if x belongs to a red interval,
- $y = a + b - x$ if x belongs to a black interval going from a to b.

Intuitively, the relation to layered permutations is that each black part resembles a layer and each red part resembles a collection of very short consecutive layers. We call a part of Π given by a black interval a *decreasing layer* and a part given by a red interval an *identity section.*

Using the results of [4, 5], it can be shown that every convergent sequence of layered permutations goes to a layered permuton; and if each of the permutations has at most k layers, then the limit has at most k decreasing layers and no identity section. (On the other hand, we note there are convergent sequences of layered permutations of unbounded number of layers whose limit has a constant number of decreasing layers and no identity section.) With the convergence on hand, it follows that for every layered permutation π, the set of all permutons maximising the density of π contains a layered permuton. Considering the length of this extended abstract, we omit the proof of the convergence, and also the proof of Theorem 2. Instead, we append a full proof for the main step towards showing Theorem 1, which is as follows.

Theorem 3. *Let π be a layered permutation with layers of sizes $(n, 1, \ell_1, \ldots, \ell_k)$. If n is sufficiently large, then every layered permuton maximising the density of π has infinitely many decreasing layers or an identity section.*

Proof of Theorem 3. For the sake of simplicity, we will just show that there exists a threshold on n rather than providing an explicit bound (inspecting the proof, we can easily get $n \geq 13$ for the case $k = 1$ and $\ell_1 = 2$); we should need that $n \geq 1 + \ell_1 + \cdots + \ell_k$ throughout the argument, and the assumption that n is sufficiently large is only used at the end of the proof.

We let d denote the maximum possible density of π in a permuton. For the sake of a contradiction, we suppose that there is a layered permuton Λ attaining this density and having finitely many layers and no identity section. (By definition, each layer of Λ has a positive size.)

We let m be the number of layers of Λ, and x_i be the size of the i-th layer from the left (so that $x_1 + \cdots + x_m = 1$). For the sake of brevity, we define $L = 1 + \ell_1 + \cdots + \ell_k$ and $N = n + L$ and
$$A = \frac{N!}{n! \prod_{j=1}^{k} (\ell_j!)}.$$

We note that

$$d = A\left(\sum_{1\le a<b<i_1<\cdots<i_k\le m} x_a^n x_b \prod_{j=1}^{k} x_{i_j}^{\ell_j}\right) \tag{1}$$

where the sum is taken over all $(k+2)$-tuples of integers $a, b, i_1, \ldots, i_k$ satisfying the inequalities. As we will repeatedly consider sampling from Λ without using its first two layers, we define the following short notions

$$\Sigma_0 = \sum_{3\le a<b<i_1<\cdots<i_k\le m} x_a^n x_b \prod_{j=1}^{k} x_{i_j}^{\ell_j},$$

$$\Sigma_1 = \sum_{3\le b<i_1<\cdots<i_k\le m} x_b \prod_{j=1}^{k} x_{i_j}^{\ell_j},$$

$$\Sigma_2 = \sum_{3\le i_1<\cdots<i_k\le m} \prod_{j=1}^{k} x_{i_j}^{\ell_j}.$$

By definition, we have $0 \le \Sigma_0 \le \Sigma_1 \le \Sigma_2$ and we observe that $\Sigma_2 \le 1$. Furthermore, we note that (1) can be expanded as follows

$$d = A\left(x_1^n x_2 \Sigma_2 + x_1^n \Sigma_1 + x_2^n \Sigma_1 + \Sigma_0\right). \tag{2}$$

We will consider five auxiliary permutons and discuss whether they improve the density of π. In particular, each consideration will give a bound on some element of (2), together leading to a contradiction. We start by showing that $x_1 \ge n x_2$. The argument is split into proving two claims.

Claim 1. *$x_1 \ge x_2$.*

Proof of Claim 1. We consider the layered permuton Λ' obtained from Λ by swapping the first two layers; that is, the first layer of Λ' has size x_2, the second has size x_1, and the i-th layer has size x_i for every $i = 3, \ldots, m$. We note that the density of π in Λ' minus the density of π in Λ is equal to

$$A\left(x_2^n x_1 \Sigma_2 - x_1^n x_2 \Sigma_2\right).$$

By the optimality of Λ, the difference of the densities is non-positive. Since $A > 0$ and $\Sigma_2 > 0$ and $n \ge 2$, we conclude that $x_1 \ge x_2$. □

Claim 2. *$x_1 \ge n x_2$.*

Proof of Claim 2. For $0 \le y < x_2$, we consider the layered permuton Λ_y whose first layer is of size $x_1 + y$, second layer is of size $x_2 - y$, and i-th layer is of size x_i for every $i = 3, \ldots, m$. (Clearly, the permuton Λ_0 is just Λ.) The idea is to view the density of π in Λ_y as a function of y, say $f(y)$, and consider the derivative of f evaluated at 0, and use the fact that this value is non-positive by the optimality of Λ. In particular, we have

$$f = A\left((x_1+y)^n (x_2-y)\Sigma_2 + (x_1+y)^n \Sigma_1 + (x_2-y)^n \Sigma_1 + \Sigma_0\right)$$

and

$$\frac{\partial f}{\partial y} = A\left(n(x_1+y)^{n-1}(x_2-y)\Sigma_2 - (x_1+y)^n\Sigma_2 + n(x_1+y)^{n-1}\Sigma_1 - n(x_2-y)^{n-1}\Sigma_1\right).$$

We evaluate the derivative at $y = 0$ and ignore the factor of A and obtain

$$\left(nx_1^{n-1}x_2 - x_1^n\right)\Sigma_2 + \left(nx_1^{n-1} - nx_2^{n-1}\right)\Sigma_1.$$

We conclude that the second term is non-negative by Claim 1, and thus the first term cannot be positive which yields $x_1 \geq nx_2$. □

Next, we show the following upper bound on Σ_1.

Claim 3. $\Sigma_1 \leq \frac{x_1}{n}$.

Proof of Claim 3. We consider the layered permuton Λ_{x_2} obtained from Λ by merging its first and second layer. (It can be viewed as the case $y = x_2$ of the permuton Λ_y defined above.) In other words, Λ_{x_2} has $m-1$ layers and the first layer is of size $x_1 + x_2$, and the i-th layer is of size x_{i+1} for every $i = 2, \ldots, m-1$. We note that the density of π in Λ_{x_2} minus the density of π in Λ is equal to

$$A\left(((x_1+x_2)^n - x_1^n - x_2^n)\Sigma_1 - x_1^n x_2 \Sigma_2\right).$$

By the optimality of Λ, this quantity is non-positive. Using Binomial theorem and the facts that $n \geq 2$ and $A > 0$ and $\Sigma_2 \leq 1$, we note that $nx_1^{n-1}x_2\Sigma_1 \leq x_1^n x_2$. The desired bound follows. □

In addition, we let π' denote the layered permutation obtained from π by removing its first layer, that is, π' has layers of sizes $(1, \ell_1, \ldots, \ell_k)$, and we consider an arbitrary layered permuton Φ such that Φ has non-zero density of π'; and we let d' denote this density. (We remark that an explicit bound on n can be obtained by choosing a particular permuton Φ and enumerating d'.) We show the following lower bound on d.

Claim 4. $d \geq \binom{N}{n}\left(\frac{n}{N}\right)^n\left(\frac{L}{N}\right)^L d'$.

Proof of Claim 4. We consider the layered permuton Λ^+ obtained from Φ by adding a long decreasing layer to the left and rescaling. More precisely, the first layer of Λ^+ is of size $\frac{n}{N}$ and the rest is given by the structure of Φ where the length of each interval is multiplied by $\frac{L}{N}$. We use the fact that the density of π in Λ^+ is lower bounded by the probability that n points are mapped to the first layer and L points are mapped to the rest and these L points form a copy of π'. This gives

$$\binom{N}{n}\left(\frac{n}{N}\right)^n\left(\frac{L}{N}\right)^L d',$$

and we recall that Λ maximises the density of π, and hence it is a lower bound on d. □

Finally, we apply Claim 2 and use permuton Φ once again, and we show the following.

Claim 5. $x_1 \leq \sqrt[L]{\frac{A}{\binom{N}{n}n^{n-L}d'}}$.

Proof of Claim 5. We consider the layered permuton Λ^* obtained from Λ by replacing the second layer with a scaled copy of Φ (multiplying the length of each interval by x_2). We use the fact that the second layer of π is a singleton, and we note that Λ^* preserves every copy of π which does not map the first layer of π into the second layer of Λ. Furthermore, there are additional copies of π in Λ^* which map the first layer of π to the first layer of Λ^* and all the rest into the newly created part. Thus, the difference between the density of π in Λ^* and the density of π in Λ is at least

$$\binom{N}{n} x_1^n x_2^L d' - A x_2^n \Sigma_1.$$

The optimality of Λ implies that

$$\binom{N}{n} x_1^n x_2^L d' \leq A x_2^n \Sigma_1,$$

and since $\Sigma_1 \leq 1$, we can write

$$\binom{N}{n} x_1^n d' \leq A x_2^{n-L}.$$

Using the assumption that $n \geq L$ and Claim 2, we note that $x_1^{n-L} \geq (nx_2)^{n-L}$ and obtain

$$\binom{N}{n} x_1^L n^{n-L} d' \leq A,$$

and the desired inequality follows. $\square$

In the remainder of the proof, we combine the ingredients and arrive at a contradiction. Recalling (2) and $\Sigma_0 \leq \Sigma_1 \leq \Sigma_2 \leq 1$, we can write

$$d \leq A \left(x_1^n x_2 + x_1^n + x_2^n + \Sigma_1 \right).$$

Furthermore, we use $x_2 \leq 1$ and Claims 1 and 3, and we get

$$d \leq A \left(3x_1^n + \frac{x_1}{n} \right).$$

In order to obtain a contradiction, we just need to show that the following inequality is satisfied for sufficiently large n.

$$\frac{d}{A} > 3x_1^n + \frac{x_1}{n} \tag{3}$$

To this end, we note that A and d can be viewed as functions of n, and we discuss what happens as n goes to infinity. We observe that $A = \Theta(n^L)$ where all implicit constants depend only on $\ell_1, \ldots, \ell_k$. Regarding the asymptotics of d, we use Claim 4 and the fact that

$$\lim_{n \to \infty} \left(\binom{N}{n} \left(\frac{n}{N} \right)^n \left(\frac{L}{N} \right)^L d' \right) = \frac{d'}{L!} \left(\frac{L}{e} \right)^L,$$

and we obtain $d = \Theta(1)$. We conclude that the left-hand side of (3) is of order $\Omega(n^{-L})$, whereas the right-hand side tends to zero much more quickly by Claim 5. Thus, inequality (3) is satisfied for sufficiently large n, a contradiction. $\square$

[1] M. H. Albert, M. D. Atkinson, C. C. Handley, D. A. Holton and W. Stromquist: On packing densities of permutations, Electron. J. Combin. 9 (2002), Research Paper 5, 20.

[2] R. W. Barton: Packing densities of patterns, Electron. J. Combin. 11 (2004), Research paper 80, 16.

[3] P. A. Hästö: The packing density of other layered permutations, Electron. J. Combin. (2002/03), Research paper 1, 16.

[4] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Ráth and R. Menezes Sampaio: Limits of permutation sequences, J. Comb. Theory, Ser. B 103 (2013), 93–113.

[5] P.-L. Méliot: A survey of the theory of graphons and permutons, `http://www.imo.universite-paris-saclay.fr/~meliot/surveys/graphon_survey.pdf`.

[6] A. L. Price: Packing densities of layered patterns, Ph.D. Thesis, University of Pennsylvania (1997).

[7] W. Stromquist: Packing layered posets into posets, `http://walterstromquist.com/POSETS.DOC`.

[8] D. Warren: Optimal packing behavior of some 2-block patterns, Ann. Comb. 8 (2004), 355–367.

[9] D. Warren: Optimizing the packing behavior of layered permutation patterns, Ph.D. Thesis, University of Florida (2005).

[10] D. Warren: Packing densities of more 2-block patterns, Adv. in Appl. Math. 36 (2006), 202–211.

A simple proof of a CLT for vincular permutation patterns for conjugation invariant permutations[1]

Mohamed Slim Kammoun

m.kammoun@lancaster.ac.uk

Lancaster University

The number of occurrences of any fixed vincular permutation pattern in a uniform permutation is known to satisfy a central limit theorem. Using a comparaison technique, we extend this result to other non-uniform permutations. The technique can be used for other statistics.

1 Main result

A random permutation σ_n is called conjugation invariant if its law is conjugation invariant i.e. if $\rho \circ \sigma_n \circ \rho^{-1} \stackrel{d}{=} \sigma_n$ for any $\rho \in \mathfrak{S}_n$. For example, the uniform permutation, a uniform cyclic permutation and (generalized) Ewens permutations are classic examples of conjugation invariant permutations.

A vincular pattern of size p is a couple (τ, X) such that $\tau \in \mathfrak{S}_p$ and $X \subset [p-1]$. Given $\sigma \in \mathfrak{S}_n$, an occurrence of (τ, X) is a list $i_1 < \cdots < i_p$ such that

- $i_{x+1} = i_x + 1$ for any $x \in X$.
- $(\sigma(i_1), \ldots, \sigma(i_p))$ is in the same relative order as $(\tau(i_1), \ldots, \tau(i_p))$.

We denote by $\mathcal{N}_{(\tau,X)}(\sigma)$ the number of occurrences of (τ, X) in σ.

For the uniform case, Bóna [2], Janson et al. [5] and Hofer [4] proved respectively a CLT for monotone, classic and vincular patterns. Féray [3] gave a generalization for the Ewens distribution. In particular, Hofer [4] proved that for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$,

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_{unif,n}) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \tag{1}$$

Here, $\sigma_{unif,n}$ is a uniform permutation of size n, $q = \mathrm{card}(X)$ and $V_{\tau,X} > 0$. We prove the folowing.

[1]The full paper is available in https://arxiv.org/pdf/2012.05845.pdf

	$\sigma_{unif,3}$	$T(\sigma_{unif,3})$	$T^2(\sigma_{unif,3})$
Id	1/6	0	0
$(1,2)$	1/6	1/18	0
$(1,3)$	1/6	1/18	0
$(2,3)$	1/6	1/18	0
$(1,2,3)$	1/6	5/12	1/2
$(1,3,2)$	1/6	5/12	1/2

Table 1: Transitions for $\sigma_{unif,3}$

Proposition 1. *Suppose that for any $n \geq 1$, σ_n is conjugation invariant random permutation of $\mathfrak{S}_n$ and the sequence of number of cycles $(\#(\sigma_n))_{n\geq 1}$ satisfies*

$$\frac{\#(\sigma_n)}{\sqrt{n}} \xrightarrow[n\to\infty]{d} 0 \tag{2}$$

Then, for any $\tau \in \mathfrak{S}_p$ and any $X \subset [p-1]$

$$\frac{\mathcal{N}_{(\tau,X)}(\sigma_n) - \frac{n^{p-q}}{p!(p-q)!}}{n^{p-q-\frac{1}{2}}} \xrightarrow[n\to\infty]{d} \mathcal{N}(0, V_{\tau,X}). \tag{3}$$

We give the proof of this result in the next section. We give then an idea of a generalization to other statistics on permutations using the same kind of proofs.

2 Idea of proof

The proof uses a coupling argument. We will define a Markov chain with an Ewens stationary measure and such that conjugation invariant random permutations with few cycles are converges to the stationary measure rapidly. Formally, let ρ_n be a conjugation invariant random permutation. The idea is to modify ρ_n to obtain a uniform cyclic permutation. We define the following Markov operator T:

- If the realization σ of ρ_n has one cycle, σ remains unchanged ($T(\sigma) = \sigma$).
- Otherwise, we choose a couple (i,j) uniformly from the nonempty set

  $$\{(i,j) : j \notin \mathcal{C}_i(\sigma)\}$$

  and we take $T(\sigma) = \sigma \circ (i,j)$. Here $\mathcal{C}_i(\sigma)$ is the cycle of σ containing i.

