Permutation Statistics of Indexed Permutations

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Abstract

The definitions of descent, excedance, major index, inversion index and Denert's statistic for the elements of the symmetric group $\mathcal{S}_d$ are generalized to indexed permutations, i.e. the elements of the group $S_d^n := \mathbf{Z}_n \wr \mathcal{S}_d$, where $\wr$ is wreath product with respect to the usual action of $\mathcal{S}_d$ by permutations of $\{1, 2, \ldots, d\}$.

It is shown, bijectively, that excedances and descents are equidistributed, and the corresponding descent polynomial, analogous to the Eulerian polynomial, is computed as the f–eulerian polynomial of a simple polynomial. The descent polynomial is shown to equal the h–polynomial (essentially the h–vector) of a certain triangulation of the unit d–cube. This is proved by a bijection which exploits the fact that the h–vector of the simplicial complex arising from the triangulation can be computed via a shelling of the complex. The famous formula $\sum_{d\geq 0} E_d \frac{x^d}{d!} = \sec x + \tan x$, where E_d is the number of alternating permutations in $\mathcal{S}_d$, is generalized in two different ways, one relating to recent work of V.I. Arnold on Morse theory. The major index and inversion index are shown to be equidistributed over S_d^n. Likewise, the pair of statistics $(d, \text{ maj})$ is shown to be equidistributed with the pair $(\epsilon, \text{ den})$, where den is Denert's statistic and ϵ is an alternative definition of excedance. A result of Stanley, relating the number of permutations with k descents to the volume of a certain "slice" of the unit d–cube, is also generalized.

1 Introduction

There is a wealth of literature on various statistics of the elements of the symmetric group $\mathcal{S}_d$ (see for example [9] and [12] for a bibliography) and some of this has recently been generalized to the hyperoctahedral group B_d (see [17]). In this paper we generalize some of these statistics to the wreath product $\mathbf{Z}_n \wr \mathcal{S}_d$ of the cyclic group on n elements by the symmetric group $\mathcal{S}_d$.

In the classical case of the symmetric group $\mathcal{S}_d$, whose elements we view as permutations of the set $[d] = \{1, 2, \ldots, d\}$, represented as words, a *descent* in $\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d$ is an i in $[d-1]$ such that $a_i > a_{i+1}$, i.e. where a letter in the word π is larger than its successor. The *descent set* $D(\pi)$ of π is the set of those $i \in [d-1]$ for which $a_i > a_{i+1}$, i.e. $D(\pi) = \{i \in [d–1] \mid a_i > a_{i+1}\}$. An *excedance* in π is an i in $[d]$ such that $a_i > i$ and the *excedance set* of π is $E(\pi) = \{i \in [d] \mid a_i > i\}$. We set $d(\pi) = \#D(\pi)$ and $e(\pi) = \#E(\pi)$. As an example, the permutation $\pi = 34521$ has $D(\pi) = \{3, 4\}$ and $E(\pi) = \{1, 2, 3\}$, and hence $d(\pi) = 2$ and $e(\pi) = 3$. We construct the *descent polynomial* $D_d(t)$ of $\mathcal{S}_d$ by defining its k–th coefficient to be the number of permutations in $\mathcal{S}_d$ with k descents and the *excedance polynomial* $E_d(t)$ of $\mathcal{S}_d$ in an analogous way. It is well known that $D_d(t) = E_d(t)$, i.e. descents and excedances are *equidistributed* over $\mathcal{S}_d$. Moreover, $D_d(t)$ equals, up to a factor of t, the d–th *Eulerian polynomial* $A_d(t)$. The Eulerian polynomials have been extensively studied in various different contexts.

Other statistics which have been much studied are the *major index* and the *inversion index* of a permutation. The major index $\mathrm{maj}(\pi)$ of $\pi = a_1 a_2 \ldots a_d$ is the sum of all i in the descent set of π. An *inversion* in π is a pair (i, j) such that $i < j$ and $a_i > a_j$. The inversion index of a permutation π is the number of inversions in π and is denoted $\mathrm{inv}(\pi)$. It is known that inv and maj are equidistributed, i.e. $\sum_{\pi \in \mathcal{S}_d} t^{inv(\pi)} = \sum_{\pi \in \mathcal{S}_d} t^{maj(\pi)}$, and Foata [8] has constructed a bijection $\phi : \mathcal{S}_d \to \mathcal{S}_d$ which satisfies $\mathrm{maj}(\pi) = \mathrm{inv}(\phi(\pi))$.

Recently, Denert [7] described a new statistic on $\mathcal{S}_d$, defined in terms of excedances (see section 4.3 here). She conjectured that the joint distribution of the pair $(d(\pi), \mathrm{maj}(\pi))$ was equal to that of $(e(\pi), \mathrm{den}(\pi))$, i.e. that $\sum_{\pi \in \mathcal{S}_d} t^{d(\pi)} x^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_d} t^{e(\pi)} x^{\mathrm{den}(\pi)}$. In [10], Foata and Zeilberger proved the conjecture, and later Han (see [13]) constructed an explicit bijection to prove this.

In this paper, we generalize the definitions of descents and excedances to the elements (which we call *indexed permutations*) of the groups $S_d^n := \mathbf{Z}_n \wr \mathcal{S}_d$, where $\wr$ is wreath product with respect to the usual action of $\mathcal{S}_d$ by permutations of $[d]$. These groups are *unitary groups generated by reflections*, i.e. the symmetry groups of certain regular complex polytopes (see [18]). The elements of S_d^n can be represented by permutation words in $\mathcal{S}_d$ where each letter a_i has a subscript z_i (its *index*), where $z_i \in \{0, 1, \ldots, n-1\}$. As an example, $2_5 4_0 1_2 3_1$ is an element of S_4^6. We show, bijectively, that excedances and descents are still equidistributed, and we compute the corresponding descent polynomials $D_d^n(t)$ as the *f–eulerian polynomial* of a simple polynomial.

We also show that the descent polynomial equals the *h–polynomial* (essentially the *h–vector*) of a certain triangulation of the unit d–cube. This is done by constructing a bijection which exploits the fact that the h–vector of the triangulation in question can be computed via a *shelling* of the simplicial complex arising from the triangulation.

Using the work of Brenti [6], we show that the descent polynomials $D_d^n(t)$ have only real roots, which implies that they are *unimodal*.

We also generalize the famous formula $\sum_{d\geq 0} E_d \frac{x^d}{d!} = \sec x + \tan x$, where E_d is the number of *alternating* permutations in $\mathcal{S}_d$, in two different ways, one of which relates to recent work of Arnold [2] on Morse theory. In each case, the resulting formula is then used to find a relation between the number of alternating (respectivley *weakly alternating*) indexed permutations in S_d^3 (respectively S_d^2) and the value of the corresponding descent polynomial at –1.

We generalize the definitions of inv and maj to S_d^n, and show that they are equidistributed, and we also generalize Denert's statistic to S_d^n and show that the pair $(d(p), \mathrm{maj}(p))$ is equidistributed with the pair $(\epsilon(p), \mathrm{den}(p))$, where $\epsilon(p)$ is an alternative definition of excedance, equidistributed with our first one.

Finally, we generalize a bijective proof of Stanley's (of a result essentially due to Laplace) which shows that the the number of permutations in $\mathcal{S}_d$ with k descents equals, up to a factor of $d!$, the volume of the subspace of the unit d–cube lying between the hyperplanes defined by $\{\mathbf{x} \in \mathbf{R}^d \mid \sum_i x_i = k\}$ and $\{\mathbf{x} \in \mathbf{R}^d \mid \sum_i x_i = k+1\}$, respectively.

Many of the statistics studied here are computed on a finer scale than just for the groups S_d^n, namely for the left cosets of a certain distinguished subgroup of S_d^n. The bijection mentioned above, which proves the equality of $D_d^n(t)$ and the h–polynomial of a triangulation of the unit d–cube, relates each of these cosets to a certain geometrically defined subcomplex of the triangulation in question.

The following notation will be adhered to throughout:

We denote by $[n]$ the set $\{1, 2, \dots n\}$ which, when relevant, is assumed endowed with its usual linear order.

The quotient $\mathbf{Z}/n\mathbf{Z}$ where $\mathbf{Z}$ is the infinite cyclic group of integers and $n \in \mathbf{Z}$ will be denoted $\mathbf{Z}_n$. We always represent the elements of $\mathbf{Z}_n$ by the elements of $\{0, 1, \dots, n-1\}$, and when we refer to an ordering of the elements of $\mathbf{Z}_n$ it is the ordering induced by the usual ordering of $\{0, 1, \dots, n-1\}$.

An element π of the symmetric group $\mathcal{S}_d$ will most often be represented as a word $\pi = a_1 a_2 \dots a_d$, where $a_i = \pi(i)$.

We use boldface letters to denote vectors, for example $\mathbf{z} = (z_1, z_2, \dots, z_d)$. In particular, $\mathbf{0} := (0, 0, \dots, 0)$.

We shall be concerned with the elements of the *wreath product* $\mathbf{Z}_n \wr \mathcal{S}_d$. For a definition and more information on wreath products, see [15].

2 Definitions and some basic results

Definition 1 *An* indexed permutation *is an element of the group* $S_d^n := \mathbf{Z}_n \wr \mathcal{S}_d$ *(where* $\wr$ *is wreath product with respect to the usual action of* $\mathcal{S}_d$ *by permutation of* $[d]$*). We represent an indexed permutation as the product* $\pi \times \mathbf{z}$ *of a permutation word* $\pi = a_1 a_2 \dots a_d \in \mathcal{S}_d$ *and a* d*–tuple* $\mathbf{z} = (z_1, z_2, \dots, z_d)$ *of integers* $z_i \in \mathbf{Z}_n$*. As a convention, we set* $a_{d+1} = d + 1$ *and* $z_{d+1} = 0$*.*

It should be pointed out that the elements of S_d^n can be taken as those matrices in $GL(n, \mathbb{C})$ which have exactly one non–zero entry in each row and column and such that each of these non–zero entries is an n–th root of unity. With this definition, the product in S_d^n is simply matrix multiplication. With the notation $\pi \times \mathbf{z}$, the product is defined by $(\pi \times \mathbf{z}) \cdot (\tau \times \mathbf{w}) = \pi\tau \times (\mathbf{z} + \pi(\mathbf{w}))$, where $\pi(\mathbf{w}) = (w_{\pi(1)}, w_{\pi(2)}, \dots, w_{\pi(d)})$ and the $+$ is coordinate-wise addition modulo n.

