The Volume of Relaxed Boolean-Quadric and Cut Polytopes

Chun-Wa Ko*
30 Newport Parkway, #809
Jersey City, New Jersey 07310 USA

Jon Lee**
Department of Mathematics
University of Kentucky
Lexington, Kentucky 40506-0027 USA

Einar Steingrímsson***
Matematiska Institutionen CTH & GU
412 96 Göteborg, SWEDEN

Abstract. For $n \geq 2$, the *boolean quadric polytope* $\mathcal{P}_n$ is the convex hull in $d := \binom{n+1}{2}$ dimensions of the binary solutions of $x_i x_j = y_{ij}$, for all $i < j$ in $N := \{1, 2, ..., n\}$. The polytope is naturally modeled by a somewhat larger polytope; namely, $\mathcal{Q}_n$ the solution set of $y_{ij} \leq x_i$, $y_{ij} \leq x_j$, $x_i + x_j \leq 1 + y_{ij}$, $y_{ij} \geq 0$, for all i, j in N. In a first step toward seeing how well $\mathcal{Q}_n$ approximates $\mathcal{P}_n$, we establish that the d-dimensional volume of $\mathcal{Q}_n$ is $2^{2n-d} n!/(2n)!$. Using a well-known connection between $\mathcal{P}_n$ and the "cut polytope" of a complete graph on $n + 1$ vertices, we also establish the volume of a relaxation of this cut polytope.

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1. Introduction. A natural approach to the unconstrained, quadratic-objective, binary program in n (≥ 2) variables

$$\max\left\{\sum_{i\in N} c_i x_i + \sum_{i<j\in N} d_{ij} x_i x_j \ : \ x_i \in \{0,1\} \ \forall \, i \in N\right\}, \tag{1}$$

where $N := \{1, 2, ..., n\}$, is to model the problem as a linearly constrained, linear-objective, binary program, through the use of $\binom{n}{2}$ auxiliary binary variables y_{ij} which model the quadratic terms $x_i x_j$. We obtain the equivalent program

$$\max \sum_{i\in N} c_i x_i + \sum_{i<j\in N} d_{ij} y_{ij}, \tag{2}$$

$$\text{subject to} \quad y_{ij} \leq x_i \quad \forall \, i < j \in N, \tag{3}$$

$$y_{ij} \leq x_j \quad \forall \, i < j \in N, \tag{4}$$

$$y_{ij} \geq 0 \quad \forall \, i < j \in N, \tag{5}$$

$$x_i + x_j \leq 1 + y_{ij} \quad \forall \, i < j \in N, \tag{6}$$

$$x_i \in \{0,1\} \quad \forall \, i \in N, \tag{7}$$

$$y_{ij} \in \{0,1\} \quad \forall \, i < j \in N. \tag{8}$$

The *boolean quadric polytope* $\mathcal{P}_n$ is the convex hull (in real $d := \binom{n+1}{2}$ space) of the set of solutions of (3-8). As the problem of solving (2-8) is NP-Hard, it is natural to consider branch-and-cut methods based on (2-6). The relaxed feasible region (3-6) is denoted by $\mathcal{Q}_n$. Padberg (1989) has made a detailed study of $\mathcal{P}_n$ and $\mathcal{Q}_n$ (also see Deza, Laurent and Poljak (1993), and Pitowsky (1991)).

It is natural to consider how good of an approximation $\mathcal{Q}_n$ is to $\mathcal{P}_n$. The Chvátal-Gomory rank (see Schrijver (1986) and Chvátal, Cook and Hartman (1989)) of $\mathcal{P}_n$ with respect to $\mathcal{Q}_n$ increases with n, so in a certain combinatorial sense, $\mathcal{Q}_n$ is a poor approximation of $\mathcal{P}_n$. In a different combinatorial sense $\mathcal{Q}_n$ is quite close to $\mathcal{P}_n$; that is, the 1-skeleton of $\mathcal{P}_n$ is a subset of the 1-skeleton of $\mathcal{Q}_n$ (the so-called *Trubin Property*) (see Padberg). Another method has been proposed to study the closeness of pairs of nested polytopes, based on the volumes of the polytopes. Lee and Morris (1994) have suggested the distance function

