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Proceedings compiled by Axel Hultman with translation assistance from Jean Bétréma, Mireille Bousquet-Mélou and Lydie Mpinganzima.

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Contributed talks and posters

Philippe Flajolet, the Father of Analytic Combinatorics

Philippe Flajolet, mathematician and computer scientist extraordinaire, suddenly passed away on March 22, 2011, at the prime of his career. He is celebrated for opening new lines of research in analysis of algorithms, developing powerful new methods, and solving difficult open problems. His research contributions will have impact for generations, and his approach to research, based on curiosity, a discriminating taste, broad knowledge and interest, intellectual integrity, and a genuine sense of camaraderie, will serve as an inspiration to those who knew him for years to come.

The common theme of Flajolet's extensive and far-reaching body of work is the scientific approach to the study of algorithms, including the development of requisite mathematical and computational tools. During his forty years of research, he contributed nearly 200 publications, with an important proportion of fundamental contributions and representing uncommon breadth and depth. He is best known for fundamental advances in mathematical methods for the analysis of algorithms, and his research also opened new avenues in various domains of applied computer science, including streaming algorithms, communication protocols, database access methods, data mining, symbolic manipulation, text-processing algorithms, and random generation. He exulted in sharing his passion: his papers had more than than a hundred different co-authors and he was a regular presence at scientific meetings all over the world.

His research laid the foundation of a subfield of mathematics, now known as analytic combinatorics. His lifework Analytic Combinatorics (Cambridge University Press, 2009, co-authored with R. Sedgewick) is a prodigious achievement that now defines the field and is already recognized as an authoritative reference.

Analytic combinatorics is a modern basis for the quantitative study of combinatorial structures (such as words, trees, mappings, and graphs), with applications to probabilistic study of algorithms that are based on these structures. It also strongly influences other scientific domains, such as statistical physics, computational biology, and information theory. With deep historic roots in classical analysis, the basis of the field lies in the work of Knuth, who put the study of algorithms on a firm scientific basis starting in the late 1960s with his classic series of books. Flajolet's work takes the field forward by introducing original approaches in combinatorics based on two types of methods: symbolic and analytic. The symbolic side is based on the automation of decision procedures in combinatorial enumeration to derive characterizations of generating functions. The analytic side treats those functions as functions in the complex plane and leads to precise characterization of limit distributions. In the last few years, Flajolet was further extending and generalizing this theory into a meeting point between information theory, probability theory and dynamical systems.

Philippe Flajolet was born in Lyon on December 1, 1948. He graduated from École Polytechnique in Paris in 1970, and was immediately recruited as a junior researcher at the Institut National de Recherche en Informatique et Automatique (INRIA), where he spent his career. Attracted by linguistics and logic, he worked on formal languages and computability with Maurice Nivat, obtaining a PhD from the University of Paris 7 in 1973. Then, following Jean Vuillemin in the footsteps of Don Knuth, he turned to the emerging field of analysis of algorithms and got a Doctorate in Sciences, both in mathematics and computer science, from the University of Paris at Orsay in 1979. At INRIA, he created and led the ALGO research group, which attracted visiting researchers from all over the world.

He held numerous visiting positions, at Waterloo, Stanford, Princeton, Wien, Barcelona, IBM and the Bell Laboratories. He received several prizes, including the Grand Science Prize of UAP (1986), the Computer Science Prize of the French Academy of Sciences (1994), and the Silver Medal of CNRS (2004). He was elected a Corresponding Member (Junior Fellow) of the French Academy of Sciences in 1994, a Member of the Academia Europaea in 1995, and a Member (Fellow) of the French Academy of Sciences in 2003.

A brilliant, insightful “honnête homme” with broad scientific interests, Philippe pursued new discoveries in computer science and mathematics and shared them with students and colleagues for over 40 years with enthusiasm, joy, generosity, and warmth. In France, he was the major reference at the interface between mathematics and computer science and founded the “Alea” meetings that bring together combinatorialists, probabilists and physicists to share problems and methods involving discrete randomness. More broadly, he was the leading figure in the development of the international “AofA” (Analysis of Algorithms) community that is devoted to research on probabilistic, combinatorial, and asymptotic methods in the analysis of algorithms. The colleagues and students who are devoted to carrying on his work form the core of his primary legacy.

May 2011
Bruno Salvy, Bob Sedgewick, Michèle Soria, Wojtek Szpankowski and Brigitte Vallée

Invited talks

FPSAC: Flajolet, Power Series and Analytic Combinatorics

François Bergeron

Université du Québec à Montréal, Canada

The Law of Aboav–Weaire and its analogue in three dimensions

Richard Ehrenborg

University of Kentucky, USA

When investigating the structure of metals it is known that the atoms lie in a lattice structure. However, the lattice property only holds locally, that is, in a three dimensional cell called a grain. Bordering the grain is a boundary where the atoms lie chaotically, and beyond that is a new grain where the lattice has a different orientation. The structure of these grains amounts to a three dimensional simple subdivision of space.

Looking at the two dimensional analogue, one observes that grains with a small number of sides tend to be surrounded by grains with a large number of sides, and vice versa. The Law of Aboav–Weaire states that the average number of sides of the neighbors of an n-sided grain should be roughly $5 + 6/n$. By introducing the correct error term we prove this law of Material Science and discuss its extension to three dimensions.

This is joint work with Menachem Lazar and Jeremy Mason. Moreover, selected work of von Neumann, MacPherson and Srolovitz will be presented.

Torus Squarings

Stefan Felsner

Technical University of Berlin, Germany

A squaring is a tiling into squares of different sizes. In a seminal paper Brooks, Smith, Stone and Tutte (1940) discussed squarings related to segment contact representations of planar quadrangulations. Regarding the squares of a squaring as vertices and edges as being defined by contacts we obtain the square dual graph. Schramm (1993) showed that 5-connected inner triangulations of a 4-gon can be represented as square duals. In this talk we review the plane situation and present some results concerning squarings of the torus and the graphs represented by them.

(joint work with E. Fusy)

An alternative approach to alternating sign matrices

Ilse Fischer

University of Vienna, Austria

Alternating sign matrices were first defined by Robbins and Rumsey in the early 1980s when they discovered that the λ-determinant, a natural generalization of the determinant, has an expansion as a sum over all alternating sign matrices, just as the ordinary determinant has an expansion as a sum over permutation matrices. Later it was observed that physicists had been studying a model for square ice that is equivalent to alternating sign matrices for a long time. Since then these square ice techniques are a standard tool to attack various enumeration problems related to alternating sign matrices.

In my talk I shall present an alternative approach to alternating sign matrix enumeration which is more in the spirit of Zeilberger's original proof of the alternating sign matrix theorem. Starting point is an operator formula for the number of monotone triangles with prescribed bottom row. Refined enumerations of alternating sign matrices with respect to a fixed set of boundary columns and rows can be expressed in terms of this operator formula. This enables us to translate certain identities for the operator formula to identities for refined enumerations of alternating sign matrices. This leads, on the one hand, to systems of linear equations that determine the numbers uniquely, and, on the other hand, to surprisingly simple linear relations between them. I will also report on recent attempts to translate these calculations into more combinatorial reasonings.

Generating functions for restricted Eulerian numbers

Ron Graham

University of California, San Diego, USA

In this talk I will describe some recent work with Fan Chung on certain joint statistics for permutations $\pi \in S_n$. These involve the number of descents of π, the maximum drop of π and the value of $\pi(n)$, and result in some new identities for restricted Eulerian numbers.

The octahedral recurrence and generalizations

Richard Kenyon

Brown University, USA

This is joint work with A. Goncharov. The octahedral recurrence, or Hirota equation, is a well-known integrable discrete dynamical system, related to alternating sign matrices and domino tilings of Aztec diamonds.

We show that there is an underlying completely integrable Hamiltonian system commuting with the octahedral recurrence. Formulas relating it to the octahedral recurrence can be written explicitly in terms of dimers.

Similar systems exist for any periodic planar graph.

Counting maps and graphs

Marc Noy

Polytechnic University of Catalonia, Spain

The theory of map enumeration was started by Tutte in the 1960s, in an attempt to shed light on the four colour problem, and since then the field has grown considerably. Many classes of maps have been enumerated, including maps on surfaces, and connections have been found to other fields, particularly to statistical physics. More recently, several classes of (unembedded) graphs have been analyzed, in particular planar graphs and graphs on surfaces, and precise asymptotic estimates have been obtained. In the talk we will review these results and the companion results on the structure of random graphs from these families. The main tool in our work is analytic combinatorics, as developed by Philippe Flajolet.

Affine and projective tree metric theorems with applications to phylogenetic reconstruction

Lior Pachter

University of California, Berkeley, USA

The tree metric theorem provides a combinatorial four point condition that characterizes dissimilarity maps derived from pairwise compatible split systems. A similar (but weaker) four point condition characterizes dissimilarity maps derived from circular split systems (Kalmanson metrics). The tree metric theorem was first discovered in the context of phylogenetics and forms the basis of many tree reconstruction algorithms, whereas Kalmanson metrics were first considered by computer scientists, and are notable in that they are a non-trivial class of metrics for which the traveling salesman problem is tractable. We will review these theorems via the unifying framework of the (tropical) space of trees and extensions to PQ- and PC-trees, and will discuss applications to phylogenetic reconstruction. In particular, we will provide an explanation for the peculiar form of the balanced minimum evolution criterion popular in phylogenetics. The work to be presented is joint with Aaron Kleinman.

Simple KLR modules

Monica Vazirani

University of California at Davis, USA

Khovanov-Lauda-Rouquier (KLR) algebras have played a fundamental role in categorifying quantum groups. I will discuss the structure of their simple modules, in particular that they carry the structure of a crystal graph. This is joint work with Aaron Lauda.

KP solitons, total positivity, and cluster algebras

Lauren Williams

University of California, Berkeley, USA

Soliton solutions of the KP equation have been studied since 1970, when Kadomtsev and Petviashvili proposed a two-dimensional nonlinear dispersive wave equation now known as the KP equation. It is well-known that the Wronskian approach to the KP equation provides a method to construct soliton solutions. More recently, several authors have focused on understanding the regular soliton solutions that one obtains in this way: these come from points of the totally non-negative Grassmannian.

In joint work with Yuji Kodama, we establish a tight connection between Postnikov's theory of total positivity for the Grassmannian, and the structure of regular soliton solutions to the KP equation. This connection allows us to apply machinery from total positivity to KP solitons. In particular, we classify the soliton graphs coming from the totally non-negative Grassmannian, when the absolute value of the time parameter is sufficiently large. We demonstrate an intriguing connection between soliton graphs and the cluster algebras of Fomin and Zelevinsky. Finally, we apply this connection towards the inverse problem for KP solitons.

Contributed talks and posters

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Supercharacters, symmetric functions in noncommuting variables, extended abstract

Marcelo Aguiar and Carlos André and Carolina Benedetti and Nantel Bergeron and Zhi Chen and Persi Diaconis and Anders Hendrickson and Samuel Hsiao and I. Martin Isaacs and Andrea Jedwab and Kenneth Johnson and Gizem Karaali and Aaron Lauve and Tung Le and Stephen Lewis and Huilan Li and Kay Magaard and Eric Marberg and Jean-Christophe Novelli and Amy Pang and Franco Saliola and Lenny Tevlin and Jean-Yves Thibon and Nathaniel Thiem and Vidya Venkateswaran and C. Ryan Vinroot and Ning Yan and Mike Zabrocki

Abstract. We identify two seemingly disparate structures: supercharacters, a useful way of doing Fourier analysis on the group of unipotent uppertriangular matrices with coefficients in a finite field, and the ring of symmetric functions in noncommuting variables. Each is a Hopf algebra and the two are isomorphic as such. This allows developments in each to be transferred. The identification suggests a rich class of examples for the emerging field of combinatorial Hopf algebras.