Definition 2 *A* descent *in* $p = \pi \times z \in S_d^n$ *is an integer* $i \in [d]$ *such that*

1) $z_i > z_{i+1}$ *OR*
2) $z_i = z_{i+1}$ *and* $a_i > a_{i+1}$.

In particular, d is a descent if and only if $z_d > 0$.

Definition 3 *An* excedance *in* p *is an integer* $i \in [d]$ *such that*

1) $a_i > i$ *OR*
2) $a_i = i$ *and* $z_i > 0$.

As an example, let $p = 321465 \times (0, 0, 3, 2, 2, 1)$. Then p has descents at 1, 3, 5 and 6 and excedances at 1, 4 and 5.

It is convenient to think of an element of S_d^n as a permutation word in which every letter has a subscript. For example, $p = 321465 \times (0, 0, 3, 2, 2, 1) \in$

S_5^4 can be represented by $3_0 2_0 1_3 4_2 6_2 5_1$. We call the subscripts *indices*. Using this, there is an alternative definition of descent. Namely, define an ordering $<_\ell$ on the alphabet $\{i_z \mid i \in [d], z \in \mathbf{Z}_n\}$ by setting $i_z <_\ell j_w$ if

i) $z < w$ OR

ii) $z = w$ and $i < j$.

Then a descent in $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}}$[1] is an i such that $a_{i+1_{z_{i+1}}} <_\ell a_{i_{z_i}}$. This ordering of the letters induces a lexicographic ordering of the indexed permutations in S_d^n, which we will later make use of.

Definition 4 *Define an ordering $<_L$ of the elements of S_d^n by setting $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}} <_L q = b_{1_{w_1}} b_{2_{w_2}} \ldots b_{d_{w_d}}$ if $a_{i_{z_i}} <_\ell b_{i_{w_i}}$ for the first i at which p and q differ.*

Definition 5 *Let p be an element of S_d^n. Let $e(p) = \#\{i \mid i$ is an excedance in $p\}$ and let $d(p) = \#\{i \mid i$ is a descent in $p\}$. Then $E_d^n(t) := \sum_{p \in S_d^n} t^{e(p)}$ is the* excedance polynomial *of S_d^n and $D_d^n(t) := \sum_{p \in S_d^n} t^{d(p)}$ is the* descent polynomial *of S_d^n. Moreover, let $E(d,n,k) := \#\{p \in S_d^n \mid p$ has k excedances$\}$ and let $D(d,n,k) := \#\{p \in S_d^n \mid p$ has k descents$\}$, so that $E_d^n(t) = \sum_{k=0}^d E(d,n,k) t^k$ and $D_d^n(t) = \sum_{k=0}^d D(d,n,k) t^k$.*

As a convention, if $n \geq 0$, we define S_0^n to consist of one (empty) indexed permutation and hence we have $E(0,n,0) = D(0,n,0) = 1$.

Note that when $n = 1$, S_d^n is essentially $\mathcal{S}_d$ and the definitions of descent and excedance coincide with the classical definitions (see, for example, [20]).

Definition 6 *Let $p \in S_d^n$ and let $D(p) = \{i \in [d] \mid i$ is a descent in $p\}$. Then $D(p)$ is the* descent set *of p.*

We will now construct a bijection $S_d^n \to S_d^n$ which takes an indexed permutation with k descents to one with k excedances. First a definition which we will frequently refer to in what follows.

Definition 7 *Let $S_\mathbf{z}$ be the set of permutation words on the letters $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$. That is, $S_\mathbf{z} = \{\pi(1_{z_1} 2_{z_2} \ldots d_{z_d}) \mid \pi \in \mathcal{S}_d\}$.*

[1]To make the notation a little less awkward, we write $a_{i_{z_i}}$ instead of $(a_i)_{z_i}$, although z_i is a subscript to a_i rather than to just the i in a_i.

As an example, if $\mathbf{z} = (1, 0, 1)$ then the elements of $S_{\mathbf{z}}$ (ordered by $<_L$) are $2_0 1_1 3_1$,$2_0 3_1 1_1$, $1_1 2_0 3_1$, $1_1 3_1 2_0$, $3_1 2_0 1_1$, $3_1 1_1 2_0$.

Note that $S_{\mathbf{0}}$ is the subgroup $\{\pi \times \mathbf{0} \mid \pi \in \mathcal{S}_d\}$ of S_d^n and $S_{\mathbf{z}}$ is the *left coset* $(\pi \times \mathbf{z})S_{\mathbf{0}}$ for any $\pi \in \mathcal{S}_d$.

Let $\mathbf{Z}_n^d$ be the direct product of d copies of $\mathbf{Z}_n$. Clearly, S_d^n is the disjoint union of the $S_{\mathbf{z}}$'s for all $\mathbf{z} \in \mathbf{Z}_n^d$. The bijection we are about to construct will actually map $S_{\mathbf{z}}$ to itself for each $\mathbf{z} \in \mathbf{Z}_n^d$. However, we need to do this in three steps.

Lemma 8 *Suppose* $\mathbf{z} = (z_1, \ldots, z_d) \in \mathbf{Z}_n^d$ *and* $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbf{Z}_n^d$ *have the same number of positive coordinates. Then there is a bijection* $\Gamma : S_{\mathbf{z}} \to S_{\mathbf{w}}$ *which preserves the descent set of* p*. In particular,* $d(\Gamma(p)) = d(p)$*.*

Proof: The ordering $<_\ell$ used in Definition 4 is a linear ordering of the letters $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$, respectively of the letters $1_{w_1}, 2_{w_2}, \ldots, d_{w_d}$. Hence there is a unique bijection $\theta : \{i_{z_i} \mid i \in [d]\} \to \{i_{w_i} \mid i \in [d]\}$ such that $\theta(i_{z_i}) <_\ell \theta(j_{z_j})$ if and only if $i_{z_i} <_\ell j_{z_j}$. In particular, since $\mathbf{z} = (z_1, \ldots, z_d)$ and $\mathbf{w} = (w_1, \ldots, w_d)$ have the same number of positive coordinates, $z_i > 0$ if and only if $w_j > 0$ where $j_{w_j} = \theta(i_{z_i})$. Now, given $p \in S_{\mathbf{z}}$, define $\Gamma : S_{\mathbf{z}} \to S_{\mathbf{w}}$ by $\Gamma(p) = \Gamma(a_{1 z_1} a_{2 z_2} \ldots a_{d z_d}) := \theta(a_{1 z_1})\theta(a_{2 z_2}) \ldots \theta(a_{d z_d})$. Then, by definition of θ, i is a descent in p if and only if i is a descent in $\Gamma(p)$. In particular, since $\mathbf{z}$ and $\mathbf{w}$ have the same number of positive coordinates, d is a descent in p if and only if d is a descent in $\Gamma(p)$. Hence, Γ preserves not only the number of descents in p but actually the descent set $D(p)$ of p. $\square$

Example 9 Let $\mathbf{z} = (1, 0, 2, 1)$. Then $<_\ell$ induces the following ordering of the letters $1_1, 2_0, 3_2, 4_1$: $2_0 <_\ell 1_1 <_\ell 4_1 <_\ell 3_2$. Hence, if, as an example, we let $p = 3_2 2_0 4_1 1_1$ and $\mathbf{w} = (0, 1, 1, 1)$, we have $\Gamma(p) = 4_1 1_0 3_1 2_1$.

In the proof of the next lemma, we make use of a bijection ϕ which is described in the appendix at the end of this paper.

Lemma 10 *Let* $\mathbf{w} = (w_1, \ldots, w_d) \in \mathbf{Z}_n^d$*. Suppose there is a* $k \in [d]$ *such that* $w_i = 0$ *for* $i < k$ *and* $w_i = 1$ *for all* $i \geq k$*. Then there is a bijection* $\Psi : S_{\mathbf{w}} \to S_{\mathbf{w}}$ *such that* $e(\Psi(p)) = d(p)$*.*

Proof: Given $p = a_{1 w_{a_1}} a_{2 w_{a_2}} \ldots a_{d w_{a_d}} \in S_{\mathbf{w}}$, map p to $\pi = a'_1 a'_2 \ldots a'_{d+1} \in \mathcal{S}_{d+1}$ where $a'_{d+1} = k$ and $a'_i = a_i$ if $a_i < k$, $a'_i = a_i + 1$ if $k \leq a_i \leq d$. Then

i is a descent in π if and only if i is a descent in p. Now apply the bijection ϕ in Theorem 51 to π to obtain $\tau = \phi(\pi)$, where $\tau = b_1 b_2 \ldots b_{d+1}$ has an excedance $b_i > i$ if and only if $\ldots b_i i \ldots$ appears as a descent in π. Let m be such that $b_m = k$, and observe that, by the definition of ϕ, $m \geq k$, so that m is not an excedance in τ. Let $i' = i$ if $i < m$ and $i' = i+1$ if $i \geq m$. Now map τ to $q = c_{1_{w_{c_1}}} c_{2_{w_{c_2}}} \ldots c_{d_{w_{c_d}}} \in S_{\mathbf{w}}$ by setting $c_i = b_{i'}$ and $w_{c_i} = 0$ if $b_{i'} < k$, $w_{c_i} = 1$ if $b_{i'} > k$. Thus, k is deleted from τ and each remaining letter of τ is mapped back to what it was in p, that is, b_i in τ is replaced by $(b_i - 1)_1$ if $b_i > k$, but otherwise b_i is replaced by $(b_i)_0$. Also, some of the "place numbers" have to be reduced, so that a letter which was in place i with $i > m$ is in place $i - 1$ in q.

We claim that i is an excedance in τ if and only if i' is an excedance in q, so that τ and q have the same number of excedances, since m was not an excedance in τ. If $i < m$ then in q we either have $(b_i)_0$ or $(b_i - 1)_1$ in place i. In either case, i is an excedance in q if and only if i is an excedance in τ. If $i > m$ then in place $i-1$ in q we again have either $(b_i)_0$ or $(b_i - 1)_1$. If $b_i \neq i$ then $i-1$ is an excedance in q if and only if i is an excedance in τ. Suppose, then, that $b_i = i$. Then, by Corollary 53, $i < k$, since k is the last letter in π. Hence, we have $i < k \leq m$, contrary to assumption, so $b_i \neq i$ and we are done. $\square$

Example 11 Let $p = 4_1 1_0 3_1 2_1$. Then $p \longmapsto 51432 \overset{\phi}{\longmapsto} 53421 \longmapsto 4_1 2_1 3_1 1_0$, so $\Psi(p) = 4_1 2_1 3_1 1_0$.

