

A Decomposition of 2-Weak Vertex-Packing Polytopes

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Abstract

The 2-weak vertex-packing polytope of a loopless graph G with d vertices is the subset of the unit d -cube satisfying $x_i + x_j \leq 1$ for every edge (i, j) of G . The dilation by 2 of this polytope is a polytope \mathcal{P} with integral vertices. We triangulate \mathcal{P} with lattice simplices of minimal volume and label the maximal simplices with elements of the hyperoctahedral group B_d . This labeling gives rise to a shelling of the triangulation $\widehat{\mathcal{P}}$ of \mathcal{P} , where the h -vector of $\widehat{\mathcal{P}}$ (and the Ehrhart h^* -vector of \mathcal{P}) can be computed as a descent statistic on a subset of B_d defined in terms of G . A recursive way of computing the h -vector of $\widehat{\mathcal{P}}$ is also given, and a recursive formula for the volume of \mathcal{P} .

1 Introduction

Let G be a graph (with no loops), d the number of vertices in G , and label the vertices of G by the integers $1, 2, \dots, d$. The *vertex-packing polytope* of G is the d -polytope $\mathcal{V}(G)$ in \mathbf{R}^d defined by the following inequalities:

$$0 \leq x_i \leq 1, \quad 1 \leq i \leq d, \quad (1)$$

$$x_{i_1} + x_{i_2} + \dots + x_{i_k} \leq 1, \quad \text{if } \{i_1, i_2, \dots, i_k\} \text{ is a clique of } G. \quad (2)$$

These polytopes have been studied by various authors, mostly from the linear programming point of view (for some background, see [4] or [8]). A simpler polytope, related to $\mathcal{V}(G)$, is the *2-weak vertex-packing polytope* of G , which we denote by $\mathcal{W}(G)$ and which is defined by these inequalities:

$$x_i \leq 1, \quad 1 \leq i \leq d, \quad (3)$$

$$0 \leq x_i, \quad 1 \leq i \leq d, \quad (4)$$

$$x_i + x_j \leq 1, \quad \text{if } (i, j) \text{ is an edge of } G. \quad (5)$$

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If G is a *bipartite* graph, then $\mathcal{W}(G)$ and $\mathcal{V}(G)$ coincide.

It is easily seen that the inequalities given by (4) and (5) and those of the inequalities (3) for which i is an isolated vertex of G are linearly independent, and thus these inequalities define the facets of $\mathcal{W}(G)$. We will show that the vertices of $\mathcal{W}(G)$ have coordinates in $\{0, \frac{1}{2}, 1\}$. Hence, if we dilate $\mathcal{W}(G)$ by 2, i.e. take the image of $\mathcal{W}(G)$ under the map $f : \mathbf{R}^d \rightarrow \mathbf{R}^d$ defined by $f(x) = 2x$, then we get a polytope $\mathcal{P}(G)$ whose vertices are integral, more precisely $\{0, 1, 2\}$ -valued. We call the polytope $\mathcal{P}(G)$ the *extended 2-weak vertex-packing polytope* of G . It is clearly defined by

$$x_i \leq 2, \quad 1 \leq i \leq d, \quad (6)$$

$$0 \leq x_i, \quad 1 \leq i \leq d, \quad (7)$$

$$x_i + x_j \leq 2, \quad \text{if } (i, j) \text{ is an edge of } G. \quad (8)$$

This paper deals with the structure of $\mathcal{P}(G)$. We triangulate $\mathcal{P}(G)$ in a certain systematic way and label the maximal simplices in the triangulation, which we denote by $\widehat{\mathcal{P}}$, with elements of the *hyperoctahedral group* B_d . This labeling allows us to *shell* $\widehat{\mathcal{P}}$ in such a way that we can compute the h -vector of $\mathcal{P}(G)$ as a *descent statistic* on a subset of B_d determined by G . Moreover, the triangulation is such that its h -vector equals the *Ehrhart h^* -vector* of $\mathcal{P}(G)$. We give a decomposition of $\widehat{\mathcal{P}}$ into maximal simplices, whose intersections with other maximal simplices we can describe explicitly. We also give a recursive formula for computing the h -vector of $\widehat{\mathcal{P}}(G)$, by descending to those facets of $\mathcal{P}(G)$ which lie on the boundary of $2C^d$ where C^d is the standard unit d -cube (these facets are extended 2-weak vertex-packing polytopes in their own right). A simplified version of this recursion gives a simple recursive formula for computing the volume of $\mathcal{P}(G)$.

2 Preliminaries

2.1 Ehrhart polynomials

Let \mathcal{P} be a d -dimensional polytope (or simplicial complex (see section 2.2)) in \mathbf{R}^n with integral (or lattice) vertices, i.e. $v_i \in \mathbf{Z}^n$ for all vertices v_i of \mathcal{P} . For $k \in \mathbf{N}$ let $k\mathcal{P} = \{kx \mid x \in \mathcal{P}\}$, i.e. $k\mathcal{P}$ is the (lattice) polytope obtained by dilating \mathcal{P} by a factor of k .

For $k \in \mathbf{N}$ define the function

$$i(\mathcal{P}, k) = \#\{x \in \mathbf{R}^n \mid x \in k\mathcal{P} \cap \mathbf{Z}^n\}.$$

Thus, $i(\mathcal{P}, k)$ is the number of lattice points contained in $k\mathcal{P}$. By Cor. 4.6.28 in [11], $i(\mathcal{P}, k)$ is a polynomial in k of degree d , called the *Ehrhart polynomial* of \mathcal{P} . Now define the generating function

$$E(\mathcal{P}, t) = \sum_{k \geq 0} i(\mathcal{P}, k) t^k.$$

By Thm. 2.1 in [10], we have

$$E(\mathcal{P}, t) = \frac{h^*(\mathcal{P}, t)}{(1-t)^{d+1}}$$

where $h^*(\mathcal{P}, t)$ is a polynomial of degree at most d with non-negative integer coefficients, called the *Ehrhart h^* -polynomial* of \mathcal{P} . Its coefficients are the coordinates of the Ehrhart h^* -vector $h^*(\mathcal{P})$ of \mathcal{P} .

2.2 Simplicial complexes

An *abstract simplicial complex* is a nonempty collection K of sets such that if $F \in K$ and $G \subset F$ then $G \in K$. An element of K is called a *face* of K . We will be mostly concerned with the *geometric realization* of simplicial complexes (for definitions and basic properties, see [7]) and we will, by abuse of notation, *not* distinguish between a simplicial complex and its geometric realization.