Résumé. Nous montrons que deux structures en apparence bien différentes peuvent être identifiées: les supercaractères, qui sont un outil commode pour faire de l'analyse de Fourier sur le groupe des matrices unipotentes triangulaires supérieures à coefficients dans un corps fini, et l'anneau des fonctions symétriques en variables non-commutatives. Ces deux structures sont des algèbres de Hopf isomorphes. Cette identification permet de traduire dans une structure les dévelopements conçus pour l'autre, et suggère de nombreux exemples dans le domaine nouveau des algèbres de Hopf combinatoires.

Keywords: supercharacters, set partitions, symmetric functions in non-commuting variables, Hopf algebras

1 Introduction

Identifying structures in seemingly disparate fields is a basic task of mathematics. An example, with parallels to the present work, is the identification of the character theory of the symmetric group with symmetric function theory. This connection is wonderfully exposited in Macdonald's book [20]. Later, Geissinger and Zelevinsky independently realized that there was an underlying structure of Hopf algebras that forced and illuminated the identification [14, 27]. We present a similar program for a "supercharacter" theory associated to the uppertriangular group and the symmetric functions in noncommuting variables.

Let $\mathrm{UT}_n(q)$ be the group of uppertriangular matrices with entries in the finite field $\mathbb{F}_q$ and ones on the diagonal. This group is a Sylow p-subgroup of $\mathrm{GL}_n(q)$. Describing the conjugacy classes or characters of $\mathrm{UT}_n(q)$ is a provably "wild" problem. In a series of papers, André developed a cruder theory that lumps together various conjugacy classes into "superclasses" and considers certain sums of irreducible characters as "supercharacters." The two structures are compatible (so supercharacters are constant on superclasses). The resulting theory is very nicely behaved — there is a rich combinatorics describing induction and restriction along with an elegant formula for the values of supercharacters on superclasses. The combinatorics is described in terms of set partitions (the symmetric group theory involves integer partitions) and the combinatorics seems akin to tableau combinatorics. At the same time, supercharacter theory is rich enough to serve as a substitute for ordinary character theory in some problems [6] .

In more detail, the group $\mathrm{UT}_n(q)$ acts on both sides of the algebra of strictly upper-triangular matrices $\mathfrak{n}_n$ (which can be thought of as $\mathfrak{n}_n = \mathrm{UT}_n(q) - 1$). The two sided orbits on $\mathfrak{n}_n$ can be mapped back to $\mathrm{UT}_n(q)$ by adding the identity matrix. These orbits form the superclasses in $\mathrm{UT}_n(q)$. A similar construction on the dual space $\mathfrak{n}_n^*$ gives a collection of class functions on $\mathrm{UT}_n(q)$ that turn out to be constant on superclasses. These orbit sums (suitably normalized) are the supercharacters. Let

$$\mathbf{SC} = \bigoplus_{n \geq 0} \mathbf{SC}_n,$$

where $\mathbf{SC}_n$ is the set of functions from $\mathrm{UT}_n(q)$ to $\mathbb{C}$ that are constant on superclasses, and $\mathbf{SC}_0 = \mathbb{C}\text{-span}\{1\}$ is by convention the set of class functions of $\mathrm{UT}_0(q) = \{\}$.

Let

$$\mathbf{\Pi} = \bigoplus_{n \geq 0} \mathbf{\Pi}_n$$

be the ring of symmetric functions in non-commuting variables. Such functions were considered by Wolf [25] and Doubilet [12]. More recent work of Sagan brought them to the forefront. A lucid introduction is given by Rosas and Sagan [22] and combinatorial applications by Gebhard and Sagan [13]. The algebra $\mathbf{\Pi}$ is actively studied as part of the theory of combinatorial Hopf algebras [3, 7, 9, 10, 17, 21]. The $\mathbf{m}_\lambda$ and thus $\mathbf{\Pi}$ are invariant under permutations of variables.

Our main result is to show that when $q = 2$, $\mathbf{SC}$ has a Hopf structure isomorphic to that of $\mathbf{\Pi}$. This construction of a Hopf algebra from the representation theory of a sequence of groups is the main contribution of this paper. It differs from previous work in that supercharacters are used. Previous work was confined to ordinary characters (e.g. [19]) and the results of [8] indicate that this is a restrictive setting. This work opens the possibility for a vast new source of Hopf algebras.

Acknowledgements

This paper developed during a focused research week at the American Institute of Mathematics in May 2010. The main results presented here were proved as a group during that meeting.

2 Background

2.1 Supercharacter theory

Supercharacters were first studied by André (e.g. [4]) and Yan [26] in relation to $\mathrm{UT}_n(q)$ in order to find a more tractable way to understand the representation theory of $\mathrm{UT}_n(q)$. Diaconis and Isaacs [11] then generalized the concept to arbitrary finite groups, and we reproduce a version of this more general definition below.

A *supercharacter theory* of a finite group G is a pair $(\mathcal{K}, \mathcal{X})$ where $\mathcal{K}$ is a partition of G and $\mathcal{X}$ is a partition of the irreducible characters of G such that

(a) Each $K \in \mathcal{K}$ is a union of conjugacy classes,

(b) $\{1\} \in \mathcal{K}$, where 1 is the identity element of G, and $\{\mathbb{1}\} \in \mathcal{X}$, where $\mathbb{1}$ is the trivial character of G.

(c) For $X \in \mathcal{X}$, the character

$$\sum_{\psi \in X} \psi(1)\psi$$

is constant on the parts of $\mathcal{K}$,

(d) $|\mathcal{K}| = |\mathcal{X}|$.

We will refer to the parts of $\mathcal{K}$ as *superclasses*, and for some fixed choice of scalars $c_X \in \mathbb{Q}$ (which are not uniquely determined), we will refer to the characters

$$\chi^X = c_X \sum_{\psi \in X} \psi(1)\psi, \qquad \text{for } X \in \mathcal{X}$$

as *supercharacters* (the scalars c_X should be picked such that the supercharacters are indeed characters). For more information on the implications of these axioms, including some redundancies in the definition, see [11].

There are a number of different known ways to construct supercharacter theories for groups, including

- Gluing together group elements and irreducible characters using outer automorphisms [11],
- Finding normal subgroups $N \lhd G$ and grafting together superchararacter theories for the normal subgroup N and for the factor group G/N to get a supercharacter theory for the whole group [16].

This paper will however focus on a technique first introduced for algebra groups [11], and then generalized to some other types of groups by André and Neto (e.g. [5]).

The group $\mathrm{UT}_n(q)$ has a natural two-sided action on the $\mathbb{F}_q$-spaces

$$\mathfrak{n} = \mathrm{UT}_n(q) - 1 \qquad \text{and} \qquad \mathfrak{n}^* = \mathrm{Hom}(\mathfrak{n}, \mathbb{F}_q)$$

given by left and right multiplication on $\mathfrak{n}$ and for $\lambda \in \mathfrak{n}^*$,

$$(u\lambda v)(x-1) = \lambda(u^{-1}(x-1)v^{-1}), \qquad \text{for } u, v, x \in \mathrm{UT}_n(q).$$

It can be shown that the orbits of these actions parametrize the superclasses and supercharacters, respectively, for a supercharacter theory. In particular, two elements $u, v \in \mathrm{UT}_n(q)$ are in the same superclass if and only if $u-1$ and $v-1$ are in the same two-sided orbit in $\mathrm{UT}_n(q)\backslash\mathfrak{n}/\mathrm{UT}_n(q)$. Since the action of $\mathrm{UT}_n(q)$ on $\mathfrak{n}$ can be viewed as applying row and column operations, we obtain a parameterization of superclasses given by

$$\left\{\begin{array}{c}\text{Superclasses}\\ \text{of } \mathrm{UT}_n(q)\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}u-1\in\mathfrak{n} \text{ with at most}\\ \text{one nonzero entry in}\\ \text{each row and column}\end{array}\right\}.$$

This indexing set is central to the combinatorics of this paper, so we give several interpretations for it. Let

$$\mathcal{S}_n(q) = \left\{\begin{array}{c}\text{Sets } \lambda \text{ of triples } i\overset{a}{\frown}j = (i,j,a) \in [n]\times[n]\times\mathbb{F}_q^{\times},\\ \text{with } i<j, \text{ and } i\overset{a}{\frown}j, k\overset{b}{\frown}l \in \lambda \text{ implies } i\neq k, j\neq l\end{array}\right\},$$

where we will refer to the elements of $\mathcal{S}_n(q)$ as $\mathbb{F}_q^{\times}$*-set partitions*. In particular,

$$\begin{array}{ccc}\mathcal{S}_n(q) & \longleftrightarrow & \left\{\begin{array}{c}u-1\in\mathfrak{n} \text{ with at most}\\ \text{one nonzero entry in}\\ \text{each row and column}\end{array}\right\}\\ \lambda = \{i\overset{\phi(i,j)}{\frown}j \mid (i,j)\in D\} & \mapsto & \displaystyle\sum_{i\overset{a}{\frown}j\in\lambda} ae_{ij},\end{array} \tag{2.1}$$

where e_{ij} is the matrix with 1 in the (i,j) position and zeroes elsewhere.

Remark 1 *Consider the map*

$$\begin{array}{cccc}\pi: & \mathcal{S}_n(q) & \to & \mathcal{S}_n(2)\\ & \lambda & \mapsto & \{i\overset{1}{\frown}j \mid i\overset{a}{\frown}j\in\lambda\},\end{array} \tag{2.2}$$

which ignores the part of the data that involves field scalars. Note that $\mathcal{S}_n(2)$ is in bijection with the set partitions of the set $\{1,2,\ldots,n\}$. Indeed, the connected components of an element $\lambda\in\mathcal{S}_n(2)$ may be viewed as the blocks of a partition of $\{1,2,\ldots,n\}$. Composing the map π with this bijection associates a set partition to an element of $\mathcal{M}_n(q)$ or $\mathcal{S}_n(q)$, which we call the underlying *set partition.*

Fix a nontrivial homomorphism $\vartheta:\mathbb{F}_q^{+}\to\mathbb{C}^{\times}$. For each $\lambda\in\mathfrak{n}^*$, construct a $\mathrm{UT}_n(q)$-module

$$V^{\lambda} = \mathbb{C}\text{-span}\{v_{\mu} \mid \mu\in -\mathrm{UT}_n(q)\cdot\lambda\}$$

with left action given by

$$uv_{\mu} = \vartheta\big(\mu(u^{-1}-1)\big)v_{u\mu}, \qquad \text{for } u\in\mathrm{UT}_n(q), \mu\in-\mathrm{UT}_n(q)\lambda.$$

Up to isomorphism, these modules depend only on the two-sided orbit in $\mathrm{UT}_n(q)\backslash\mathfrak{n}^*/\mathrm{UT}_n(q)$ of λ. Furthermore, there is an injective function $\iota:\mathcal{S}_n(q)\to\mathfrak{n}^*$ given by

$$\begin{array}{cccc}\iota(\lambda): & \mathfrak{n} & \longrightarrow & \mathbb{F}_q\\ & X & \mapsto & \displaystyle\sum_{i\overset{a}{\frown}j\in\lambda} aX_{ij}\end{array}$$

that maps $\mathcal{S}_n(q)$ onto a natural set of orbit representatives in $\mathfrak{n}^*$. We will identify $\lambda \in \mathcal{S}_n(q)$ with $\iota(\lambda) \in \mathfrak{n}^*$.