Lemma 12 *Suppose* $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ *has* $w_k > 0$ *for some* k *and* $w_j = 0$ *for some* j *and that* $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ *satisfies* $z_k = 0$, $z_j > 0$ *and* $z_i = w_i$ *for* $i \notin \{k, j\}$. *Then there is a bijection* $\Phi' : S_{\mathbf{w}} \to S_{\mathbf{z}}$ *such that* $e(\Phi'(p)) = e(p)$.

Proof: A positive coordinate w_i of $\mathbf{w}$ affects excedances in $p = \pi \times \pi(\mathbf{w})$ in a way which is independent of whether $w_i = 1$ or $w_i > 1$. Hence we may assume, without loss of generality, that $z_i, w_i \in \{0, 1\}$ for all i. Then, $z_j = w_k = 1$ and $z_k = w_j = 0$. That is, $\mathbf{z}$ is obtained from $\mathbf{w}$ by transposing w_k and w_j. Let $p = \pi \times \pi(\mathbf{w})$ where $\pi = a_1 a_2 \ldots a_d$. We define $\Phi' : S_{\mathbf{w}} \to S_{\mathbf{z}}$ by defining a certain bijection $\phi' : \mathcal{S}_d \to \mathcal{S}_d$ and setting $\Phi'(\pi \times \pi(\mathbf{w})) = \phi'(\pi) \times \phi'(\pi)(\mathbf{z})$. $\phi'(\pi)$ is defined by the following trichotomy.

(1) For all $\pi \in \mathcal{S}_d$ such that π either fixes both j and k or neither, i.e. either $a_j = j$ and $a_k = k$ or $a_j \neq j$ and $a_k \neq k$, we let $\phi'(\pi) = \pi$. Hence,

for such p, $q = \Phi'(p)$ is obtained from p simply by interchanging the indices of k and j in p (i.e. k gets the index w_j and j the index w_k). Consequently, the number of excedances is preserved, for in the first case we are moving an excedance from k to j and in the latter case no excedances will be affected since $a_j \neq j$ and $a_k \neq k$. As an example, if $k = 2$ and $j = 5$, we have $\Phi'(3_0 2_1 4_1 1_0 5_0) = 3_0 2_0 4_1 1_0 5_1$ and $\Phi'(5_1 4_0 2_1 1_0 3_0) = 5_1 4_0 2_0 1_0 3_0$. Clearly, this is injective, for $\phi'(\pi) = \phi'(\tau)$ if and only if $\pi = \tau$.

(2) Suppose $a_k = k$ and $a_j \neq j$. We then define $\phi'(\pi) = \tau = b_1 b_2 \ldots b_d$ in the following way. Let $b_j = j$. Let F be the set of fixed points of π, i.e. $F = \{i \in [d] \mid a_i = i\}$. In particular, $k \in F$ and $j \notin F$. Given a set S, let S_i denote $S \setminus \{i\}$ and let S^i denote $S \cup \{i\}$. Let $D = [d] \setminus F$. Set $b_j = j$ and set $b_i = i$ for all $i \in F_k$. By definition, the restriction of π to D is a *derangement* of D, i.e. $a_i \neq i$ for all $i \in D$. We have already defined b_i for all $i \in F_k^j$ by declaring such i to be fixed points of τ. Hence, for all $i \in F_k$, i is an excedance in $\Phi'(p)$ if and only if i is an excedance in p, because $a_i = b_i$ and $w_i = z_i$. Moreover, k is an excedance in p and j is an excedance in $\Phi'(p)$. Thus, so far, we have the same number of excedances in p and $\Phi'(p)$.

What remains to be defined is how τ permutes the elements of D_j^k.

There is a unique order preserving bijection $\theta : D \to D_j^k$, i.e. θ maps the smallest element of D to the smallest element of D_j^k, the next smallest element of D to the next smallest element of D_j^k and so on. In other words, $\theta(i) > \theta(m)$ if and only if $i > m$. Now, if $i \in D_j^k$, we set $b_i = \theta(a_{\theta^{-1}(i)})$. Note that this defines a bijection $\tau \mid_{D_j^k} \colon D_j^k \to D_j^k$, as required. This further guarantees that $b_i \neq i$ for all $i \in D_j^k$, in particular $b_k \neq k$, and, moreover, that $b_i > i$ precisely when $a_{\theta^{-1}(i)} > \theta^{-1}(i)$. Note also that whether $i \in D_j^k$ is an excedance in $\Phi'(p)$ is not dependent on w_{b_i} since $b_i \neq i$. The same is true of $\theta^{-1}(i)$ and p (and $z_{\theta^{-1}(i)}$), so i is an excedance in $\Phi'(p)$ if and only if $\theta^{-1}(i)$ is an excedance in p.

Let us illustrate this by an example. Let $k = 2, j = 5$ and $q = 3_1 2_1 1_0 4_1 6_0 5_0 7_0$ so that $\pi = 3214657$. Then $F = \{2, 4, 7\}$ and $D = \{1, 3, 5, 6\}$. Hence, τ fixes $4, 5$ and 7. θ maps $\{1, 3, 5, 6\}$ to $\{1, 2, 3, 6\}$ by sending 1 to 1, 3 to 2, 5 to 3 and 6 to 6. Hence, $\tau = 2164537$, so $\Phi'(p) = 2_0 1_0 6_0 4_1 5_1 3_1 7_0$.

Again, this is injective because if $\phi'(\pi) = \phi'(\tau)$ then $\phi'(\pi)$ and $\phi'(\tau)$ have the same fixed points, and hence π and τ have the same fixed points, so π and τ must be identical on the remaining elements of $[d]$, because the bijection θ was unique.

(3) The case when $a_k \neq k$ and $a_j = j$ is similar to (2). As a matter of fact, it turns out that the similar argument results in this: If $p \in \{q = \pi \times \pi(\mathbf{w}) \in S_{\mathbf{w}} \mid a_k \neq k \text{ and } a_j = j\}$ then $\Phi'(p) = (\phi')^{-1}(\pi) \times (\phi')^{-1}(\pi)(\mathbf{z})$, which is well defined, because $(\phi')^{-1}(\pi)$ is (implicitly) defined in (2).

As an example, since we had $\Phi'(3_1 2_1 1_0 4_1 6_0 5_0 7_0) = 2_0 1_0 6_0 4_1 5_1 3_1 7_0$, we have
$\Phi'(2_1 1_0 6_0 4_1 5_0 3_1 7_0) = 3_1 2_0 1_0 4_1 6_0 5_1 7_0$.

It is obvious that $S_{\mathbf{w}}$ is the disjoint union of the domains described in (1), (2) and (3) and that $S_{\mathbf{z}}$ is the disjoint union of the images in (1), (2), and (3). $\square$

Example 13 Let $p = 4_1 2_1 3_1 1_0$ and let $\mathbf{z} = (1, 0, 2, 1)$ (so $\mathbf{z}$ is as in Example 9). Then $\Phi'(p) = 1_1 4_1 3_2 2_0$.

By repeated applications of Φ' we get the following, more general result:

Lemma 14 *Suppose* $\mathbf{w} = (w_1, w_2, \ldots, w_d)$ *and* $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ *have the same number of positive coordinates. Then there is a bijection* $\Phi : S_{\mathbf{w}} \to S_{\mathbf{z}}$ *such that* $e(\Phi(p)) = e(p)$. $\square$

We now use these lemmas to construct a bijection $S_{\mathbf{z}} \to S_{\mathbf{z}}$ which takes an indexed permutation with k descents to one with k excedances. Suppose $\mathbf{z}$ has exactly m positive coordinates. Let $\mathbf{w}$ be defined by $w_i = 0$ if $i \leq d-m$ and $w_i = 1$ if $i > d - m$. Then the composition

$$S_{\mathbf{z}} \xrightarrow{\Gamma} S_{\mathbf{w}} \xrightarrow{\Psi} S_{\mathbf{w}} \xrightarrow{\Phi} S_{\mathbf{z}}$$

is a bijection which takes a $p \in S_{\mathbf{z}}$ with k descents to a $q \in S_{\mathbf{z}}$ with k excedances. There follows

Theorem 15 *For all* $n \geq 1$ *and for all* $d \geq 0$, $E_d^n(t) = D_d^n(t)$. $\square$

Let $A_d(t) = tD_d^1(t)$. It has long been known that $A_d(t)$ satisfies $\frac{A_d(t)}{(1-t)^{d+1}} = \sum_{k\geq 1} k^d t^k$ and the polynomial $A_d(t)$ is called the d-th *Eulerian polynomial.* Theorem 17 generalizes this relation to our descent polynomials $D_d^n(t)$. First, a lemma. By analyzing the effect of inserting d_m, for $0 \leq m \leq n-1$, in a permutation in S_{d-1}^n, one finds the following recurrence:

Lemma 16 *The coefficients of* $E_d^n(t)$*, and hence those of* $D_d^n(t)$*, satisfy*

$$E(d,n,k) = (nk+1)E(d-1,n,k) + (n(d-k)+(n-1))E(d-1,n,k-1). \square$$

As a generalization of the Eulerian polynomials, a polynomial $P(t)$ which satisfies $\frac{P(t)}{(1-t)^{d+1}} = \sum_{k\geq 0} f(k)t^k$, where f is a polynomial of degree d, is called the *f-eulerian polynomial.* Using the recurrence in Lemma 16, we get

Theorem 17 $\dfrac{E_d^n(t)}{(1-t)^{d+1}} = \sum_{i\geq 0}(ni+1)^d t^i$, *i.e.* $E_d^n(t)$ *is the f-eulerian polynomial where* $f(i) = (ni+1)^d$. $\square$

There is a way of proving the preceding theorem *combinatorially* when $E_d^n(t)$ is replaced by $D_d^n(t)$. Actually, we can derive the theorem from a finer computation of $D_d^n(t)$. Namely, given $\mathbf{z} \in \mathbf{Z}_n^d$, we compute the descent polynomial $D_{\mathbf{z}}(t) := \sum_{p\in S_{\mathbf{z}}} t^{d(p)}$. The proof of the following theorem (given in [22]) is a modification of the proof of Lemma 4.5.1 and of the proof of Theorem 4.5.14 in [20] in the special case where the poset in question is an antichain.