$$\rho_d(\mathcal{Q}_n, \mathcal{P}_n) := \left(\frac{\mathrm{vol}_d(\mathcal{Q}_n)}{\mathrm{vol}_d(B^d)}\right)^{1/d} - \left(\frac{\mathrm{vol}_d(\mathcal{P}_n)}{\mathrm{vol}_d(B^d)}\right)^{1/d},$$

where B^d is the d-dimensional Euclidean ball, and vol_d denotes d-dimensional Lebesgue measure. For polytope pairs contained in $[0,1]^d$, ρ_d is at most $O(\sqrt{d})$. In some interesting cases of sets of polytope pairs, ρ_d may increase more slowly than this upper bound, in other situations the bound is sharp (see Lee and Morris). In Section 2, as a step toward determining the asymptotic behavior of $\rho_d(\mathcal{Q}_n, \mathcal{P}_n)$, we calculate $\mathrm{vol}_d(\mathcal{Q}_n)$.

There is a well-known connection between P_n and the "cut polytope" of a complete graph on $n+1$ vertices. In Section 3, we determine the volume of a natural relaxation of this cut polytope.

2. The Volume of a Relaxed Boolean-Quadric Polytope. Let $\mathcal{Q}'_n := 2\mathcal{Q}_n$, that is, the polytope $\mathcal{Q}_n$ magnified by a factor of 2. Clearly, $\mathrm{vol}_d(\mathcal{Q}'_n) = 2^d \mathrm{vol}_d(\mathcal{Q}_n)$. Padberg demonstrated that $\mathcal{Q}'_n$ is a lattice polytope (i.e., its extreme points are lattice points). For simplicity, we work with $\mathcal{Q}'_n$, which is defined by the inequalities

$$y_{ij} \leq x_i \quad \forall\, i < j \in N, \tag{9}$$

$$y_{ij} \leq x_j \quad \forall\, i < j \in N, \tag{10}$$

$$y_{ij} \geq 0 \quad \forall\, i < j \in N, \tag{11}$$

$$x_i + x_j \leq 2 + y_{ij} \quad \forall\, i < j \in N. \tag{12}$$

Our first step in calculating $\mathrm{vol}_d(\mathcal{Q}'_n)$ is to reduce the problem to that of calculating the volume of a subset of $\mathcal{Q}'_n$. Points in Euclidean d-space will be denoted by $(x, y) = (x_1, x_2, \ldots, x_n, y_{12}, y_{13}, \ldots, y_{n-1,n})$. For $a \in \{0,1\}^n$, let

$$C_a := \{(x, y) \in \mathcal{Q}'_n \;:\; a \leq x \leq a + \mathbf{1}\},$$

where $\mathbf{1}$ is the n-vector $(1, 1, ..., 1)$. Clearly, $\mathcal{Q}'_n$ is the union of all such polytopes C_a. Furthermore, $\mathrm{vol}_d(C_a \cap C_b) = 0$ for $a \neq b$, so $\mathrm{vol}_d(\mathcal{Q}'_n) = \sum_a \mathrm{vol}_d(C_a)$.

Proposition 1. $\mathrm{vol}_d(C_a) = \mathrm{vol}_d(C_{\mathbf{0}})$, *for all* $a \in \{0,1\}^n$.

Proof: It suffices to demonstrate that if binary n-vectors a and b differ in precisely one coordinate, then $\mathrm{vol}_d(C_a) = \mathrm{vol}_d(C_b)$. Suppose, without loss of generality, that $a_j = b_j$ for $j \neq i$, $a_i = 0$, and $b_i = 1$. We define a map $\Phi_i : C_a \mapsto C_b$ as follows: Φ_i is a composition of coordinate maps $\{\phi_i, \phi_j, \phi_{ki}, \phi_{ij}, \phi_{kj} \;:\; 1 \leq k < i < j \leq n\}$, where $\phi_i(x_i) := 2 - x_i$, $\phi_j(x_j) := x_j$ for $j \neq i$, $\phi_{kj}(y_{kj}) := y_{kj}$, $\phi_{ij}(y_{ij}) := x_j - y_{ij}$, and $\phi_{ki}(y_{ki}) := x_k - y_{ki}$. To see that the range of Φ_i is contained in C_b, we only need to consider ϕ_{ij}; the analysis for ϕ_{ki} is similar. Clearly,