The traces of the modules V^λ for $\lambda \in \mathcal{S}_n(q)$ are the supercharacters of $\mathrm{UT}_n(q)$, and they have a nice supercharacter formula given by

$$\chi^\lambda(u_\mu) = \begin{cases} \dfrac{q^{\#\{(i,j,k) \mid i<j<k, i \overset{a}{\frown} k \in \lambda\}}}{q^{\#\{(i \overset{a}{\frown} l, j \overset{b}{\frown} k) \in \lambda \times \mu \mid i<j<k<l\}}} \displaystyle\prod_{\substack{i \overset{a}{\frown} l \in \lambda \\ i \overset{b}{\frown} l \in \mu}} \vartheta(ab), & \text{if } i \overset{a}{\frown} k \in \lambda \text{ and } i<j<k \\ & \text{implies } i \overset{b}{\frown} j, j \overset{b}{\frown} k \notin \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{2.3}$$

where u_μ has superclass type μ [6]. Note that the degree of the supercharacter is

$$\chi^\lambda(1) = \prod_{i \overset{a}{\frown} k \in \lambda} q^{k-i-1}. \tag{2.4}$$

Define

$$\mathbf{SC} = \bigoplus_{n \geq 0} \mathbf{SC}_n, \qquad \text{where} \qquad \mathbf{SC}_n = \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{S}_n(q)\},$$

and let $\mathbf{SC}_0 = \mathbb{C}\text{-span}\{\chi^\emptyset\}$. By convention, we write $1 = \chi^\emptyset$, since this element will be the identity of our Hopf algebra. Note that since $\mathbf{SC}_n$ is in fact the space of superclass functions of $\mathrm{UT}_n(q)$, it also has another distinguished basis, the superclass characteristic functions,

$$\mathbf{SC}_n = \mathbb{C}\text{-span}\{\kappa_\mu \mid \mu \in \mathcal{S}_n(q)\}, \qquad \text{where} \qquad \kappa_\mu(u) = \begin{cases} 1, & \text{if } u \text{ has superclass type } \mu, \\ 0, & \text{otherwise,} \end{cases}$$

and $\kappa_\emptyset = \chi^\emptyset$. Section 3 will explore a Hopf structure for this space.

2.2 Representation theoretic functors on SC

We will focus on two representation theoretic operations on the space $\mathbf{SC}$. For $J = (J_1|J_2|\cdots|J_\ell)$ any set composition of $\{1, 2, \ldots, n\}$, let

$$\mathrm{UT}_J(q) = \{u \in \mathrm{UT}_n(q) \mid u_{ij} \neq 0 \text{ with } i<j \text{ implies } i, j \text{ are in the same part of } J\}.$$

In the remainder of the paper we will need variants of a straightening map on set compositions. For each set composition $J = (J_1|J_2|\cdots|J_\ell)$, let

$$\mathrm{st}_J([n]) = \mathrm{st}_{J_1}(J_1) \times \mathrm{st}_{J_2}(J_2) \times \cdots \times \mathrm{st}_{J_\ell}(J_\ell), \tag{2.5}$$

where for $K \subseteq [n]$, $\mathrm{st}_K : K \longrightarrow [|K|]$ is the unique order preserving map. For example, $\mathrm{st}_{(14|3|256)}([6]) = \{1,2\} \times \{1\} \times \{1,2,3\}$.

We can extend this straightening map to a canonical isomorphism

$$\mathrm{st}_J : \mathrm{UT}_J(q) \longrightarrow \mathrm{UT}_{|J_1|}(q) \times \mathrm{UT}_{|J_2|}(q) \times \cdots \times \mathrm{UT}_{|J_\ell|}(q) \tag{2.6}$$

by reordering the rows and columns according to (2.5). For example, if $J = (14|3|256)$, then

$$\mathrm{UT}_J(q) \ni \begin{pmatrix} 1 & 0 & 0 & a & 0 & 0 \\ 0 & 1 & 0 & 0 & b & c \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & d \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{\mathrm{st}_J}{\longmapsto} \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, (1), \begin{pmatrix} 1 & b & c \\ 0 & 1 & d \\ 0 & 0 & 1 \end{pmatrix} \right) \in \mathrm{UT}_2(q) \times \mathrm{UT}_1(q) \times \mathrm{UT}_3(q).$$

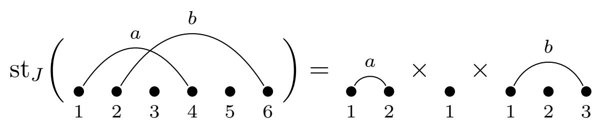
Combinatorially, if $J = (J_1|J_2|\cdots|J_\ell)$ we let

$$\mathcal{S}_J(q) = \{\lambda \in \mathcal{S}_n(q) \mid i \overset{a}{\frown} j \in \lambda \text{ implies } i, j \text{ are in the same part in } J\}.$$

Then we obtain the bijection

$$\mathrm{st}_J : \mathcal{S}_J(q) \longrightarrow \mathcal{S}_{|J_1|}(q) \times \mathcal{S}_{|J_2|}(q) \times \cdots \times \mathcal{S}_{|J_\ell|}(q) \tag{2.7}$$

that relabels the indices using the straightening map (2.5). For example, if $J = 14|3|256$, then



Note that $\mathrm{UT}_m(q) \times \mathrm{UT}_n(q)$ is an algebra group, so it has a supercharacter theory with the standard construction [11] such that

$$\mathbf{SC}(\mathrm{UT}_m(q) \times \mathrm{UT}_n(q)) \cong \mathbf{SC}_m \otimes \mathbf{SC}_n.$$

The combinatorial map (2.7) preserves supercharacters across this isomorphism.

The first operation of interest is restriction

$$\begin{array}{rccc} {}^J\mathrm{Res}^{\mathrm{UT}_n(q)}_{\mathrm{st}_J(\mathrm{UT}_J(q))} : & \mathbf{SC}_n & \longrightarrow & \mathbf{SC}_{|J_1|} \otimes \mathbf{SC}_{|J_2|} \otimes \cdots \otimes \mathbf{SC}_{|J_\ell|} \\ & \chi & \mapsto & \mathrm{Res}^{\mathrm{UT}_n(q)}_{\mathrm{UT}_J(q)}(\chi) \circ \mathrm{st}_J^{-1}, \end{array}$$

or

$${}^J\mathrm{Res}^{\mathrm{UT}_n(q)}_{\mathrm{st}_J(\mathrm{UT}_J(q))}(\chi)(u) = \chi(\mathrm{st}_J^{-1}(u)), \qquad \text{for } u \in \mathrm{UT}_{|J_1|}(q) \times \cdots \times \mathrm{UT}_{|J_\ell|}(q).$$

Remark 2 *There is an algorithmic method for computing restrictions of supercharacters (and also tensor products of characters) [23, 24]. This has been implemented in Sage.*

For an integer composition $(m_1, m_2, \ldots, m_\ell)$ of n, let

$$\mathrm{UT}_{(m_1,m_2,\ldots,m_\ell)}(q) = \mathrm{UT}_{(1,\ldots,m_1|m_1+1,\ldots,m_1+m_2|\cdots|n-m_\ell+1,\ldots,n)}(q) \subseteq \mathrm{UT}_{m_1+\cdots+m_\ell}(q).$$

There is a surjective homomorphism $\tau : \mathrm{UT}_n(q) \to \mathrm{UT}_{(m_1,m_2,\ldots,m_\ell)}(q)$ such that $\tau^2 = \tau$ (τ fixes the subgroup $\mathrm{UT}_{(m_1,m_2,\ldots,m_\ell)}(q)$ and sends the normal complement to 1). We now obtain the inflation map

$$\mathrm{Inf}^{\mathrm{UT}_n(q)}_{\mathrm{UT}_{(m_1,m_2,\ldots,m_\ell)}(q)} : \mathbf{SC}_{m_1} \otimes \mathbf{SC}_{m_2} \otimes \cdots \otimes \mathbf{SC}_{m_\ell} \longrightarrow \mathbf{SC}_n,$$

where

$$\mathrm{Inf}^{\mathrm{UT}_n(q)}_{\mathrm{UT}_{(m_1,m_2,\ldots,m_\ell)}(q)}(\chi)(u) = \chi(\tau(u)), \qquad \text{for } u \in \mathrm{UT}_n(q).$$

2.3 *The Hopf algebra* $\mathbf{\Pi}$

Symmetric polynomials in a set of commuting variables X are the invariants of the action of the symmetric group $\mathfrak{S}_X$ of X by automorphisms of the polynomial algebra $\mathbb{K}[X]$ over a field $\mathbb{K}$.

When $X = \{x_1, x_2, \ldots\}$ is infinite, we let $\mathfrak{S}_X$ be the set of bijections on X with finitely many nonfixed points. Then the subspace of $\mathbb{K}[[X]]^{\mathfrak{S}_X}$ of formal power series with bounded degree is the algebra of symmetric functions $\mathrm{Sym}(X)$ over $\mathbb{K}$. It has a natural bialgebra structure defined by

$$\Delta(f) = \sum_k f'_k \otimes f''_k, \tag{2.8}$$

where the f'_k, f''_k are defined by the identity

$$f(X' + X'') = \sum_k f'_k(X') f''_k(X''), \tag{2.9}$$

and $X' + X''$ denotes the disjoint union of two copies of X. The advantage of defining the coproduct in this way is that Δ is clearly coassociative and that it is obviously a morphism for the product. For each *integer* partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, the monomial symmetric function corresponding to λ is the sum

$$m_\lambda(X) = \sum_{x^\alpha \in O(x^\lambda)} x^\alpha \tag{2.10}$$

over elements of the orbit $O(x^\lambda)$ of $x^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_\ell^{\lambda_\ell}$ under $\mathfrak{S}_X$, and the monomial symmetric functions form a basis of $\mathrm{Sym}(X)$. The coproduct of a monomial function is

$$\Delta(m_\lambda) = \sum_{\mu \cup \nu = \lambda} m_\mu \otimes m_\nu\,. \tag{2.11}$$

The dual basis m^*_λ of m_λ is a multiplicative basis of the graded dual Sym^*, which turns out to be isomorphic to Sym via the identification $m^*_n = h_n$ (the complete homogeneous function, the sum of all monomials of degree n).

The case of noncommuting variables is very similar. Let A be an alphabet, and consider the invariants of $\mathfrak{S}_A$ acting by automorphisms on the free algebra $\mathbb{K}\langle A\rangle$. Two words $a = a_1 a_2 \cdots a_n$ and $b = b_1 b_2 \cdots b_n$ are in the same orbit whenever $a_i = a_j$ if and only if $b_i = b_j$. Thus, orbits are parametrized by set partitions in at most $|A|$ blocks. Assuming as above that A is infinite, we obtain an algebra based on all set partitions, defining the monomial basis by

$$\mathbf{m}_\lambda(A) = \sum_{w \in O_\lambda} w, \tag{2.12}$$

where O_λ is the set of words such that $w_i = w_j$ if and only if i and j are in the same block of λ.

One can introduce a bialgebra structure by means of the coproduct

$$\Delta(f) = \sum_k f'_k \otimes f''_k \qquad \text{where} \qquad f(A' + A'') = \sum_k f'_k(A') f''_k(A''), \tag{2.13}$$

and $A' + A''$ denotes the disjoint union of two mutually commuting copies of A. The coproduct of a monomial function is

$$\Delta(\mathbf{m}_\lambda) = \sum_{J \subseteq [\ell(\lambda)]} \mathbf{m}_{\mathrm{st}(\lambda_J)} \otimes \mathbf{m}_{\mathrm{st}(\lambda_{J^c})}. \tag{2.14}$$

This coproduct is cocommutative. With the unit that sends 1 to $m_\emptyset$ and the counit $\varepsilon(f(A)) = f(0, 0, \ldots)$, we have that $\mathbf{\Pi}$ is a connected graded bialgebra and therefore a graded Hopf algebra.