Theorem 18 *Suppose* $\mathbf{z} \in \mathbf{Z}_n^d$ *has exactly* m *positive coordinates and let* $D_{\mathbf{z}}(t) := \sum_{p\in S_{\mathbf{z}}} t^{d(p)}$. *Then*

$$\sum_{k\geq 0}(k+1)^{d-m}k^m t^k \quad = \quad \frac{D_{\mathbf{z}}(t)}{(1-t)^{d+1}}\,. \qquad \square$$

From these expressions for $D_{\mathbf{z}}(t)$ and $D_d^n(t)$, we get some further interesting results about these polynomials. In [6], Brenti shows that if a polynomial $f(n)$ has all its roots in the interval $[-1, 0]$, then its f-eulerian polynomial $W(t) = w_0+w_1t+\cdots+w_dt^d$ (defined above) has only real zeros (see Theorems 4.4.4 and 2.3.3 in [6]). That, in turn, implies that the sequence $w_0, w_1, \ldots, w_d$ of coefficients of $W(t)$ is *unimodal*, i.e. $w_0 \leq w_1 \leq \cdots \leq w_k \geq w_{k+1} \geq \cdots \geq w_d$ for some k with $0 \leq k \leq d$. Thus, the following theorem is an obvious consequence of Theorems 17 and 18.

Theorem 19 *For any d and n, the polynomial* $D_d^n(t)$ *has only real zeros. In particular, there is a* $k \in \{0, 1, \ldots, d\}$ *such that*

$$D(d, n, 0) \leq D(d, n, 1) \leq \cdots \leq D(d, n, k) \geq D(d, n, k+1) \geq \cdots \geq D(d, n, d).$$

The same is true of the polynomial $D_{\mathbf{z}}(t)$ *for any* $\mathbf{z} \in \mathbf{Z}_n^d$. $\square$

It was known already to Euler that the polynomials $A_d(t) = tD^1_d(t)$ satisfy

$$t^{-1}\sum_{d\geq 0} A_d(t)\frac{x^d}{d!} \quad = \frac{(1-t)e^{x(1-t)}}{1-te^{x(1-t)}}.$$

This can be derived in a way which trivially generalizes to the derivation for $D^n_d(t)$:

$$\sum_{d\geq 0}\frac{D^n_d(t)}{(1-t)^{d+1}}\frac{x^d}{d!} = \sum_{d\geq 0}\left(\sum_{k\geq 0}(nk+1)^d t^k\right)\frac{x^d}{d!} = \sum_{k\geq 0} t^k \sum_{d\geq 0}\frac{(nk+1)^d x^d}{d!} = \sum_{k\geq 0} t^k e^{(nk+1)x}.$$

Now, multiply both sides by $(1-t)$ and replace k by d in the RHS to get

$$\sum_{d\geq 0}\frac{D^n_d(t)}{(1-t)^d}\frac{x^d}{d!} \quad = \quad (1-t)\sum_{d\geq 0} t^d e^{(nd+1)x} \quad = \quad (1-t)e^x\frac{1}{1-te^{nx}}.$$

Finally, replace x by $x(1-t)$ to obtain

Theorem 20 $\displaystyle\sum_{d\geq 0} D^n_d(t)\frac{x^d}{d!} \quad = \quad \frac{(1-t)e^{x(1-t)}}{1-te^{nx(1-t)}}.$ □

3 The geometric connection

What is perhaps most interesting about Theorem 17 is that it suggests a connection between our descent polynomials and the *Ehrhart polynomials* of certain integral polytopes. This, in turn, leads to the observation that if the dilation nC^d of the unit d–cube by n could be triangulated by d–simplices of volume $1/d!$, then the *h–polynomial* of the triangulation would equal $D^n_d(t)$. This is discussed in detail in [22].

3.1 Background

A simplicial complex K is *pure* if all its maximal faces have the same dimension $d = \dim(K)$. If K is a pure simplicial complex of dimension d, then a *facet* of K is a d-face, i.e. a d-dimensional face, of K. The *h-vector* $h(K) = (h_0, h_1, \ldots, h_d)$ of a simplicial complex K of dimension d-1 is defined as follows: Let $f_i = f_i(K)$ be the number of i-dimensional faces in K, where we set $f_{-1} = 1$ (corresponding to the empty set), and define $h(K) = (h_0, h_1, \ldots, h_d)$ by setting

$$\sum_{i=0}^{d} f_{i-1}(x-1)^{d-i} = \sum_{i=0}^{d} h_i x^{d-i}.$$

We define the *h-polynomial* $h(K,t)$ of K by $h(K,t) = h_0 + h_1 t + \cdots + h_d t^d$. For further information about h-vectors, see [21].

Let σ be a simplex. In what follows we will, by abuse of notation, also let σ denote the complex consisting of σ and all its faces and, in case σ has a *geometric realization* in the euclidean space $\mathbf{R}^d$, the subspace of $\mathbf{R}^d$ realizing σ.

Definition 21 *Let K be a finite pure simplicial complex of dimension d. An ordering $F_1, F_2, \ldots, F_n$ of the facets of K is called a* shelling *if, for all k with $1 < k \leq n$, $F_k \cap \bigcup_{i=1}^{k-1} F_i$ is a pure complex of dimension (d-1). A complex K is said to be* shellable *if there exists a shelling of K.*

That is, a complex is shellable if it can be built up by adding one facet at a time in such a way that, for $k > 1$, the intersection of each F_k with the complex generated by the previous F_i's is a nonempty union of (d-1)-faces of F_k.

As it turns out, the h-vector of a shellable complex can be computed from the shelling. The following theorem is essentially due to McMullen [16].

Theorem 22 *Let $F_1, F_2, \ldots, F_n$ be a shelling of K and let $c(k)$ be the number of $(d-1)$–faces of F_k contained in $\bigcup_{i<k} F_i$. Then we have the following formula:*

$$h(K,t) = \sum_{i=1}^{n} t^{c(i)}. \qquad \square$$

Thus, given a shelling $F_1, F_2, \ldots, F_n$ of a simplicial complex K, we can compute the h-polynomial $h(K,t)$ of K via Theorem 22. In doing that, we say that a facet F_i of K *contributes to the k-th coefficient of $h(K,t)$* if $c(i) = k$.

Our goal is to find a shellable triangulation of nC^d, whose h-polynomial equals $D_d^n(t)$. For this purpose, we need a couple of lemmas.

Definition 23 *Let C^d be the standard unit d-cube. For each permutation word $\pi = a_1a_2\dots a_d$ in $\mathcal{S}_d$, let $\sigma_\pi = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in C^d \mid 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\}$. We call σ_π the* path simplex defined by π.

The reason for calling σ_{pi} a *path* simplex is that if $\pi = a_1a_2\dots a_d$ then σ_π can be defined as the convex hull of the path traveling through vertices $\mathbf{0}$, $\mathbf{e}_{a_1}$, $\mathbf{e}_{a_1} + \mathbf{e}_{a_2}$, ..., $\mathbf{e}_{a_1} + \mathbf{e}_{a_2} + \cdots + \mathbf{e}_{a_d}$, where $\mathbf{e}_i$ is the i-th standard basis vector in $\mathbf{R}^d$.

The collection $\{\sigma_\pi \mid \pi \in \mathcal{S}_d\}$ of path simplices induces a simplicial subdivision of the unit d-cube C^d. Namely, their union covers C^d and the intersection of any two of the path simplices is a face of each one.

Remark 24 Let $\pi = a_1a_2\dots a_d$, so that $\sigma_\pi = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in C^d \mid 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\}$ is a path simplex. A k-dimensional face of σ_π is defined by replacing $d-k$ of the $\geq$'s by $=$'s, i.e. by replacing $d-k$ of the linear inequalities defining σ_π by their boundary equalities.

For example, the 2-faces of $\sigma_{213} = \{\mathbf{x} = (x_1, x_2, x_3) \in C^3 \mid 1 \geq x_2 \geq x_1 \geq x_3 \geq 0\}$ are $\{\mathbf{x} \in C^3 \mid 1 = x_2 \geq x_1 \geq x_3 \geq 0\}, \{\mathbf{x} \in C^3 \mid 1 \geq x_2 = x_1 \geq x_3 \geq 0\}, \{\mathbf{x} \in C^3 \mid 1 \geq x_2 \geq x_1 = x_3 \geq 0\}, \{\mathbf{x} \in C^3 \mid 1 \geq x_2 \geq x_1 \geq x_3 = 0\}$.

The following lemma is a straightforward consequence of Remark 24.

Lemma 25 *Two path simplices intersect maximally if and only if their corresponding permutations differ by a single transposition $\dots a_ia_{i+1} \dots \to \dots a_{i+1}a_i \dots$ of adjacent letters.* □

Lemma 26 *Let K_d be the collection $\{\sigma_\pi \mid \pi \in \mathcal{S}_d\}$ of path simplices which triangulate the unit d-cube. Order the simplices in K_d by the lexicographic ordering of their corresponding permutation words. This ordering is a shelling of the unit d-cube.*

Proof: Let B_d be the Boolean algebra on d elements. Then K_d is the *order complex* of B_d and the lemma is just a special case of *lexicographic shellability* (see [3]). A direct proof of the lemma is given in [22]. □

3.2 The triangulation and shelling of nC^d

We will now construct a triangulation $\widehat{nC^d}$ of nC^d and then shell that triangulation. The shelling will give rise to a bijection associating an indexed permutation in S_d^n with k descents to a facet of $\widehat{nC^d}$ that contributes to the k-th coefficient of $h(nC^d, t)$ when the h-polynomial is computed from the shelling.

Embed nC^d in $\mathbf{R}^d$ so that the coordinates of its vertices are all d-tuples which consist of only 0's and n's. That is, nC^d is the image of the standard unit d-cube under the map $f : \mathbf{R}^d \to \mathbf{R}^d$ defined by $f(x) = nx$. Subdivide nC^d into n^d cubes of volume 1 in the obvious way, i.e. given any vector $\mathbf{v} = (v_1, v_2, \ldots, v_d)$ such that $v_i \in \{0, 1, \ldots, n-1\}$, we obtain a unique d-cube contained in nC^d by translating the standard unit d-cube by this vector. We label each of these cubes with the corresponding vector, so that the standard unit cube is $c_{\mathbf{0}}$ and $c_{\mathbf{v}} = c_{\mathbf{0}} + \mathbf{v}$. Subdivide $c_{\mathbf{0}}$ into the path simplices defined in 23. This induces a simplicial subdivision of $c_{\mathbf{0}}$. The other cubes are subdivided in an analogous way, so that a triangulation of a cube labeled with $\mathbf{v}$ coincides with the translation by $\mathbf{v}$ of the triangulated standard unit cube. This induces a simplicial subdivision of nC^d which we call $\widehat{nC^d}$.