For example, for $n = 3$, transition probabilities of T are given in Figure 1. We denote by $T^k(\rho_n)$ the random permutation obtained after applying k times the operator T. It is the random permutation obtained after k steps of the uniform random walk on $\mathcal{G}_{\mathfrak{S}_n}$ with initial state ρ_n. Table 1 sums up the evolution of the random walk if we start from the uniform distribution on $\mathfrak{S}_3$. We have then the following :

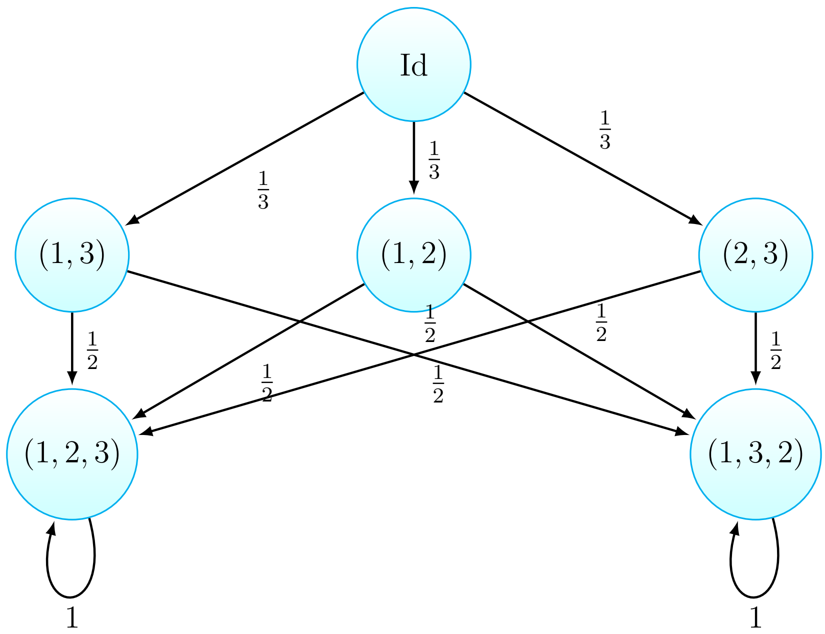


Figure 1: The transition probabilities of T for $n = 3$

- If σ_n is conjugation invariant then $T(\sigma_n)$ is conjugation invariant and $\#(T(\sigma_n)) = \max(\#(T(\sigma_n))-1, 1)$. Consequently, $T^{n-1}(\rho_n) \stackrel{d}{=} \sigma_{Ew,0,n}$. Where $\sigma_{Ew,0,n}$ is a uniform cyclic permutation of length n.

- Almost surely,
$$|\mathcal{N}_{(\tau,X)}(\rho_n) - \mathcal{N}_{(\tau,X)}(T^{n-1}(\rho_n))| \leq n^{p-q}\#(\rho_n).$$

Choosing first ρ_n a uniform permutation, we obtain that (1) is equivalent to (3) for $\sigma_n = \sigma_{Ew,0,n}$. In a second step, we choose ρ_n a conjugation invariant random permutation satisfying (2). In the case, the convergence in (3) (for $\sigma_n = \rho_n$) is again equivalent to the same convergence in the particle case $\sigma_n = \sigma_{Ew,0,n}$ which concludes the proof.

3 Generalization

This technique is not specific to permutation patterns. Given $n \geq 1$ and $E \subset \mathfrak{S}_n$, we define

$$\operatorname{next}(E) := \{\rho \circ (i,j); \rho \in E,\ \#(\rho \circ (i,j)) = \#(\rho) - 1\} \cup \{\rho \in E; \#(\rho) = 1\}$$

and

$$\operatorname{final}(\sigma) := \begin{cases} \operatorname{next}^{\#(\sigma)-1}(\{\sigma\}) & \text{if } \#(\sigma) > 1 \\ \{\sigma\} & \text{otherwise} \end{cases}.$$

In other words, $\operatorname{next}(E)$ is the set of permutations obtained by concatenating, if possible, two cycles of some $\sigma \in E$, and $\operatorname{final}(\sigma)$ is the set of permutations obtained by concatenating all the cycles of σ. In particular,

$$\operatorname{final}(\sigma) \subset \mathfrak{S}_n^0 := \{\sigma \in \mathfrak{S}_n;\ \#(\sigma) = 1\}.$$

Let f be a function defined on $\mathfrak{S}_\infty := \cup_{i=1}^{\infty}\mathfrak{S}_n$ and taking its values in some metric space (F, d_F), for example $\mathbb{Z}$, $\mathbb{R}$, or $\mathbb{R}^d$. We define for $1 \leq k \leq n$,

$$\varepsilon'_{n,k}(f) := \max_{\sigma\in\mathfrak{S}_n, \#(\sigma)=k} \max_{\rho\in\mathrm{final}(\sigma)} d_F(f(\sigma), f(\rho)).$$

We present now our main result.

Theorem 2. *Assume that for any $n \geq 1$, (σ_n) and $(\sigma_{ref,n})$ are conjugation invariant permutations of size n. Suppose that there exists $x \in F$ such that*

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{\mathbb{P}} x, \tag{4}$$

$$\varepsilon'_{n,\#(\sigma_{ref,n})}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0 \tag{5}$$

$$\text{and that} \quad \varepsilon'_{n,\#(\sigma_n)}(f) \xrightarrow[n\to\infty]{\mathbb{P}} 0. \tag{6}$$

Then

$$f(\sigma_n) \xrightarrow[n\to\infty]{\mathbb{P}} x. \tag{7}$$

Moreover, if the assumptions (4)–(6) hold true for the $\mathbb{L}^p$ convergence for some $p \geq 1$ instead of the convergence in probability, then so does (7).

When $F = \mathbb{R}^d$, we obtain also the convergence in distribution.

Theorem 3. *Assume that $F = \mathbb{R}^d$ and that for any $n \geq 1$, (σ_n) and $(\sigma_{ref,n})$ are conjugation invariant permutations of size n. Suppose that (5) and (6) hold true and that there exists a random variable X supported on F such that*

$$f(\sigma_{ref,n}) \xrightarrow[n\to\infty]{d} X.$$

Then

$$f(\sigma_n) \xrightarrow[n\to\infty]{d} X.$$

This result can be applied to many statistics including the descent process, the shape of a permutation by RSK, the number of exceedences and the longest increasing (decreasing, alternating, common) subsequence. We detailed those applications in the full version of this work. We give here only one example to illustrate this result.

Given $\sigma \in \mathfrak{S}_n$, a subsequence $(\sigma(i_1), \ldots, \sigma(i_k))$ is an increasing (resp. decreasing) subsequence of σ of length k if $i_1 < \cdots < i_k$ and $\sigma(i_1) < \cdots < \sigma(i_k)$ (resp. $\sigma(i_1) > \cdots > \sigma(i_k)$). We denote by $\mathrm{LIS}(\sigma)$ (resp. $\mathrm{LDS}(\sigma)$) the length of the longest increasing (resp. decreasing) subsequence of σ. For example,

$$\text{if} \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 2 & 1 & 4 \end{pmatrix}, \ \mathrm{LIS}(\sigma) = 2 \ \text{and} \ \mathrm{LDS}(\sigma) = 4.$$

Corollary 4. *Suppose that for any $n \geq 1$, σ_n is conjugation invariant random permutation of $\mathfrak{S}_n$ and*

$$\frac{\#(\sigma_n)}{\sqrt[6]{n}} \xrightarrow[n\to\infty]{d} 0.$$

Then,

$$\mathbb{P}\left(\frac{\mathrm{LIS}(\sigma_n) - 2\sqrt{n}}{n^{\frac{1}{6}}} \leq s\right) \xrightarrow[n\to\infty]{} F_2(s),$$

where F_2 is the cumulative distribution function of the GUE Tracy-Widom distribution.

This result generalizes that of Baik, Deift and Johansson[1] who proved these fluctuations for the uniform case.

[1] J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the longest increasing subsequence of random permutations. *J. Amer. Math. Soc.*, 12(4):1119–1178, 1999.

[2] M. Bóna. Permutation patterns: On three different notions of monotone subsequences. *arXiv: Combinatorics*, 2010.

[3] V. Féray. Asymptotic behavior of some statistics in Ewens random permutations. *Electron. J. Probab.*, 18:no. 76, 32, 2013.

[4] L. Hofer. A central limit theorem for vincular permutation patterns. *Discret. Math. Theor. Comput. Sci.*, 19, 2017.

[5] S. Janson, T. Luczak, and A. Rucinski. *Random graphs*, volume 45. John Wiley & Sons, 2011.

Qubonacci words

Sergey Kirgizov

sergey.kirgizov@u-bourgogne.fr

LIB, Université de Bourgogne Franche-Comté, Dijon, France

(This talk is based on joint work with Jean-Luc Baril and Vincent Vajnovszki.)

An n-length binary word is *q-decreasing*, $q \in \mathbb{N}^+$, if every of its length maximal factor of the form 0^a1^b satisfies $a = 0$ or $q \cdot a > b$. The set of q-decreasing words of length n is denoted by $\mathcal{W}_n^q$. For example we have

$\mathcal{W}_4^1 = \{0000, 0001, 0010, 1000, 1001, 1100, 1110, 1111\}$,
$\mathcal{W}_4^2 = \{0000, 0001, 0010, 0011, 0100, 0101, 1000, 1001, 1010, 1100, 1101, 1110, 1111\}$.

Denote by $\mathcal{B}_n(1^k), k \geq 2$, the set of all n-length binary words containing no occurrences of factor 1^k. The fact that $\mathcal{B}_n(1^k)$ is enumerated by k-generalized Fibonacci numbers is widely known [3, p. 286]. For $q \geq 1$ we construct a bijection between $\mathcal{B}_n(1^{q+1})$ and $\mathcal{W}_n^q$, and give bivariate generating functions according to the length of the words and the number of 1s. In general, there are more 1s in $\mathcal{W}_n^q$ than in $\mathcal{B}_n(1^{q+1})$.

The *1s frequency* of a set is the ratio between the total number of 1s and the overall number of bits in the words of the set. Alternatively, it is the expected value of a random bit in a random word of the set. It can be shown, for $q \geq 1$, that the 1s frequencies of $\mathcal{W}_n^q$ and of $\mathcal{B}_n(1^{q+1})$ converge, when n tends to infinity, to a same limit whose value is related to the generalized golden ratio φ_{q+1}, the root of largest modulus of the famous Fibonacci polynomial $x^{q+1} - x^q - \cdots - x - 1$. In particular, we show that the common limit of the 1s frequencies of $\mathcal{W}_n^1$ and of $\mathcal{B}_n(11)$ is $(2 - \varphi)/(3 - \varphi) \approx 0.27639$ with $\varphi = (1 + \sqrt{5})/2$ the golden ratio.

For any $q \geq 1$, we provide an efficient exhaustive generating algorithm for q-decreasing words in lexicographic order. We show the existence of a 3-Gray code and explain how a generating algorithm for this Gray code can be obtained. Moreover, we give the construction of a more restrictive 1-Gray code for 1-decreasing words, which in particular settles a conjecture about the Hamiltonicity of certain hypercube subgraphs stated recently by Eğecioğlu and Iršič [2]. We conjecture the existence of 1-Gray code for any $q \geq 2$. The extended version of this work is available on the arXiv [1].

The case $q \in \mathbb{Q}^+$ may be an interesting lane to explore.

[1] Baril, J.-L., Kirgizov, S., Vajnovszki, V. "Gray codes for Fibonacci q-decreasing words". arXiv:2010.09505 [Math.CO] (2020)

[2] Eğecioğlu, O., Iršič, V. "Fibonacci-run graphs I: Basic properties". Discrete Applied Mathematics, 295, 70–84 (2021)

[3] Knuth, D. "The Art of Computer Programming, Volume 3: Sorting and Searching", 2nd ed. (1998)

Permutation groups and permutation patterns

Erkko Lehtonen

e.lehtonen@fct.unl.pt

Universidade Nova de Lisboa

We look at permutations from two different points of view: the algebraic one of permutation groups and the combinatorial one of permutation patterns. Permutation groups are a classical topic in algebra and require no further explanation. Permutation patterns have been an active field of study in the past decades. At first sight, these two well-established notions may not seem to have much in common, but there is a perhaps surprising connection that we will explain below.

We will use the standard terminology related to permutation patterns and permutation groups. We denote the set of all positive integers by $\mathbb{N}_+$. For any $n \in \mathbb{N}_+$, the set $\{1, \dots, n\}$ is denoted by $[n]$. The set S_n of all permutations of $[n]$, endowed with the operation of composition, constitutes the *symmetric group* (*of degree* n). For any $S \subseteq S_n$, the subgroup of S_n generated by S is denoted by $\langle S \rangle$. We write $G \leq H$ to express that G is a subgroup of H. Concerning permutation patterns, we write $\pi \leq \tau$ if π is a pattern of τ (or τ involves π), and we write $\pi \nleq \tau$ if τ avoids π.

Let $\ell, n \in \mathbb{N}_+$ with $\ell \leq n$. For $\tau \in S_n$, we denote by $\operatorname{Pat}^{(\ell)} \tau$ the set of all ℓ-patterns of π, i.e., $\operatorname{Pat}^{(\ell)} \tau := \{\pi \in S_\ell : \pi \leq \tau\}$. We say that a permutation $\tau \in S_n$ is *compatible* with a set $S \subseteq S_\ell$ of ℓ-permutations if $\operatorname{Pat}^{(\ell)} \tau \subseteq S$. For $S \subseteq S_\ell$, $T \subseteq S_n$, we write

$$\operatorname{Comp}^{(n)} S := \{\tau \in S_n \mid \operatorname{Pat}^{(\ell)} \tau \subseteq S\}, \qquad \operatorname{Pat}^{(\ell)} T := \bigcup_{\tau \in T} \operatorname{Pat}^{(\ell)} \tau.$$

As observed in [4, Section 3], the operators $\operatorname{Comp}^{(n)}$ and $\operatorname{Pat}^{(\ell)}$ are the upper and lower adjoints of the monotone Galois connection (residuation) between the power sets $\mathcal{P}(S_\ell)$ and $\mathcal{P}(S_n)$ induced by the pattern avoidance relation $\nleq$. This means that $\operatorname{Comp}^{(n)}$ and $\operatorname{Pat}^{(\ell)}$ are monotone operators satisfying $\operatorname{Comp}^{(n)} S \subseteq T$ if and only if $S \subseteq \operatorname{Pat}^{(\ell)} T$, for all $S \subseteq S_\ell$ and $T \subseteq S_n$. Moreover, $\operatorname{Pat}^{(\ell)} \operatorname{Comp}^{(n)}$ and $\operatorname{Comp}^{(n)} \operatorname{Pat}^{(\ell)}$ are kernel and closure operators, respectively, i.e., for all $S \subseteq S_\ell$ and $T \subseteq S_n$, it holds that

$$\begin{aligned} \operatorname{Pat}^{(\ell)} \operatorname{Comp}^{(n)} S &\subseteq S, & \operatorname{Comp}^{(n)} S &= \operatorname{Comp}^{(n)} \operatorname{Pat}^{(\ell)} \operatorname{Comp}^{(n)} S, \\ T &\subseteq \operatorname{Comp}^{(n)} \operatorname{Pat}^{(\ell)} T, & \operatorname{Pat}^{(\ell)} T &= \operatorname{Pat}^{(\ell)} \operatorname{Comp}^{(n)} \operatorname{Pat}^{(\ell)} T. \end{aligned}$$

The following observation is key to our work and builds the connection between permutation groups and permutation patterns.

Lemma 1 (Lehtonen, Pöschel [4, Lemma 2.6(ii)]). *Let $\pi, \tau \in S_n$, and let $\ell \leq n$. Then* $\operatorname{Pat}^{(\ell)} \pi\tau \subseteq (\operatorname{Pat}^{(\ell)} \pi)(\operatorname{Pat}^{(\ell)} \tau) := \{\sigma\sigma' \mid \sigma \in \operatorname{Pat}^{(\ell)} \pi,\ \sigma' \in \operatorname{Pat}^{(\ell)} \tau\}$.

As a consequence of Lemma 1, we obtain the following result.

Proposition 2 (Lehtonen, Pöschel [4, Proposition 3.1])**.** *If S is a subgroup of S_ℓ, then* $\operatorname{Comp}^{(n)} S$ *is a subgroup of S_n.*

Using the standard terminology and notation of the theory of permutation patterns, Proposition 2 can be rephrased as follows.