Remark 3 *We note that $\mathbf{\Pi}$ is often denoted in the literature as $\mathbf{NCSym}$ or $\mathbf{WSym}$.*

3 A Hopf algebra realization of $\mathbf{SC}$

In this section we describe a Hopf structure for the space

$$\begin{aligned} \mathbf{SC} &= \bigoplus_{n \geq 0} \mathbf{SC}_n \\ &= \mathbb{C}\text{-span}\{\kappa_\mu \mid \mu \in \mathcal{S}_n(q), n \in \mathbb{Z}_{\geq 0}\} \\ &= \mathbb{C}\text{-span}\{\chi^\lambda \mid \lambda \in \mathcal{S}_n(q), n \in \mathbb{Z}_{\geq 0}\}. \end{aligned}$$

The product and coproduct are defined representation theoretically by the inflation and restriction operations of Section 2.2,

$$\chi \cdot \psi = \mathrm{Inf}_{\mathrm{UT}_{(a,b)}(q)}^{\mathrm{UT}_{a+b}(q)}(\chi \times \psi), \qquad \text{where } \chi \in \mathbf{SC}_a, \psi \in \mathbf{SC}_b, \tag{3.1}$$

and

$$\Delta(\chi) = \sum_{\substack{J=(A|A^c)\\ A \subset [n]}} {}^{J}\mathrm{Res}_{\mathrm{UT}_{|A|}(q) \times \mathrm{UT}_{|A^c|}(q)}^{\mathrm{UT}_n(q)}(\chi), \qquad \text{for } \chi \in \mathbf{SC}_n. \tag{3.2}$$

For a combinatorial description of the Hopf structure of $\mathbf{SC}$ it is most convenient to work with the superclass characteristic functions.

Proposition 3.1

(a) For $\mu \in \mathcal{S}_k(q)$, $\nu \in \mathcal{S}_{n-k}(q)$,

$$\kappa_\mu \cdot \kappa_\nu = \sum_{\substack{\lambda = \mu \sqcup \gamma \sqcup (k+\nu) \in \mathcal{S}_n(q) \\ i \overset{a}{\frown} l \in \gamma \text{ implies } i \leq k < l}} \kappa_\lambda,$$

where $(k+\nu) = \{(k+i) \overset{a}{\frown} (k+j) \mid i \overset{a}{\frown} j \in \nu\}$ and $\sqcup$ denotes disjoint union.

(b) For $\lambda \in \mathcal{S}_n(q)$,

$$\Delta(\kappa_\lambda) = \sum_{\substack{\lambda = \mu \sqcup \nu \\ \mu \in \mathcal{S}_A(q), \nu \in \mathcal{S}_{A^c}(q) \\ A \subseteq \{1,2,\ldots,n\}}} \kappa_{\mathrm{st}_A(\mu)} \otimes \kappa_{\mathrm{st}_{A^c}(\nu)}.$$

Example 1 *We have*

$$\kappa_{\{1\overset{a}{\frown}3\},[3]}\cdot\kappa_{\{1\overset{b}{\frown}2,\,2\overset{c}{\frown}4\},[4]} = \kappa_{\{1\overset{a}{\frown}3,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]} + \sum_{d\in\mathbb{F}_q^\times}\Big(\kappa_{\{1\overset{a}{\frown}3,\,3\overset{d}{\frown}4,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]} + \kappa_{\{1\overset{a}{\frown}3,\,2\overset{d}{\frown}4,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]} + \kappa_{\{1\overset{a}{\frown}3,\,3\overset{d}{\frown}6,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]} + \kappa_{\{1\overset{a}{\frown}3,\,2\overset{d}{\frown}6,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]}\Big)$$
$$+ \sum_{d,e\in\mathbb{F}_q^\times}\Big(\kappa_{\{1\overset{a}{\frown}3,\,2\overset{d}{\frown}6,\,3\overset{e}{\frown}4,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]} + \kappa_{\{1\overset{a}{\frown}3,\,2\overset{d}{\frown}4,\,3\overset{e}{\frown}6,\,4\overset{b}{\frown}5,\,5\overset{c}{\frown}7\},[7]}\Big).$$

and

$$\Delta\Big(\kappa_{\{1\overset{a}{\frown}4\},[4]}\Big) = \kappa_{\{1\overset{a}{\frown}4\},[4]}\otimes\kappa_\emptyset + 2\kappa_{\{1\overset{a}{\frown}3\},[3]}\otimes\kappa_{[1]} + \kappa_{\{1\overset{a}{\frown}2\},[2]}\otimes\kappa_{[2]}$$
$$+ \kappa_{[2]}\otimes\kappa_{\{1\overset{a}{\frown}2\},[2]} + 2\kappa_{[1]}\otimes\kappa_{\{1\overset{a}{\frown}3\},[3]} + \kappa_\emptyset\otimes\kappa_{\{1\overset{a}{\frown}4\},[4]}.$$

By comparing Proposition 3.1 to (2.14) and the product on monomials, we obtain the following theorem.

Theorem 3.2 *For $q=2$, the map*

$$\begin{array}{rccc} \mathrm{ch}: & \mathbf{SC} & \longrightarrow & \mathbf{\Pi} \\ & \kappa_\mu & \mapsto & \mathbf{m}_\mu \end{array}$$

is a Hopf algebra isomorphism.

Note that although we did not assume for the theorem that $\mathbf{SC}$ is a Hopf algebra, the fact that ch preserves the Hopf operations implies that $\mathbf{SC}$ for $q=2$ is indeed a Hopf algebra.

Corollary 3.3 *The algebra $\mathbf{SC}$ with product given by (3.1) and coproduct given by (3.2) is a Hopf algebra.*

Remark 4

(a) Note that the isomorphism of Theorem 3.2 is not in any way canonical. In fact, the automorphism group of $\mathbf{\Pi}$ is rather large, so there are many possible isomorphisms. For our chosen isomorphism, there is no nice interpretation for the image of the supercharacters under the isomorphism of Theorem 3.2. Even less pleasant, when one composes it with the map

$$\mathbf{\Pi}\longrightarrow\mathrm{Sym}$$

that allows variables to commute (see [12, 22]), one in fact obtains that the supercharacters are not Schur positive. But, exploration with Sage suggests that it may be possible to choose an isomorphism such that the image of the supercharacters are Schur positive.

(b) Although the antipode is determined by the bialgebra structure of $\mathbf{\Pi}$, explicit expressions are not well understood. However, there are a number of forthcoming papers (e.g. [2, 18]) addressing this situation.

The Hopf algebra $\mathbf{SC}$ has a number of natural Hopf subalgebras. One of particular interest is the subspace spanned by linear characters (characters with degree 1). In fact, for this supercharacter theory every linear character of U_n is a supercharacter and by (2.4) these are exactly indexed by the set

$$\mathcal{L}_n = \{\lambda \in \mathcal{S}_n(q) \mid i \overset{a}{\frown} j \in \lambda \text{ implies } j = i+1\}.$$

Corollary 3.4 *For $q = 2$, the Hopf subalgebra*

$$\mathbf{LSC} = \mathbb{C}\text{-span}\{\chi^\lambda \mid i \overset{1}{\frown} j \in \lambda \textit{ implies } j = i+1\},$$

is isomorphic to the Hopf algebra of noncommutative symmetric functions $\mathbf{Sym}$ *studied in [15].*

Remark 5 *In fact, for each $k \in \mathbb{Z}_{\geq 0}$ the space*

$$\mathbf{SC}^{(k)} = \mathbb{C}\text{-span}\{\chi^\lambda \mid i \frown j \in \lambda \textit{ implies } j - i \leq k\}$$

is a Hopf subalgebra. This gives an unexplored filtration of Hopf algebras which interpolate between $\mathbf{LSC}$ *and* $\mathbf{SC}$.

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Lagrange's Theorem for Hopf Monoids in Species

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Abstract. We prove Lagrange's theorem for Hopf monoids in the category of connected species. We deduce necessary conditions for a given subspecies $\mathbf{k}$ of a Hopf monoid $\mathbf{h}$ to be a Hopf submonoid: each of the generating series of $\mathbf{k}$ must divide the corresponding generating series of $\mathbf{k}$ in $\mathbb{N}[\![x]\!]$. Among other corollaries we obtain necessary inequalities for a sequence of nonnegative integers to be the sequence of dimensions of a Hopf monoid. In the set-theoretic case the inequalities are linear and demand the non negativity of the binomial transform of the sequence.

Résumé. Nous prouvons le théorème de Lagrange pour les monoïdes de Hopf dans la catégorie des espèces connexes. Nous déduisons des conditions nécessaires pour qu'une sous-espèce $\mathbf{k}$ d'un monoïde de Hopf $\mathbf{h}$ soit un sous-monoïde de Hopf: chacune des séries génératrices de $\mathbf{k}$ doit diviser la série génératrice correspondante de $\mathbf{h}$ dans $\mathbb{N}[\![x]\!]$. Parmi d'autres corollaires nous trouvons des inégalités nécessaires pour qu'une suite d'entiers soit la suite des dimensions d'un monoïde de Hopf. Dans le cas ensembliste les inégalités sont linéaires et exigent que la transformée binomiale de la suite soit non négative.

Keywords: Hopf monoids, species, graded Hopf algebras, Lagrange's theorem, generating series

Introduction

Lagrange's theorem states that for any subgroup K of a group H, $H \cong K \times Q$ as (left) K-sets, where $Q = H/K$. In particular, if H is finite, $|K|$ divides $|H|$. Passing to group algebras over a field $\Bbbk$, we have that $\Bbbk H \cong \Bbbk K \otimes \Bbbk Q$ as (left) $\Bbbk K$-modules, or that $\Bbbk H$ is free as a $\Bbbk K$-module. Kaplansky [6] conjectured that the same statement holds for Hopf algebras (group algebras being principal examples). It turns out that the result does not not in general, as shown by Oberst and Schneider [13, Proposition 10] and [11, Example 3.5.2]. On the other hand, the result does hold for large classes of Hopf algebras, including the finite dimensional ones by a theorem Nichols and Zoeller [12], and the pointed ones by a theorem of Radford [16]. More information can be found in Sommerhäuser's survey [15].

The main result of this paper (Theorem 7) is a version of Lagrange's theorem for Hopf monoids in the category of connected species. (Hopf algebras are Hopf monoids in the category of vector spaces.) An immediate application is a test for Hopf submonoids (Corollary 12): if any one of the generating series for

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a subspecies $\mathbf{k}$ does not divide the corresponding generating series for the Hopf monoid $\mathbf{h}$ (in the sense that the quotient has negative coefficients), then $\mathbf{k}$ is not a Hopf submonoid of $\mathbf{h}$. A similar test also holds for connected graded Hopf algebras (Corollary 4). The proof of Theorem 7 (for Hopf monoids in species) parallels Radford's proof (for Hopf algebras).

This abstract is organized as follows. In Section 1, we recall Lagrange's theorem for several classes of Hopf algebras. In Section 2, we recall the basics of Hopf monoids in species and prove Lagrange's theorem in this setting. We conclude in Section 3 with some examples and applications involving the associated generating series. Among these, we derive necessary conditions for a sequence of nonnegative integers to be the sequence of dimensions of a connected Hopf monoid in species.

1 Lagrange's theorem for Hopf algebras

We begin by recalling a couple of versions of this theorem. (All vector spaces are over a fixed field $\Bbbk$.)

Theorem 1 *Let H be a finite dimensional Hopf algebra over a field $\Bbbk$. If $K \subseteq H$ is any Hopf subalgebra, then H is a free left (and right) K-module.*

This is the Nichols-Zoeller theorem [12]; see also [11, Theorem 3.1.5]. We will not make direct use of this result, but rather the related results discussed below.