A simplicial complex K is *pure* if all its maximal faces have the same dimension $d = \dim(K)$. If K is a pure simplicial complex of dimension d , then a *facet* of K is a d -face, i.e. a d -dimensional face, of K . In other words, a facet is a maximal simplex of K . When a complex K triangulates a polytope \mathcal{P} , the facets of K are d -dimensional, but the facets of \mathcal{P} have dimension $d-1$. This can cause confusion, which we will try to avoid in subsequent sections by referring to the facets of K as maximal simplices.

The *h -vector* $h(K) = (h_0, h_1, \dots, h_d)$ of a simplicial complex K of dimension $d-1$ is defined as follows: Let $f_i = f_i(K)$ be the number of i -dimensional faces in K , where we set $f_{-1} = 1$ (corresponding to the empty set), and define $h(K) = (h_0, h_1, \dots, h_d)$ by setting

$$\sum_{i=0}^d f_{i-1} (x-1)^{d-i} = \sum_{i=0}^d h_i x^{d-i}. \quad (9)$$

We define the *h -polynomial* $h(K, t)$ of K by $h(K, t) = h_0 + h_1 t + \dots + h_d t^d$. For further information about h -vectors, see [13].

Let K be a pure simplicial lattice complex of dimension d . If all facets of K have volume $1/d!$ (see section 2.4) then we say that K is *primitively triangulated*. The following theorem is essentially a consequence of Cor. 2.5 in [10], whose conclusion is expressed in greater generality in Thm. 2 in [1].

Theorem 1 *Suppose K is a primitively triangulated simplicial lattice complex. Then $h^*(K, t) = h(K, t)$, where $h^*(K, t)$ is the Ehrhart h^* -polynomial of K . ■*

For certain pure simplicial complexes K the coefficients of $h(K, t)$ can be interpreted in a way that partitions the facets of K according to how they intersect other facets. We will briefly review this now. For further information see [2] and [3].

Definition 2 *Let K be a finite pure simplicial complex of dimension d . If F is a face of K , let \overline{F} be the complex consisting of F and all its faces. An ordering F_1, F_2, \dots, F_n of the facets of K is called a shelling if, for all k with $1 < k \leq n$, $\overline{F}_k \cap \bigcup_{i < k} \overline{F}_i$ is a nonempty union of $(d-1)$ -faces of \overline{F}_k . A complex K is said to be shellable if there exists a shelling of K .*

That is, a complex is shellable if it can be built up by adding one facet at a time in such a way that, for $k > 1$, the intersection of each \overline{F}_k with the complex generated by the previous \overline{F}_i 's is a nonempty union of $(d-1)$ -faces of \overline{F}_k .

As it turns out, the h -vector of a shellable complex can be computed from the shelling. The following theorem is essentially due to McMullen [6].

Theorem 3 *Let F_1, F_2, \dots, F_n be a shelling of a d -dimensional complex K and let $c(k)$ be the number of $(d-1)$ -faces of \overline{F}_k contained in $\bigcup_{i < k} \overline{F}_i$. Then we have the following formula for the h -polynomial of K :*

$$h(K, t) = \sum_{k=1}^n t^{c(k)}. \quad \blacksquare$$

Thus, given a shelling F_1, F_2, \dots, F_n of a simplicial complex K , we can compute the h -polynomial $h(K, t)$ of K via Theorem 3. That is, the k -th coefficient of $h(K, t)$ equals the number of F_i with $c(i) = k$. We then say that a facet F_i of K *contributes to the k -th coefficient of $h(K, t)$ if $c(i) = k$.*

If K is a simplicial complex and p a vertex not in K , then the *cone with apex p over K* (or *with base K*), denoted $p * K$, is the simplicial complex whose i -faces are the i -faces of K and $\{p \cup f \mid f \text{ an } (i-1)\text{-face of } K\}$. Geometrically, a cone can be defined as follows. If K is a $(d-1)$ -dimensional simplicial (or polytopal) complex in \mathbf{R}^n and p is a point in \mathbf{R}^n such that each ray emanating from p intersects K in at most one point, then the cone $p * K$ consists of K and p and the new i -faces, for $1 \leq i \leq d$, obtained by taking, for each $(i-1)$ -face f in K , the union of all line segments connecting p to points in f .

The following theorem is straightforward to prove from equation (9).

Theorem 4 *Suppose the simplicial complex K is a cone with apex p over B , i.e. $K = p * B$. Then $h(K, t) = h(B, t)$. ■*

2.3 The hyperoctahedral group

The *hyperoctahedral group* B_d is the group of symmetries of the d -dimensional *cross polytope* (or *hyperoctahedron*) defined as the convex hull of the points $\pm e_i$ where e_i is a standard basis vector in \mathbf{R}^d . We represent the elements of the hyperoctahedral group B_d by *signed permutation words*, i.e. ordinary permutation words in which each letter has a sign. To simplify the notation, we write $\bar{1}, \bar{2}, \bar{3}$ etc. for $-1, -2, -3$, and simply $1, 2, 3$ etc. for $+1, +2, +3$. For example, $B_2 = \{12, 21, \bar{1}2, 2\bar{1}, 1\bar{2}, \bar{2}1, \bar{1}\bar{2}, \bar{2}\bar{1}\}$.

As we will not be concerned with ordinary permutations, we simply call the elements of B_d permutations. When we refer to the letters in a permutation, we regard them as integers (with the sign incorporated) and order them as such, i.e. $\dots \bar{3} < \bar{2} < \bar{1} < 0 < 1 < 2 < 3 \dots$. Let $[d] := \{1, 2, \dots, d\}$.

Definition 5 *A descent in $\pi \in B_d$ is an $i \in [d]$ such that one of the following holds:*

- 1) $i < d$ and $a_i > a_{i+1}$,
- 2) $i = d$ and $a_d > 0$.

The number of descents in $\pi \in B_d$ is $\text{des}(\pi)$.

For any subset S of B_d , the descent polynomial of S is $D(S, t) := \sum_{\pi \in S} t^{\text{des}(\pi)}$.