To order the simplices of $\widehat{nC^d}$ we proceed as follows: A facet σ of the cube $c_{\mathbf{0}}$ is labeled by $\pi \times \mathbf{0}$ where π is the permutation defining σ (cf. Definition 23). For $\mathbf{z} \neq \mathbf{0}$, if σ is a facet in the cube $c_{\mathbf{z}}$ and $\sigma = \sigma_{\pi\times\mathbf{0}} + \mathbf{z}$ (i.e. σ is the translation by $\mathbf{z}$ of the path simplex defined by π), then σ is labeled by $\pi \times \pi(\mathbf{z})$. Note that by permuting the coordinates of $\mathbf{z}$ in this way, so that the i-th coordinate of $\mathbf{z}$ follows i, we are actually labeling the facets of the cube $c_{\mathbf{z}}$ by all the permutation words on the letters $1_{z_1}, 2_{z_2}, \ldots, d_{z_d}$, that is, by the elements of $S_{\mathbf{z}}$ (See Definition 7).

Let $<$ denote the lexicographic ordering of vectors of the same length. That is, if $\mathbf{z} = (z_1, z_2, \ldots, z_d)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_d)$, then $\mathbf{z} < \mathbf{w}$ if and only if $z_i < w_i$ for the first i at which $\mathbf{z}$ and $\mathbf{w}$ differ. We now order the facets of $\widehat{nC^d}$ as follows:

Definition 27 *Let $\mathcal{O}$ be the following ordering $<_{\mathcal{O}}$ of the facets of $\widehat{nC^d}$:*
$\mathcal{O}1$) If $\mathbf{z} < \mathbf{w}$ then $\sigma_{\pi\times\pi(\mathbf{z})} <_{\mathcal{O}} \sigma_{\tau\times\tau(\mathbf{w})}$ for all π and τ.
$\mathcal{O}2$) If $\pi \times \pi(\mathbf{z}) <_L \tau \times \tau(\mathbf{z})$ then $\sigma_{\pi\times\pi(\mathbf{z})} <_{\mathcal{O}} \sigma_{\tau\times\tau(\mathbf{z})}$.

Thus, a facet in $c_{\mathbf{z}}$ comes before any facet in $c_{\mathbf{w}}$ if $\mathbf{z} < \mathbf{w}$. The ordering of the facets in a single cube $c_{\mathbf{z}}$ is a permutation of the shelling order described in

Lemma 26. Moreover, it is induced by permuting the coordinate axes in $\mathbf{R}^d$. Hence, this ordering must also be a shelling of the cube in question, because the shelling in Lemma 26 is clearly independent of how the coordinate axes are labeled. We thus have:

Lemma 28 *The restriction of the ordering $\mathcal{O}$ to the facets of a cube $c_{\mathbf{z}}$ in $\widehat{nC^d}$ is a shelling of that cube.* $\square$

For the next lemma, we need the following remark.

Remark 29 Let $\pi = a_1a_2\ldots a_d$ and let $\mathbf{z} = (z_1, z_2, \ldots, z_d)$. The facet $\sigma_{\pi\times\pi(\mathbf{z})}$ satisfies

$$\sigma_{\pi\times\pi(\mathbf{z})} = \{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_d} - z_{a_d} \geq 0\}.$$

Lemma 30 *Let σ_p be a facet of $c_{\mathbf{z}}$ in $\widehat{nC^d}$ and let $p = \pi \times \pi(\mathbf{z})$ where $\pi = a_1a_2\ldots a_d$. Then σ_p has two (d-1)-faces which lie on the boundary of $c_{\mathbf{z}}$. These faces are defined by $\sigma_p^0 := \{\mathbf{x} \in C^d \mid 1 \geq x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} = 0\} + \mathbf{z}$ and $\sigma_p^1 := \{\mathbf{x} \in C^d \mid 1 = x_{a_1} \geq x_{a_2} \geq \cdots \geq x_{a_d} \geq 0\} + \mathbf{z}$, respectively. If $z_{a_d} \geq 1$ then σ_p^0 is a (d-1)-face of a facet of the cube $c_{\mathbf{z}-\mathbf{e}_{a_d}} = c_{\mathbf{z}} - \mathbf{e}_{a_d}$. If $z_{a_1} \leq n-2$ then σ_p^1 is a (d-1)-face of a facet of the cube $c_{\mathbf{z}+\mathbf{e}_{a_1}} = c_{\mathbf{z}} + \mathbf{e}_{a_1}$.*

Moreover, the intersection of σ_p with any cube $c_{\mathbf{w}} \neq c_{\mathbf{z}}$ is contained in the union of σ_p^0 and σ_p^1. More specifically, if $\mathbf{w} < \mathbf{z}$ then $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^0$ and if $\mathbf{w} > \mathbf{z}$ then $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^1$

Proof: Clearly, σ_p^0 and σ_p^1 are (d-1)-faces of σ_p. Since each lies in a hyperplane supporting the cube $c_{\mathbf{z}}$, they must lie on the boundary of $c_{\mathbf{z}}$. Now, if $z_{a_d} \geq 1$ then

$$\sigma_p^0 = \{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_d} - z_{a_d} = 0\} =$$

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 = x_{a_d} - z_{a_d} + 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_{d-1}} - z_{a_{d-1}} \geq 0\} = \sigma_{\pi'\times\pi'(\mathbf{z}')}$$

where $\pi' = a_da_1a_2\ldots a_{d-1}$ and $\mathbf{z}' = \mathbf{z} - \mathbf{e}_{a_d}$, so $\sigma_{\pi'\times\pi'(\mathbf{z}')} \subset c_{\mathbf{z}'}$. Similar reasoning shows that if $z_{a_1} \leq n-2$ then $\sigma_p^1 = \sigma_r^0$ where $r = \pi'' \times \pi''(\mathbf{z} + \mathbf{e}_{a_1})$ and $\pi'' = a_2a_3\ldots a_da_1$. To show that $\sigma_p \cap c_{\mathbf{w}} \subset \sigma_p^0 \cup \sigma_p^1$ for any $\mathbf{w} \neq \mathbf{z}$, observe that a point $\mathbf{x}_0 = (x_1, x_2, \ldots, x_d) \in \sigma_p \cap c_{\mathbf{w}}$ must lie on the boundary of $c_{\mathbf{z}}$ and must have $x_i = z_i$ (if $\mathbf{w} < \mathbf{z}$) or $x_i = z_i + 1$ (if $\mathbf{w} > \mathbf{z}$), where i is the first coordinate in which $\mathbf{w}$ and $\mathbf{z}$ differ. Suppose $a_j = i$. Then, if $\mathbf{w} < \mathbf{z}$, $\mathbf{x_0}$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_j} - z_{a_j} = x_{a_{j+1}} - z_{a_{j+1}} = \cdots = x_{a_d} - z_{a_d} = 0\} \subset \sigma_p^0$$

and, if $\mathbf{w} > \mathbf{z}$, $\mathbf{x_0}$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 = x_{a_1} - z_{a_1} = x_{a_2} - z_{a_2} = \cdots = x_{a_j} - z_{a_j} \geq x_{a_{j+1}} - z_{a_{j+1}} \geq \cdots \geq x_{a_d} - z_{a_d} \geq 0\} \subset \sigma_p^1,$$

as claimed. □

Theorem 31 *The ordering $\mathcal{O}$ defines a shelling of $\widehat{nC^d}$.*

Proof: Let σ_p be a facet of the cube $c_{\mathbf{z}}$ in $\widehat{nC^d}$ with $p = \pi \times \pi(\mathbf{z}) = a_1 a_2 \ldots a_d \times (z_{a_1}, z_{a_2}, \ldots, z_{a_d})$. If $\mathbf{z} = \mathbf{0}$ then we are done, by Lemma 26. So assume $\mathbf{z} \neq \mathbf{0}$.

We need to show that $I_p := \sigma_p \cap \bigcup_{q<p} \sigma_q$ is a nonempty union of (d-1)-faces of σ_p (where, by abuse of notation, $q < p$ means $q <_{\mathcal{O}} p$). By Lemma 28, since the restriction of $\mathcal{O}$ to the cube $c_{\mathbf{z}}$ is a shelling of $c_{\mathbf{z}}$, the intersection $I_{\mathbf{z}} := I_p \cap c_{\mathbf{z}}$ of σ_p with those facets in $c_{\mathbf{z}}$ which are prior to σ_p must be a union (possibly empty) of (d-1)-faces of σ_p. If this union is empty, p must be the least indexed permutation in $S_{\mathbf{z}}$, so $z_{a_d} > 0$ since $\mathbf{z} \neq \mathbf{0}$. Hence, σ_p^0 belongs to a facet of the cube $c_{\mathbf{z}'} = c_{\mathbf{z} - \mathbf{e}_{a_d}}$, so $I_p = \sigma_p^0$, a (d-1)-face of σ_p as desired.

If $I_{\mathbf{z}} \neq \emptyset$, then, by Lemma 28, $I_{\mathbf{z}}$ is a union of (d-1)-faces of σ_p, so what remains to be taken into account is how σ_p intersects other small cubes than its own. Obviously, we need only check those cubes $c_{\mathbf{w}}$ for which $\mathbf{w} < \mathbf{z}$. By Lemma 30, we need only check how σ_p^0 intersects such small cubes. Now, if $z_{a_d} \neq 0$ then, by Lemma 30, $\sigma_p^0 = \sigma_q^1$ for some $q < p$, so I_p is a union of $(d-1)$-faces of σ_p, viz. $I_p = I_z \cup \sigma_p^0$.

Suppose, then, that $z_{a_d} = 0$ and that σ_p^0 intersects $c_{\mathbf{w}}$ where $\mathbf{w} < \mathbf{z}$. Then, for each $i \in [d]$, w_i can differ by at most 1 from z_i. Let i be the first coordinate in which $\mathbf{w}$ and $\mathbf{z}$ differ. Then, since $\mathbf{w} < \mathbf{z}$, we must have $w_i = z_i - 1$. Hence, any point $\mathbf{x_0}$ in $\sigma_p^0 \cap c_{\mathbf{w}}$ must belong to the set

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_i - z_i = x_{i+1} - z_{i+1} = \cdots = x_{a_d} - z_{a_d} = 0\},$$

because $0 \leq x_i - w_i \leq 1$, so $0 \leq x_i - z_i + 1 \leq 1$, and therefore $x_i - z_i = 0$.