A Hopf algebra H is *pointed* if all its simple subcoalgebras are one dimensional. Equivalently, the group-like elements of H linearly span the coradical of H. Given a subspace K of H, let

$$K_+ := K \cap \ker(\epsilon)$$

where $\epsilon : H \to \Bbbk$ is the counit of H. Also, K_+H denotes the right H-ideal generated by K_+.

Theorem 2 *Let H be a pointed Hopf algebra. If $K \subseteq H$ is any Hopf subalgebra, then H is a free left (and right) K-module. Moreover, $H \cong K \otimes (H/K_+H)$ as left K-modules.*

The first statement is due to Radford [16, Section 4] and the second (stronger) statement is due to Schneider [14, Remark 4.14]. See Sommerhaüser's survey [15] for further generalizations.

A Hopf algebra H is *graded* if there is given a decomposition

$$H = \bigoplus_{n \geq 0} H_n$$

into linear subspaces that is preserved by all operations. It is *connected* if in addition H_0 is linearly spanned by the unit element.

Theorem 3 *Let H be a graded connected Hopf algebra. If $K \subseteq H$ is a graded Hopf subalgebra, then H is a free left (and right) K-module. Moreover,*

$$H \cong K \otimes (H/K_+H)$$

as left K-modules and as graded vector spaces.

Proof: Since H is connected graded, its coradical is $H_0 = \Bbbk$, so H is pointed and Theorem 2 applies. Radford's proof shows that there exists a graded vector space Q such that

$$H \cong K \otimes Q$$

as left K-modules and as graded vector spaces. (The argument we give in the parallel setting of Theorem 7 makes this clear.) Note that $K_+ = \bigoplus_{n \geq 1} K_n$, and K_+H and H/K_+H inherit the grading of H. To complete the proof, it suffices to show that $Q \cong H/K_+H$ as graded vector spaces. $\square$

Given a graded Hopf algebra H, let $\mathcal{P}_H(x) \in \mathbb{N}[\![x]\!]$ denote its *Poincaré series*—the formal power series enumerating the dimensions of its graded components,

$$\mathcal{P}_H(x) := \sum_{n \geq 0} \dim H_n \, x^n.$$

Suppose H is graded connected and K is a graded Hopf subalgebra. In this case, their Poincaré series are of the form $1 + a_1 x + a_2 x^2 + \cdots$ with $a_i \in \mathbb{N}$ and the quotient $\mathcal{P}_H(x)/\mathcal{P}_K(x)$ is a well-defined power series in $\mathbb{Z}[\![x]\!]$.

Corollary 4 *Let H be a connected graded Hopf algebra. If $K \subseteq H$ is any graded Hopf subalgebra, then the quotient $\mathcal{P}_H(x)/\mathcal{P}_K(x)$ of Poincaré series is nonnegative, i.e., belongs to $\mathbb{N}[\![x]\!]$.*

Proof: By Theorem 3, $H \cong K \otimes Q$ as graded vector spaces, where $Q = H/K_+H$. Hence $\mathcal{P}_H(x) = \mathcal{P}_K(x)\,\mathcal{P}_Q(x)$ and the result follows. $\square$

Example 5 Consider the Hopf algebra *QSym* of quasisymmetric functions in countably many variables, and the Hopf subalgebra *Sym* of symmetric functions. They are graded connected, so by Theorem 3, *QSym* is a free module over *Sym*. Garsia and Wallach prove this same fact for the algebras QSym_n and Sym_n of (quasi) symmetric functions in n variables [4]. These are not Hopf algebras when n is finite, so Theorem 3 does not yield the result of Garsia and Wallach. The papers [4] and [8] provide information on the space Q_n entering in the decomposition $\mathit{QSym}_n \cong \mathit{Sym}_n \otimes Q_n$.

2 Lagrange's theorem for Hopf monoids in species

We first review the notion of Hopf monoid in the category of species, following [2], and then prove Lagrange's theorem in this setting. We restrict attention to the case of connected Hopf monoids.

2.1 Hopf monoids in species

The notion of species was introduced by Joyal [5]. It formalizes the notion of combinatorial structure and provides a framework for studying the generating functions which enumerate these structures. The book [3] by Bergeron, Labelle and Leroux expounds the theory of (set) species.

Joyal's work indicates that species may also be regarded as algebraic objects; this is the point of view adopted in [2] and in this work. To this end, it is convenient to work with vector species.

A *(vector) species* is a functor $\mathbf{q}$ from finite sets and bijections to vector spaces and linear maps. Specifically, it is a family of vector spaces $\mathbf{q}[I]$, one for each finite set I, together with linear maps $\mathbf{q}[\sigma] : \mathbf{q}[I] \to \mathbf{q}[J]$, one for each bijection $\sigma : I \to J$, satisfying

$$\mathbf{q}[\mathrm{id}_I] = \mathrm{id}_{\mathbf{q}[I]} \quad \text{and} \quad \mathbf{q}[\sigma \circ \tau] = \mathbf{q}[\sigma] \circ \mathbf{q}[\tau]$$

whenever σ and τ are composable bijections. A species $\mathbf{q}$ is *finite dimensional* if each vector space $\mathbf{q}[I]$ is finite dimensional. In this paper, all species are finite dimensional. A morphism of species is a natural transformation of functors. Let Sp denote the category of (finite dimensional) species.

We give two elementary examples that will be useful later.

Example 6 Let $\mathbf{E}$ be the *exponential species*, defined by $\mathbf{E}[I] = \Bbbk\{*_I\}$ for all I. The symbol $*_I$ denotes an element canonically associated to the set I (for definiteness, we may take $*_I = I$). Thus, $\mathbf{E}[I]$ is a one dimensional space with a distinguished basis element. A richer example is provided by the species $\mathbf{L}$ of *linear orders*, defined by $\mathbf{L}[I] = \Bbbk\{\text{linear orders on } I\}$ for all I (a space of dimension $n!$ when $|I| = n$).

We use $\cdot$ to denote the *Cauchy product* of two species. Specifically,

$$\big(\mathbf{p} \cdot \mathbf{q}\big)[I] := \bigoplus_{S \sqcup T = I} \mathbf{p}[S] \otimes \mathbf{q}[T] \quad \text{for all finite sets } I.$$

The notation $S \sqcup T = I$ indicates that $S \cup T = I$ and $S \cap T = \emptyset$. The sum runs over all such *ordered decompositions* of I, or equivalently over all subsets S of I: there is one term for $S \sqcup T$ and another for $T \sqcup S$. The Cauchy product turns Sp into a symmetric monoidal category. The braiding simply switches the tensor factors. The unit object is the species $\mathbf{1}$ defined by

$$\mathbf{1}[I] := \begin{cases} \Bbbk & \text{if } I \text{ is empty,} \\ 0 & \text{otherwise.} \end{cases}$$

A *monoid* in the category $(\mathsf{Sp}, \cdot)$ is a species $\mathbf{m}$ together with a morphism of species $\mu : \mathbf{m} \cdot \mathbf{m} \to \mathbf{m}$, i.e., a family of maps

$$\mu_{S,T} : \mathbf{m}[S] \otimes \mathbf{m}[T] \to \mathbf{m}[I],$$

one for each ordered decomposition $I = S \sqcup T$, satisfying appropriate associativity and unital conditions, and naturality with respect to bijections. Briefly, to each $\mathbf{m}$-structure on S and $\mathbf{m}$-structure on T, there is assigned an $\mathbf{m}$-structure on $S \sqcup T$. The analogous object in the category gVec of graded vector spaces is a graded algebra.

The species $\mathbf{E}$ has a monoid structure defined by sending the basis element $*_S \otimes *_T$ to the basis element $*_I$. For $\mathbf{L}$, a monoid structure is provided by concatenation of linear orders: $\mu_{S,T}(\ell_1 \otimes \ell_2) = (\ell_1, \ell_2)$.

A *comonoid* in the category $(\mathsf{Sp}, \cdot)$ is a species $\mathbf{c}$ together with a morphism of species $\Delta : \mathbf{c} \to \mathbf{c} \cdot \mathbf{c}$, i.e., a family of maps

$$\Delta_{S,T} : \mathbf{c}[I] \to \mathbf{c}[S] \otimes \mathbf{c}[T],$$

one for each ordered decomposition $I = S \sqcup T$, which are natural, coassociative and counital.

For $\mathbf{E}$, a comonoid structure is defined by sending the basis vector $*_I$ to the basis vector $*_S \otimes *_T$. For $\mathbf{L}$, a comonoid structure is provided by restricting a total order ℓ on I: $\Delta_{S,T}(\ell) = \ell|_S \otimes \ell|_T$.

We assume that our species $\mathbf{q}$ are *connected*, i.e., $\mathbf{q}[\emptyset] = \Bbbk$. In this case, the (co)unital conditions for a (co)monoid force the maps $\mu_{S,T}$ and $\Delta_{S,T}$ to be the canonical identifications if either S or T is empty.

A *Hopf monoid* in the category $(\mathsf{Sp}, \cdot)$ is a monoid and comonoid whose two structures are compatible in an appropriate sense, and which carries an antipode. In this paper we only consider connected Hopf monoids. For such Hopf monoids, the existence of the antipode is guaranteed. The species $\mathbf{E}$ and $\mathbf{L}$, with the structures outlined above, are two important examples of (connected) Hopf monoids.

For further details on Hopf monoids in species, see Chapter 8 of [2]. The theory of Hopf monoids in species is developed in Part II of this reference; several examples are discussed in Chapters 12 and 13.

2.2 Lagrange's theorem for connected Hopf monoids

Given a connected Hopf monoid $\mathbf{k}$ in species, we let $\mathbf{k}_+$ denote the species defined by

$$\mathbf{k}_+[I] = \begin{cases} \mathbf{k}[I] & \text{if } I \neq \emptyset, \\ 0 & \text{if } I = \emptyset. \end{cases}$$

If $\mathbf{k}$ is a submonoid of a monoid $\mathbf{h}$, then $\mathbf{k}_+\mathbf{h}$ denotes the right ideal of $\mathbf{h}$ generated by $\mathbf{k}_+$. In other words,

$$(\mathbf{k}_+\mathbf{h})[I] = \sum_{\substack{S \sqcup T = I \\ S \neq \emptyset}} \mu_{S,T}\big(\mathbf{k}[S] \otimes \mathbf{h}[T]\big).$$

Theorem 7 *Let $\mathbf{h}$ be a connected Hopf monoid in the category of species. If $\mathbf{k}$ is a Hopf submonoid of $\mathbf{h}$, then $\mathbf{h}$ is a free left $\mathbf{k}$-module. Moreover,*

$$\mathbf{h} \cong \mathbf{k} \cdot (\mathbf{h}/\mathbf{k}_+\mathbf{h})$$

as left $\mathbf{k}$-modules (and as species).

The proof is given after a series of preparatory results. Our argument parallels Radford's proof of Theorem 2 [16, Section 4]. The main ingredient is a result of Larson and Sweedler [7] known as the fundamental theorem of Hopf modules [11, Thm. 1.9.4]. It states that if (M, ρ) is a left Hopf module over K, then M is free as a left K-module and in fact is isomorphic to the Hopf module $K \otimes Q$, where Q is the space of *coinvariants* for the coaction ρ. Takeuchi extends this result to the context of Hopf monoids in a braided monoidal category with equalizers [19, Thm. 3.4]; a similar result (in a more restrictive setting) is given by Lyubashenko [9, Thm. 1.1]. As a special case of Takeuchi's result, we have the following.

Proposition 8 *Let $\mathbf{m}$ be a left Hopf module over a connected Hopf monoid $\mathbf{k}$ in species. There is an isomorphism $\mathbf{m} \cong \mathbf{k} \cdot \mathbf{q}$ of left Hopf modules, where*

$$\mathbf{q}[I] := \big\{ m \in \mathbf{m}[I] \mid \ \rho_{S,T}(m) = 0 \textit{ for } S \sqcup T = I,\, T \neq I \big\}.$$

In particular, $\mathbf{m}$ is free as a left $\mathbf{k}$-module.