For example, the descents of $2\bar{3}\bar{4}1$ are 1, 2 and 4, so $\text{des}(2\bar{3}\bar{4}1) = 3$. If $S = \{\bar{3}\bar{2}\bar{1}, 1\bar{2}\bar{3}, 213\}$ then $D(S, t) = 1 + 2t^2$.

For more information on descent (and other) statistics of B_d , see [9] or [14].

2.4 Volumes

When we talk about volume in \mathbf{R}^d we mean the usual d -dimensional volume, which we denote $\text{vol}_d(\cdot)$. If S is a subset of a d -dimensional coordinate subspace of \mathbf{R}^n , then by $\text{vol}_d(S)$ we mean the volume of S in that subspace. If S is a union of such subsets S_i then by $\text{vol}_d(S)$ we mean the sum of the volumes of the S_i . In particular, a polytope \mathcal{P} of dimension less than d has $\text{vol}_d(\mathcal{P}) = 0$. For convenience, we make the following definition.

Definition 6 *If \mathcal{P} is a d -dimensional polytope or simplicial complex in \mathbf{R}^n such that $\text{vol}_d(\mathcal{P})$ is defined, then the normalized volume of \mathcal{P} is $\text{Nvol}(\mathcal{P}) := d! \cdot \text{vol}_d(\mathcal{P})$.*

Hence, for any polytope (or simplicial complex) \mathcal{P} of positive dimension, $\text{Nvol}(\mathcal{P})$ is positive. The rationale behind this definition is that the least volume a lattice d -simplex can have is $1/d!$. In particular, the normalized volume of a primitively triangulated complex equals its number of maximal simplices.

3 Main Theorems

The following proposition is intuitively obvious, but we give a proof anyway.

Proposition 7 *Let \mathbf{p} be a point in the polytope \mathcal{P} and let $\mathcal{P}_{\mathbf{p}}$ be the union of those facets of \mathcal{P} which do not contain \mathbf{p} . Then \mathcal{P} is a cone with apex \mathbf{p} over $\mathcal{P}_{\mathbf{p}}$.*

Proof: If \mathbf{p} is contained in the interior of \mathcal{P} this is trivial, and the same is true if the dimension of \mathcal{P} is 1. Also, by convexity, any ray from \mathbf{p} can intersect $\mathcal{P}_{\mathbf{p}}$ in at most one point.

Assume, then, that \mathbf{p} lies in some facet of \mathcal{P} and $\dim(\mathcal{P}) \geq 2$. Let \mathbf{x} be a point in \mathcal{P} . It suffices to show that \mathbf{x} lies on a line segment between \mathbf{p} and a point $\mathbf{q} \in G$ for some facet $G \subset \mathcal{P}_{\mathbf{p}}$. Let R be the extension beyond \mathbf{x} of the line segment $\mathbf{p}\mathbf{x}$ and let \mathbf{q} be the point on R such that every point on R not on $\mathbf{p}\mathbf{q}$ lies outside \mathcal{P} .

It is clear that if \mathbf{x} does not belong to any of the facets of \mathcal{P} containing \mathbf{p} then neither does \mathbf{q} , by convexity, and we would be done. Suppose, therefore, that \mathbf{x} and \mathbf{p} belong to the same facet F of \mathcal{P} . We now proceed by induction, so we may further assume that \mathbf{q} belongs to a $(d-2)$ -face f of F such that $\mathbf{p} \notin f$. Let G be the unique facet of \mathcal{P} different from F and containing f . We claim that G does not contain \mathbf{p} , which will complete the proof. If G did contain \mathbf{p} , then $G \cap F$ would contain the $(d-1)$ -dimensional set $\mathbf{p} * f$, which is absurd. ■

Throughout this paper, given a graph G , $\mathcal{P}(G)$ will be the extended 2-weak vertex-packing polytope of G .

It is clear from the definition of $\mathcal{P}(G)$ that it is a subset of $2C^d$, the dilation of the unit d -cube by 2.

Theorem 8 *Let G be a graph and let $\mathcal{P}'(G) = \mathcal{P}(G) \cap \partial(2C^d)$, i.e. $\mathcal{P}'(G)$ is the union of those facets of $\mathcal{P}(G)$ which lie on the boundary of $2C^d$. Let $\mathbf{p} = (1, 1, \dots, 1)$. Then $\mathcal{P}(G) = \mathbf{p} * \mathcal{P}'(G)$.*

Proof: The facets of $\mathcal{P}(G)$ are defined by the inequalities (7) and (8) and those of the inequalities (6) for which i is an isolated vertex in G (or, more precisely, by the corresponding boundary equalities). Hence, those facets of $\mathcal{P}(G)$ which do not lie on $\partial(2C^d)$ are precisely those which are defined by $x_i + x_j = 2$ for some (i, j) . Every

such facet contains the point \mathbf{p} and no other facets of $\mathcal{P}(G)$ do. Hence, by Prop. 7, $\mathcal{P}(G) = \mathbf{p} * \mathcal{P}'(G)$. \blacksquare

To triangulate $\mathcal{P}(G)$ we first triangulate $2C^d$ in the following way. $2C^d$ is embedded in \mathbf{R}^d so that its vertices are all points whose coordinates are either 0 or 2. In particular, its center (of symmetry) is the point $\mathbf{p} = (1, 1, \dots, 1)$. We subdivide $2C^d$ into the 2^d unit cubes all of whose vertices are lattice points. Each of these small cubes contains \mathbf{p} and a unique vertex which is a vertex of $2C^d$. We label each small cube by that vertex of $2C^d$ which it contains. As an example, the standard unit d -cube is labeled by $\mathbf{0} = (0, 0, \dots, 0)$ and denoted $c_{\mathbf{0}}$.

Next, we triangulate each of these small cubes. Let $c_{\mathbf{z}}$ be the small cube labeled by \mathbf{z} . Then every maximal simplex in the triangulation of $c_{\mathbf{z}}$ contains \mathbf{p} and \mathbf{z} and is defined as the convex hull of a *path along edges of $c_{\mathbf{z}}$ from \mathbf{p} to \mathbf{z}* , as follows.

Let $\mathbf{p}_0 = \mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d = \mathbf{z}$ be a sequence of vertices of $c_{\mathbf{z}}$ such that $\mathbf{p}_k = \mathbf{p}_{k-1} \pm e_j$ where e_j is the vector $(0, \dots, 0, 1, 0, \dots, 0)$ with a 1 in the j -th place and 0's elsewhere. It follows that in the sequence of \mathbf{p}_k 's the i -th coordinate must change precisely once, from 1 to z_i , because we start out from $\mathbf{p} = (1, 1, \dots, 1)$ and $\mathbf{z} = (z_1, z_2, \dots, z_d)$ is a vertex of $2C^d$, so $z_i \in \{0, 2\}$ for each i .