Corollary 3. *Let $\ell, n \in \mathbb{N}_+$. For any permutation group $G \leq S_\ell$, the class* $\mathsf{Av}_n(S_\ell \setminus G)$ *is a permutation group.*

In view of Proposition 2, it makes sense to modify the Galois connection $(\operatorname{Comp}^{(n)}, \operatorname{Pat}^{(\ell)})$ and turn it into a pair of mappings between the subgroup lattices $\operatorname{Sub}(S_\ell)$ and $\operatorname{Sub}(S_n)$ of the symmetric groups S_ℓ and S_n. Thus, for subgroups $G \leq S_\ell$, $H \leq S_n$, we define

$$\operatorname{gComp}^{(n)} G := \langle \operatorname{Comp}^{(n)} G \rangle = \operatorname{Comp}^{(n)} G, \qquad \operatorname{gPat}^{(\ell)} H := \langle \operatorname{Pat}^{(\ell)} H \rangle.$$

The operators $\operatorname{gComp}^{(n)}$ and $\operatorname{gPat}^{(\ell)}$ do indeed constitute a Galois connection between $\operatorname{Sub}(S_\ell)$ and $\operatorname{Sub}(S_n)$ (see [4, Lemma 3.3]).

The operators $\operatorname{Comp}^{(n)}$ and $\operatorname{Pat}^{(\ell)}$, as well as $\operatorname{gComp}^{(n)}$ and $\operatorname{gPat}^{(\ell)}$ satisfy the following "transitive property".

Lemma 4 ([3, Lemma 2.9])**.** *Assume that $\ell \leq m \leq n$. Then for all subsets $S \subseteq S_\ell$, $T \subseteq S_n$,*

$$\operatorname{Comp}^{(n)} \operatorname{Comp}^{(m)} S = \operatorname{Comp}^{(n)} S, \qquad \operatorname{Pat}^{(\ell)} \operatorname{Pat}^{(m)} T = \operatorname{Pat}^{(\ell)} T$$

and for all subgroups $G \leq S_\ell$, $H \leq S_n$,

$$\operatorname{gComp}^{(n)} \operatorname{gComp}^{(m)} G = \operatorname{gComp}^{(n)} G, \qquad \operatorname{gPat}^{(\ell)} \operatorname{gPat}^{(m)} H = \operatorname{gPat}^{(\ell)} H.$$

Subgroups of S_n of the form $\operatorname{Comp}^{(n)} S$ for some subset $S \subseteq S_\ell$ are called *ℓ-pattern subgroups* of S_n. As Proposition 2 asserts, $\operatorname{Comp}^{(n)} S$ is a subgroup of S_n whenever S is a subgroup of S_ℓ. However, $\operatorname{Comp}^{(n)} S$ may be a group even if S is not a group. There do even exist groups of the form $\operatorname{Comp}^{(n)} S$ for some subset $S \subseteq S_\ell$ that are not of the form $\operatorname{Comp}^{(n)} G$ for any subgroup $G \leq S_\ell$.

This raises the question which subgroups of S_n are ℓ-pattern subgroups, and for which subsets $S \subseteq S_\ell$ the set $\operatorname{Comp}^{(n)} S$ is a group. A conclusive answer was provided for small values of ℓ.

Proposition 5 (Lehtonen, Pöschel [4, Proposition 3.6])**.** *Let $\ell, n \in \mathbb{N}_+$ with $\ell \leq n$ and $\ell \leq 3$, and let S be a subset of S_ℓ. Then* $\operatorname{Comp}^{(n)} S$ *is a subgroup of S_n if and only if S is a subgroup of S_ℓ.*

For arbitrary ℓ, the ℓ-pattern subgroups of S_n can be described as invariant groups of ℓ-ary relations of a certain prescribed form. The subgroups of the form $\operatorname{Comp}^{(n)} G$ for a subgroup $G \leq S_\ell$ admit a simpler description as invariant groups of a single ℓ-ary relation of a special form. For further details, we refer the reader to the paper [4].

Let us introduce at this point a few special permutations that will be used many times in what follows:

- the *identity permutation* $\iota_n := 12 \dots n$,
- the *descending permutation* $\delta_n := n(n-1)\dots 1$,
- the *natural cycle* $\zeta_n := 23 \dots n1 = (1\ 2\ \cdots\ n)$.

The subgroup $\langle \zeta_n \rangle$ of S_n generated by the natural cycle ζ_n is called the *natural cyclic group* of degree n and is denoted by Z_n. The subgroup $\langle \zeta_n, \delta_n \rangle$ is called the *natural dihedral group* of degree n and is denoted by D_n. The *alternating group* of degree n is denoted by A_n.

A closely related interplay between permutation groups and permutation patterns was studied by Atkinson and Beals in their papers [1, 2] on *group classes,* i.e., permutation classes in which every level is a permutation group. They determined the asymptotic behaviour of group classes, and we rephrase their result in Theorem 6 below. In this description, $S_n^{a,b}$ denotes the group of all permutations in S_n that map each one of the intervals $[1,a]$ and $\delta_n([1,b]) = [n-b+1,n]$ onto itself and fix the points in $[a+1,n-b]$. Note that $S_n^{1,1}$ is the trivial subgroup of S_n. (Since we are only concerned about the asymptotical behaviour of the level sequence, it does not matter how $S_n^{a,b}$ is defined when $n < a+b$.)

Theorem 6 (Atkinson, Beals [1, 2])**.** *If C is a permutation class in which every level $C^{(n)}$ is a permutation group, then the level sequence $C^{(1)}, C^{(2)}, \dots$ eventually coincides with one of the following families of groups ("stable sequences"):*

(1) *the groups $S_n^{a,b}$ for some fixed $a, b \in \mathbb{N}_+$,*
(2) *the natural cyclic groups Z_n,*
(3) *the full symmetric groups S_n,*
(4) *the groups $\langle G_n, \delta_n \rangle$, where $(G_n)_{n \in \mathbb{N}}$ is one of the above families (with $a = b$ in (1)).*

Moreover, Atkinson and Beals completely and explicitly described those group classes in which every level is a transitive group. Following the terminology of [2], a permutation group $G \leq S_n$ is *anomalous* if $\zeta_n \in G$ and $\langle \mathrm{Pat}^{(n-1)} G \rangle \neq S_{n-1}$. The natural cyclic and dihedral groups are examples of anomalous groups. On the other hand, the symmetric and alternating groups are not anomalous even though they contain ζ_n.

Theorem 7 (Atkinson, Beals [2, Theorem 2])**.** *Let C be a permutation class in which every level $C^{(n)}$ is a transitive group. Then, with the exception of at most two levels, one of the following holds.*

(1) *$C^{(n)} = S_n$ for all $n \in \mathbb{N}_+$.*
(2) *For some $M \in \mathbb{N}$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, and $C^{(n)} = D_n$ for $n > M$.*
(3) *For some $M, N \in \mathbb{N}$ with $M \leq N$, $C^{(n)} = S_n$ for $1 \leq n \leq M$, $C^{(n)} = D_n$ for $M+1 \leq n \leq N$, and $C^{(n)} = Z_n$ for $n > N$.*

The exceptions, if any, may occur in the second and third cases and are of the following two possible types:

(1) *$C^{(M+1)} = A_{M+1}$ and $C^{(M+2)}$ is an anomalous group that is neither D_{M+2} nor Z_{M+2}, or*
(2) *$C^{(M+1)}$ is a proper overgroup of Z_{M+1} but is not D_{M+1}.*

It should be noted that every group mentioned in Theorem 7 contains the natural cycle (of the appropriate degree). For group classes with an intransitive group at some level, only the asymptotic behaviour is revealed by Atkinson and Beals's results.

This led us to analysing more carefully the behaviour of the sequence

$$G,\ \operatorname{Comp}^{(n+1)} G,\ \operatorname{Comp}^{(n+2)} G,\ \ldots \tag{1}$$

for an arbitrary subgroup G of S_n. This sequence constitutes the levels of a group class (with full symmetric groups in the levels below the n-th one), so Atkinson and Beals's results are applicable here, and we may ask how fast the sequence reaches one of the stable sequences predicted in Theorem 6? Can we say something more specific about the local behaviour of the sequence? With these questions in mind, we set about studying classes of permutations avoiding the complement of a permutation group. Due to space limitations and our desire to avoid technicalities, we present here only a brief outline of our results; for further details, we refer the interested reader to the paper [3].

Note that, in order to determine the sequence (1), it suffices, by Lemma 4, to determine $\operatorname{Comp}^{(n+1)} G$ for every $n \in \mathbb{N}_+$ and for every group $G \leq S_n$. The entire sequence (1) can then be determined by applying such results repeatedly, one level at a time. Unfortunately, in some cases, we were not able to determine $\operatorname{Comp}^{(n+1)} G$ exactly, but then we can take a suitable subgroup H_1 and an overgroup H_2 of G. The monotonicity of $\operatorname{Comp}^{(n+1)}$ implies that $\operatorname{Comp}^{(n+1)} H_1 \leq \operatorname{Comp}^{(n+1)} G \leq \operatorname{Comp}^{(n+1)} H_2$, and we may still be able to obtain good lower and upper bounds for $\operatorname{Comp}^{(n+1)} G$. With these results, it is also possible to determine (or estimate) how fast the sequence (1) reaches one of the stable sequences predicted by Theorem 6.

For the analysis, we need to consider separately various types of permutation groups, as different groups behave in quite different ways regarding compatible permutations. We first deal with the symmetric, alternating, and trivial groups, as well as the group generated by the descending permutation δ_n. For the remaining groups, we first make a distinction between those groups that contain the natural cycle ζ_n and those that do not. The groups not containing ζ_n are then divided into intransitive and transitive ones. The transitive groups without ζ_n are further subdivided into imprimitive and primitive ones. (Recall that a permutation group is *transitive* if it has only one orbit; otherwise it is *intransitive.* A transitive subgroup of S_n is *primitive* if it preserves no nontrivial partition of $[n]$; otherwise it is *imprimitive.*)

The following statement provides a very high-level description of the sequence (1) for a wide range of groups.

Proposition 8. *Let G be a subgroup of S_n. Then, unless $\zeta_n \notin G$ and G is intransitive or imprimitive, the smallest $i \in \mathbb{N}$ for which $\operatorname{Comp}^{(n+i)} G$ coincides with one of the stable sequences of Theorem 6 is at most 2.*

In the case when G is an intransitive or imprimitive group, the behaviour of the sequence (1) is more complicated. In order to explain the situation, let us introduce a special family of permutation groups that will recur in what follows. Let Π be a partition of $[n]$, and define S_Π to be the set of all n-permutations that map each Π-block onto itself, i.e.,

$$S_\Pi := \{\pi \in S_n \mid \pi(B) = B \text{ for every block } B \in \Pi\}.$$

It is clear that S_Π is a group, and if Π has at least two blocks, then S_Π is intransitive. Note that the group $S_n^{a,b}$ of Theorem 6 equals $S_{\Pi_n^{a,b}}$, where

$$\Pi_n^{a,b} := \{[1,a], \delta_n([1,b])\} \cup \{\{i\} \mid a < i < \delta_n(b)\}.$$

Given a partition Π of $[n]$, we define a partition Π' of $[n+1]$ using the following procedure. First, let I_Π be the coarsest interval partition of $[n]$ that refines Π. The blocks of Π' are then obtained from the blocks of I_Π as follows. For each block B of I_Π, we let $\{\min B\}$ and $B \setminus \{\min B\}$ be blocks of Π' (we take the latter only if it is nonempty), with the following exceptions: if $1 \in B$ and $n \in B$ (i.e., $\Pi = \{[n]\}$), then we let $\Pi' := \{[n+1]\}$; if $1 \in B$ and $n \notin B$ then we let B be a block of Π'; if $1 \notin B$ and $n \in B$ then we let $\{\min B\}$ and $B \setminus \{\min B\} \cup \{n+1\}$ be blocks of Π'. Let us denote $\Pi^{(1)} := \Pi'$ and $\Pi^{(i+1)} = (\Pi^{(i)})'$ for $i \geq 1$.

Proposition 9. *For any partition Π of $[n]$ and for any $i \geq 1$, it holds that*

$$\operatorname{Comp}^{(n+i)} S_\Pi = \begin{cases} S_{\Pi^{(i)}}, & \text{if } \delta_n \notin S_\Pi, \\ \langle S_{\Pi^{(i)}}, \delta_{n+i} \rangle, & \text{if } \delta_n \in S_\Pi. \end{cases}$$

If a and b are the largest integers α and β such that $\Pi_n^{\alpha,\beta}$ is a refinement of Π, then the sequence (1) for $G = S_\Pi$ reaches the stable sequence $S_{n+i}^{a,b}$ or $\langle S_{n+i}^{a,b}, \delta_{n+i} \rangle$ in a number of steps that can be easily determined from the block structure of Π.

Let now G be an arbitrary intransitive subgroup of S_n. Let Π be the partition of $[n]$ into orbits of G, and let a and b be the largest integers α and β such that $S_n^{\alpha,\beta} \leq G$. Then we have $S_n^{a,b} \leq G \leq S_\Pi$, and the monotonicity of $\operatorname{Comp}^{(n+i)}$ implies that

$$S_{n+i}^{a,b} = \operatorname{Comp}^{(n+i)} S_n^{a,b} \leq \operatorname{Comp}^{(n+i)} G \leq \operatorname{Comp}^{(n+i)} S_\Pi \leq \langle S_{\Pi^{(i)}}, \delta_{n+i} \rangle$$

for all $i \geq 1$. Furthermore, we can show that the sequence $\operatorname{Comp}^{(n+i)} G$ reaches the stable sequence $S_{n+i}^{a,b}$ or $\langle S_{n+i}^{a,b}, \delta_{n+i} \rangle$, and this happens in at most a certain number of steps that depends on the block structure of Π and the numbers a and b.

The analysis of imprimitive groups leads to somewhat similar conclusions as in the case of intransitive groups, but with many more technicalities. The idea is to first consider the compatible permutations of $\operatorname{Aut} \Pi$, the automorphism group of the partition Π. An imprimitive group G is a transitive subgroup of a group of the form $\operatorname{Aut} \Pi$, where Π is a nontrivial partition with blocks of equal size. By the monotonicity of $\operatorname{Comp}^{(n+i)}$, we can bound $\operatorname{Comp}^{(n+i)} G$ above by $\operatorname{Comp}^{(n+1)} \operatorname{Aut} \Pi$, and in this way we can determine the asymptote and estimate how fast it is reached.

[1] M.D. Atkinson and R. Beals. Permuting mechanisms and closed classes of permutations. In: C.S. Calude and M.J. Dinneen (eds.). *Combinatorics, Computation & Logic,* Proc. DMTCS '99 and CATS '99 (Auckland). Aust. Comput. Sci. Commun., 21, No. 3, pp. 117–127. Springer, Singapore, 1999.

[2] M.D. Atkinson and R. Beals. Permutation involvement and groups. *Q. J. Math.*, 52:415–421, 2001.

[3] E. Lehtonen. Permutation groups arising from pattern involvement. *J. Algebraic Combin.*, 52:251–298, 2020.

[4] E. Lehtonen and R. Pöschel. Permutation groups, pattern involvement, and Galois connections. *Acta Sci. Math. (Szeged)*, 83:355–375, 2017.

Pattern Avoidance in Cyclic Permutations

Jinting Liang

liangj26@msu.edu

Michigan State University

(This talk is based on joint work with Dr. Bruce Sagan, Rachel Domagalski, Quinn Minnich, Jamie Schmidt and Alexander Sietsema.)

A cyclic permutation is the set of all rotations of a linear permutation. We say a cyclic permutation $[\sigma]$ avoids a given pattern $[\pi]$ if there is no rotation of σ contains the pattern π. In this work, we studied the enumeration of cyclic permutations that avoid different pairs and triples of length 4 patterens.

[1] David Callan. Pattern avoidence in circular permutations. Preprint *arXiv*: 0210014.

Triangular permutation matrices

Vadim Lozin

V.Lozin@warwick.ac.uk

University of Warwick

Abstract

We introduce a new way of representing permutations by matrices that are resistant to row or column swaps, i.e. by matrices each of which uniquely represents a permutation regardless of the order of its columns and rows.

Keywords: permutation matrices; permutation graphs

1 Introduction

The theory of permutations is a rapidly growing field of research with a vast literature (see e.g. the book [4] and the references therein), with connections to many other fields of mathematics (e.g. graph theory [2]), and with applications in various areas far beyond mathematics (e.g. textile art [8]).