Here $\rho : \mathbf{m} \to \mathbf{k} \cdot \mathbf{m}$ denotes the comodule structure, which consists of maps

$$\rho_{S,T} : \mathbf{m}[I] \to \mathbf{k}[S] \otimes \mathbf{m}[T],$$

one for each ordered decomposition $I = S \sqcup T$.

Given a comonoid $\mathbf{h}$ and two subspecies $\mathbf{u}, \mathbf{v} \subseteq \mathbf{h}$, their *wedge* is defined by

$$\mathbf{u} \wedge \mathbf{v} := \Delta^{-1}(\mathbf{u} \cdot \mathbf{h} + \mathbf{h} \cdot \mathbf{v}).$$

The remaining ingredients needed for the proof are supplied by the following lemmas.

Lemma 9 *Let $\mathbf{h}$ be a comonoid in species. If $\mathbf{u}$ and $\mathbf{v}$ are subcomonoids of $\mathbf{h}$, then:* (i) *$\mathbf{u} \wedge \mathbf{v}$ is a subcomonoid of $\mathbf{h}$ and $\mathbf{u} + \mathbf{v} \subseteq \mathbf{u} \wedge \mathbf{v}$;* (ii) $\mathbf{u} \wedge \mathbf{v} = \Delta^{-1}\big(\mathbf{u} \cdot (\mathbf{u} \wedge \mathbf{v}) + (\mathbf{u} \wedge \mathbf{v}) \cdot \mathbf{v}\big)$.

Proof: (i) Proofs of analogous statements for coalgebras, given in [1, Section 3.3], extend to this setting.

(ii) From the definition, $\Delta^{-1}\big(\mathbf{u}\cdot(\mathbf{u}\wedge\mathbf{v})+(\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{v}\big)\subseteq\mathbf{u}\wedge\mathbf{v}$. Now, since $\mathbf{u}\wedge\mathbf{v}$ is a subcomonoid,

$$\Delta(\mathbf{u}\wedge\mathbf{v})\subseteq\big((\mathbf{u}\wedge\mathbf{v})\cdot(\mathbf{u}\wedge\mathbf{v})\big)\cap(\mathbf{u}\cdot\mathbf{h}+\mathbf{h}\cdot\mathbf{v})=\mathbf{u}\cdot(\mathbf{u}\wedge\mathbf{v})+(\mathbf{u}\wedge\mathbf{v})\cdot\mathbf{v},$$

since $\mathbf{u},\mathbf{v}\subseteq\mathbf{u}\wedge\mathbf{v}$. This proves the converse inclusion. □

Lemma 10 *Let* $\mathbf{h}$ *be a Hopf monoid in species and* $\mathbf{k}$ *a submonoid. Let* $\mathbf{u},\mathbf{v}\subseteq\mathbf{h}$ *be subspecies which are left* $\mathbf{k}$*-submodules of* $\mathbf{h}$*. Then* $\mathbf{u}\wedge\mathbf{v}$ *is a left* $\mathbf{k}$*-submodule of* $\mathbf{h}$*.*

Proof: Since $\mathbf{h}$ is a Hopf monoid, the coproduct $\Delta:\mathbf{h}\to\mathbf{h}\cdot\mathbf{h}$ is a morphism of left $\mathbf{h}$-modules, where $\mathbf{h}$ acts on $\mathbf{h}\cdot\mathbf{h}$ via Δ. Hence it is also a morphism of left $\mathbf{k}$-modules. By hypothesis, $\mathbf{u}\cdot\mathbf{h}+\mathbf{h}\cdot\mathbf{v}$ is a left $\mathbf{k}$-submodule of $\mathbf{h}\cdot\mathbf{h}$. Hence, $\mathbf{u}\wedge\mathbf{v}=\Delta^{-1}(\mathbf{u}\cdot\mathbf{h}+\mathbf{h}\cdot\mathbf{v})$ is a left $\mathbf{k}$-submodule of $\mathbf{h}$. □

Lemma 11 *Let* $\mathbf{h}$ *be a Hopf monoid in species and* $\mathbf{k}$ *a Hopf submonoid. Let* $\mathbf{c}$ *be a subcomonoid of* $\mathbf{h}$ *and a left* $\mathbf{k}$*-submodule of* $\mathbf{h}$*. Then* $(\mathbf{k}\wedge\mathbf{c})/\mathbf{c}$ *is a left* $\mathbf{k}$*-Hopf module.* □

We are nearly ready for the proof of the main result. First, recall the *coradical filtration* of a connected comonoid in species [2, §8.10]. Given a connected comonoid $\mathbf{c}$, define subspecies $\mathbf{c}_{(n)}$ by

$$\mathbf{c}_{(0)}=\mathbf{1}\quad\text{and}\quad\mathbf{c}_{(n)}=\mathbf{c}_{(0)}\wedge\mathbf{c}_{(n-1)}\ \text{ for all } n\geq 1.$$

We then have

$$\mathbf{c}_{(0)}\subseteq\mathbf{c}_{(1)}\subseteq\cdots\subseteq\mathbf{c}_{(n)}\subseteq\cdots\mathbf{c}\quad\text{and}\quad\mathbf{c}=\bigcup_{n\geq 0}\mathbf{c}_{(n)}.$$

Proof of Theorem 7: We show that there is a species $\mathbf{q}$ such that $\mathbf{h}\cong\mathbf{k}\cdot\mathbf{q}$ as left $\mathbf{k}$-modules. As in the proof of Theorem 3, one then argues that $\mathbf{q}\cong\mathbf{h}/\mathbf{k}_+\mathbf{h}$.

Define a sequence $\mathbf{k}^{(n)}$ of subspecies of $\mathbf{h}$ by

$$\mathbf{k}^{(0)}=\mathbf{k}\quad\text{and}\quad\mathbf{k}^{(n)}=\mathbf{k}\wedge\mathbf{k}^{(n-1)}\ \text{ for all } n\geq 1.$$

Each $\mathbf{k}^{(n)}$ is a subcomonoid and a left $\mathbf{k}$-submodule of $\mathbf{h}$. This follows from Lemmas 9 and 10, by induction on n. Then, by Lemma 11, for all $n\geq 1$ the quotient species $\mathbf{k}^{(n)}/\mathbf{k}^{(n-1)}$ is a left Hopf $\mathbf{k}$-module. Therefore, by Proposition 8, each $\mathbf{k}^{(n)}/\mathbf{k}^{(n-1)}$ is a free left $\mathbf{k}$-module.

We claim that there exists a sequence of species $\mathbf{q}_n$ ($n\geq 0$) such that

$$\mathbf{k}^{(n)}\cong\mathbf{k}\cdot\mathbf{q}_n$$

as left $\mathbf{k}$-modules (so that each $\mathbf{k}^{(n)}$ is a free left $\mathbf{k}$-module). Moreover, the $\mathbf{q}_n$ can be chosen so that

$$\mathbf{q}_0\subseteq\mathbf{q}_1\ \subseteq\cdots\subseteq\mathbf{q}_n\subseteq\cdots$$

and the above isomorphisms are compatible with the inclusions $\mathbf{q}_{n-1}\subseteq\mathbf{q}_n$ and $\mathbf{k}^{(n-1)}\subseteq\mathbf{k}^{(n)}$. This may be proven by induction on n.

Finally, since $\mathbf{h}$ is connected, $\mathbf{h}_{(0)}=\mathbf{1}\subseteq\mathbf{k}=\mathbf{k}^{(0)}$, and by induction, $\mathbf{h}_{(n)}\subseteq\mathbf{k}^{(n)}$ for all $n\geq 0$. Hence,

$$\mathbf{h}=\bigcup_{n\geq 0}\mathbf{h}_{(n)}=\bigcup_{n\geq 0}\mathbf{k}^{(n)}\cong\bigcup_{n\geq 0}\mathbf{k}\cdot\mathbf{q}_n\cong\mathbf{k}\cdot\mathbf{q}\ \text{ where }\ \mathbf{q}=\bigcup_{n\geq 0}\mathbf{q}_n.$$

Thus, $\mathbf{h}$ is free as a left $\mathbf{k}$-module. □

3 Applications and examples

3.1 A test for Hopf submonoids

Two important power series associated to a (finite dimensional) species $\mathbf{q}$ are its *exponential generating series* $\mathcal{E}_{\mathbf{q}}(x)$ and *type generating series* $\mathcal{T}_{\mathbf{q}}(x)$. They are given by

$$\mathcal{E}_{\mathbf{q}}(x) = \sum_{n\geq 0} \dim \mathbf{q}[n] \, \frac{x^n}{n!} \quad \text{and} \quad \mathcal{T}_{\mathbf{q}}(x) = \sum_{n\geq 0} \dim \mathbf{q}[n]_{S_n} \, x^n,$$

where

$$\mathbf{q}[n]_{S_n} = \mathbf{q}[n]/\Bbbk\{v - \sigma v \mid v \in \mathbf{q}[n],\ \sigma \in S_n\}.$$

Both are specializations of the *cycle index series* $\mathcal{Z}_{\mathbf{q}}(x_1, x_2, \dots)$; see [3, §1.2] for definitions. Specifically,

$$\mathcal{E}_{\mathbf{q}}(x) = \mathcal{Z}_{\mathbf{q}}(x, 0, \dots) \qquad \text{and} \qquad \mathcal{T}_{\mathbf{q}}(x) = \mathcal{Z}_{\mathbf{q}}(x, x^2, \dots).$$

The cycle index series is multiplicative under Cauchy product: if $\mathbf{h} = \mathbf{k} \cdot \mathbf{q}$, then $\mathcal{Z}_{\mathbf{h}}(x_1, x_2, \dots) = \mathcal{Z}_{\mathbf{k}}(x_1, x_2, \dots)\, \mathcal{Z}_{\mathbf{q}}(x_1, x_2, \dots)$; see [3, §1.3]. Thus, the same is true for $\mathcal{E}_{\mathbf{q}}(x)$ and $\mathcal{T}_{\mathbf{q}}(x)$.

Let $\mathbb{Q}_{\geq 0}$ denote the nonnegative rational numbers. A consequence of Theorem 7 is the following.

Corollary 12 *Let* $\mathbf{h}$ *and* $\mathbf{k}$ *be connected Hopf monoids in species. If* $\mathbf{k}$ *is either a Hopf submonoid or a quotient Hopf monoid of* $\mathbf{h}$*, then the quotient* $\mathcal{Z}_{\mathbf{h}}(x_1, x_2, \dots)/\mathcal{Z}_{\mathbf{k}}(x_1, x_2, \dots)$ *of cycle index series is nonnegative, i.e., belongs to* $\mathbb{Q}_{\geq 0}[\![x_1, x_2, \dots]\!]$*. Likewise for the quotients* $\mathcal{E}_{\mathbf{h}}(x)/\mathcal{E}_{\mathbf{k}}(x)$ *and* $\mathcal{T}_{\mathbf{h}}(x)/\mathcal{T}_{\mathbf{k}}(x)$*.*

Given a connected Hopf monoid $\mathbf{h}$ in species, we may use Corollary 12 to determine if a given species $\mathbf{k}$ may be a Hopf submonoid (or a quotient Hopf monoid).

Example 13 A *partition* of a set I is an unordered collection of disjoint nonempty subsets of I whose union is I. The notation $ab.c$ is shorthand for the partition $\big\{\{a,b\},\ \{c\}\big\}$ of $\{a,b,c\}$.