The points \mathbf{p}_i are easily seen to be geometrically independent and thus they are the vertices of a d -dimensional simplex, namely their convex hull. It is also clear that the intersection of any two maximal simplices is the convex hull of their common vertices, so this is indeed a simplicial complex. (In the case of the unit d -cube, this triangulation is well known. See, for example, [12]).

Such a sequence of vertices defining a maximal simplex can be coded by a permutation in the hyperoctahedral group B_d . Namely, we define $\pi = a_1 a_2 \dots a_d$ by setting $a_i = k$ if $\mathbf{p}_i - \mathbf{p}_{i-1} = e_k$ and $a_i = -k$ if $\mathbf{p}_i - \mathbf{p}_{i-1} = -e_k$. For example, the sequence $(1, 1, 1), (0, 1, 1), (0, 1, 2), (0, 0, 2)$ of points in $c_{(0,0,2)}$ corresponds to the permutation $\bar{1}3\bar{2}$. Conversely, every $\pi \in B_d$ determines a unique path from \mathbf{p} to a vertex \mathbf{z} of $2C^d$ and hence a unique d -simplex, which we denote by σ_π , contained in $c_{\mathbf{z}}$. The number of distinct paths from \mathbf{p} to \mathbf{z} is $d!$, and the following lemma is now immediate.

Lemma 9 *Let $\{\pi_i \mid 1 \leq i \leq d!\}$ be the permutations labeling the maximal simplices in a cube $c_{\mathbf{z}}$. Then each integer k in $[d]$ appears with the same sign in every π_i . More precisely, the sign of $k \in [d]$ in such a permutation is $+$ or $-$ according as the k -th coordinate of \mathbf{z} is 2 or 0. Conversely, if each $k \in [d]$ appears with the same sign in two permutations π and τ , then σ_π and σ_τ belong to the same cube $c_{\mathbf{z}}$. \blacksquare*

For example, the paths in the cube $c_{(0,0,2)}$ are $(1, 1) \rightarrow (1, 2) \rightarrow (0, 2)$ and $(1, 1) \rightarrow (0, 1) \rightarrow (0, 2)$, corresponding to the permutations $2\bar{1}$ and $\bar{1}2$, respectively.

Proposition 10 *The collection $\{\sigma_\pi \mid \pi \in B_d\}$ covers $2C^d$. Any two of these simplices are isometric, in particular each has volume $1/d!$ and hence $\text{Nvol}(\sigma_\pi) = 1$ for each π .*

Proof: Given a cube $c_{\mathbf{z}}$, there is an orientation-preserving isometry which fixes $(1, 1, \dots, 1)$, takes \mathbf{z} to $\mathbf{0}$, and each vertex of $c_{\mathbf{z}}$ to a vertex of $c_{\mathbf{0}}$ (there are many such isometries, of course). Such an isometry sends $\sigma_\pi \subset c_{\mathbf{z}}$ to some $\sigma_\tau \subset c_{\mathbf{0}}$. Given $\sigma_\alpha \subset c_{\mathbf{0}}$, a suitable permutation of the coordinate axes in \mathbf{R}^d sends σ_τ to σ_α , so σ_τ and σ_α are isometric. If $\sigma_\tau \subset c_{\mathbf{0}}$ then all the letters of τ are negative, say $\tau = -a_1 - a_2 \cdots - a_d$. It follows from the definition of σ_τ that $\sigma_\tau = \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbf{R}^d \mid 0 \leq x_{a_1} \leq x_{a_2} \leq \cdots \leq x_{a_d} \leq 1\}$. Clearly, every point in $c_{\mathbf{0}}$ is contained in a subset defined by some τ . There are $d!$ maximal simplices $\sigma_\tau \subset c_{\mathbf{0}}$ and the intersection of any two distinct such has dimension at most $d - 1$, so $\text{vol}_d(\sigma_\tau) = 1/d!$. ■

Thus the collection $\{\sigma_\pi \mid \pi \in B_d\}$ triangulates $2C^d$. We denote this triangulation by $\widehat{2C^d}$. We can now give a succinct characterization of the permutations corresponding to the maximal simplices of $\widehat{2C^d}$ contained in \mathcal{P} . First a definition.

Definition 11 *Let G be a graph. The set of permissible permutations with respect to G is $\Pi(G) = \{\pi \in B_d \mid \sigma_\pi \subset \mathcal{P}(G)\}$. A permutation π is permissible w.r.t. G if $\pi \in \Pi(G)$.*

Theorem 12 *A permutation $\pi \in B_d$ is permissible w.r.t. G if and only if it satisfies the following condition:*

If (i, j) is an edge in G and $+i$ appears in π , then $-j$ must precede $+i$ in π .

Proof: Suppose that $\mathbf{p}_0 = \mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d = \mathbf{z}$ is a sequence of points defining a maximal simplex $\sigma_\pi \subset \mathcal{P}(G)$. If $+i$ appears in π and is not preceded by $-j$, then some of these points will have coordinates $x_i = 2$ and $x_j = 1$, so $x_i + x_j = 3$, which is not allowed if (i, j) is an edge in G . Conversely, a permutation satisfying the condition determines a maximal simplex σ_π which is the convex hull of points whose coordinates satisfy the defining inequalities of $\mathcal{P}(G)$. ■

Suppose K is a simplicial complex and \mathcal{P} a polytope (or any subset), both embedded in \mathbf{R}^n . By $K \cap \mathcal{P}$ we mean the subcomplex of K contained in \mathcal{P} , assuming that this is well defined (which will always be the case here).

For the remainder of this section, fix a graph G and let \mathcal{P} denote its extended 2-weak vertex-packing polytope. We define $\widehat{\mathcal{P}}$ to be the triangulation of \mathcal{P} induced by the triangulation $\widehat{2C^d}$ of $2C^d$. That is, $\widehat{\mathcal{P}} = \mathcal{P} \cap \widehat{2C^d}$. We need to show, of course, that this actually triangulates \mathcal{P} , i.e. that $\widehat{\mathcal{P}}$ is well defined.