Let j be such that $z_{a_j} > 0$ and $z_{a_k} = 0$ for all $k > j$. Such a j must exist, since $\mathbf{z} \neq 0$ and $z_{a_d} = 0$. Also, $a_j \geq i$, since $z_i = w_i + 1 \geq 1$. But then

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_i - z_i = x_{i+1} - z_{i+1} = \cdots = x_{a_d} - z_{a_d} = 0\} \subset$$

$$\{\mathbf{x} \in \mathbf{R}^d \mid 1 \geq x_{a_1} - z_{a_1} \geq x_{a_2} - z_{a_2} \geq \cdots \geq x_{a_j} - z_{a_j} = x_{a_{j+1}} - z_{a_{j+1}} \geq \cdots \geq x_{a_d} - z_{a_d} = 0\}.$$

This last set is a (d-1)-face of σ_p and of $\sigma_q = \sigma_{\tau \times \tau(\mathbf{z})}$ where $\tau = a_1 a_2 \dots a_{j+1} a_j \dots a_d$, so $\tau(\mathbf{z}) = (z_{a_1}, z_{a_2}, \dots, z_{a_{j+1}}, z_{a_j}, \dots, z_{a_d})$. Hence, since $z_{a_j} > 0$ and $z_{a_{j+1}} = 0$, q is prior to p in $\mathcal{O}$, so the (d-1)-face $\sigma_p \cap \sigma_q$ of σ_p is contained in I_p and we have shown that any $\mathbf{x}_0 \in \sigma_p^0 \cap c_{\mathbf{w}}$ lies in this face. Hence, I_p is a union of (d-1)-faces of σ_p and the proof is complete. $\square$

Recall that by Theorem 22 we can compute the h-polynomial of a simplicial complex K from a shelling of K. Namely, if $F_1, F_2, \dots, F_n$ is a shelling of K and $c(i)$ is as in Theorem 22, then $h_k = \#\{i \mid c(i) = k\}$, where h_k is the k-th coefficient of the h-polynomial of K. That is, h_k equals the number of facets F_i such that F_i intersects $\bigcup_{j=1}^{i-1} F_j$ in k distinct faces of dimension $(d-1)$. Now, in the shelling of $\widehat{nC^d}$ the facets in a single cube $C_{\mathbf{z}}$ were ordered so that $\sigma_q = \sigma_{\tau \times \tau(\mathbf{z})}$ was prior to $\sigma_p = \sigma_{\pi \times \pi(\mathbf{z})}$ if and only if $q <_L p$ in the lexicographic ordering of indexed permutations. Also, by Lemma 25, σ_p intersects σ_q maximally if and only if π and τ (hence p and q) differ by a single transposition. Suppose now that σ_p and σ_q intersect maximally. Then, if $p = a_{1_{z_1}} a_{2_{z_2}} \dots a_{d_{z_d}}$, we must have $q = a_{1_{z_1}} a_{2_{z_2}} \dots a_{k+1_{z_{k+1}}} a_{k_{z_k}} \dots a_{d_{z_d}}$ for some $k \in [d-1]$. If σ_q is prior to σ_p then we must have that $a_{k+1_{z_{k+1}}} <_\ell a_{k_{z_k}}$ and hence that k constituted a descent in p. Conversely, every *internal* descent k (i.e. $k \in [d-1]$) in p corresponds to a facet σ_s in $c_{\mathbf{z}}$ which intersects σ_p maximally and for which $s <_L p$. That is, there is a one-to-one correspondence between internal descents in p and facets in $c_{\mathbf{z}}$ which are prior to σ_p and which intersect σ_p maximally.

The only other facets of $\widehat{nC^d}$ which σ_p intersects maximally are those which contain σ_p^0 and σ_p^1. A facet containing σ_p^1 must come after σ_p. A facet containing σ_p^0 must be prior to σ_p and belong to the cube $c_{\mathbf{z}-\mathbf{e}_{a_d}}$, which exists in $\widehat{nC^d}$ *if and only if* $z_{a_d} > 0$, i.e. if and only if d is a descent in p. Hence, the number of descents in p equals the number of facets in $\widehat{nC^d}$ which are prior to σ_p and which intersect σ_p maximally. This number must equal the number of $(d-1)$-faces in $\sigma_p \cap \bigcup_{q<p} \sigma_q$, because nC^d is a manifold with boundary, so a $(d-1)$-face can belong to at most two facets. We have proved:

Theorem 32 *For all $d \geq 0$ and for all $n \geq 1$, $D_d^n(t) = h(\widehat{nC^d, t})$.* □

4 Other statistics

4.1 Alternating permutations

In the classical case of the symmetric group, a permutation $\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d$ is said to be *alternating* if it has descent set $D(\pi) = \{1, 3, 5, \ldots, d-1\}$ for d even and $D(\pi) = \{1, 3, 5, \ldots, d-2\}$ for d odd, so that $a_1 > a_2 < a_3 > \ldots$. A permutation is *reverse alternating* if $a_1 < a_2 > a_3 < \ldots$. There is a one-to-one correspondence between alternating and reverse alternating permutations, viz. $a_1 a_2 \ldots a_d \to b_1 b_2 \ldots b_d$ where $b_i = d + 1 - a_i$. The number E_d of alternating permutations in $\mathcal{S}_d$ is called an *Euler number* and there is a remarkable formula, due to André [1], related to these. Namely, we have $\sum_{d \geq 0} E_d \frac{x^d}{d!} = \tan(x) + \sec(x)$.

It seems that to generalize the definition of alternating permutation to our S_d^n, one ought to consider the descent/ascent at d, and we will do this later. However, such a definition isn't altogether satisfying, beause it means that in the case of S_d^1, i.e. essentially the symmetric group $\mathcal{S}_d$, there would be alternating permutations only for even d and reverse alternating only for odd d. Moreover, there is something to be gained from the definition which ignores the descent/ascent at d and thus has the classical case as a specialization.

Definition 33 *An indexed permutation $p \in S_d^n$ is* weakly alternating *if, for $i \in [d\text{-}1]$, i is a descent precisely when i is odd.*

Thus, $2_1 3_0 4_2 1_1$ and $2_1 3_0 4_2 1_0$ are both weakly alternating, because we are ignoring the descent/ascent at $d = 4$.

This definition allows us to generalize the mysterious formula of André in a very simple way. For a proof of the following theorem, see [22].

Theorem 34 *Let W_d^n be the number of weakly alternating permutations in S_d^n. Then $\sum_{d \geq 0} W_d^n \frac{x^d}{d!} = \tan(nx) + \sec(nx)$.* □

An interesting formula relating Euler numbers to the Eulerian polynomials states that $E_{2d+1} = (-1)^{d+1} A_{2d+1}(-1)$ (where $A_d(t)$ is the d-th Eulerian polynomial) or, in terms of our descent polynomials, $E_{2d+1} = (-1)^d D^1_{2d+1}(-1)$. We can extend this to the case $n = 2$.

Theorem 35 *Let W^2_d be the number of weakly alternating permutations in S^2_d. Then $W^2_{2d} = (-1)^d D^2_{2d}(-1)$.*

Proof: By Theorem 20, $\displaystyle\sum_{d\geq 0} D^2_d(t)\frac{x^d}{d!} = \frac{(1-t)e^{x(1-t)}}{1-te^{2x(1-t)}}$. Substitute -1 for t to get

$\displaystyle\sum_{d\geq 0} D^2_d(-1)\frac{x^d}{d!} = \frac{2e^{2x}}{1+e^{4x}}$. Hence, if $i = \sqrt{-1}$, we have

$$\sum_{d\geq 0} D^2_d(-1)\frac{(ix)^d}{d!} = \frac{2e^{2ix}}{1+e^{4ix}} = \frac{2}{e^{-2ix}+e^{2ix}} = \frac{1}{\cos(2x)} = \sec(2x).$$

But, since $D^2_d(t)$ is symmetric, with $D(d,2,k) = D(d,2,d\text{–}k)$, we have $D^2_{2d+1}(-1) = 0$, so $\sum_{d\geq 0}(-1)^d D^2_{2d}(-1)\frac{x^{2d}}{(2d)!} = \sec(2x)$. Comparing this with Theorem 34 (and the Taylor expansion of $\sec x$ and $\tan x$ at 0) yields the theorem. $\square$

We now turn to a new definition of alternating indexed permutations.

Definition 36 *An indexed permutation $p \in S^n_d$ is* alternating *if, for $i \in [d]$, i is a descent if and only if i is even. p is* reverse alternating *if, for $i \in [d]$, i is a descent if and only if i is odd.*

Note that this interchanges the definitions from the classical case.

We now turn to computing the distribution of alternating indexed permutatins. Once again, we will do this not for S^n_d as a whole, but for each coset $S_{\mathbf{z}}$.

Consider the following triangle, defined by setting $a^0_0 = 1$ and, in general, $a^k_d = \sum_{i=k}^{d-1} a^i_{d-1}$ for d even and $a^k_d = \sum_{i=0}^{k-1} a^i_{d-1}$ for d odd. The first line is number 0 and a^k_d is the entry number k from the right in line d, where the rightmost entry in line d is a^0_d.

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 0 & & & & \\
 & & & 0 & & 1 & & 1 & & & \\
 & & 2 & & 2 & & 1 & & 0 & & \\
 & 0 & & 2 & & 4 & & 5 & & 5 & \\
16 & & 16 & & 14 & & 10 & & 5 & & 0
\end{array}$$

$$\vdots$$

This triangle appears in [2], where it is called the *Bernoulli-Euler triangle*. We will show shortly that the numbers on the diagonal edges of the triangle are the Euler numbers. In [2], Arnold states that each line in the triangle defines *finite mass distributions* and he shows, among other things, that the Euler number E_d is the number of *maximal morsifications* of the function x^{d+1}. The following theorem is proved indirectly in [2]. A direct proof is given in [22].

Theorem 37 *Suppose $\mathbf{z} \in \mathbf{Z}_n^d$ has $d-k$ positive coordinates. Then a_d^k is the number of alternating indexed permutations in $S_{\mathbf{z}}$ and a_d^{d-k} is the number of reverse alternating indexed permutations in $S_{\mathbf{z}}$. In particular, if d is even then $a_d^0 = E_d$ and if d is odd then $a_d^d = E_d$, where E_d is the d-th Euler number.* $\square$

One can derive several recurrence relations between the entries in the *BE*-triangle, but there is a particular one which we will need. If we cut off the first $d+1$ lines of the triangle and turn this initial segment upside down, then we can express the entries a_d^k in the top line as a polynomial in k. Let us say that we take the first 5 lines and turn them upside down. If we then change the sign of every entry in lines 3 and 4 from the top, we get the following triangle

$$\begin{array}{ccccccccc}
5 & & 5 & & 4 & & 2 & & 0 \\
& 0 & & -1 & & -2 & & -2 & \\
& & -1 & & -1 & & 0 & & \\
& & & 0 & & 1 & & & \\
& & & & 1 & & & &
\end{array}$$

which constitutes a *difference table*, i.e. each entry is the difference between the entries just above it. More precisely, if we have $\begin{smallmatrix} a & & b \\ & c & \end{smallmatrix}$ then $c = b - a$. This yields a formula for the entries a_4^k now sitting in the top line: $a_4^k = 5 + 0\binom{k}{1} - 1\binom{k}{2} - 0\binom{k}{3} + 1\binom{k}{4}$. In general (see, e.g., [20], Proposition 1.4.2), the entries on the far left diagonal constitute the coefficients of a polynomial in k *in the basis* $\{\binom{k}{i} \mid i \in \mathbb{N}\}$. Making use of the fact that every other entry on this diagonal is 0 we get the following result.