One of the common ways to represent a permutation is a permutation matrix, which is a 0-1 matrix containing exactly one 1 in each column and each row (see Figure 1 (left)). This representation inspired many important notions in the theory of permutations, for example, monotone grid classes of permutations [1].

Any permutation matrix uniquely represents a permutation and any swap of two rows or columns results in a matrix representing a different permutation. In Section 2, we introduce a new way of representing permutations by 0-1 matrices that are resistant to row/column swaps, i.e. by matrices each of which uniquely represents a permutation regardless of the order of its columns and rows. In Section 3, we discuss possible applications of the proposed representation.

2 Triangular permutation matrices

Let $P = p(i, j)$ be the permutation matrix of a permutation π of n elements. The unique 1 in each column and each row of P will be called a permutation point corresponding to this column and row. Let $T = t(i, j)$ be a lower triangular matrix of order n, i.e. $t(i, j) = 1$ for $i \geq j$ and $t(i, j) = 0$ for $i < j$, and let $A = a(i, j)$ be the symmetric difference of P and T, i.e. $a(i, j) = p(i, j) \oplus t(i, j)$, where $\oplus$ is addition modulo 2. Figure 1 (right)

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Figure 1: Permutation matrix (left) and triangular permutation matrix (right)

represents matrix A for the permutation matrix P on the left of Figure 1. Clearly, A also represents π. However, some permutation points are represented in A by 0s.

We call any matrix obtained from A by any shuffle of its rows or columns a *triangular permutation* matrix and show that each triangular permutation matrix uniquely represents a permutation.

Theorem 1. *Let $A = a(i,j)$ be the symmetric difference of a permutation matrix P and a lower triangular matrix T, and let B any shuffle of A, i.e. a matrix obtained from A by permuting its rows and columns. Then A can be restored from B in a unique way.*

Proof. To restore A from B we need to determine, for each row i and each column j of B, the original position $r(i)$ of i and $c(j)$ of j in the matrix A. We prove the theorem by induction on the order n of A. For $n = 2$, the statement is trivial, since in this case A contains either a single 1 or a single 0. For $n > 2$, we either determine the correct values of $r(i)$, $c(j)$, or identify indices $i_1, \ldots, i_k$ such that the submatrix of B in rows $i_1, \ldots, i_k$ and columns $i_1, \ldots, i_k$, which we denote by $B[i_1, \ldots, i_k]$, is a shuffle of a triangular permutation matrix of order $k < n$. In the latter case, the result follows by induction.

If the Hamming weight of row i in B (i.e. the number of 1s in the row) is k, then $r(i) = k - 1$ or $r(i) = k + 1$. The problem is that two different rows (and no more than two) can have the same Hamming weight, in which case we say that the two rows are *matched*. We use a similar terminology for columns.

If B has neither matched rows nor matched columns, then restoring A from B is a trivial task. In what follows we analyze the problem in the presence of matched rows and columns. Without loss of generality we assume that any unmatched row or column is located in the only possible position for it, we call such a row or column *fixed*, and that any two matched rows or columns of B are located, randomly, in the only two possible positions available for them. An interval of consecutive rows (columns) in B will be called fixed if any row (column) in this interval either is fixed or has a matched partner in the same interval. The task is to determine for each matched pair a correct assignment.

If in row k of A the permutation point is a 1, we denote it by $\overline{1}_k$, and if the permutation point is a 0, we denote it by $\overline{0}_k$. Similarly, if in column k of A the permutation point is a 1, we denote it by $\underline{1}_k$, and if the permutation point is a 0, we denote it by $\underline{0}_k$.

We observe that if two rows k, $k+2$ are matched in B, then one them contains $\overline{1}_k$ and the other $\overline{0}_{k+2}$, and if two columns k, $k+2$ are matched, then one of them contains $\underline{0}_k$ and the other $\underline{1}_{k+2}$.

Let k be the minimum integer such that row or column k is matched with row or column $k+2$. We analyze the case of two matched rows k and $k+2$, as the case of columns is similar, and distinguish between the following two sub-cases: (i) columns k and $k+2$ are matched too, and (ii) column k is fixed.

Case (i). Assume first that rows $k+1, k+3$ are also matched, as well as columns $k+1, k+3$. In this case, $B[k, k+1, k+2, k+3]$ is simply a shuffle of $A[k, k+1, k+2, k+3]$. If one of the permutation points corresponding to rows $k, k+1, k+2, k+3$ or columns $k, k+1, k+2, k+3$ does not belong to $A[k, k+1, k+2, k+3]$ (and hence to $B[k, k+1, k+2, k+3]$), then we can identify it as follows. If, for instance, $\overline{1}_k$ does not belong to $B[k, k+1, k+2, k+3]$, then it belongs to a column in the interval $k+4, \dots, n$, which is fixed, and hence $\overline{1}_k$ is the unique 1 in rows $k, k+2$ and columns $k+4, \dots, n$ of B. By definition, the row of B containing $\overline{1}_k$ is row k in A. If all the permutation points corresponding to rows $k, k+1, k+2, k+3$ or columns $k, k+1, k+2, k+3$ belong to $B[k, k+1, k+2, k+3]$, then $B[k, k+1, k+2, k+3]$ is a triangular permutation matrix, in which case we apply induction.

Now suppose that column $k+1$ is fixed (the case of a fixed row is similar). As before, we can easily identify any of the permutations points $\overline{1}_k$, $\overline{0}_{k+2}$, $\underline{1}_{k+2}$ that does not belong to $A[k, k+1, k+2]$. If all of them belong to $A[k, k+1, k+2]$, then either

- $\overline{1}_k = \underline{1}_{k+2}$, in which case either column $k+1$ (which is fixed) distinguishes rows k and $k+2$ in B with a 1 belonging to row $k+2$ of A, or rows k and $k+2$ are identical in B and we can fix them arbitrarily, or
- $\overline{1}_k \neq \underline{1}_{k+2}$, in which case rows $k, k+2$ and columns $k, k+2$ are identical in B outside of the submatrix $B[k, k+2]$, and the unique correct position of the elements of $B[k, k+2]$ (in A) is with two 1s on the main diagonal and two 0s on the other diagonal.

Case (ii). We may assume that $\overline{0}_{k+2}$ does not belong to columns $1, \dots, k$ in A, because they are fixed in B, allowing an easy identification of $\overline{0}_{k+2}$. Therefore, $\overline{0}_{k+2}$ is either in column $k+1$ or in column $k+2$ in A, implying that $\overline{1}_k$ does not belong to columns $k+1$, $k+2$, since otherwise rows k and $k+2$ are identical, in which case we can fix them arbitrarily. This implies that rows k and $k+2$ differ in exactly two columns, which we call coloured, and one of them contains $\overline{1}_k$. It is not difficult to see that if a coloured column belongs to the interval $k+5, \dots, n$ in B or has a matched partner in this interval, then this column contains $\overline{1}_k$. Therefore, we may assume that both coloured columns belong to the interval $k+1, \dots, k+4$ in B and they do not have matched partners in the interval $k+5, \dots, n$. As a result, $\overline{1}_k$ does not belong to columns $k+5, \dots, n$ in A, leaving us with two options for $\overline{1}_k$: $\overline{1}_k$ is either in column $k+3$ or in column $k+4$ in A. Finally, we may assume that neither of the two coloured columns is fixed in B, since otherwise we can easily distinguish between rows k and $k+2$. Under these assumptions we are left with the following fours cases to analyze, where B^1 denotes the submatrix of B in rows $k, k+2$ and columns $k+1, k+3$, and B^2 denotes the submatrix of B in rows $k, k+2$ and columns $k+2, k+4$.

(a) In A, $\overline{0}_{k+2}$ is in column $k+1$ and $\overline{1}_k$ is in column $k+3$, implying that in B, each of B^1 and B^2 contains a single 1.

(b) In A, $\overline{0}_{k+2}$ is in column $k+1$ and $\overline{1}_k$ is in column $k+4$, implying that in B, B^1 is a zero matrix and B^2 contains two 1s and two 0s.

(c) In A, $\overline{0}_{k+2}$ is in column $k+2$ and $\overline{1}_k$ is in column $k+3$, implying that in B, B^1 contains two 1s and two 0s and B^2 is a zero matrix.

(d) In A, $\overline{0}_{k+2}$ is in column $k+2$ and $\overline{1}_k$ is in column $k+4$, implying that in B, each of B^1 and B^2 contains a single 1.

To distinguish between cases (a) and (d) in B, we observe that in both cases, according to our assumptions, columns $k+1$ and $k+3$ are matched and hence we can look for $\underline{1}_{k+3}$. In case (a), $\underline{1}_{k+3} = \overline{1}_k$, and in case (d), $\underline{1}$ belongs either to rows $1, \ldots, k-1$, in which case it can be easily identified in B, because these rows are fixed, or to row $k+1$ in A. This allows us to distinguish between cases (a) and (d) by looking at the submatrix $B[k+1, k+3]$. If rows $r+3$ and $r+5$ are not matched, then $B[k+1, k+3]$ contains a single 0 in case (a) and no 0s in case (d). If rows $r+3$ and $r+5$ are matched, then row $r+1$ is fixed and it distinguishes columns $k+1$ and $k+3$ in case (a) and it does not distinguish columns $k+1$ and $k+3$ in case (d).

The above discussion allows as to distinguish between all four cases (a), (b), (c), (d). If we are in case (a), then the only 1 in B^1 is $\overline{1}_k$, allowing to fix both rows $k, k+2$ and columns $k+1, k+3$. Similarly, if we are in case (d), then the only 1 in B^2 is $\overline{1}_k$, allowing to fix both rows $k, k+2$ and columns $k+2, k+4$. If we are in case (b), then rows k and $k+2$ differ in columns $k+2, k+4$ and coincide in all other columns. Moreover, we know that in A the two 0s of B^2 must be located on the main diagonal. Therefore, we can fix rows $k, k+2$ randomly and continue the procedure. If, when the procedure finishes by fixing all rows and columns, the two 0s of B^2 are not on the main diagonal, then we know that our random choice of the positions for rows k and $k+2$ was wrong and we can swap them, which changes the structure of B^2 only. Otherwise, our choice was correct and we have nothing to change. A similar argument applies in case (c) with respect to the matrix B^1. □

3 Applications and open problems

The motivation behind the question studied in this paper is modeling permutations by graphs. The usual way of representing permutations by graphs, known as permutation graphs [7], is not injective in the sense that it can map different permutations to isomorphic graphs. The result presented in the paper provides an injective mapping from permutations to coloured bipartite graphs, i.e. bipartite graphs in which the vertices in different parts have different colours, say white, corresponding to rows, and black, corresponding to columns (see Figure 2). Alternatively, we can create a clique on the set of black vertices, in which case colours are not needed and every permutation is represented by a unique (up to an isomorphism) graph. Representing permutations by graphs makes them amenable to the techniques of graph theory, which allows a structural analysis of permutations by means of various graph parameters.

We observe that the bipartite graphs represented by triangular matrices are known as half-graphs [3] and they appear in model theory as an instance of the order property [6]. We ask

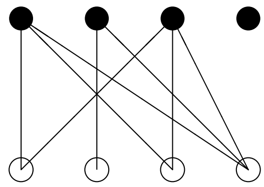


Figure 2: The graph representing the triangular permutation matrix on the right of Figure 1

what other types of graphs allow to efficiently "cipher" a permutation? This question also suggests that the result obtained in this paper can be of interest in cryptology, because if a permutation contains a valuable information, then by presenting it as a "shuffled" trinagular permutation matrix we keep it in a secrete form.

One more interesting feature of the proposed representation is that it allows us to introduce the notion of an *inessential permutation point*, i.e. a point which is not needed to restore the permutation. The following example illustrates the idea.

> *Example.* In the triangular permutation matrix represented on the right of Figure 1 the last permutation point is inessential in the sense that the last row of the matrix can be removed and the permutation can be restored even after reshuffling the remaining part of the matrix.
>
> To show this, let us denote the first three rows of the matrix by a, b, c and the columns by x, y, z, o (in the oder in which they appear in the matrix). Since the Hamming weight of b is 1, it can only by row 2, i.e. $b = 2$. Similarly, a and c can appear only in rows 1 and 3, and since a and b are identical, we can assume without loss of generality that $a = 1$ and $c = 3$.
>
> Again, since the Hamming weight of b is 1, a permutation point in b appears in one of the first two columns, i.e. $y = 1$ or $y = 2$. If $y = 1$, then row a contains two 1s outside of the first column, which is impossible. Therefore, $y = 2$. Obviously, $o = 4$, since otherwise the 4-th column contains two 1s in the first three rows, which is not possible. Finally, since x and z are identical, we can assume without loss of generality that $x = 1$ and $z = 3$. Therefore, the permutation point in row 4 is located in column 4 and hence the last row is 1110.

With the restriction to permutations in special classes, this idea can lead to more inessential points, i.e. a permutation point may become inessential if we assume that our permutation belongs to a particular class. This idea also suggests the notion of *specification number*, i.e. the minimum number of points needed to uniquely specify a permutation within a permutation class. This notion is well-studied in the context of learning Boolean functions. For instance, it is known [5] that within the class of threshold Boolean functions every linear read-once function of n variable can be specified by defining it in at most $n + 1$ (out of 2^n) points.

Finally, the proposed representation raises a number of algorithmic questions: how difficult it is to recognize triangular permutation matrices, and what is the complexity of "deciphering" a permutation from a "shuffled" trinagular permutation matrix?

[1] M. Albert, A. Atminas, R. Brignall, Characterising inflations of monotone grid classes of permutations. *J. Combin. Theory Ser. A* 154 (2018), 444–463.

[2] B. Alecu, V. Lozin, D. de Werra, V. Zamaraev, Letter graphs and geometric grid classes of permutations: characterization and recognition. *Discrete Applied Math.* 283 (2020), 482–494.

[3] D. Conlon, J. Fox, Bounds for graph regularity and removal lemmas. *Geom. Funct. Anal.* 22 (2012), 1191–1256.

[4] S. Kitaev, Patterns in permutations and words. Monographs in Theoretical Computer Science. An EATCS Series. Springer, Heidelberg, 2011. xxii+494 pp.

[5] V. Lozin, I. Razgon, V. Zamaraev, E. Zamaraeva, N. Zolotykh, Linear read-once and related Boolean functions. *Discrete Appl. Math.* 250 (2018), 16–27.

[6] M. Malliaris, S. Shelah, Regularity lemmas for stable graphs. *Trans. Amer. Math. Soc.* 366 (2014), 1551–1585.

[7] A. Pnueli, A. Lempel, S. Even, Transitive orientation of graphs and identification of permutation graphs. *Canadian J. Math.* 23 (1971), 160–175.

[8] http://www.theowright.co.uk/permutations.html

Pattern-avoiding rectangulations and permutations

Arturo Merino

merino@math.tu-berlin.de

TU Berlin

(This talk is based on joint work with Torsten Mütze (University of Warwick).)

Pattern-avoidance is a central topic in combinatorics, and in this work we initiate the systematic study of pattern-avoidance in rectangulations, which are partitions of a rectangle into finitely many interior-disjoint rectangles. We approach these objects from the point of view of combinatorial generation, by developing a versatile algorithmic framework for generating a large variety of different classes of generic rectangulations [3], i.e., rectangulations with the property that no four rectangles meet in a point.

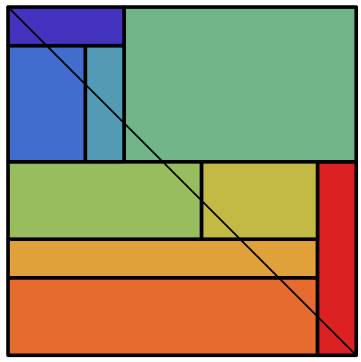
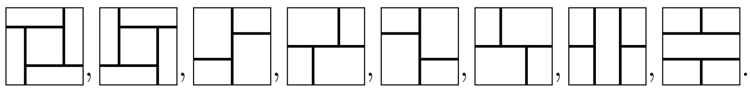


Figure 1: A diagonal rectangulation with 9 rectangles.

Our algorithms are obtained by applying the recently developed permutation language framework [2], and they allow generating and counting a large number of rectangulation classes from the literature that can be characterized by avoiding patterns such as



For example, avoiding the first two 'windmill' patterns yields rectangulations that can be subdivided into their constituent rectangles by a sequence of vertical or horizontal guillotine cuts. Similarly, avoiding the third and fourth pattern in this list yields so-called diagonal rectangulations, in which every rectangle intersects the main diagonal from the top-left corner to the bottom-right corner; see Figure 1.