Proposition 13 *Let σ_π be a maximal simplex in $c_{\mathbf{z}}$. If \mathcal{P} intersects the interior of σ_π , then $\sigma_\pi \subset \mathcal{P}$. Hence, $\widehat{\mathcal{P}} = \widehat{2C^d} \cap \mathcal{P}$ is a triangulation of \mathcal{P} .*

Proof: Let $\mathbf{p}_0, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_d$ be the sequence of points defining σ_π and suppose (i, j) is an edge of G . Let $\pi = a_1 a_2 \cdots a_d$ and let a_k be the first occurrence in π of either $\pm i$ or $\pm j$. If $a_k > 0$ (so a_k equals $+i$ or $+j$) then every one of the points \mathbf{p}_m satisfies $x_i + x_j \geq 2$ and if $a_k < 0$ then all of the \mathbf{p}_m satisfy $x_i + x_j \leq 2$. In either case, the convex hull of the \mathbf{p}_m 's, namely σ_π , lies entirely on one side of the hyperplane $H_{i,j}$ defined by $x_i + x_j = 2$. Hence, σ_π must lie entirely on one side of each of the hyperplanes defining \mathcal{P} , so either $\sigma_\pi \subset \mathcal{P}$ or else \mathcal{P} intersects σ_π only in the boundary of σ_π . ■

Remark 14 *By Proposition 10, the triangulation $\widehat{\mathcal{P}}$ of \mathcal{P} is primitive. Hence, $N\text{vol}(\mathcal{P}) = \#\Pi(G)$ and $h^*(\mathcal{P}, t) = h(\widehat{\mathcal{P}}, t)$.*

Since \mathcal{P} is triangulated by the simplicial complex $\widehat{\mathcal{P}}$, each vertex of \mathcal{P} must be a vertex of a simplex in $\widehat{\mathcal{P}}$, and thus have integral coordinates, since the vertices of the simplices σ_π are clearly integral. But we can do better than this. Namely, we can completely characterize all vertices of \mathcal{P} .

Theorem 15 *Let $\mathbf{v} = (v_1, v_2, \dots, v_d)$ be a point in \mathcal{P} with integral coordinates (so $v_i \in \{0, 1, 2\}$ for all i) and let $S = \{i \in [d] \mid v_i = 1\}$. Let G_S be the subgraph of G induced by S . Then \mathbf{v} is a vertex of \mathcal{P} iff $S = \emptyset$ or each connected component of G_S contains an odd cycle.*

Proof: Suppose $S \neq \emptyset$ and that some component of G_S contains no odd cycle. Call this component C and let T be its vertex set. Then C is bipartite and thus two-colorable, so we can color the vertices of C with the numbers 0 and 2 in such a way that no two adjacent vertices get the same color. Let $c(i)$ be the color of i . Define $\mathbf{x} = (x_1, x_2, \dots, x_d)$ and $\mathbf{y} = (y_1, y_2, \dots, y_d)$ by $x_i = y_i = v_i$ if $i \notin T$ and $x_i = c(i)$, $y_i = 2 - c(i)$ if $i \in T$. Then $\mathbf{x}, \mathbf{y} \in \mathcal{P}$ and $\mathbf{v} = \frac{1}{2}(\mathbf{x} + \mathbf{y})$, so \mathbf{v} is not a vertex of \mathcal{P} .

Conversely, suppose $S = \emptyset$ or that every component of G_S contains an odd cycle. If $S = \emptyset$ then \mathbf{v} is a vertex of $2C^d$ and thus of \mathcal{P} . Assume then that $S \neq \emptyset$ and that every component of G_S contains an odd cycle, so no component of G_S is two-colorable. Assume also that \mathbf{v} is *not* a vertex of \mathcal{P} . Then we must have a, b with $0 < a, b < 1$ and $a + b = 1$ and distinct points $\mathbf{x}, \mathbf{y} \in \mathcal{P}$ such that $a\mathbf{x} + b\mathbf{y} = \mathbf{v}$. It is easy to see that $x_i = y_i = v_i$ for each i such that $v_i \in \{0, 2\}$. Hence, since \mathbf{x} and \mathbf{y} must be different from \mathbf{v} , there must be an i such that $x_i, y_i \neq v_i = 1$. Hence, $x_i > 1$ or $y_i > 1$, since otherwise $ax_i + by_i < 1 = v_i$. Assume, without loss of generality, that $x_i > 1$. Now, i belongs to some component of G_S and that

component contains an odd cycle, so there is a path beginning at i and traveling through the entire cycle. Assume, w.l.o.g., that the vertices in this path are labeled (in order) by $i, i+1, \dots, i+m$. Since $x_i > 1$, we have $x_{i+1} < 1$, as the vertices x_i and x_{i+1} are adjacent, so $x_i + x_{i+1} \leq 2$. But then $y_{i+1} > 1$ since otherwise $ax_{i+1} + by_{i+1} < 1$. This implies that $y_{i+2} < 1$, so $x_{i+2} > 1$ and the argument can be repeated to show that $x_{i+r} > 1$ iff r is even. But this amounts to a two-coloring of the path, including the odd cycle, which is a contradiction, so \mathbf{v} is indeed a vertex of \mathcal{P} . \blacksquare

Our goal is to find a shelling of $\widehat{\mathcal{P}}$. To that end, we order the permutations in B_d *lexicographically*, i.e. a permutation $\pi = a_1a_2 \cdots a_d$ precedes $\tau = b_1b_2 \cdots b_d$ if $a_i < b_i$ for the first i at which π and τ differ. Abusing notation, we use $<$ to denote this ordering of the elements of B_d . For example, $\bar{2}3\bar{1} < 3\bar{2}\bar{1}$ and $23\bar{1} < 231$.

We will show that the ordering of maximal simplices in $\widehat{\mathcal{P}}$ induced by the lexicographic ordering of their corresponding permutations is a shelling of $\widehat{\mathcal{P}}$. We say that two d -simplices *intersect maximally* if they have a $(d-1)$ -face in common.