Let $\mathbf{\Pi}$ be the species of set partitions, i.e., $\mathbf{\Pi}[I]$ is the vector space with basis the set of all partitions of I. Let $\mathbf{\Pi}'$ denote the subspecies linearly spanned by set partitions with distinct block sizes. For example,

$$\mathbf{\Pi}[\{a,b,c\}] = \Bbbk\big\{abc, a.bc, ab.c, a.bc, a.b.c\big\} \quad \text{and} \quad \mathbf{\Pi}'[\{a,b,c\}] = \Bbbk\big\{abc, a.bc, ab.c, a.bc\big\}.$$

The sequences $(\dim \mathbf{\Pi}[n])_{n\geq 0}$ and $(\dim \mathbf{\Pi}'[n])_{n\geq 0}$ appear in [17] as A000110 and A007837, respectively. We have

$$\mathcal{E}_{\mathbf{\Pi}}(x) = \exp\big(\exp(x) - 1\big) = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \cdots$$

and

$$\mathcal{E}_{\mathbf{\Pi}'}(x) = \prod_{n\geq 1}\big(1 + \frac{x^n}{n!}\big) = 1 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{5}{24}x^4 + \cdots .$$

If a Hopf monoid structure on $\mathbf{\Pi}$ existed for which $\mathbf{\Pi}'$ were a Hopf submonoid, then the quotient of their exponential generating series would be nonnegative, by Corollary 12. However, we have

$$\mathcal{E}_{\mathbf{\Pi}}(x)/\mathcal{E}_{\mathbf{\Pi}'}(x) = 1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{2}x^4 - \frac{11}{30}x^5 + \cdots ,$$

so no such structure exists. In [2, §12.6], a Hopf monoid structure on $\mathbf{\Pi}$ is defined. By the above, there is no way to embed $\mathbf{\Pi}'$ as a Hopf submonoid.

We remark that the type generating series quotient for the pair of species in Example 13 is positive:

$$\begin{aligned} \mathcal{T}_{\mathbf{\Pi}}(x) &= 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 15x^7 + \cdots, \\ \mathcal{T}_{\mathbf{\Pi}'}(x) &= 1 + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 4x^6 + 5x^7 + \cdots, \\ \mathcal{T}_{\mathbf{\Pi}}(x) \big/ \mathcal{T}_{\mathbf{\Pi}'}(x) &= 1 + x^2 + 2x^4 + 3x^6 + 5x^8 + 7x^{10} + \cdots. \end{aligned}$$

This can be understood by appealing to the Hopf algebra *Sym*. A basis for its nth graded piece is indexed by integer partitions, so $\mathcal{P}_{\mathit{Sym}}(x) = \mathcal{T}_{\mathbf{\Pi}}(x)$. Moreover, $\mathcal{T}_{\mathbf{\Pi}'}(x)$ enumerates the integer partitions with odd part sizes and *Sym* does indeed contain a Hopf subalgebra with this Poincaré series. It is the algebra of Schur-Q functions. See [10, III.8]. Thus $\mathcal{T}_{\mathbf{\Pi}}(x) \big/ \mathcal{T}_{\mathbf{\Pi}'}(x)$ is nonnegative by Corollary 4.

3.2 A test for Hopf monoids

Let $(a_n)_{n\geq 0}$ be a sequence of nonnegative integers with $a_0 = 1$. Does there exist a connected Hopf monoid $\mathbf{h}$ with $\dim \mathbf{h}[n] = a_n$ for all n? The next result provides conditions that the sequence $(a_n)_{n\geq 0}$ must satisfy for this to be the case.

Corollary 14 (The (ord/exp)-test) *For any connected Hopf monoid in species* $\mathbf{h}$*,*

$$\Big(\sum_{n\geq 0} \dim \mathbf{h}[n]\, x^n\Big) \Big/ \Big(\sum_{n\geq 0} \frac{\dim \mathbf{h}[n]}{n!}\, x^n\Big) \in \mathbb{N}[\![x]\!].$$

Proof: We make use of the *Hadamard product* of Hopf monoids [2, Corollary 8.59]. The exponential species $\mathbf{E}$ is the unit for this operation.

Consider the canonical morphism of Hopf monoids $\mathbf{L} \twoheadrightarrow \mathbf{E}$; it maps any linear order $\ell \in \mathbf{L}[I]$ to the basis element $*_I \in \mathbf{E}[I]$ [2, Section 8.5]. The Hadamard product then yields a morphism of Hopf monoids

$$\mathbf{L} \times \mathbf{h} \twoheadrightarrow \mathbf{E} \times \mathbf{h} \cong \mathbf{h}.$$

By Corollary 12, $\mathcal{E}_{\mathbf{L}\times\mathbf{h}}(x)/\mathcal{E}_{\mathbf{h}}(x) \in \mathbb{N}[\![x]\!]$. Since $\mathcal{E}_{\mathbf{L}\times\mathbf{h}}(x) = \sum_{n\geq 0} \dim \mathbf{h}[n]\, x^n$, the result follows. □

Let $a_n = \dim \mathbf{h}[n]$. Corollary 14 states that the ratio of the ordinary to the exponential generating function of the sequence $(a_n)_{n\geq 0}$ must be nonnegative. This translates into a sequence of polynomial inequalities, the first of which are as follows:

$$5a_3 \geq 3a_2a_1, \quad 23a_4 + 12a_2a_1^2 \geq 20a_3a_1 + 6a_2^2.$$

In particular, not every nonnegative sequence arises as the sequence of dimensions of a Hopf monoid.

3.3 A test for Hopf monoids over $\mathbf{E}$

Our next result is a necessary condition for a Hopf monoid in species to contain or surject onto the exponential species $\mathbf{E}$.

Given a sequence $(a_n)_{n\geq 0}$, its *binomial transform* $(b_n)_{n\geq 0}$ is defined by

$$b_n := \sum_{i=0}^{n} \binom{n}{i} (-1)^i\, a_{n-i}.$$

Corollary 15 (The E-test) *Suppose* $\mathbf{h}$ *is a connected Hopf monoid that either contains the species* $\mathbf{E}$ *or surjects onto* $\mathbf{E}$ *(in both cases as a Hopf monoid). Let* $a_n = \dim \mathbf{h}[n]$ *and* $\overline{a}_n = \dim \mathbf{h}[n]_{S_n}$*. Then the binomial transform of* $(a_n)_{n\geq 0}$ *must be nonnegative and* $(\overline{a}_n)_{n\geq 0}$ *must be nondecreasing.*

More plainly, in this setting, we must have the following inequalities:

$$a_1 \geq a_0, \qquad a_2 \geq 2a_1 - a_0, \qquad a_3 \geq 3a_2 - 3a_1 + a_0, \;\; \ldots$$

and $\overline{a}_n \geq \overline{a}_{n-1}$ for all $n \geq 1$.

Proof: Since $\mathcal{E}_{\mathbf{E}}(x) = \exp(x)$, the quotient $\mathcal{E}_{\mathbf{h}}(x)/\mathcal{E}_{\mathbf{E}}(x)$ is given by

$$b_0 + b_1 x + b_2 \frac{x^2}{2} + b_3 \frac{x^3}{3!} + \cdots ,$$

where $(b_n)_{n\geq 0}$ is the binomial transform of $(a_n)_{n\geq 0}$. It is nonnegative by Corollary 12. Replacing exponential for type generating functions yields the result for $(\overline{a}_n)_{n\geq 0}$, since $\mathcal{T}_{\mathbf{E}}(x) = \frac{1}{1-x}$. □

We make a further remark regarding connected *linearized* Hopf monoids. These are Hopf monoids of a set theoretic nature. See [2, §8.7] for details. Briefly, there are set maps $\mu_{A,B}(x,y)$ and $\Delta_{A,B}(z)$ that produce single structures (written $(x,y) \mapsto x \cdot y$ and $z \mapsto (z|_A, z/_A)$, respectively), which are compatible at the level of set maps and which produce a Hopf monoid in species when linearized. It follows that if $\mathbf{h}$ is a linearized Hopf monoid, then there is a unique morphism of Hopf monoids from $\mathbf{h}$ onto $\mathbf{E}$. Thus, Corollary 15 provides a test for existence of a linearized Hopf monoid structure on $\mathbf{h}$.

Example 16 We return to the species $\mathbf{\Pi}'$ of set partitions into distinct block sizes. We might ask if this can be made into a linearized Hopf monoid in some way (after Example 13, this would *not* be as a Hopf submonoid of $\mathbf{\Pi}$). With a_n and b_n as above, we have:

$$\begin{aligned} (a_n)_{n\geq 0} &= 1,\ 1,\ 1,\ 4,\ \ 5,\ 16,\ \ 82,\ \ 169,\ 541, \ldots , \\ (b_n)_{n\geq 0} &= 1,\ 0,\ 0,\ 3,\ -8,\ 25,\ -9,\ -119,\ 736, \ldots . \end{aligned}$$

Thus $\mathbf{\Pi}'$ fails the $\mathbf{E}$-test and the answer to the above question is negative.

3.4 A test for Hopf monoids over $\mathbf{L}$

Let $\mathbf{h}$ be a connected Hopf monoid in species. Let $a_n = \dim \mathbf{h}[n]$ and $\overline{a}_n = \dim \mathbf{h}[n]_{S_n}$. Note that the analogous integers for the species $\mathbf{L}$ of linear orders are $b_n = n!$ and $\overline{b}_n = 1$. Now suppose that $\mathbf{h}$ contains $\mathbf{L}$ or surjects onto $\mathbf{L}$ as a Hopf monoid. An obvious necessary condition for this situation is that $a_n \geq n!$ and $\overline{a}_n \geq 1$. Our next result provides stronger conditions.

Corollary 17 (The L-test) *Suppose* $\mathbf{h}$ *is a connected Hopf monoid that either contains the species* $\mathbf{L}$ *or surjects onto* $\mathbf{L}$ *(in both cases as a Hopf monoid). If* $a_n = \dim \mathbf{h}[n]$ *and* $\overline{a}_n = \dim \mathbf{h}[n]_{S_n}$*, then*

$$a_n \geq n a_{n-1} \quad \textit{and} \quad \overline{a}_n \geq \overline{a}_{n-1} \quad (\forall\, n \geq 1).$$

Proof: It follows from Corollary 12 that both $\mathcal{E}_{\mathbf{h}}(x)/\mathcal{E}_{\mathbf{L}}(x)$ and $\mathcal{T}_{\mathbf{h}}(x)/\mathcal{T}_{\mathbf{L}}(x)$ are nonnegative. These yield the first and second set of inequalities, respectively. □

Before giving an application of the corollary, we introduce a new Hopf monoid in species. A *composition* of a set I is an ordered collection of disjoint nonempty subsets of I whose union is I. The notation $ab|c$ is shorthand for the composition $\big(\{a,b\},\ \{c\}\big)$ of $\{a,b,c\}$.

Let $\mathbf{Pal}$ denote the species of set compositions whose sequence of block sizes is palindromic. So, for instance,

$$\mathbf{Pal}[\{a,b\}] = \Bbbk\big\{ab,\ a|b,\ b|a\big\}$$

and

$$\mathbf{Pal}[\{a,b,c,d,e\}] = \Bbbk\big\{abcde,\ a|bcd|e,\ ab|c|de,\ a|b|c|d|e,\ \ldots\big\}.$$

The latter space has dimension $171 = 1 + 5\binom{4}{3} + \binom{5}{2}3 + 5!$ and $\dim \mathbf{Pal}[5]_{S_5} = 4$ for the four types of palindromic set compositions shown above.