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j = \phi_j(x_j),$$

and

$$\phi_{ij}(y_{ij}) = x_j - y_{ij} \leq x_j - x_i - x_j + 2 = 2 - x_i = \phi_i(x_i).$$

Also,

$$\phi_i(x_i) + \phi_j(x_j) = 2 - x_i + x_j \leq 2 - y_{ij} + x_j = 2 + \phi_{ij}(y_{ij}).$$

Thus, we have shown that Φ_i is, indeed, a map from C_a into C_b.[1] It is trivial to check that Φ_i is an involution. Consequently, Φ_i is bijective and unimodular, and thus measure preserving, so $\mathrm{vol}_d(C_a) = \mathrm{vol}_d(C_b)$. Now, given an *arbitrary* binary n-vector a, the composition of the maps in $\{\Phi_i \ : \ a_i = 1\}$ gives a measure preserving bijection from $C_{\mathbf{0}}$ to C_a, so $\mathrm{vol}_d(C_a) = \mathrm{vol}_d(C_{\mathbf{0}})$. ∎

Corollary 2. $\mathrm{vol}_d(\mathcal{Q}'_n) = 2^n \mathrm{vol}_d(C_{\mathbf{0}})$. ∎

Let $(S_n, \prec)$ denote the poset (partially ordered set) on $S_n := \{x_i \ : \ 1 \leq i \leq n\} \cup \{y_{ij} \ : \ 1 \leq i < j \leq n\}$ having $y_{ij} \prec x_i$ and $y_{ij} \prec x_j$. Let $e(S_n, \prec)$ denote the number of *(linear) extensions* of $(S_n, \prec)$, i.e., the number of order-preserving bijections from S_n to $D := \{1, 2, ..., d\}$, where the order on D is the usual one.

Proposition 3. $\mathrm{vol}_d(C_{\mathbf{0}}) = e(S_n, \prec)/d!$
Proof: By definition,

$$C_{\mathbf{0}} = \{(x, y) \in \mathcal{Q}'_n \ : \ 0 \leq x_i \leq 1 \ (1 \leq i \leq n)\} \ .$$

It follows that $C_{\mathbf{0}}$ is defined by the inequalities $(9 - 11)$ and

$$x_i \leq 1, \quad 1 \leq i \leq n \ , \tag{13}$$

with (12) rendered vacuous. $C_{\mathbf{0}}$ is the *order polytope* (see Stanley (1986)) of the poset $(S_n, \prec)$. The result follows by Corollary 4.2 of Stanley. ∎

Theorem 4. $e(S_n, \prec) = n!d!2^n/(2n)!$.
Proof: We regard extensions of $(S_n, \prec)$ as permutations of the set S_n. That is, given a bijection $\pi : S_n \to D$, we represent π by the permutation $\pi^{-1}(d)\pi^{-1}(d-1)\cdots\pi^{-1}(1)$. Define an *ordered extension* of $(S_n, \prec)$ to be an extension of $(S_n, \prec)$ such that x_i appears to the left of x_k, for $1 \leq i < k \leq n$. That is, we regard an ordered extension of $(S_n, \prec)$ as

[1] The map Φ_i is called a "switching" and is a standard tool in the analysis of the "cut polytope"(see Deza and Laurent (1992) and Pitowsky (1991)).

a permutation of S_n in which x_i appears to the left of x_k, and y_{ik} appears to the right of both x_i and x_k, for $1 \le i < k \le n$. Clearly the number of extensions equals $n!$ times the number of ordered extensions.

Next, we proceed to count the number of ordered extensions of $(S_n, \prec)$. Suppose that $\{y_{il} \ : k+1 \le l \le n\}$ have already been positioned, for some fixed $i < k+1$. We see, now, how to place $\{y_{lk} \ : 1 \le l \le k-1\}$. The element y_{1k} should be placed to the right of x_k. As there are already $f_k := n-k+1+\binom{n}{2}-\binom{k}{2}$ elements of S_n placed after x_k, there are f_k possible positions for y_{1k}. Then, there are f_k+1 possible positions for y_{2k}, up through f_k+k-1 possible positions for $y_{k-1,k}$. In total, the number of ordered extensions is equal to

$$\begin{aligned}\prod_{k=2}^{n}\prod_{i=0}^{k-2}(f_k+i) &= \prod_{k=2}^{n}\frac{(f_k+k-2)!}{(f_k-1)!}\\ &= \prod_{k=2}^{n}\frac{\left(\binom{n+1}{2}-\binom{k}{2}-1\right)!}{\left(\binom{n+1}{2}-\binom{k+1}{2}\right)!}\\ &= \frac{2^n\cdot d!}{(2n)!}\end{aligned}$$