Lemma 16 *Suppose $\sigma_\pi \subset c_{\mathbf{z}} \cap \widehat{\mathcal{P}}$, where $\pi = a_1a_2 \cdots a_d$, and suppose that i is a descent in π . If i is an internal descent in π , i.e. $a_i > a_{i+1}$, for some $i \leq d-1$, then $\sigma_{\pi'} \subset c_{\mathbf{z}} \cap \widehat{\mathcal{P}}$, where $\pi' = a_1a_2 \cdots a_{i+1}a_i \cdots a_d$. If $i = d$, i.e. $a_d > 0$, then $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$, where $\pi' = a_1a_2 \cdots -a_d$. In either case, $\pi' < \pi$ and σ_π and $\sigma_{\pi'}$ intersect maximally. Moreover, if two maximal simplices σ_π and $\sigma_{\pi'}$ in $\widehat{\mathcal{P}}$ intersect maximally, then π and π' either differ only by a single transposition or only by the sign of their last letter.*

Proof: If $i < d$, so $\pi' = a_1 \cdots a_{i+1}a_i \cdots a_d$, then, by Lemma 9, $\sigma_{\pi'} \subset c_{\mathbf{z}}$. If both a_i and a_{i+1} are negative, then there are no restrictions on the order in which they can appear in a permissible permutation. If both a_i and a_{i+1} are positive, then, since π is permissible, $x_{a_i} + x_{a_{i+1}} \leq 2$ can't be a defining inequality of \mathcal{P} , so again there are no restrictions on the order in which a_i and a_{i+1} can appear in a permissible permutation. Since i is a descent, the only remaining possibility is $a_i > 0$ and $a_{i+1} < 0$. That implies that π' must be permissible if π is.

If $i = d$, so $a_d > 0$, then clearly $\pi' = a_1a_2 \cdots -a_d$ is permissible if π is.

Now, regardless of whether $i < d$ or $i = d$, suppose that $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_d$ is the sequence of points defining σ_π . Then the sequence defining $\sigma_{\pi'}$ is $\mathbf{q}_0, \mathbf{q}_1, \dots, \mathbf{q}_d$, where $\mathbf{q}_j = \mathbf{p}_j$, except for $j = i$. Hence, σ_π and $\sigma_{\pi'}$ intersect in the $(d-1)$ -face defined by the convex hull of the points $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, \dots, \mathbf{p}_d$ (where possibly $i = d$).

If two maximal simplices intersect maximally, then their defining sequences of points differ by just one point. It is then straightforward to check that their corresponding permutations can differ only as described. \blacksquare

Theorem 17 *Order the maximal simplices in $\widehat{\mathcal{P}}$ so that σ_τ precedes σ_π if $\tau < \pi$. This ordering is a shelling of $\widehat{\mathcal{P}}$.*

Proof: Let σ_π be a maximal simplex in $\widehat{\mathcal{P}}$. If π is the (lexicographically) first permutation in $\Pi(G)$ then there is nothing to prove. Otherwise, we must show that $\sigma_\pi \cap \bigcup_{\tau < \pi} \sigma_\tau$ is a nonempty union of $(d-1)$ -faces of σ_π . It suffices to show that if σ_π intersects a maximal simplex $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_τ precedes σ_π , then $\sigma_\pi \cap \sigma_\tau$ is contained in some $(d-1)$ -face f of σ_π such that $f = \sigma_\pi \cap \sigma_{\pi'}$ for some $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$ with $\sigma_{\pi'}$ preceding σ_π .

Suppose $\sigma_\pi, \sigma_\tau \subset \widehat{\mathcal{P}}$ and that σ_τ precedes σ_π , so $\tau < \pi$. Let i be the first place where π and τ differ. If $i = d$ then σ_π and σ_τ intersect maximally, by Lemma 16, and we are done. Assume therefore that $i < d$. Let $\pi = a_1 a_2 \cdots a_d$ and $\tau = a_1 a_2 \cdots a_{i-1} b_i \cdots b_d$. Let k be the first descent in π after $i-1$. Such a k must exist, because otherwise we would have $a_i < a_{i+1} < \cdots < a_d < 0$ so that π was the first permutation in B_d beginning with $a_1 a_2 \cdots a_{i-1}$, contradicting $\tau < \pi$.

Let $\mathbf{p}_0, \mathbf{p}_1, \dots, \mathbf{p}_d$ be the sequence of points defining σ_π . We claim that $\mathbf{p}_k \notin \sigma_\tau$. If σ_τ did contain \mathbf{p}_k then we would have $\{a_1, a_2, \dots, a_k\} = \{b_1, b_2, \dots, b_k\}$, in particular $\{a_i, a_{i+1}, \dots, a_k\} = \{b_i, b_{i+1}, \dots, b_k\}$, so $k > i$. But then, since k was the first descent in π after $i-1$, so that $a_i < a_{i+1} < \cdots < a_k$, we must have $b_i > a_i$, contradicting the assumption $\tau < \pi$, so $\mathbf{p}_k \notin \sigma_\tau$.

If $k < d$, let $\pi' = a_1 \cdots a_{k+1} a_k \cdots a_d$. Then $\pi' < \pi$ and $\sigma_\pi \cap \sigma_{\pi'}$ is the convex hull of $\mathbf{p}_0, \widehat{\mathbf{p}}_1, \dots, \mathbf{p}_{k-1}, \mathbf{p}_{k+1}, \dots, \mathbf{p}_d$. By Lemma 16, σ_π and $\sigma_{\pi'}$ intersect maximally and $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$. Moreover, since $\mathbf{p}_k \notin \sigma_\tau$, $\sigma_\pi \cap \sigma_\tau \subset \sigma_\pi \cap \sigma_{\pi'}$, as desired.

If $k = d$, then $a_d > 0$. Let $\pi' = a_1 a_2 \cdots - a_d$. Then $\pi' < \pi$ and, by Lemma 16, σ_π and $\sigma_{\pi'}$ intersect maximally and $\sigma_{\pi'} \subset \widehat{\mathcal{P}}$. Since $\mathbf{p}_d \notin \sigma_\tau$, we have $\sigma_\pi \cap \sigma_\tau \subset \sigma_\pi \cap \sigma_{\pi'}$, as desired. ■

Theorem 18 *The h -polynomial of $\widehat{\mathcal{P}}$ equals the descent polynomial of the set of permissible permutations with respect to G . That is, $h(\widehat{\mathcal{P}}, t) = D(\Pi(G), t)$.*

Proof: By Theorem 3, we need to show that for each descent in $\pi \in \Pi(G)$ there is a unique maximal simplex $\sigma_\tau \in \widehat{\mathcal{P}}$ such that σ_π and σ_τ intersect maximally and such that $\tau < \pi$. First suppose that i is an internal descent in π , i.e. $1 \leq i \leq d-1$ and let $\pi = a_1 a_2 \cdots a_d$, so $a_i > a_{i+1}$. By Lemma 16, two maximal simplices σ_π and σ_τ in the same cube $c_{\mathbf{z}}$ intersect maximally if and only if π and τ differ by a single transposition. Let $\tau = a_1 a_2 \cdots a_{i+1} a_i \cdots a_d$. Then τ precedes π , $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_π and σ_τ intersect maximally. Conversely, if σ_π and σ_τ in $c_{\mathbf{z}}$ intersect maximally then they differ by a single transposition and if $\tau < \pi$ then π has a descent between the two letters involved in that transposition.