Lemma 38 $a_{2d}^k = \sum_{i=0}^{d} (-1)^i \binom{k}{2i} a_{2d-2i}^0$ *and* $a_{2d+1}^k = \sum_{i=0}^{d} (-1)^i \binom{k\square}{2i+1} a_{2d-2i}^0$.

Note that this expresses a_d^k in terms of Euler numbers, since $a_{2d}^0 = E_{2d}$ by Theorem 37.

Theorem 39 *Let A_d^n be the number of alternating indexed permutations in S_d^n and R_d^n the number of reverse alternating such. Then*

$$\sum_{d\geq 0} A_d^n \frac{x^d}{d!} = \frac{\cos x + \sin x}{\cos(nx)} \quad \textit{and} \quad \sum_{d\geq 0} R_d^n \frac{x^d}{d!} = \frac{\cos((n\text{-}1)x) + \sin((n\text{-}1)x)}{\cos(nx)}.$$

Proof: Because $\frac{1}{\cos(nx)}$ has only terms of even degree, the theorem claims, among other things, that $\sum_{d\geq 0} A_{2d}^n \frac{x^{2d}}{(2d)!} = \frac{\cos x}{\cos(nx)}$. We will prove this. The other three cases are similar.

By Theorem 37, and the fact that $\sum E_{2d}\frac{x^d}{d!} = \sec x$, $\sec(nx) = \sum_{d\geq 0} n^{2d} a_{2d}^0 \frac{x^{2d}}{(2d)!}$, so $\frac{\cos x}{\cos(nx)} = \sum_{d\geq 0}\left(\sum_{k=0}^{d}(-1)^{d-k}\binom{2d}{2k} n^{2k} a_{2k}^0\right)\frac{x^{2d}}{(2d)!}$.

Also, $A_{2d}^n = \sum_{k=0}^{2d}\binom{2d}{k}(n-1)^{2d-k} a_{2d}^k$, because a_{2d}^k is the number of alternating permutations in $S_{\mathbf{z}} \subset S_{2d}^n$ if $\mathbf{z}$ has exactly $2d-k$ positive coordinates, and there are exactly $\binom{2d}{k}(n-1)^{2d-k}$ such $\mathbf{z}$. Hence, we need to show

$$\sum_{k=0}^{2d}\binom{2d}{k}(n-1)^{2d-k} a_{2d}^k \quad = \quad \sum_{k=0}^{d}(-1)^{d-k}\binom{2d}{2k} n^{2k} a_{2k}^0 \, .$$

Let $m = n-1$ and use Lemma 38 to replace this by

$$\sum_{k=0}^{2d}\binom{2d}{k} m^{2d-k}\sum_{i=0}^{d}(-1)^i\binom{k}{2i} a_{2d-2i}^0 \quad = \quad \sum_{k=0}^{d}(-1)^{d-k}\binom{2d}{2k} a_{2k}^0 \sum_{i=0}^{2k}\binom{2k}{i} m^i \, . \tag{1}$$

Clearly, each side of (1) is a polynomial in m, so it suffices to show that the coefficient to m^j is the same on both sides for each j. Let L_j be the coefficient to m^j in the LHS and let R_j be the coefficient to m^j in the RHS. Then we have $L_{2d-k} = \binom{2d}{k}\sum_{i=0}^{d}(-1)^i\binom{k}{2i} a_{2d-2i}^0$, so $L_j = \binom{2d}{2d-j}\sum_{i=0}^{d}(-1)^i\binom{2d-j}{2i} a_{2d-2i}^0$. Now, using the identity $\binom{a}{b}\binom{b}{c} = \binom{a}{c}\binom{a-c}{b-c}$ we get

$$L_j \quad = \quad \sum_{i=0}^{d}(-1)^i\binom{2d}{2i}\binom{2d-2i}{2d-j-2i} a_{2d-2i}^0 \quad = \quad \sum_{i=0}^{d}(-1)^i\binom{2d}{2i}\binom{2d-2i}{j} a_{2d-2i}^0 \, . \tag{2}$$

As for the right hand side we have

$$R_j \quad = \quad \sum_{k=0}^{d} (-1)^{d-k} \binom{2d}{2k} \binom{2k}{j} a^0_{2k} \quad = \quad \sum_{k=0}^{d} (-1)^{k} \binom{2d}{2k} \binom{2d-2k}{j} a^0_{2d-2k} \, ,$$

which agrees with (2) as desired. $\square$

Theorem 39 yields the following result, with a proof similar to that of Theorem 35, where $\lfloor m \rfloor$ is the largest integer smaller than or equal to m:

Theorem 40 $$(-1)^{\lfloor \frac{d+1}{2} \rfloor} D^3_d(-1) = A^3_d. \qquad \square$$

4.2 Major index and inversions

Apart from descents and excedances, there are two other statistics of the elements of the symmetric group $\mathcal{S}_d$ that have been extensively studied. These are the *inversion index* and the *major index* of $\pi \in \mathcal{S}_d$. An *inversion* in $\pi = a_1 a_2 \ldots a_d$ is a pair (i, j) such that $i < j$ and $a_i > a_j$. The inversion index $\mathrm{inv}(\pi)$ of π is the number of inversions in π. The major index $\mathrm{maj}(\pi)$ of π is the sum of the elements of the descent set $D(\pi)$ of π.

Foata [8] has constructed a bijection $\phi : \mathcal{S}_d \to \mathcal{S}_d$ such that $\mathrm{maj}(\pi) = \mathrm{inv}(\phi(\pi))$, which shows that maj and inv are equidistributed over $\mathcal{S}_d$. For a nice description of ϕ, see [5].

By definition, Foata's bijection ϕ has the property that if $\pi = a_1 a_2 \ldots a_d$ and $\phi(\pi) = b_1 b_2 \ldots b_d$, then $a_d = b_d$. Hence the following.

Remark 41 Let $k \in [d]$ and let $A_{d,k} := \{\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d \mid a_d = k\}$. Then

$$\sum_{\pi \in A_{d,k}} t^{\mathrm{maj}(\pi)} = \sum_{\pi \in A_{d,k}} t^{\mathrm{inv}(\pi)}.$$

Definition 42 *For* $p \in S^n_d$*, the* major index of p *is* $\mathrm{maj}(p) = \sum_{j \in D(p)} j$.

Definition 43 *For* $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}} \in S^n_d$*, an* inversion *in* p *is a pair* (i, j) *such that* $1 \le i < j \le d+1$ *and* $a_{j_{z_j}} <_\ell a_{i_{z_i}}$*. The* inversion index of p, $\mathrm{inv}(p)$*, is the number of inversins in* p.

Note that this differs from the classical definition in that we consider an indexed permutation in S_d^n to have $(a_{d+1})_{z_{d+1}} = (d+1)_0$ so $(i, d+1)$ is an inversion for any i such that $z_i > 0$. For example, $2_0 3_1 1_0$ has three inversions, namely $(1,3)$, $(2,3)$, and $(2,4)$.

Theorem 44 *For any* $\mathbf{z} \in \mathbf{Z}_n^d$,
$$\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{maj}(p)} = \sum_{p \in S_{\mathbf{z}}} t^{\mathrm{inv}(p)}. \text{ Hence, } \sum_{p \in S_d^n} t^{\mathrm{maj}(p)} = \sum_{p \in S_d^n} t^{\mathrm{inv}(p)}.$$

Proof: Suppose $\mathbf{z}$ has $z_i = 0$ for exactly k–1 values of i. Let $\theta : \{i_{z_i} \mid i \in [d+1]\} \to [d+1]$ be the bijection which takes the i-th element of $\{i_{z_i} \mid i \in [d+1]\}$ (in the ordering $<_\ell$) to i. In particular, $\theta((d+1)_{z_{d+1}}) = k$. Let $A_{d,k}$ be as in Remark 41 and define $\Theta : S_{\mathbf{z}} \to A_{d+1,k}$ by $\Theta(a_{1_{z_{a_1}}} a_{2_{z_{a_2}}} \ldots a_{d_{z_{a_d}}}) = \theta(a_{1_{z_{a_1}}})\theta(a_{2_{z_{a_2}}}) \ldots \theta(a_{d_{z_{a_d}}})k$. It follows that i is a descent in p *iff* i is a descent in $\Theta(p)$ and that (i,j) is an inversion in p *iff* (i,j) is an inversion in $\Theta(p)$. Hence, $\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{maj}(p)} = \sum_{\pi \in A_{d+1,k}} t^{\mathrm{maj}(\pi)}$ and $\sum_{p \in S_{\mathbf{z}}} t^{\mathrm{inv}(p)} = \sum_{\pi \in A_{d+1,k}} t^{\mathrm{inv}(\pi)}$. By Remark 41, this implies the desired result. $\square$

4.3 Denert's statistic

In [7], Denert defined a new statistic on the symmetric group $\mathcal{S}_d$. For convenience, we use the equivalent definition of Foata and Zeilberger [10, Theorem 2].

Let $\mathrm{exc}(\pi)$ be the *excedance subword* of π, i.e. the word $a_{i_1} a_{i_2} \ldots a_{i_r}$ such that $\{i_1, i_2, \ldots, i_r\} = E(\pi)$, and let $\mathrm{nex}(\pi)$ be the *non-excedance subword* of π, i.e. $\mathrm{nex}(\pi) = a_{j_1} a_{j_2} \ldots a_{j_s}$ such that $\{j_1, j_2, \ldots, j_s\} = [d] \setminus E(\pi)$. Also, extending our definition of inv, for any word $w = a_1 a_2 \ldots a_r$ such that each a_i is an integer, let $\mathrm{inv}(w)$ be the number of inversions in w, i.e. the number of pairs (i,j) such that $1 \le i < j \le r$ and $a_i > a_j$. Then Denert's statistic is defined by

$$\mathrm{den}(\pi) = \left(\sum_{i \in E(\pi)} i \right) + \mathrm{inv}(exc(\pi)) + \mathrm{inv}(nex(\pi)).$$

As an example, $\mathrm{den}(326541) = (1+3+4) + \mathrm{inv}(365) + \mathrm{inv}(241) = 8+1+2 = 11$.