We implemented these generation algorithms in C++ and made the code available for download and experimentation [1]. This talk will cover the basics of our generation framework for pattern-avoiding rectangulations, and will mention a number of intriguing conjectures and open questions that arise from our experiments with these programs, and that connect pattern-avoiding rectangulations to pattern-avoiding permutations.

[1] The Combinatorial Object Server: Generate rectangulations. http://www.combos.org/rect.

[2] E. Hartung, H. P. Hoang, T. Mütze, and A. Williams. Combinatorial generation via permutation languages. I. Fundamentals. To appear in *Trans. Amer. Math. Soc.*; preprint available at https://arxiv.org/abs/1906.06069, 2021.

[3] N. Reading. Generic rectangulations. *European J. Combin.*, 33(4):610–623, 2012.

A Formula for Counting the Number of Permutations with a Fixed Pinnacle Set

Quinn Minnich

minnichq@msu.edu

Michigan State University

(This talk is based on joint work with Rachel Domagalski, Jinting Liang, Bruce Sagan, Jamie Schmidt, and Alexander Sietsema.)

Abstract: Let S_n be the symmetric group. The *peak set* of a permutation $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ is a permutation statistic defined to be the set $\{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$. This statistic, which arises from occurrences of the consecutive patterns $\underline{132}$ and $\underline{231}$, is well studied and shows up in countless theorems in combinatorics. A slight variation of this statistic is the *pinnacle set* which is defined to be the set $\{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$. In 2018, Davis et al. introduced several techniques in [1] for counting the number of permutations with a given pinnacle set for a fixed n, but most of their formulas were recursive. In 2021, Diaz-Lopez et al. did further work in [2] and found a faster formula which did not use recursion, but that was still very slow for permutations in which the pinnacles were far apart. In this talk I will introduce a new formula for counting the number of permutations with a fixed pinnacle set that, in general, is substantially more efficient than the one introduced in [2] and that does not depend of the relative values of the pinnacles.

[1] Davis, Robert and Nelson, Sarah A. and Kyle Petersen, T. and Tenner, Bridget E. The pinnacle set of a permutation. *Discrete Mathematics*, 341(11):3249—3270, 2018.

[2] Diaz-Lopez, Alexander and Harris, Pamela E. and Huang, Isabella and Insko, Erik and Nilsen, Lars. A formula for enumerating permutations with a fixed pinnacle set. *Discrete Mathematics*, 344(6):112375, 2021

An automatic direct enumeration of Av(1342)

Émile Nadeau

emile19@ru.is

Reykjavik University

(This talk is based on joint work with Christian Bean, Jay Pantone and Henning Ulfarsson.)

The tilescope algorithm has proven very successful in the enumeration of numerous permutation classes. The approach relies on creating a massive universe of combinatorial rules for a permutation class. We then use a search algorithm (the prune algorithm) to determine if the universe contains a combinatorial specification for the permutation class. However, we have observed that sometimes, even if the combinatorial rules of the universe are not sufficient to create a specification, the universe still "holds" enough information to find the generating function of a permutation class.

Two key components of the tilescope algorithm are the prune algorithm and the combinatorial specification. The combinatorial specification is the output of the algorithm. It is a nice, human readable structural decomposition of the permutation class. The prune algorithm is an efficient way to decide if the universe has a combinatorial specification. In this talk, we will touch on how those two key components can be adapted to the situation where the universe does not contain a specification but still "holds" enough information to enumerate the permutation class. To do so, we will rely on the use of new constructions such as *combinatorial complement* ($\mathcal{A} = \mathcal{B} - \mathcal{C}$) and *cartesian quotient* ($\mathcal{A} = \mathcal{B}/\mathcal{C}$) that are in some sense the reverse of the traditional combinatorial sum and cartesian product.

We will conclude by considering Av(1342). This class was first enumerated by Bóna (1997) using a bijection to a certain type of labeled trees. It was subsequently enumerated by Bloom and Elizalde (2013) using a bijection to certain rook placements on Ferrers boards. Finally, a new polynomial time counting algorithm was provided by Biers-Ariel (2020) using enumeration schemes and structural arguments. We will present the first direct enumeration of the class found automatically by our new technique.

[1] M. Bóna. Exact enumeration of 1342-avoiding permutations: a close link with labeled trees and planar maps, *J. Combin. Theory Ser. A*, 80:257–272, 1997.

[2] J. Bloom and S. Elizalde. Patterns in matchings and rook placements. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AS, 25th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2013), 2013

[3] Y. Biers-Ariel. Flexible schemes and beyond: experimental enumeration of pattern avoidance classes *PhD Thesis*, Rutgers University, 2020

Hardness of $\mathcal{C}$-Permutation Pattern Matching

Michal Opler

opler@iuuk.mff.cuni.cz

Computer Science Institute, Charles University, Prague, Czechia

(This talk is based on joint work with Vít Jelínek and Jakub Pekárek.)

In this talk, we investigate the hardness of deciding whether a permutation π is contained in a permutation τ given that both π and τ belong to a fixed permutation class $\mathcal{C}$.

1 Introduction

Permutation Pattern Matching, or PPM , is one of the most fundamental decision problems related to permutations. In PPM, the input consists of two permutations: τ, referred to as the 'text', and π, referred to as the 'pattern', and the goal is to determine whether the text τ contains the pattern σ. Bose, Buss and Lubiw [2] have shown that the PPM problem is NP-complete. Thus, most recent research into the complexity of PPM focuses either on parametrized or superpolynomial algorithms or on restricted instances of the problem.

These restrictions usually take the form of specifying that the pattern must belong to a prescribed permutation class $\mathcal{C}$, called $\mathcal{C}$-Pattern PPM problem, or that both the pattern and the text must belong to $\mathcal{C}$, called $\mathcal{C}$-PPM problem. The most commonly considered choices for $\mathcal{C}$ are the principal classes $\mathrm{Av}(\sigma)$.

The hardness of $\mathrm{Av}(\sigma)$-Pattern PPM has been resolved by Jelínek and Kynčl [5] for all choices of σ. They showed that $\mathrm{Av}(\sigma)$-Pattern PPM is polynomial-time solvable for $\sigma \in \{1, 12, 21, 132, 213, 231, 312\}$ and NP-complete for any other σ. Their results also imply the hardness of $\mathrm{Av}(\sigma)$-PPM when σ is large enough, in particular when $|\sigma| \geq 10$. On the positive side, $\mathrm{Av}(\sigma)$-PPM is known to be polynomial-time solvable for any σ of length at most 3 (see [1, 3]). Our work aims to narrow this gap between polynomial-time solvable and NP-complete cases.

An important role in our work is played by grid classes. A matrix $\mathcal{M}$ whose entries are (possibly empty) permutation classes is called a *gridding matrix*. Additionally, we say that $\mathcal{M}$ is *monotone* if its every entry is either empty or equal to one of $\mathrm{Av}(21)$, $\mathrm{Av}(12)$. We say that a permutation π has an $\mathcal{M}$*-gridding* if its plot can be partitioned, by horizontal and vertical cuts, into an array of rectangles, where each rectangle induces in π a subpermutation from the permutation class in the corresponding cell of $\mathcal{M}$. The permutation class $\mathrm{Grid}(\mathcal{M})$ then consists of all the permutations that have an $\mathcal{M}$-gridding. We say that a permutation class $\mathcal{C}$ is *monotone-griddable* if $\mathcal{C}$ is a subclass of $\mathrm{Grid}(\mathcal{M})$ for some monotone gridding matrix $\mathcal{M}$.

To a gridding matrix $\mathcal{M}$, we associate a *cell graph*, denoted by $G_{\mathcal{M}}$, which is the graph whose vertices are the entries in $\mathcal{M}$ that correspond to infinite classes, with two vertices being adjacent if they belong to the same row or column of $\mathcal{M}$ and there is no other infinite entry of $\mathcal{M}$ between them.

2 Our results

We develop a general type of hardness reduction, applicable to any permutation class that contains a suitable grid subclass. Roughly, we show that $\mathcal{C}$-PPM is NP-complete for a class $\mathcal{C}$ whenever $\mathcal{C}$ contains, for each n and a fixed $\epsilon > 0$, a grid subclass whose cell graph is a path of length n, and at least ϵn of its cells are equal to a non-monotone-griddable class $\mathcal{D}$.

This hardness reduction, apart from being more general than previous results, has also the advantage of being more efficient: if we additionally assume that $\mathcal{D}$ contains some monotone juxtaposition, we can reduce an instance of 3-SAT of size m to an instance of $\mathcal{C}$-PPM of size $O(m \log m)$. This implies, assuming the exponential time hypothesis (ETH) that 3-SAT cannot be solved in time $2^{o(n)}$, that $\mathrm{Av}(\sigma)$-PPM cannot be solved in time $2^{o(n/\log n)}$. Previously, this conditional lower bound was not known to hold even for the unconstrained PPM problem.

From this general reduction, we can deduce several hardness results. First, we show that even for gridding matrices $\mathcal{M}$ that are in some sense close to being monotone, $\mathrm{Grid}(\mathcal{M})$-PPM can be NP-complete.

Theorem 1. *If $\mathcal{M}$ is a gridding matrix such that $G_{\mathcal{M}}$ contains a cycle with no three vertices in the same row or column and one entry equal to a non-monotone-griddable class $\mathcal{D}$ then* $\mathrm{Grid}(\mathcal{M})$-PPM *is NP-complete, and unless ETH fails, there cannot be an algorithm for* $\mathrm{Grid}(\mathcal{M})$-PPM *running*

- *in time $2^{o(n/\log n)}$ if $\mathcal{D}$ moreover contains any monotone juxtaposition and is either sum-closed or skew-closed,*
- *in time $2^{o(\sqrt{n})}$ otherwise.*

It turns out that most classes of the form $\mathrm{Av}(\sigma)$ for σ of length at least 5 contain such suitable grid subclass. Combining it with certain staircase-like grid subclasses allows us to resolve most of the unsolved cases. In particular, we show that $\mathrm{Av}(\sigma)$-PPM is NP-complete for all permutations of length at least 4 except for one symmetry type of length 5 and for three out of seven symmetry types of length 4. As $\mathrm{Av}(\sigma)$-PPM is polynomial-time solvable for any σ of length at most 3, these are, in fact, the only cases left unsolved.

Theorem 2. *If σ is a permutation of length at least 4 that is not symmetric to any of* $3412, 3142, 4213, 4123$ *or* 41352*, then* $\mathrm{Av}(\sigma)$-PPM *is NP-complete, and unless ETH fails, it cannot be solved in time $2^{o(n/\log n)}$.*

On the positive side, we complement Theorem 1 by showing that $\mathcal{C}$-PPM can be solved in polynomial time whenever $\mathcal{C}$ itself is monotone-griddable.

Theorem 3. $\mathcal{C}$-PPM *is polynomial-time solvable for any monotone-griddable class* $\mathcal{C}$.

We remark that Theorem 1 and Theorem 3 together substantially narrow the unsolved cases of Grid($\mathcal{M}$)-PPM. As an example, consider the following two gridding matrices

$$\mathcal{M}_1 = \begin{pmatrix} \mathrm{Av}(12) & \mathrm{Av}(21) \\ \mathrm{Av}(12) & \mathrm{Av}(21) \end{pmatrix} \qquad \mathcal{M}_2 = \begin{pmatrix} \mathrm{Av}(21) & \mathrm{Av}(21) \\ \mathrm{Av}(21) & \oplus 21 \end{pmatrix}$$

where $\oplus 21$ is the class of permutations $\pi = \pi_1 \pi_2 \cdots \pi_n$ satisfying $|\pi_i - i| \leq 1$ for every i. A characterization of monotone-griddable classes by Huczynska and Vatter [4] implies that $\oplus 21$ is non-monotone-griddable. Therefore, Grid($\mathcal{M}_2$)-PPM is NP-complete by Theorem 1 while Grid($\mathcal{M}_1$)-PPM is polynomial-time solvable by Theorem 3.

[1] Michael Albert, Marie-Louise Lackner, Martin Lackner, and Vincent Vatter. The complexity of pattern matching for 321-avoiding and skew-merged permutations. *Discrete Math. Theor. Comput. Sci.*, 18(2):Paper No. 11, 17, 2016.

[2] Prosenjit Bose, Jonathan F. Buss, and Anna Lubiw. Pattern matching for permutations. *Inform. Process. Lett.*, 65(5):277–283, 1998.

[3] Sylvain Guillemot and Stéphane Vialette. Pattern matching for 321-avoiding permutations. In *Algorithms and computation*, volume 5878 of *Lecture Notes in Comput. Sci.*, pages 1064–1073. Springer, Berlin, 2009.

[4] Sophie Huczynska and Vincent Vatter. Grid classes and the Fibonacci dichotomy for restricted permutations. *Electron. J. Combin.*, 13(1):Research Paper 54, 14, 2006.

[5] Vít Jelínek and Jan Kynčl. Hardness of permutation pattern matching. In *Proceedings of the Twenty-Eighth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 378–396. SIAM, Philadelphia, PA, 2017.

Feasible regions and permutation patterns

Raul Penaguiao

raul.penaguiao@math.uzh.ch

San Francisco State University

(This talk is based on joint work with Jacopo Borga.)

Abstract

We study proportions of consecutive occurrences of permutation patterns of a given size by analysing the *feasible region*. We show that this feasible region is a polytope, more precisely the cycle polytope of a specific graph called *overlap graph*. This allows us to compute the dimension, vertices and faces of the polytope. We further develop the analysis of the feasible region by introducing the *consecutive patterns feasible reagion for* $\mathcal{C}$, where $\mathcal{C}$ is a permutation class. We give a precise description of this feasible regions whenever $\mathcal{C} = \mathrm{Av}(\tau)$ for τ a permutation of size 3 and for τ a monotone permutation. We also show that, whenever $\mathcal{C} = \mathrm{Av}(\tau)$, the resulting feasible region has dimension $|\mathrm{Av}_k(\tau)| - |\mathrm{Av}_{k-1}(\tau)|$. Finally, we conjecture that in these cases, the feasible region is a polytope.

1 Classical feasible regions

We denote by $\mathcal{S}_n$ the set of permutations of size n, by $\mathcal{S}$ the space of all permutations, and by $\widetilde{\mathrm{occ}}(\pi, \sigma)$ (resp. $\widetilde{\text{c-occ}}(\pi, \sigma)$) the proportion of classical occurrences (resp. consecutive occurrences) of a permutation π in σ. In [4], the feasible region for **classical** patterns was defined:

$$clP_k := \left\{ \vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\mathrm{occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \tag{1}$$
$$= \left\{ (\Delta_\pi(P))_{\pi \in \mathcal{S}_k} \middle| P \text{ is a permuton} \right\} ,$$

The feasible region was first studied for some particular families of patterns instead of the whole $\mathcal{S}_k$. More precisely, given a list of finite sets of permutations $(\mathcal{P}_1, \dots, \mathcal{P}_\ell)$, the authors in [4] considered the *feasible region* for $(\mathcal{P}_1, \dots, \mathcal{P}_\ell)$:

$$\left\{ \vec{v} \in [0,1]^\ell \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \sum_{\tau \in \mathcal{P}_i} \widetilde{\mathrm{occ}}(\tau, \sigma^m) \to \vec{v}_i, \text{ for } i \in [\ell] \right\} .$$

They first studied the simplest case when $\mathcal{P}_1 = \{12\}$ and $\mathcal{P}_2 = \{123, 213\}$ showing that the corresponding feasible region for $(\mathcal{P}_1, \mathcal{P}_2)$ is the region of the square $[0,1]^2$ bounded from below by the curve parameterized by $(2t - t^2, 3t^2 - 2t^3)_{t\in[0,1]}$ and from above by the curve parameterized by $(1 - t^2, 1 - t^3)_{t\in[0,1]}$ (see [4, Theorem 13]).