The only other maximal simplices σ_π can intersect maximally are those belonging to other cubes than $c_{\mathbf{z}}$. By Lemma 16, if σ_τ is such a simplex and $\pi = a_1 a_2 \cdots a_d$,

then $\tau = a_1 a_2 \cdots - a_d$, so, for τ to precede π , we must have $a_d > 0$, i.e. d is a descent in π . Conversely, if d is a descent in π then $a_d > 0$, so if $\tau = a_1 a_2 \cdots - a_d$ then $\tau < \pi$, $\sigma_\tau \subset \widehat{\mathcal{P}}$ and σ_π and σ_τ intersect maximally. ■

4 Applications

As we shall be much concerned with that part of the boundary of \mathcal{P} which is contained in $\partial(2C^d)$, we make the following definition. (Unless otherwise specified, \mathcal{P} and $\widehat{\mathcal{P}}$ have the same meaning as before).

Definition 19 $\mathcal{B} := \mathcal{P} \cap \partial(2C^d)$ and $\widehat{\mathcal{B}} := \widehat{\mathcal{P}} \cap \partial(2C^d)$. A facet of \mathcal{B} is a facet of \mathcal{P} contained in \mathcal{B} .

Theorem 20 $h(\widehat{\mathcal{P}}, t) = h(\widehat{\mathcal{B}}, t)$. Hence, $h^*(\mathcal{P}, t) = h^*(\widehat{\mathcal{B}}, t) = h^*(\mathcal{B}, t)$.

Proof: $\widehat{\mathcal{P}}$ is a cone over $\widehat{\mathcal{B}}$, which yields the equality of h -polynomials, by Theorem 4. The latter two equalities are then implied by Theorem 1 and the fact that $\widehat{\mathcal{P}}$ (and hence $\widehat{\mathcal{B}}$) is primitively triangulated. ■

Corollary 21 $\text{vol}_d(\mathcal{P}) = \text{vol}_{d-1}(\mathcal{B})$. Equivalently, $\text{Nvol}(\mathcal{P}) = \text{Nvol}(\mathcal{B})$.

Proof: $\text{Nvol}(\mathcal{P})$ equals the number of maximal simplices in $\widehat{\mathcal{P}}$, which in turn equals the number of maximal simplices in $\widehat{\mathcal{B}}$, since $\widehat{\mathcal{P}}$ is a cone over $\widehat{\mathcal{B}}$. ■

Corollary 21 yields a recursive formula for the volume of \mathcal{P} , because each facet of \mathcal{B} is an extended 2-weak vertex-packing polytope. More precisely, the facet of \mathcal{B} obtained by setting $x_i = 0$ (which we denote $\mathcal{B}_{x_i=0}$) is the extended 2-weak vertex-packing polytope of the graph obtained by removing x_i from G . If x_i is an isolated vertex of G then $\mathcal{B}_{x_i=2}$ is isometric to $\mathcal{B}_{x_i=0}$ (since then $\mathcal{P} = \mathcal{P}_{x_i=0} \times [0, 2]$), but otherwise $\mathcal{B}_{x_i=2}$ has dimension less than $d-1$ and thus $\text{vol}_{d-1}(\mathcal{B}_{x_i=2}) = 0$.

If $d = a_1 + a_2 + \cdots + a_k$, let $\binom{d}{a_1, a_2, \dots, a_k}$ be the multinomial coefficient defined by $\binom{d}{a_1, a_2, \dots, a_k} = \frac{d!}{a_1! a_2! \cdots a_k!}$.

Abusing notation, we will, in what follows, write $\text{Nvol}(G)$ instead of $\text{Nvol}(\mathcal{P}(G))$, where G is a graph and $\mathcal{P}(G)$ its extended 2-weak vertex-packing polytope.

Theorem 22 Let C_1, C_2, \dots, C_k be the connected components of G , with $a_i = \#C_i$ for each i , and $d = \#G$. Then $\text{Nvol}(G) = \binom{d}{a_1, a_2, \dots, a_k} \prod_{i=1}^k \text{Nvol}(C_i)$. In particular, if G has an isolated vertex x and G_x is the graph obtained by removing x from G , then $\text{Nvol}(G) = 2 \cdot d \cdot \text{Nvol}(G_x)$.

Proof: For the sake of simplicity, assume that G has only two connected components, C_1 and C_2 , with vertices x_1, x_2, \dots, x_m and $x_{m+1}, x_{m+2}, \dots, x_d$, respectively. A permutation π is permissible with respect to G if and only if the subword of π corresponding to each component is permissible with respect to that component. Hence, given permutations $\pi_1 \in \Pi(C_1)$ and $\pi_2 \in \Pi(C_2)$ ¹, every “shuffling” of π_1 and π_2 is a permissible permutation with respect to G . That is, we can select any m positions out of $1, 2, \dots, d$ and place the letters of π_1 , preserving their order, in these positions and the letters of π_2 , preserving their order, in the remaining $d - m$ positions. This means that $\#\Pi(G) = \binom{d}{m} \cdot \#\Pi(C_1) \cdot \#\Pi(C_2)$ and the argument obviously generalizes to any number of components with $\binom{d}{m}$ replaced by the multinomial coefficient $\binom{d}{a_1, a_2, \dots, a_k}$. ■

In lieu of the preceding theorem, we can now restrict our attention to connected graphs, in particular to graphs without isolated vertices.