Denert conjectured that the joint distribution of the pair (d, maj) was equal to that of (e, den), i.e. that $\sum_{\pi \in \mathcal{S}_d} t^{d(\pi)} x^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_d} t^{e(\pi)} x^{\mathrm{den}(\pi)}$. In

[10], Foata and Zeilberger proved the conjecture, and later Han (see [13]) constructed an explicit bijection to prove this. Han's bijection has the property that it fixes the last letter of each permutation $\pi \in \mathcal{S}_d$ ([11]). We thus have the following:

Remark 45 Let $k \in [d]$ and let $A_{d,k} = \{\pi = a_1 a_2 \ldots a_d \in \mathcal{S}_d \mid a_d = k\}$. Then

$$\sum_{\pi \in A_{d,k}} t^{d(\pi)} x^{\mathrm{maj}(\pi)} = \sum_{\pi \in A_{d,k}} t^{e(\pi)} x^{\mathrm{den}(\pi)}.$$

Much as in the previous section, we can use this to extend the results of Foata and Zeilberger on (e, den). First, however, we need to modify our definition of excedance.

Definition 46 *Fix* $\mathbf{z} = (z_1, z_2, \ldots, z_d) \in \mathbf{Z}^n$ *and suppose that* $\mathbf{z}$ *has exactly* $k-1$ *coordinates equal to* 0*. Let* Θ *be as in the proof of Theorem 44.*

An ϵ–excedance *in* $p = a_{1_{z_1}} a_{2_{z_2}} \ldots a_{d_{z_d}} \in S_d^n$ *is an* $i \in [d]$ *such that* $\theta(a_{i_{z_{a_i}}}) > i$*. Moreover, we let* $\mathcal{E}(p) := \{i \in [d] \mid \theta(a_{i_{z_{a_i}}}) > i\}$ *and* $\epsilon(p) := \#\mathcal{E}(p)$*.*

By construction, the ϵ–excedance set $\mathcal{E}(p)$ of p is equal to $E(\pi)$ where $\pi = \Theta(p)$, so in particular $\epsilon(p) = e(\pi)$. As an example, let $p = 4_1 5_0 1_2 3_0 2_3$. Then, applying θ to each letter of p (including the invisible 6_0), we get $\pi = 425163$, so $\mathcal{E}(p) = E(\pi) = \{1, 3, 5\}$. Also, by definition, Θ preserves the inversions of p and, in particular, it preserves the inversions of $\mathrm{exc}(p)$ and $\mathrm{nex}(p)$, respectively, where $\mathrm{exc}(p)$ and $\mathrm{nex}(p)$ are defined in the obvious way. Consequently, Θ simultaneously preserves $\epsilon(p)$, $\mathrm{inv}(\mathrm{exc}(p))$ and $\mathrm{inv}(\mathrm{nex}(p))$. We now define $\mathrm{den}(p)$ analogously to the definition of $\mathrm{den}(\pi)$.

Definition 47 *For* $p \in S_d^n$, $\mathrm{den}(p) := \left(\sum_{i \in \mathcal{E}(p)} i \right) + \mathrm{inv}(\mathrm{exc}(p)) + \mathrm{inv}(\mathrm{nex}(p))$.

It is evident that for $p \in S_d^n$, $\mathrm{den}(p) = \mathrm{den}(\Theta(p))$ and, hence, we have

$$\sum_{p \in S_{\mathbf{z}}} t^{\epsilon(p)} x^{\mathrm{den}(p)} = \sum_{\pi \in A_{d+1,k}} t^{e(\pi)} x^{\mathrm{den}(\pi)} = \sum_{\pi \in A_{d+1,k}} t^{d(\pi)} x^{\mathrm{maj}(\pi)},$$

the last equality by Remark 45. Since Θ preserves both $d(\cdot)$ and $\mathrm{maj}(\cdot)$ (see the proof of Theorem 44), so does $\Theta^{-1} : A_{d+1,k} \to S_{\mathbf{z}}$, which yields this:

Theorem 48 *For any* $\mathbf{z} \in \mathbf{Z}_n^d$, $\displaystyle\sum_{p \in S_{\mathbf{z}}} t^{\epsilon(p)} x^{\mathrm{den}(p)} = \sum_{p \in S_{\mathbf{z}}} t^{d(p)} x^{\mathrm{maj}(p)}$. □

Note that this implies that ϵ–excedances are equidistributed with excedances.

Corollary 49 *The pairs* (ϵ, den) *and* (d, maj) *are equidistributed over* S_d^n. □

4.4 Volumes

In [19], Stanley, answering a question posed by Foata, showed that the Eulerian numbers $A(n,k)$, which are the coefficients of $D_d^1(t)$, equal, up to a factor of $d!$, the volume of the subspace of the unit d–cube lying between the hyperplanes $\{\mathbf{x} \in \mathbf{R}^d \mid \sum_i x_i = k\}$ and $\{\mathbf{x} \in \mathbf{R}^d \mid \sum_i x_i = k+1\}$. We generalize this to $D_d^n(t)$ and certain subspaces of nC^d.

Fix $n \geq 0$ and $d > 0$. Let A_k be the union (in nC^d) of all path simplices σ_p such that p has k descents and set $x_{d+1} := 1$. Then $A_k = \{\mathbf{x} \in nC^d \mid x_i > x_{i+1}$ for k values of $i \in [d]\}$ (see Remark 29). Also, for $0 \leq k \leq d$, let S_k be the "slice" of nC^d consisting of all points $\mathbf{x}$ satisfying

$$(k-1)n + 1 \leq \sum_{i=1}^{d} x_i \leq kn + 1.$$

Thus, $S_0 = \{\mathbf{x} \in nC^d \mid 0 \leq \displaystyle\sum_{i=1}^{d} x_i \leq 1\}$ and $S_d = \{\mathbf{x} \in nC^d \mid (d-1)n + 1 \leq \displaystyle\sum_{i=1}^{d} x_i \leq dn\}$.

Let $K := \{\mathbf{x} \in nC^d \mid x_1 \neq x_2 \neq \cdots \neq x_d \neq 1 \text{ and } 0 < x_i < n\}$. Clearly $nC^d \setminus K$ has measure zero. We define a map $\phi : K \to nC^d$ by setting $\phi(x_1, x_2, \ldots, x_d) = (y_1, y_2, \ldots, y_d)$ where

$$y_i = \begin{cases} x_{i+1} - x_i & \text{if } x_i < x_{i+1} \\ n + x_{i+1} - x_i & \text{if } x_i > x_{i+1}. \end{cases}$$

It is easy to check that the map ϕ is injective. Namely, if $\phi(x_1, x_2, \ldots, x_d) = \phi(v_1, v_2, \ldots, v_d)$, then $x_d = v_d$ because either $1 - x_d = 1 - v_d$, and we are done, or $1 - x_d = 1 - v_d \pm n$, so $x_d = v_d \pm n$. In that case we either have $x_d = n$ and $v_d = 0$ or vice versa, so $\mathbf{x}$ and $\mathbf{v}$ lie outside the domain of ϕ. But now, similarly, $x_{d-1} = v_{d-1}$ etc., so that $\mathbf{x} = \mathbf{v}$.

It follows from the definition of ϕ that, where defined, it can be represented by the affine linear transformation $\Phi(\mathbf{x}) = n(\epsilon_1, \epsilon_2, \ldots, \epsilon_d) + A\mathbf{x}$, where

$$\epsilon_i = \begin{cases} 0 & \text{if } x_i < x_{i+1} \\ 1 & \text{if } x_i > x_{i+1} \end{cases}$$

and A is the $d \times d$ matrix with entries a_{ij} satisfying

$$a_{ij} = \begin{cases} -1 & \text{if } i = j \\ 1 & \text{if } i = j - 1 \\ 0 & \text{otherwise.} \end{cases}$$

A is thus upper triangular with -1's on the diagonal, so $\det A = (-1)^d$. Hence, ϕ is volume-preserving and, since it is defined on all of nC^d except for a set of measure 0, its coimage in nC^d must have measure zero. It is easily seen that ϕ maps $A_k \cap K$ into S_k. Thus, the restriction of ϕ to $\phi_k : A_k \cap K \to S_k$ is volume preserving and a bijection except for a set of measure 0 in S_k.

Now, since a path simplex σ_p has volume $1/d!$, the volume of A_k equals $D(d, n, k)/d!$, where $D(d, n, k)$ is the k–th coefficient of $D_d^n(t)$, i.e. the number of indexed permutations in S_d^n with k descents. As a consequence we have the following:

Theorem 50 $vol(S_k) = D(d, n, k)/d!$. □

Appendix

A new bijection proving the equidistribution of excedances and descents in the symmetric group

We describe here a new bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ which satisfies $d(\pi) = e(\phi(\pi))$. For a proof of the following theorem, see [22].

Theorem 51 *The following algorithm defines a bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ such that $d(\pi) = e(\phi(\pi))$: Let $\pi = a_1 a_2 \ldots a_n$ be a permutation word in $\mathcal{S}_n$. We define a bijection $\phi : \mathcal{S}_n \to \mathcal{S}_n$ by constructing $\tau = b_1 b_2 \ldots b_n = \phi(\pi)$ from π*

as follows: Define $a_0 := 0$ and let $k \in [n]$. If $a_k > a_m$ for some $m > k$ then a_k is in place number a_{k+1} in τ (i.e. $b_{a_{k+1}} := a_k$). Otherwise, find the first (rightmost) number in π which is less than a_k (and hence is to the left of a_k). If this number is a_i then a_k is in place number a_{i+1} in τ. In particular, this means that $\ldots ab \ldots$ is a descent in π if and only if a is in the b-th place in τ and hence constitutes an excedance in τ. $\square$

Example 52 $\phi(35142) = 54123$. 5 goes to the first place and 4 to the second because 51 and 42 are descents in 35142. To place 1, since no number less than 1 appears after 1 we trace back until we hit 0. The successor of 0 is 3 so 1 goes to the third place. 2; trace back to 1, whose successor is 4 so 2 goes to the fourth place. 3 has smaller numbers to its right so 3 goes to the fifth place, 5 being the successor of 3.

The following is a straightforward consequence of Theorem 51.

Corollary 53 *Let $\pi = a_1 a_2 \ldots a_n$ and let $\tau = \phi(\pi) = b_1 b_2 \ldots b_n$. Suppose that τ fixes i, i.e. $b_i = i$, and let k be such that $a_k = i$. Then $a_m > a_k$ for any $m > k$ and $a_{k-1} < a_k$. In particular, if $a_n = i$ then $a_m \neq m$ for any $m > i$.* $\square$

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