The set clP_k was also studied in [6], even though with a different goal. There, the notion of permutons was leveraged to establish a lower bound for the dimension of this feasible region. There is still no improvement on this bound, but an upper bound was indirectly established in [7] as the number of so-called *Lyndon permutations* of size at most k, whose set we denote $\mathcal{L}_k$. By analyzing smaller cases, we conjecture that this upper bound is tight:

Conjecture 1. *The feasible region clP_k is full-dimensional inside a manifold of dimension $|\mathcal{L}_k|$.*

2 Feasible regions for consecutive patterns

We introduce the parallel study of feasible regions using **consecutive** occurences as our statistic. This has several motivations and advantages. First, the study of consecutive occurrences led to a notion of permutation limit in [3], called *random infinite rooted shift-invariant permutations*, much in the same way as the study of classical permutation patterns led to the permuton limit of permutations, introduced in [5]. Thus, we expect to be able to leverage this new notion of permutation limit to extract results for the feasible region.

Second, the resulting feasible region is much more tractable, and arises as a suitable application of **cycle polytopes**, a polytope construction for graphs.

The feasible region for consecutive occurrences is thus defined in [9] as:

$$\begin{aligned} P_k &:= \left\{\vec{v} \in [0,1]^{\mathcal{S}_k} \middle| \exists (\sigma^m)_{m\in\mathbb{N}} \in \mathcal{S}^{\mathbb{N}} \text{ s.t. } |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k \right\} \qquad (2) \\ &= \left\{(\Gamma_\pi(\sigma^\infty))_{\pi\in\mathcal{S}_k} \middle| \sigma^\infty \text{ is a random infinite rooted } \textit{shift-invariant} \text{ permutation}\right\}. \end{aligned}$$

Definition 2.1. The graph $\mathcal{O}v(k)$ is a directed multigraph with labeled edges, where the vertices are elements of $\mathcal{S}_{k-1}$ and for every $\pi \in \mathcal{S}_k$ there is an edge labeled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

The overlap graph $\mathcal{O}v(4)$ is displayed in fig. 1. Recall that a simple cycle on a directed graph is a cycle that dos not repeat vertices.

Definition 2.2. Let $G = (V, E)$ be a directed multigraph. For each non-empty cycle $\mathcal{C}$ in G, define $\vec{e}_{\mathcal{C}} \in \mathbb{R}^E$ so that

$$(\vec{e}_{\mathcal{C}})_e := \frac{\#\text{ of occurrences of } e \text{ in } \mathcal{C}}{|\mathcal{C}|}, \quad \text{for all} \quad e \in E.$$

We define $P(G) := \text{conv}\{\vec{e}_{\mathcal{C}} | \mathcal{C} \text{ is a simple cycle of } G\}$, the *cycle polytope* of G.

Our first main result relates the feasible region for consecutive occurrences P_k with the cycle polytope of the overlap graph. An example for $k = 3$ is displayed in fig. 2.

Theorem 2. *P_k is the cycle polytope of the overlap graph $\mathcal{O}v(k)$. Its dimension is $k! - (k-1)!$ and its vertices are given by the simple cycles of $\mathcal{O}v(k)$.*

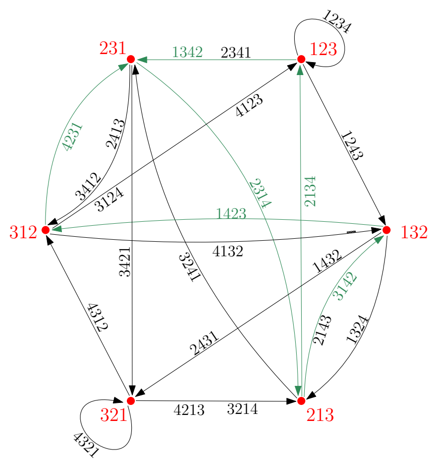


Figure 1: The overlap graph $\mathcal{O}v(4)$. The six vertices are painted in red and the edges are drawn as labeled arrows. Note that in order to obtain a clearer picture we did not draw multiple edges, but we use multiple labels.

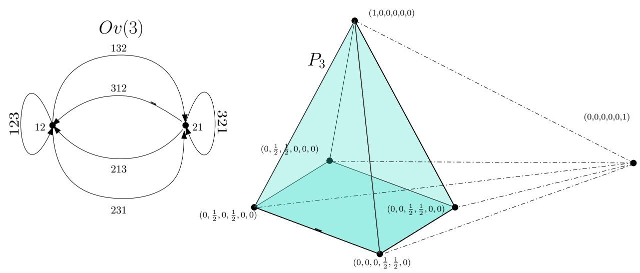


Figure 2: **Left:** The overlap graph $\mathcal{O}v(3)$ (see definition 2.1). **Right:** The four-dimensional polytope P_3 given by the six patterns of size three (see (2) for a precise definition). We highlight in light-blue one of the six three-dimensional faces of P_3. This face is a pyramid with square base. The polytope itself is a four-dimensional pyramid, whose base is the highlighted face. From theorem 2 we have that P_3 is the cycle polytope of $\mathcal{O}v(3)$.

3 Pattern avoidance

We propose here a further development of the study of feasible regions, where the feasible points are now obtained from a sequence of permutations in a specified permutation class. The feasible regions thus obtained, called *consecutive patterns feasible region*, are smaller.

Let us now consider B a collection of permutations. We introduced the *consecutive patterns feasible region for* $\mathrm{Av}(B)$, defined by

$$P_k^B := \{\vec{v} \in [0,1]^{\mathcal{S}_k} \,\big|\, \exists (\sigma^m)_{m\in\mathbb{Z}_{\geq 1}} \in \mathrm{Av}(B)^{\mathbb{Z}_{\geq 1}} \text{ such that} \\ |\sigma^m| \to \infty \text{ and } \widetilde{\text{c-occ}}(\pi, \sigma^m) \to \vec{v}_\pi, \forall \pi \in \mathcal{S}_k\}.$$

In words, the region P_k^B is formed by the $k!$-dimensional vectors $\vec{v}$ for which there exists a sequence of permutations in $\mathrm{Av}(B)$ whose size tends to infinity and whose proportion of consecutive patterns of size k tends to $\vec{v}$. For simplicity, whenever $B = \{\tau\}$ we simply write P_k^τ for $P_k^{\{\tau\}}$ (and we use the same convention for related notation). Recall that $\oplus$ and $\ominus$ are classical binary operations on permutations.

Theorem 3.1. *[8, Theorem 1.1] Fix $k \in \mathbb{Z}_{\geq 1}$ and a set of patterns $B \subset \mathcal{S}$ such that the family* $\mathrm{Av}(B)$ *is closed either for the $\oplus$ operation or $\ominus$ operation. The feasible region P_k^B is closed and convex. Moreover,*

$$\dim(P_k^B) = |\,\mathrm{Av}_k(B)| - |\,\mathrm{Av}_{k-1}(B)|.$$

Remark that, whenever $B = \{\tau\}$ is a singleton, this permutation class is either closed for the $\oplus$ operation (whenever τ is $\oplus$ indecomposible) or closed for the $\ominus$ operation (whenever τ is $\ominus$ indecomposible). Therefore, this and the following theorems apply when B is a singleton.

Given that we have a dimension, it is natural to wonder if the restricted feasible regions can be described geometrically. Thus, we get the following conjecture:

Conjecture 3.2. *Fix $k \in \mathbb{Z}_{\geq 1}$ and a sets of patterns $B \subset \mathcal{S}$ such that the family* $\mathrm{Av}(B)$ *is closed either for the $\oplus$ operation or $\ominus$ operation. The feasible region P_k^B is a polytope.*

The following section will be dedicated to establishing this conjecture for some particular cases. As we will see latter in theorem 4.3, even these particular cases do not enjoy of a simple polytopal description so this conjecture is rather ambitious.

4 Pattern avoidance: 312 and monotone patterns

We now introduce a new overlap graph in order to study the restricted feasible regions. Of particular interest are the feasible regions that arise for $B = \{\tau\}$.

Definition 4.1. Fix a set of patterns $B \subset \mathcal{S}$ and $k \in \mathbb{Z}_{\geq 1}$. The *overlap graph* $\mathcal{O}v^B(k)$ is a directed multigraph with labelled edges, where the vertices are elements of $\mathrm{Av}_{k-1}(B)$ and for every $\pi \in \mathrm{Av}_k(B)$ there is an edge labelled by π from the pattern induced by the first $k-1$ indices of π to the pattern induced by the last $k-1$ indices of π.

For an example with $k = 3$ see the top of fig. 3.

Theorem 4.2. *[8, Theorem 1.14] Fix $k \in \mathbb{Z}_{\geq 1}$. The feasible region P_k^{312} is the cycle polytope of the overlap graph $\mathcal{O}v^{312}(k)$.*

To help us with the case where τ is a monotone permutation, a different overlap graph $\mathfrak{C}\mathcal{O}v^{\nu_n}(k)$ was introduced in [8]. This new graph is the one that helps us establish our main result for the monotone patterns case, which differs slightly from the result on P_k^{312} above. Indeed, in [8] we observe that in general $P_k^{\nu_n}$ is different from the cycle polytope of the overlap graph $\mathcal{O}v^{\nu_n}(k)$.

Theorem 4.3. *[8, Theorem 1.16] Fix $\nu_n = n \cdots 1$ for $n \in \mathbb{Z}_{\geq 2}$. There exists a projection map Π, explicitly described in [8], such that the consecutive patterns feasible region $P_k^{\nu_n}$ is the Π-projection of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{\nu_n}(k)$. That is,*

$$P_k^{\nu_n} = \Pi(P(\mathfrak{C}\mathcal{O}v^{\nu_n}(k)))\,.$$

An instance of the result stated in theorem 4.3 is depicted on the bottom part of fig. 3. We remark that theorem 4.3 highlights what kind of difficulties can be encountered in proving conjecture 3.2, because it shows that we cannot hope for a general description of the feasible reagion via cyclic polytopes.

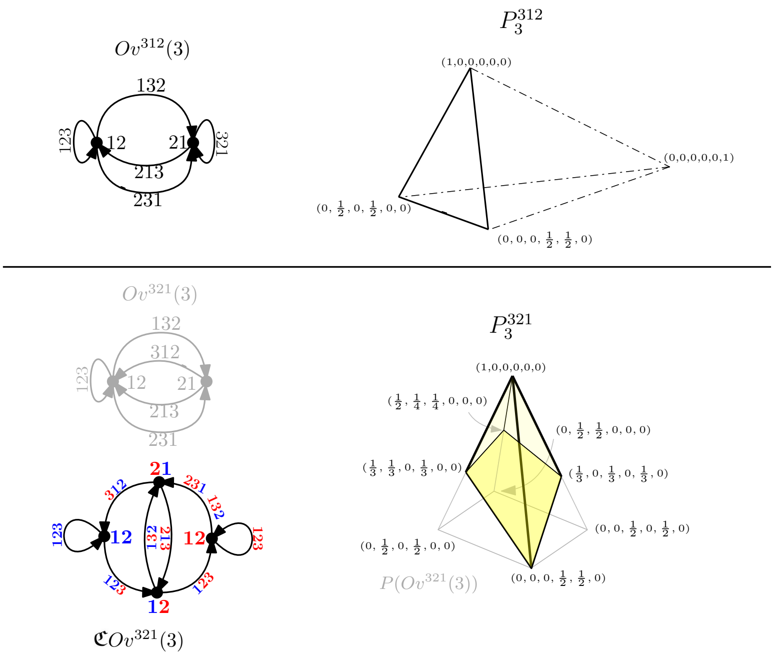


Figure 3: **Top:** The overlap graph $\mathcal{O}v^{312}(3)$ and the three-dimensional polytope P_3^{312}. Note that $P_3^{312} \subset P_3$. From theorem 4.2 we have that P_3^{312} is the cycle polytope of $\mathcal{O}v^{312}(3)$. **Bottom:** In light grey the overlap graph $\mathcal{O}v^{321}(3)$ and the corresponding three-dimensional cycle polytope $P(\mathcal{O}v^{321}(3))$, that is strictly larger than P_3^{321}. The latter feasible region is highlighted in yellow. From theorem 4.3 we have that P_k^{321} is the projection of the cycle polytope of the coloured overlap graph $\mathfrak{C}\mathcal{O}v^{321}(3)$. This graph is plotted in the bottom-left side. Note that $P_3^{321} \subset P_3$.

[1] F. Bassino, M. Bouvel, V. Féray, L. Gerin, and A. Pierrot, "The Brownian limit of separable permutations," *Ann. Probab.*, vol. 46, no. 4, pp. 2134–2189, 2018.

[2] D. Bevan, "Permutations with few inversions are locally uniform," *arXiv preprint:1908.07277*, 2019.

[3] J. Borga, "Local convergence for permutations and local limits for uniform ρ-avoiding permutations with $|\rho| = 3$," *Probab. Theory Related Fields*, vol. 176, no. 1-2, pp. 449–531, 2020.

[4] R. Charles, K. Daniel, W. Peter, K. Richard, "Permutations with fixed pattern densities," *Random Structures Algorithms*, 2015.

[5] C. Hoppen, Y. Kohayakawa, C. G. Moreira, B. Ráth, and R. Menezes Sampaio, "Limits of permutation sequences," *J. Combin. Theory Ser. B*, vol. 103, no. 1, pp. 93–113, 2013.

[6] H. Carlos, K. Daniel, L. Hong, G. Roman, K. Tereza, K. Yoshiharu, "Densities in large permutations and parameter testing," *European J. Combin.*, 2017.

[7] Y. Vargas, "Hopf algebra of permutation pattern functions," *HAL preprint: hal-01207565*, 2015.

[8] J. Borga, R. Penaguiao, "The feasible region for consecutive patterns of permutations is a cycle polytope," *arXiv preprint: 1910.02233*, 2020.

[9] J. Borga, R. Penaguiao, "The feasible region for consecutive patterns of permutations is a cycle polytope," *Algebraic Combinatorics, Volume 3 no. 6, pp. 1259-1281.* , 2020.

The Bivariate Generating Function on the Statistics Peak and Des for Cyclic Permutations on $[n+2]$ which avoid the patterns $[1324]$ and $[1423]$

James Schmidt

schmi710@msu.edu

Michigan State University

(This talk is based on joint work with Bruce Sagan, Alex Sietsema, Jinting Liang, Quinn Minnich, Rachel Domagalski.)

A cyclic permutation is a permutation with no defined start or end. For example, the cyclic permutation [1243] is the same as the cyclic permutation [2431]. A peak of a permutation σ is a position i such that $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$, and $\mathrm{peak}(\sigma)$ is the number of peaks in σ. The cyclic analogues of $\mathrm{des}(\sigma)$ and $\mathrm{peak}(\sigma)$ are $\mathrm{cdes}(\sigma)$ and $\mathrm{cpeak}(\sigma)$, respectively, which are the number of descents and peaks in a cyclic permutation σ, again respectively. In this talk, we will demonstrate an example of our work, and show that $xy(x+1)^n$ is the bivariate generating function for cyclic permutations on $[n+2]$ which avoid the patterns [1324] and [1423] where a term cx^ay^b means that there are c such permutations σ with $\mathrm{cdes}(\sigma) = a$ and $\mathrm{cpeak}(\sigma) = b$.

A New Algorithm for Counting The Admissible Orderings of a Pinnacle Set

Alexander Sietsema

sietsem6@msu.edu

Michigan State University

(This talk is based on joint work with Rachel Domagalski, Jinting Liang, Quinn Minnich, Bruce Sagan, and Jamie Schmidt.)

Abstract: Let S_n be the symmetric group. The *peak set* of a permutation $\pi = \pi_1\pi_2 \cdots \pi_n \in S_n$ is a permutation statistic defined to be the set $\{i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$. This statistic, which arises from occurrences of the consecutive patterns $\underline{132}$ and $\underline{231}$, is well studied and shows up in countless theorems in combinatorics. A variation of this statistic introduced in [1] is the *pinnacle set* which is defined to be the set $\operatorname{Pin}\pi = \{\pi_i \mid \pi_{i-1} < \pi_i > \pi_{i+1}\}$. In essence, the peak set describes the locations of the peaks, while the pinnacle set describes the values of those peaks. We call a set S that is the pinnacle set for some permutation π an *admissible pinnacle set*. For example, the set $S = \{3, 5\}$ is an admissible pinnacle set since it is the pinnacle set of the permutation $\pi = 13254$ (among others), while any set T containing 2 is not admissible, as for 2 to be a pinnacle it would need a smaller element on both sides.