Theorem 23 *Let G be a graph without isolated vertices, $\#G = d$, and let G_x denote the graph obtained by removing the vertex x from G . Then*

$$\text{Nvol}(G) = \sum_{x \in G} \text{Nvol}(G_x).$$

Proof: As pointed out above, if G has no isolated vertices then all the facets of $\mathcal{B} = \mathcal{P} \cap 2C^d$ are of the form $\mathcal{B}_{x_i=0}$ and so $\mathcal{B} = \bigcup_{x \in G} \mathcal{B}_{x=0}$. Clearly $\text{Nvol}(\mathcal{B}) = \sum_{x \in G} \text{Nvol}(\mathcal{B}_{x=0})$, which, together with Corollary 21, yields the theorem. ■

We will now give a few examples of how one can use the recurrence afforded by Theorems 22 and 23 to compute the volume of extended 2-weak vertex-packing polytopes. To get the recursion off the ground, observe that if G consists of a single vertex, then $\mathcal{P}(G) = [0, 2] \subset \mathbf{R}$, so $\text{Nvol}(G) = 2$.

Example 24 $\text{Nvol}(\bullet \bullet \bullet) = 2 \cdot \text{Nvol}(\bullet \bullet) + \text{Nvol}(\bullet \bullet) = 2 \cdot 2 \cdot \text{Nvol}(\bullet) + \binom{2}{1} \cdot (\text{Nvol}(\bullet))^2 = 8 + 2 \cdot 2^2 = 16$.

Example 25 $\text{Nvol}(\bullet \bullet \bullet) = 2 \cdot \text{Nvol}(\bullet \bullet \bullet) + 2 \cdot \text{Nvol}(\bullet \bullet) = 2 \cdot 16 + 6 \cdot \text{Nvol}(\bullet \bullet) = 32 + 6 \cdot 4 = 56$.

Example 26 If G_d is the graph $\bullet \bullet \cdots \bullet \bullet$ with d vertices, so G_d is the comparability graph of the *fence poset* on elements x_1, \dots, x_d (with relations $x_1 < x_2 > x_3 < x_4 > \cdots$), then it is well known (see [12]) that the volume of

¹We are being a little sloppy with the notation here.

the 2-weak vertex-packing polytope $\mathcal{W}(G_d)$ of G_d is given by the d -th coefficient of the Taylor series of $\tan x + \sec x$. We can compute the corresponding result for $\mathcal{P}(G_d)$ in the following way. Using Theorems 22 and 23, we get this recurrence for $A_d = \text{Nvol}(G_d)$: For $d \geq 1$, $A_{d+1} = \sum_{i=0}^d \binom{d}{i} A_i \cdot A_{d-i}$ and hence $\sum_{d \geq 1} A_{d+1} \frac{x^d}{d!} = \left(\sum_{d \geq 0} A_d \frac{x^d}{d!} \right)^2$. Setting $F(x) = \sum_{d \geq 0} A_d \frac{x^d}{d!}$ yields $F'(x) - 1 = (F(x))^2$, which, together with $F(0) = 1$, has the unique solution $F(x) = \tan(2x) + \sec(2x)$.

We give one example of how one can compute $h(\widehat{\mathcal{P}}(G), t)$ from $\Pi(G)$.

Example 27 Let K_d be the complete graph on d vertices. Then $\Pi(K_d)$ consists of all permutations $\pi = a_1 a_2 \cdots a_d \in B_d$ such that $a_i < 0$ for all i except, perhaps, for $i = d$. Let Π_- be the set of permutations in $\Pi(G)$ all of whose letters are negative, and let Π_+ be the set of those permutations in $\Pi(G)$ whose last letter is positive (but all others negative). It is easy to see that $D(\Pi_-, t)$ equals the *usual* descent polynomial of the symmetric group S_d , consisting of all permutations of the letters $\{1, 2, \dots, d\}$ (no signs involved and never a descent at d). This polynomial is well known and called the d -th *Eulerian polynomial* (see, e.g. [11]) and often denoted by $A_d(t)$.² In fact, $A_d(t)$ equals the h -polynomial of our triangulation of c_0 . As for Π_+ , we see that there is always a descent at d , never a descent at $d - 1$, and the first $d - 1$ letters in a $\pi \in \Pi_+$ behave just like a permutation in S_{d-1} . Hence each permutation in Π_+ corresponds to a permutation in S_{d-1} , but has an extra descent, namely the one at d . There are d possible choices for the last letter of $\pi \in \Pi_+$ and the descent polynomial of Π_+ is thus equal to $d \cdot t A_{d-1}(t)$. Hence, $h(\widehat{\mathcal{P}}(K_d), t) = A_d(t) + d \cdot t A_{d-1}(t)$.

Note also that $\text{Nvol}(K_d) = \#\Pi(K_d) = 2 \cdot d!$ This volume was previously computed by Lee and Morris [5].

Finally, we record, without proof, two theorems which give a recursive algorithm for computing $h(\widehat{\mathcal{P}}, t)$. For proofs, see Theorems 38 and 35 in [15], which also contains many examples and details not found here. Recall that $[d] = \{1, 2, \dots, d\}$.

Definition 28 Let G be a graph with vertex set $[d]$ and \mathcal{P} defined as usual in terms of G . Let $S \subset [d]$. Then $\mathcal{P}_S := \{\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathcal{P} \mid x_i = 0 \text{ if } i \in S\}$.

That is, \mathcal{P}_S is isomorphic to $\mathcal{P}(G_S)$, where G_S is the subgraph of G induced by $[d] \setminus S$. We also define $\widehat{\mathcal{P}}_S$ similarly, i.e. $\widehat{\mathcal{P}}_S := \widehat{\mathcal{P}} \cap \mathcal{P}_S$.

Theorem 29 Let G be a graph with vertex set $[d]$ and no isolated vertices. Then $h(\widehat{\mathcal{P}}, t) = \sum_S h(\widehat{\mathcal{P}}_S, t)(t-1)^{\#S-1}$, where S ranges over all nonempty subsets of $[d]$. ■

²Usually, though, $A_d(t)$ is defined so that $D(\Pi_-, t) = t^{-1} A_d(t)$.

Theorem 30 *Let G be a graph with $d - 1$ vertices and denote by G' the graph obtained by adding to G an isolated vertex. Suppose $h(\widehat{\mathcal{P}}(G), t) = a_0 + a_1t + \cdots + a_{d-1}t^{d-1}$. Then $h(\widehat{\mathcal{P}}(G'), t) = b_0 + b_1t + \cdots + b_{d-1}t^{d-1}$, where $b_k = (2k + 1)a_k + (2d - 2k + 1)a_{k-1}$. ■*

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