Hence, we get that $e(S_n, \prec) = \dfrac{n!\cdot d!\cdot 2^n}{(2n)!}$. ∎

Corollary 2, Proposition 3, and Theorem 4 now yield

Theorem 5. $\mathrm{vol}_d(\mathcal{Q}_n) = 2^{2n-d}n!/(2n)!$. ∎

We note that by Stirling's formula, $\mathrm{vol}_d(\mathcal{Q}'_n) = 2^{-1/2}(e/n)^n(1+o(1))$. Hence, for example, if it could be shown that $\mathrm{vol}_d(2\mathcal{P}_n) = 2^{-1/2}(e/n)^d(1+o(1))$, then we could conclude that $\rho_d(\mathcal{Q}_n, \mathcal{P}_n)$ behaves like $\sqrt{d}$.

3. The Volume of a Relaxed Cut-Polytope of a Complete Graph . Let G be a simple undirected graph with vertex set $V(G) := \{0, 1, 2, ..., n\} = N \cup \{0\}$ and edge set $E(G)$. A *cut* of G is any set of edges that crosses a nontrivial partition of $V(G)$; that is, $F \subset E(G)$ is a cut if F is the set of edges with exactly one endpoint in some nonempty proper subset W of $V(G)$. Associated with every cut of G is its incidence vector $z \in \{0,1\}^{E(G)}$. Let the *cut polytope* of G be the convex hull of the incidence vectors of cuts of G. For the complete graph K_{n+1}, we denote the cut polytope by $\mathcal{C}_{n+1}$. We immediately notice that $\mathcal{P}_n$ and $\mathcal{C}_{n+1}$ both have dimension $d = \binom{n+1}{2}$. As has been observed by many authors (see Deza and Laurent), there is a linear bijective transformation τ from $\mathcal{P}_n$ to

$\mathcal{C}_{n+1}$; namely,

$$x_i := z_{0i} \qquad \forall\, i \in N,$$
$$y_{ij} := \frac{1}{2}(z_{0i} + z_{0j} - z_{ij}) \qquad \forall\, i < j \in N.$$

We may apply this same transformation to the relaxed Boolean-quadric polytope and define $\mathcal{D}_{n+1} := \tau(\mathcal{Q}_n)$, a natural relaxation of C_{n+1}. The polytope $\mathcal{D}_{n+1}$ is the solution set of

$$\begin{aligned} z_{0i} - z_{0j} - z_{ij} &\leq 0 \qquad \forall\, i < j \in N, \\ -z_{0i} + z_{0j} - z_{ij} &\leq 0 \qquad \forall\, i < j \in N, \\ -z_{0i} - z_{0j} + z_{ij} &\leq 0 \qquad \forall\, i < j \in N, \\ z_{0i} + z_{0j} + z_{ij} &\leq 2 \qquad \forall\, i < j \in N. \end{aligned}$$

Let $x := \{x_1, x_2, \ldots, x_n\}^T$, $z^0 := (z_{01}, z_{02}, \ldots, z_{0n})^T$, $z := \binom{z^0}{z^N}$, and define y and z^N so that y_{ij} occupies the same position in y as z_{ij} does in z^N, $i < j \in N$. In matrix terms, we can view the transformation τ as

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} I & 0 \\ \frac{1}{2}A^T & -\frac{1}{2}I \end{pmatrix} \begin{pmatrix} z^0 \\ z^N \end{pmatrix},$$

where A is a vertex-edge incidence matrix of K_n on vertex set N. The absolute value of the determinant of the transformation matrix is 2^{n-d}, so we can conclude the following result.

Theorem 6. $\mathrm{vol}_d(\mathcal{D}_{n+1}) = 2^n n!/(2n)!$. ∎

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