Given an admissible S, a permutation σ of S is called an *admissible ordering* if there is a $\pi \in S_n$ with $\operatorname{Pin}\pi = S$ and the pinnacles of π occur in the same order as they do in σ. For example, if $S = \{3, 5, 7\}$ then $\sigma = 537$ is admissible as shown by $\pi = 4513276$. But $\tau = 375$ is not admissible since in order for 6 not to be a pinnacle, it must be directly to the left or right of 7 and both choices lead to a contradiction. Rusu and Tenner [2] asked for a function to count the number of admissible orderings of a given pinnacle set. In my talk, I will introduce such a new function to count admissible orderings, and discuss its relationship to another new algorithm from our work [4] to count the number of permutations with a fixed pinnacle set.

[1] Davis, Robert and Nelson, Sarah A. and Kyle Petersen, T. and Tenner, Bridget E. The pinnacle set of a permutation. *Discrete Mathematics*, 341(11):3249—3270, 2018.

[2] Rusu, Irina and Tenner, Bridge Eileen. Admissible Pinnacle Orderings. arXiv:2001.08185 [math.CO], 2020

[3] Diaz-Lopez, Alexander and Harris, Pamela E. and Huang, Isabella and Insko, Erik and Nilsen, Lars. A formula for enumerating permutations with a fixed pinnacle set. *Discrete Mathematics*, 344(6):112375, 2021

[4] Domagalski, Rachel and Liang, Jinting and Minnich, Quinn and Sagan, Bruce E. and Schmidt, Jamie and Sietsema, Alexander. Pinnacle Set Properties. arXiv:2105.10388 [math.CO], 2021

Fixed points of permutations avoiding increasing patterns

Erik Slivken

slivkene@uncw.edu

University of North Carolina Wilmington

(This talk is based on joint work with Christopher Hoffman and Douglas Rizzolo.)

Let ρ_d denote the increasing pattern $1, 2, \cdots, d, d+1$ and $Av_n(\rho_d)$ the set of permutations that avoid ρ_d. We give the limiting distribution for the number of of fixed points for a permutation in $Av_n(\rho_d)$.

In [1] the authors show that permutations sampled uniformly from $Av_n(\rho_d)$ can be decomposed into a family of d decreasing sequences that, when scaled and normalized properly, converge to a family of paths described by Brownian bridges conditioned to be non-intersecting and sum to zero. Each of the d decreasing sequences will contain at most one fixed point. We use the spacing between the paths provided by the above convergence to show that the total number of fixed points of $\sigma \in Av_n(\rho_d)$ converges in distribution to the sum of d independent Bernoulli($1/2d$) random variables.

[1] C. Hoffman and D. Rizzolo and E. Slivken. Scaling limits of permutations avoiding long decreasing sequences. arXiv:1911.04982, 2020

[2] C. Hoffman and D. Rizzolo and E. Slivken. Fixed points of permutations avoiding long increasing sequences. *In Preparation.*

On pattern avoidance in matchings and involutions

Justin M. Troyka

jutroyka@davidson.edu

Davidson College

(This talk is based on joint work with Jonathan J. Fang and Zachary Hamaker.)

This extended abstract is an abbreviated form of our paper [FHT20], in which we study the relationship between two notions of pattern avoidance for involutions in the symmetric group and their restriction to fixed-point-free involutions. The first is classical, while the second generalizes the notion of pattern avoidance for perfect matchings studied by Jelínek [Jel07]. The first notion can always be expressed in terms of the second, and we give an effective algorithm to do so. We also give partial results characterizing the families of involutions where the converse holds. As a consequence, we prove two conjectures of McGovern [McG19a] characterizing (rational) smoothness of certain varieties. We also give new exact enumerations and explore the asymptotic landscape.

1 Notation and main concepts

Let $\mathfrak{S}_n$ be the symmetric group and $\mathfrak{S} = \bigsqcup_n \mathfrak{S}_n$. For $\Pi \subseteq \mathfrak{S}$, let $\mathfrak{S}(\Pi)$ be the subset of $\mathfrak{S}$ avoiding every $\pi \in \Pi$ and $\mathfrak{S}_n(\Pi) = \mathfrak{S}(\Pi) \cap \mathfrak{S}_n$. We are interested in pattern avoidance for involutions. Let

$$\mathfrak{I}_n = \{\pi \in \mathfrak{S}_n : \pi = \pi^{-1}\} \quad \text{and} \quad \mathfrak{F}_{2n} = \{\tau \in \mathfrak{I}_{2n} : \tau(i) \neq i \quad \forall i \in [2n]\}$$

be the set of involutions and fixed-point-free involutions in $\mathfrak{S}_n$. Additionally, let $\mathfrak{I} = \bigsqcup_n \mathfrak{I}_n$ and $\mathfrak{F} = \bigsqcup_n \mathfrak{F}_{2n}$. The poset of pattern containment restricts naturally to involutions. Given $\Pi \subseteq \mathfrak{S}$, we define $\mathfrak{I}_{\mathfrak{S}}(\Pi) = \mathfrak{I} \cap \mathfrak{S}(\Pi)$ and $\mathfrak{F}_{\mathfrak{S}}(\Pi) = \mathfrak{F} \cap \mathfrak{S}(\Pi)$, which are the set of involutions and fixed-point-free involutions that avoid Π, respectively. These notions are well studied [SS85, BHPV16].

In our paper [FHT20] we study a stronger notion of pattern containment on $\mathfrak{I}$ and $\mathfrak{F}$ that requires cycle structure to be preserved. The first usage of $\mathfrak{I}$-avoidance we are aware of is in recent work by the Hamaker, Marberg and Pawlowski [HMP18, HMP19], where it is used to classify algebraic properties of certain polynomials associated to involutions.

We define **$\mathfrak{I}$-containment** as the transitive closure on $\mathfrak{I}$ of the following three relations: ρ is less than τ if it can be obtained by

(1) deleting a 2-cycle from τ and standardizing,

(2) deleting a fixed point from τ and standardizing;

(3) deleting one entry from the 2-cycle $(i, i+1)$ and standardizing.

For example, $\tau = 21647358 = (12)(36)(4)(57)(8)$ $\mathfrak{I}$-contains the involution $\rho = 1432 = (1)(24)(3)$ since ρ can be obtained from τ by deleting the 2-cycle (57), the fixed point (8) and one entry from the 2-cycle (12), then standardizing. However, 65872143 $\mathfrak{I}$-avoids the involution 2143, despite containing it in the ordinary sense.

We also define **$\mathfrak{F}$-containment** as the restriction of $\mathfrak{I}$-containment to $\mathfrak{F}$; only relation (1) is applicable here since there are no fixed points. $\mathfrak{F}$-containment is the same thing as pattern containment for (perfect) matchings [Jel07], which generalizes work on non-crossing and non-nesting ordered set partitions [CDD07].

By analogy with permutation classes, an **involution class** or **$\mathfrak{I}$-class** (resp. **$\mathfrak{F}$-class**) is an order ideal in $\mathfrak{I}$ (resp. $\mathfrak{F}$) under $\mathfrak{I}$-containment (resp. $\mathfrak{F}$-containment). The notions of $\mathfrak{I}$-basis, $\mathfrak{F}$-basis, $\mathfrak{I}$-avoidance, and $\mathfrak{F}$-avoidance extend accordingly.

2 Main results

Since $\mathfrak{I}$-containment and $\mathfrak{F}$-containment are more coarse than pattern containment, if follows for $\Pi \subseteq \mathfrak{S}$ that $\mathfrak{I}_{\mathfrak{S}}(\Pi)$ is an $\mathfrak{I}$-class and $\mathfrak{F}_{\mathfrak{S}}(\Pi)$ is an $\mathfrak{F}$-class.

Theorem 1. *Let $\Pi \subseteq \bigsqcup_{k=1}^{n} \mathfrak{S}_k$. Then the $\mathfrak{I}$-basis of $\mathfrak{I}_{\mathfrak{S}}(\Pi)$ is contained in $\bigsqcup_{m=1}^{2n} \mathfrak{I}_m$. Likewise, the $\mathfrak{F}$-basis of $\mathfrak{F}_{\mathfrak{S}}(\Pi)$ is contained in $\bigsqcup_{m=1}^{n} \mathfrak{F}_{2m}$.*

As a consequence, if $\Pi \subseteq \mathfrak{S}$ is finite, then $\mathfrak{I}_{\mathfrak{S}}(\Pi)$ and $\mathfrak{F}_{\mathfrak{S}}(\Pi)$ have finite bases as well. If n is the largest size of a permutation in Π, then checking up to size $2n$ is an effective algorithm for computing the $\mathfrak{I}$-basis of $\mathfrak{I}_{\mathfrak{S}}(\Pi)$ and the $\mathfrak{F}$-basis of $\mathfrak{F}_{\mathfrak{S}}(\Pi)$. We used this algorithm to compute the bases in Table 1.

The converse of Theorem 1 cannot hold. For $\mathcal{T} \subseteq \mathfrak{I}$, let $\mathfrak{I}(\mathcal{T})$ be the set of involutions $\mathfrak{I}$-avoiding each $\tau \in \mathcal{T}$, and for $\mathcal{T} \subseteq \mathfrak{F}$ define $\mathfrak{F}(\mathcal{T})$ analogously. We observe $\mathfrak{F}_{2n}(2143) = \{\tau \in \mathfrak{F}_{2n} : \tau(i) > n \, \forall i \in [n]\}$, which are sometimes called the **permutational matchings** since they are in easy bijection with $\mathfrak{S}_n$. Therefore $|\mathfrak{F}_{2n}(2143)| = n!$, and so $\mathfrak{F}(2143) \neq \mathcal{C} \cap \mathfrak{F}$ for any $\mathcal{C} \subseteq \mathfrak{S}$ (by the Marcus–Tardos Theorem). Since $\mathfrak{I}_n(12)$ is in bijection with $\mathfrak{S}_{\lfloor n/2 \rfloor}$, the same applies for $\mathfrak{I}$-avoidance.

Our second main result shows in some sense that these are the only obstructions for $\mathfrak{I}/\mathfrak{F}$-classes whose $\mathfrak{I}/\mathfrak{F}$-bases are singletons.

basis	$\mathfrak{I}$-basis	$\mathfrak{F}$-basis
123	123, 14523, 34125, 351624, 456123	214365, 341265, 215634, 351624, 456123
132	132, 35142, 465132	2143, 465132
213	213, 42513, 546213	2143, 546213
231 or 312	3412, 4231	3412, 632541
321	321	4321

Table 1: $\mathfrak{F}$-bases and $\mathfrak{I}$-bases for permutations in S_3.

Theorem 2. *Let $\tau \in \mathfrak{I}$, $\rho \in \mathfrak{F}$. Then $\tau \in \mathfrak{I}(12)$ (or $\rho \in \mathfrak{F}(2143)$) if and only if*

$$\mathfrak{I}(\tau) = \mathfrak{I}_{\mathfrak{S}}(\tau) \quad \textit{(or} \quad \mathfrak{F}(\rho) = \mathfrak{F}_{\mathfrak{S}}(\rho)\textit{)}.$$

In fact, if $\mathfrak{I}(\tau)$ is realizable as $\mathcal{C} \cap \mathfrak{I}$ for some permutation class $\mathcal{C}$, then τ is the basis of $\mathcal{C}$. This is not quite true for $\mathfrak{F}(\tau)$ since the basis of $\mathcal{C}$ may not be fixed-point-free. However, in both cases we obtain the following dichotomy.

Corollary 3. *Let $\tau \in \mathfrak{I}$, $\rho \in \mathfrak{F}$. The sequences $\{|\mathfrak{I}_{2n}(\tau)|\}, |\{\mathfrak{I}_{2n+1}(\tau)\}|$ and $\{|\mathfrak{F}_n(\rho)|\}$ are either bounded above by an exponential function or bounded below by $n!$.*

For $\mathfrak{I}/\mathfrak{F}$-classes with larger $\mathfrak{I}/\mathfrak{F}$-bases, the relationship to permutation classes is more subtle, and we are unable to give a complete characterization. Instead, we can give the following partial result.

Theorem 4. *Let $\mathcal{T} \subseteq \mathfrak{I}$ and $\mathcal{R} \subseteq \mathfrak{F}$ be non-empty. Then $(A) \Rightarrow (B) \Rightarrow (C)$:*

(A) $\mathcal{T} \subseteq \mathfrak{I}(12)$ *(respectively $\mathcal{R} \subseteq \mathfrak{F}(2143)$).*

(B) $\mathfrak{I}(\mathcal{T}) = \mathfrak{I}_{\mathfrak{S}}(\mathcal{T})$ *(respectively $\mathfrak{F}(\mathcal{R}) = \mathfrak{F}_{\mathfrak{S}}(\mathcal{R})$).*

(C) $\mathcal{T} \cap \mathfrak{I}(12)$ *(respectively $\mathcal{R} \cap \mathfrak{F}(2143)$) is non-empty.*

Since we are the first to systematically study $\mathfrak{I}$-avoidance, basic questions on its enumerative properties remain open. As a first step, we enumerate $\mathfrak{I}(\tau)$ for $\tau \in \mathfrak{I}_3$.

Theorem 5. *For $n \geq 1$,*

$$(a)\ |\mathfrak{I}_n(321)| = \binom{n}{\lfloor n/2 \rfloor}, \quad (b)\ |\mathfrak{I}_n(213)| = |\mathfrak{I}_n(132)| = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} k!,$$

$$(c)\ |\mathfrak{I}_n(123)| = \sum_{k=1}^{n} \lfloor \tfrac{k}{2} \rfloor!\ \lfloor \tfrac{n-k}{2} \rfloor!\ \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{n-k}.$$

Some remarks on this theorem:

- Theorem 2 implies that $\mathfrak{I}(321) = \mathfrak{I}_{\mathfrak{S}}(321)$; the enumeration of $\mathfrak{I}_{\mathfrak{S}}(321)$ appears in [SS85], and we give their formula in Theorem 5 (a).
- The first equality in (b) is because the sets $\mathfrak{I}(132)$ and $\mathfrak{I}(213)$ are images of each other under the reverse–complement symmetry (since this symmetry also preserves cycle structure).
- The numbers in Theorem 5 (b) are in the OEIS [OEIS, A122852]. Han showed [Han20] this generating function is the $q = -1$ evaluation of a q-analogue of the Euler numbers. Pan and Zeng found that these numbers count certain labeled Motzkin paths called André paths [PZ19].
- The numbers in Theorem 5 (c) do not appear in OEIS. We prove the formula by characterizing an involution in $\mathfrak{I}(123)$ as the union of two involutions in $\mathfrak{I}(12)$. This is analogous to the classical characterization of $\mathfrak{S}(123)$ as the set of permutations that are a union of two permutations in $\mathfrak{S}(12)$, i.e. decreasing subsequences. A natural next step would be to extend this analysis to unions of k involutions in $\mathfrak{I}(12)$ for $k > 2$.

3 McGovern's conjectures

One early application of pattern avoidance appears in the context of Schubert calculus. One of the motivations of our work is to highlight the naturality of $\mathfrak{I}$-avoidance and $\mathfrak{F}$-avoidance from a geometric perspective. Let $B(n)$ be the group of invertible lower-triangular $n \times n$ matrices over $\mathbb{C}$. Schubert varieties are defined in terms of the action of $B(n)$ on the (complete) flag variety $FL(n)$, and Schubert varieties are indexed by elements of $\mathfrak{S}_n$. If we instead consider $FL(n)$ under the action of the orthogonal group $O(n)$ or the symplectic group $Sp(n)$ (n even), we obtain varieties $\{Y_\tau\}_{\tau \in \mathfrak{I}_n}$ and $\{Z_\rho\}_{\rho \in \mathfrak{F}_n}$ indexed by $\mathfrak{I}_n$ and $\mathfrak{F}_n$ respectively. Such varieties are generally referred to as K-orbits. McGovern's results are some of the first to characterize geometric properties of these varieties in terms of pattern avoidance. Hamaker, Marberg and Pawlowski have shown many other properties of K-orbits are governed by $\mathfrak{I}$ and $\mathfrak{F}$-avoidance [HMP18, HMP19, HMP20].

McGovern has given the following characterizations of smoothness (the variety is a manifold) and rational smoothness (a technical condition that is roughly equivalent to the variety satisfying Poincaré duality) for these orbits.

Theorem 6 ([McG11a, Theorem 1]). *For $\rho \in \mathfrak{F}_{2n}$, the orbit Z_ρ is rationally smooth if and only if $\rho \in \mathfrak{F}(\Pi')$ where*

$$\begin{aligned}\Pi' = \{&351624, 64827153, 57681324, 53281764, 43218765, 65872143, 21654387,\\ &21563487, 34127856, 43217856, 34128765, 36154287, 21754836, 63287154,\\ &54821763, 46513287, 21768435\}.\end{aligned}$$

Let $\mathfrak{I}(21 * 43)$ be the set of involutions τ so that for every pair of cycles (a, b) and (c, d) of τ with $a < b < c < d$, there are an odd number of fixed points of τ between b and c.

Theorem 7 ([McG20, Theorem 1]). *For $\tau \in \mathfrak{I}_n$, the orbit Y_τ is rationally smooth if and only if $\tau \in \mathfrak{I}(\Pi) \cap \mathfrak{I}(21 * 43)$ where*

$$\begin{aligned}\Pi = \{&14325, 21543, 32154, 154326, 124356, 351624, 132546, 426153, 153624, 351426,\\ &1243576, 2135467, 2137654, 4321576, 5276143, 5472163, 1657324, 4651327,\\ &57681324, 65872143, 13247856, 34125768, 34127856, 64827153\}.\end{aligned}$$

One direction of Theorem 7 first appeared as [McG11b, Theorem 1]. A similar result also applies for smoothness.

Theorem 8 ([McG20, Theorem 2]). *For $\tau \in \mathfrak{I}_n$, the orbit Y_τ is smooth if and only if $\tau \in \mathfrak{I}(\Pi \cup \{2143, 1324\})$ with Π as in Theorem 7.*

We prove McGovern's $\mathfrak{I}'$ and $\mathfrak{F}$-avoidance characterizations in Theorem 6 and Theorem 8 hold when using ordinary pattern avoidance.

Corollary 9. *Let Π be as in Theorem 7 and Π' as in Theorem 6. Then*

1. $\mathfrak{I}(\Pi \cup \{2143, 1324\}) = \mathfrak{I}_{\mathfrak{S}}(\Pi \cup \{2143, 1324\})$.
2. $\mathfrak{F}(\Pi') = \mathfrak{F}_{\mathfrak{S}}(\Pi')$.

The results in Corollary 9 were presented as conjectures at [McG19a]. We used our Theorem 1 to reduce these conjectures to finite computations.

τ	$\lvert\mathfrak{I}_n(\tau)\rvert$
12	$= (n/2)!$
132	$\sim \frac{1}{2}e^{1/8}\, e^{\sqrt{n/2}}\,(n/2)!$
2143	$\sim \frac{1}{2}e^{-1/2}\, e^{\sqrt{2n}}\,(n/2)!$
123	$\sim \sqrt{\frac{2\pi}{27}n}\left(\frac{2}{\sqrt{3}}\right)^n (n/2)!$

Table 2: Asymptotic formula for $|\mathfrak{I}_n(\tau)|$, for various involutions τ, valid for even n.

4 Asymptotic enumeration

Let a_n be a sequence and let $c \geq 0$. We say that a_n is **of exponential order** c^n, written $a_n \bowtie c^n$, if $\limsup_{n\to\infty} |a_n|^{1/n} = c$. This condition is equivalent to

$$c = \inf\left\{ r > 0 : \lim_{n\to\infty} \frac{|a_n|}{c^n} = 0 \right\}.$$

This is a coarse measure of asymptotic growth that can only distinguish exponential growth of different bases.

By Corollary 3, for each $\tau \in \mathfrak{I}$, either $|\mathfrak{I}_n(\tau)| \bowtie c^n$ for some finite c or $|\mathfrak{I}_n(\tau)|$ is bounded below by $\lfloor n/2 \rfloor!$. The former case happens when $\tau \in \mathfrak{I}(12)$, and here $\mathfrak{I}$-avoidance coincides with ordinary pattern avoidance by Theorem 2. In the latter case, $\mathfrak{I}$-avoidance is a distinct phenomenon. Our paper [FHT20] enumerates four $\mathfrak{I}$ classes (up to geometric symmetry) that fall into this latter case: $\mathfrak{I}(12)$, $\mathfrak{I}(132)$, $\mathfrak{I}(2143)$, and $\mathfrak{I}(123)$.

For these four classes and in general, it is natural to ask: if the enumeration is greater than $\lfloor n/2 \rfloor$, then how much greater is it? Since $\frac{|\mathfrak{I}_n|}{\lfloor n/2 \rfloor} \bowtie \left(\sqrt{2}\right)^n$, we know that the exponential order of $\frac{|\mathfrak{I}_n(\tau)|}{\lfloor n/2 \rfloor}$ is between 1^n and $\left(\sqrt{2}\right)^n$. Table 2 lists the asymptotic enumerations for the four classes, restricting to even n. For $\tau \in \{12, 132, 2143\}$, we find that $\frac{|\mathfrak{I}_n(\tau)|}{\lfloor n/2 \rfloor} \bowtie 1^n$; that is, $|\mathfrak{I}_n(\tau)|$ is a sub-exponential function times $\lfloor n/2 \rfloor$. However, $\frac{|\mathfrak{I}_n(123)|}{\lfloor n/2 \rfloor} \bowtie \left(\frac{2}{\sqrt{3}}\right)^n$. Why should the first three all be within a sub-exponential factor of each other while the fourth is higher?

If there is an underlying mathematical reason for this, it may become clearer if we look at the asymptotic enumeration of $\mathfrak{I}(\tau)$ for other small involutions τ that $\mathfrak{I}$-contain 12. A reasonable next step would be to take τ of size 4 that $\mathfrak{I}$-contains 12, in which case the remaining patterns to investigate (up to geometric symmetry) are 1234, 1243, 1324, 1432, and 4231. In particular, we can hope that the enumeration of $\mathfrak{I}(1324)$ and $\mathfrak{I}(4231)$ is easier than the notoriously intractable $\mathfrak{S}(1324)$ and $\mathfrak{S}(4231)$.

[BHPV16] Miklós Bóna, Cheyne Homberger, Jay Pantone, and Vincent Vatter, *Pattern-avoiding involutions: exact and asymptotic enumeration*, Australasian Journal of Combinatorics **64** (2016), no. 1, 88–119.

[CDD07] William Chen, Eva Deng, Rosena Du, Richard Stanley, and Catherine Yan, *Crossings and nestings of matchings and partitions*, Transactions of the American Mathematical Society **359** (2007), no. 4, 1555–1575.

[FHT20] Jonathan J Fang, Zachary Hamaker, and Justin M Troyka, *On pattern avoidance in matchings and involutions*, arXiv preprint arXiv:2009.00079 (2020), submitted for publication.

[HMP18] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski, *Involution words: counting problems and connections to Schubert calculus for symmetric orbit closures*, Journal of Combinatorial Theory, Series A **160** (2018), 217–260.

[HMP19] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski, *Schur-positivity and involution Stanley symmetric functions*, International Mathematics Research Notices **2019** (2019), no. 17, 5389–5440.

[HMP20] Zachary Hamaker, Eric Marberg, and Brendan Pawlowski, *Fixed-point-free involutions and Schur P-positivity*, Journal of Combinatorics **11** (2020), no. 1, 65–110.

[Han20] Guo-Niu Han, *Hankel continued fractions and Hankel determinants of the Euler numbers*, Transactions of the American Mathematical Society (2020).

[Jel07] Vít Jelínek, *Dyck paths and pattern-avoiding matchings*, European Journal of Combinatorics **28** (2007), no. 1, 202–213.

[McG11a] William M McGovern, *Closures of K-orbits in the flag variety for SU(2n)*, Representation Theory **15** (2011), no. 16, 568–573.

[McG11b] William M McGovern, *Closures of K-orbits in the flag variety for GL(2n)*, arXiv preprint arXiv:1107.0284 (2011).

[McG19a] William M McGovern, *Closures of O(n) orbits in flag varieties of type A.*, AMS Fall Southeastern Sectional Meeting, Special Session on Combinatorial Lie Theory, 2019.

[McG20] William M McGovern, *Closures of O_n-orbits in the flag variety for GL_n, II*, arXiv preprint arXiv:2010.07114 (2020).

[PZ19] Qiong Qiong Pan and Jiang Zeng, *Bränden's (p, q)-Eulerian polynomials, André permutations and continued fractions*, arXiv preprint arXiv:1910.01747 (2019).

[SS85] Rodica Simion and Frank W Schmidt, *Restricted permutations*, European Journal of Combinatorics **6** (1985), no. 4, 383–406.

[OEIS] Neil JA Sloane et al., *The on-line encyclopedia of integer sequences*, Published electronically at http://oeis.org.

The Density of Costas Arrays Decays Exponentially

Lutz Warnke

warnke@math.gatech.edu

Georgia Institute of Technology

(This talk is based on joint work with Bill Correll and Christopher Swanson.)

Costas arrays arise in radar and sonar engineering: formally they are simply permutation matrices with the extra property that vectors joining pairs of ones are all distinct. Using ideas from random graph theory we prove that the density of Costas arrays among permutation matrices decays exponentially, solving a core theoretical problem for Costas arrays. Many intriguing questions remain open for this interdisciplinary topic at the intersection of combinatorics, probability and enumeration.

Based on joint work in progress with B. Correll and C. Swanson.

On SIF permutations avoiding a pattern

Michael D. Weiner

mdw8@psu.edu

Penn State Altoona

(This talk is based on joint work with Juan B. Gil.)

In 2004 David Callan introduced the notion of a stabilized-interval-free (SIF) permutation. This is a permutation that does not fix any proper subinterval of $[n]$. There are 3 Wilf equivalence classes for patterns of size 3, and 8 classes for pairs of patterns of size 3.

Pattern σ	$\lvert A_n^{\text{sif}}(\sigma)\rvert$
123	1, 1, 2, 5, 14, 44, 150, 496, 1758, 6018, 21782, 76414, 280448, ...
132, 213, 321	1, 1, 2, 5, 14, 42, 132, 429, 1430, ... (Catalan)
231, 312	1, 1, 1, 2, 6, 18, 54, 170, 551, 1817, 6092, 20722, 71,325, 248055, ...

(σ_1, σ_2)	$\lvert A_n^{\text{sif}}(\sigma_1, \sigma_2)\rvert$	OEIS
123, 132	1, 1, 2, 3, 6, 9, 18, 27, ...	A182522
123, 231	1, 1, 1, 1, 2, 3, 3, 5, ...	N/A
123, 321	1, 1, 2, 3, 0, 0, 0, 0, ...	finite
132, 213	1, 1, 2, 5, 8, 17, 26, 53, ...	A62318
132, 231	1, 1, 1, 2, 4, 8, 16, 32, ...	A011782
132, 321	1, 1, 2, 3, 4, 5, 6, 7, ...	A028310
231, 312	1, 1, 0, 0, 0, 0, 0, 0, ...	finite
231, 321	1, 1, 1, 1, 1, 1, 1, 1, ...	A000012

In this (work-in-progress) report we will focus on the SIF permutations that avoid the pattern 231 and the ones that avoid the pair of patterns $(123, 231)$.

Conjecture 1. *The generating function for the number of* SIF *permutations that avoid the pattern 231 is given by the continuing fraction*

$$f_i(x) = \cfrac{x}{1 + c_1x^2(x+1) - \frac{x}{1+c_2x^3(x+1)-\frac{x}{1+c_3x^4(x+1)-\dots}}}$$

where c_n is the n-th Catalan number.

Theorem 2. *For $n > 2$, the number of SIF permutations that avoid both patterns 123 and 231 is given by*

$$\begin{cases} \frac{(n+12)(n-2)}{24} & \text{if } n \equiv 0 \mod 6 \\ \frac{(n+5)(n-1)}{24} & \text{if } n \equiv 1 \mod 6 \\ \frac{(n+12)(n-2)}{24} & \text{if } n \equiv 2 \mod 6 \\ \frac{(n+3)(n+1)}{24} & \text{if } n \equiv 3 \mod 6 \\ \frac{(n+12)(n-2)}{24} - \frac{1}{3} & \text{if } n \equiv 4 \mod 6 \\ \frac{(n+3)(n+1)}{24} & \text{if } n \equiv 5 \mod 6 \end{cases}.$$

The generating function is

$$f(x) = 1 + x + x^2 - \frac{1 - 2x^2 - 2x^3 + 2x^5}{(1-x)^3(1+x)^2(1+x+x^2)}.$$

[1] D. Callan, Counting stabilized-interval-free permutations, *J. Integer Seq.* **7** (2004), Article 04.1.8.

New Refinements of a Classical Formula in Consecutive Pattern Avoidance

Yan Zhuang

yazhuang@davidson.edu

Davidson College

Let $\mathrm{Av}_n(\pi)$ denote the set of all permutations of length n avoiding π as a *consecutive* pattern. In 1962, David and Barton [1] gave the exponential generating function formula

$$\sum_{n=0}^{\infty} |\mathrm{Av}_n(12\cdots m)| \frac{x^n}{n!} = \left[\sum_{n=0}^{\infty} \Big(\frac{x^{mn}}{(mn)!} - \frac{x^{mn+1}}{(mn+1)!}\Big)\right]^{-1}$$

for counting permutations avoiding the monotone consecutive pattern $12\cdots m$. David and Barton's formula can be proven in a multitude of ways, one of which is the celebrated cluster method of Goulden and Jackson [5], adapted for permutations by Elizalde and Noy [3]. More recently, Elizalde [2] gave a q-analogue of the Goulden–Jackson cluster method which he then used to derive a refinement of David and Barton's formula that also keeps track of the inversion number.

In this talk, we present several refinements of David and Barton's formula for "inverse statistics". Given a permutation statistic st, let us define its *inverse statistic* ist by $\mathrm{ist}(\pi) = \mathrm{st}(\pi^{-1})$. For example, if des is the descent number statistic, then the inverse descent number is given by $\mathrm{ides}(\pi) = \mathrm{des}(\pi^{-1})$. One of our new results is the following. Define $A_{m,n}(t,q)$ by

$$A_{m,n}(t,q) \coloneqq \sum_{\pi \in \mathrm{Av}_n(12\cdots m)} t^{\mathrm{ides}(\pi)+1} q^{\mathrm{imaj}(\pi)}$$

for $n \geq 1$ and $A_{m,0}(t,q) \coloneqq 1$. Then if $m \geq 2$, we have

$$\sum_{n=0}^{\infty} \frac{A_{m,n}(t,q)}{\prod_{i=0}^{n}(1-tq^i)} x^n = 1 + \sum_{k=0}^{\infty} \left[\sum_{j=0}^{\infty} \left(\binom{k+jm-1}{k-1}_q x^{jm} - \binom{k+jm}{k-1}_q x^{jm+1} \right)\right]^{-1} t^k.$$

We also prove analogous formulas for the "inverse peak number" and "inverse left peak number" statistics, and present real-rootedness conjectures for the associated polynomials.

To derive these new formulas, we first prove a lifting of the Goulden–Jackson cluster method to the Malvenuto–Reutenauer algebra **FQSym** (also known as the algebra of free quasisymmetric functions). By applying appropriate homomorphisms to the cluster method in **FQSym**, we can recover both the cluster method for permutations as well as Elizalde's q-analogue, so the cluster method in **FQSym** can be thought of as a unifying generalization of both. Additional homomorphisms can be obtained from the theory of shuffle-compatibility developed by Gessel and Zhuang [4], as every "shuffle-compatible" permutation statistic st induces a homomorphism on **FQSym** which can be used to count permutations by ist. By applying these new homomorphisms to the result of applying the cluster method in **FQSym** to the monotone consecutive pattern $12\cdots m$, we obtain our formulas.

[1] F. N. David and D. E. Barton. *Combinatorial Chance*. Lubrecht & Cramer Ltd, 1962.

[2] S. Elizalde. A survey of consecutive patterns in permutations. In *Recent Trends in Combinatorics*, volume 159 of *IMA Vol. Math. Appl.*, pages 601–618. Springer, [Cham], 2016.

[3] S. Elizalde and M. Noy. Clusters, generating functions and asymptotics for consecutive patterns in permutations. *Adv. in Appl. Math.*, 49(3-5):351–374, 2012.

[4] I. M. Gessel and Y. Zhuang. Shuffle-compatible permutation statistics. *Adv. Math.*, 332:85–141, 2018.

[5] I. P. Goulden and D. M. Jackson. An inversion theorem for cluster decompositions of sequences with distinguished subsequences. *J. London Math. Soc. (2)*, 20(3):567–576, 1979.