Given a palindromic set composition $\pi = \pi_1|\cdots|\pi_r$, we view it as a triple $\pi = (\pi^-, \pi^0, \pi^+)$, where π^- is the initial sequence of blocks, π^0 is the central block if this exists (if the number of blocks is odd) and otherwise it is the empty set, and π^+ is the final sequence of blocks. That is,

$$\pi^- = \pi_1|\cdots|\pi_{\lfloor r/2\rfloor}, \qquad \pi^0 = \begin{cases} \pi_{\lfloor r/2\rfloor+1} & \text{if } r \text{ is odd,} \\ \emptyset & \text{if } r \text{ is even,} \end{cases} \qquad \pi^+ = \pi_{\lceil r/2+1\rceil}|\cdots|\pi_r.$$

Given $S \subseteq I$, let

$$\pi|_S := \pi_1 \cap S \mid \pi_2 \cap S \mid \cdots \mid \pi_r \cap S\,,$$

where empty intersections are deleted. Then $\pi|_S$ is a composition of S. It is not always the case that $\pi|_S$ is palindromic. Let us say that S is *admissible* for π when it is, i.e.,

$$\#\big(\pi_i \cap S\big) = \#\big(\pi_{r+1-i} \cap S\big) \quad \text{for all } i = 1, \ldots, r.$$

In this case, both $\pi|_S$ and $\pi|_{I\setminus S}$ are palindromic.

We now define product and coproduct operations on $\mathbf{Pal}$. Fix a decomposition $I = S \sqcup T$.

Product. Given palindromic set compositions $\pi \in \mathbf{Pal}[S]$ and $\sigma \in \mathbf{Pal}[T]$, we put

$$\mu_{S,T}(\pi \otimes \sigma) := \big(\pi^-|\sigma^-, \pi^0 \cup \sigma^0, \sigma^+|\pi^+\big).$$

In other words, we concatenate the initial sequences of blocks of π and σ in that order, merge their central blocks, and concatenate their final sequences in the opposite order. The result is a palindromic composition of I. For example, with $S = \{a,b\}$ and $T = \{c,d,e,f\}$,

$$(a|b) \otimes (c|de|f) \mapsto a|c|de|f|b.$$

Coproduct. Given a palindromic set composition $\tau \in \mathbf{Pal}[I]$, we put

$$\Delta_{S,T}(\tau) := \begin{cases} \tau|_S \otimes \tau|_T & \text{if } S \text{ is admissible for } \pi, \\ 0 & \text{otherwise.} \end{cases}$$

For example, with S and T as above,

$$ad|b|e|cf \mapsto 0 \quad \text{and} \quad e|abcd|f \mapsto (ab) \otimes (e|cd|f).$$

These operations endow $\mathbf{Pal}$ with the structure of Hopf monoid, as may be easily checked.

Example 18 We ask if $\mathbf{Pal}$ contains (or surjects onto) the Hopf monoid $\mathbf{L}$. Both Hopf monoids are cocommutative and not commutative. Writing $a_n = \dim \mathbf{Pal}[n]$, we have:

$$(a_n)_{n\geq 0} \;=\; 1,\; 1,\; 3,\; 7,\; 43,\; 171,\; 1581,\; 8793,\; 108347, \ldots\,.$$

Every linear order is a palindromic set composition with singleton blocks. Thus $a_n \geq n!$ for all n and the question has some hope for an affirmative answer. However,

$$(a_n - na_{n-1})_{n\geq 1} \;=\; 0,\; 1,\; -2,\; 15,\; -44,\; 555,\; -2274,\;\; 38003, \ldots\,,$$

so $\mathbf{Pal}$ fails the $\mathbf{L}$-test and the answer to the above question is negative.

3.5 Examples of nonnegative quotients

We comment on a few examples where the quotient power series $\mathcal{E}_{\mathbf{h}}(x)/\mathcal{E}_{\mathbf{k}}(x)$ is not only nonnegative but is known to have a combinatorial interpretation as a generating function.

Example 19 Consider the Hopf monoid $\mathbf{\Pi}$ of set partitions. It contains $\mathbf{E}$ as a Hopf submonoid via the map that sends $*_I$ to the partition of I into singletons. We have

$$\mathcal{E}_{\mathbf{\Pi}}(x)/\mathcal{E}_{\mathbf{E}}(x) = \exp\bigl(\exp(x) - x - 1\bigr),$$

which is the exponential generating function for the number of set partitions into blocks of size strictly bigger than 1. This fact may also be understood with the aid of Theorem 7, as follows. The I-component of the right ideal $\mathbf{E}_+\mathbf{\Pi}$ is linearly spanned by elements of the form $*_S \cdot \pi$ where $I = S \sqcup T$ and π is a partition of T. Now, since $*_S = *_{\{i\}} \cdot *_{S\setminus\{i\}}$ (for any $i \in S$), we have that $\mathbf{E}_+\mathbf{\Pi}[I]$ is linearly spanned by elements of the form $*_{\{i\}} \cdot \pi$ where $i \in I$ and π is a partition of $I \setminus \{i\}$. But these are precisely the partitions with at least one singleton block.

Example 20 Let $\mathbf{\Sigma}$ be the Hopf monoid of set compositions defined in [2, Section 12.4]. It contains $\mathbf{L}$ as a Hopf submonoid via the map that views a linear order as a composition into singletons. This and other morphisms relating $\mathbf{E}$, $\mathbf{L}$, $\mathbf{\Pi}$ and $\mathbf{\Sigma}$, as well as other Hopf monoids, are discussed in [2, Section 12.8].

The sequence $(\dim \mathbf{\Sigma}[n])_{n\geq 0}$ is A000670 in [17]. We have

$$\mathcal{E}_{\mathbf{\Sigma}}(x) = \frac{1}{2 - \exp(x)}.$$

Moreover, it is known from [18, Exercise 5.4.(a)] that

$$\frac{1-x}{2-\exp(x)} = \sum_{n\geq 0} \frac{s_n}{n!}\, x^n$$

where s_n is the number of *threshold* graphs with vertex set $[n]$ and no isolated vertices. Together with Theorem 7, this suggests the existence of a basis for $\mathbf{\Sigma}/\mathbf{L}_+\mathbf{\Sigma}$ indexed by such graphs.

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Gelfand–Tsetlin Polytopes and Feigin–Fourier–Littelmann–Vinberg Polytopes as Marked Poset Polytopes

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Abstract. Stanley (1986) showed how a finite partially ordered set gives rise to two polytopes, called the order polytope and chain polytope, which have the same Ehrhart polynomial despite being quite different combinatorially. We generalize his result to a wider family of polytopes constructed from a poset P with integers assigned to some of its elements.

Through this construction, we explain combinatorially the relationship between the Gelfand–Tsetlin polytopes (1950) and the Feigin–Fourier–Littelmann–Vinberg polytopes (2010, 2005), which arise in the representation theory of the special linear Lie algebra.

We then use the generalized Gelfand–Tsetlin polytopes of Berenstein and Zelevinsky (1989) to propose conjectural analogues of the Feigin–Fourier–Littelmann–Vinberg polytopes corresponding to the symplectic and odd orthogonal Lie algebras.

Résumé. Stanley (1986) a montré que chaque ensemble fini partiellement ordonné permet de définir deux polyèdres, le polyèdre de l'ordre et le polyèdre des chaînes. Ces polyèdres ont le même polynôme de Ehrhart, bien qu'ils soient tout à fait distincts du point de vue combinatoire. On généralise ce résultat à une famille plus générale de polyèdres, construits à partir d'un ensemble partiellement ordonné ayant des entiers attachés à certains de ses éléments.

Par cette construction, on explique en termes combinatoires la relation entre les polyèdres de Gelfand-Tsetlin (1950) et ceux de Feigin-Fourier-Littelmann-Vinberg (2010, 2005), qui apparaissent dans la théorie des représentations des algèbres de Lie linéaires spéciales. On utilise les polyèdres de Gelfand-Tsetlin généralisés par Berenstein et Zelevinsky (1989) afin d'obtenir des analogues (conjecturés) des polytopes de Feigin-Fourier-Littelmann-Vinberg pour les algèbres de Lie symplectiques et orthogonales impaires.

Keywords: poset, polytope, semisimple Lie algebra, PBW filtration

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1 Introduction

Consider the simple complex Lie algebra $\mathfrak{sl}_n$. The irreducible representations of $\mathfrak{sl}_n$ are parametrized up to isomorphism by dominant integral weights, i.e., weakly decreasing n-tuples of integers determined up to adding multiples of $(1,\dots,1)$. Given a dominant integral weight λ, let $V(\lambda)$ denote the corresponding irreducible $\mathfrak{sl}_n$-module. The module $V(\lambda)$ has a distinguished basis, the Gelfand–Tsetlin [5] basis, parametrized by the points with integral coordinates ("integral points" or "lattice points" for short) in the Gelfand–Tsetlin polytope $\mathrm{GT}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$.

Recently, Feigin, Fourier, and Littelmann [3] constructed a different basis of $V(\lambda)$, conjecturally announced by Vinberg [8]. This basis is related to the Poincaré–Birkhoff–Witt basis of the universal enveloping algebra $U(\mathfrak{n}^-)$, where $\mathfrak{n}^-$ is the span of the negative root spaces. Again, the basis elements are parametrized by the integral points in a certain polytope $\mathrm{FFLV}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$.

Feigin, Fourier, and Littelmann used two subtle algebraic arguments to prove that their basis indeed spans $V(\lambda)$ and is linearly independent. When they had only produced the first half of the proof, they asked the second author of this paper:

Question 1.1. *[4] Is there a combinatorial explanation for the fact that* $\mathrm{GT}(\lambda)$ *and* $\mathrm{FFLV}(\lambda)$ *contain the same number of lattice points?*

This question provided the motivation for this paper. We answer it by generalizing a result of Stanley [6] on poset polytopes, as we now describe. Let P be a finite poset. Let A be a subset of P which contains all minimal and maximal elements of P. Let $\lambda = (\lambda_a)_{a\in A}$ be a vector in $\mathbf{R}^A$, which we think of as a marking of the elements of A with real numbers. We call such a triple (P,A,λ) a **marked poset**.

Definition 1.2. The **marked order polytope** of (P,A,λ) is

$$\begin{aligned}\mathscr{O}(P,A)_\lambda = \{x \in \mathbf{R}^{P-A} \mid x_p \le x_q \text{ for } p<q, \quad \lambda_a \le x_p \text{ for } a<p,\\ x_p \le \lambda_a \text{ for } p<a\},\end{aligned}$$

where p and q represent elements of $P-A$, and a represents an element of A. The **marked chain polytope** of (P,A,λ) is

$$\begin{aligned}\mathscr{C}(P,A)_\lambda = \{x \in \mathbf{R}_{\ge 0}^{P-A} \mid x_{p_1} + \cdots + x_{p_k} \le \lambda_b - \lambda_a\\ \text{for } a < p_1 < \cdots < p_k < b\},\end{aligned}$$

where a,b represent elements of A, and $p_1,\dots,p_k$ represent elements of $P-A$.

For any polytope with integer coordinates Q there exists a polynomial $E_Q(t)$, the **Ehrhart polynomial** of Q, with the following property: for every positive integer n, the n-th dilate nQ of Q contains exactly $E_Q(n)$ lattice points (see [7]). With this notion, our answer to Question 1.1 is given by the following two results.

Theorem 1.3. *For any marked poset (P,A,λ) with $\lambda \in \mathbf{Z}^A$, the marked order polytope $\mathscr{O}(P,A)_\lambda$ and the marked chain polytope $\mathscr{C}(P,A)_\lambda$ have the same Ehrhart polynomial.*

Theorem 1.4. *For every partition λ there exists a marked poset (P,A,λ) such that* $\mathrm{GT}(\lambda) = \mathscr{O}(P,A)_\lambda$ *and* $\mathrm{FFLV}(\lambda) = \mathscr{C}(P,A)_\lambda$.

We also consider the extension of these constructions to other Lie algebras. Berenstein and Zelevinsky proposed a construction of generalized Gelfand–Tsetlin polytopes [1] for other semisimple Lie algebras. For the symplectic and odd orthogonal Lie algebras, their polytopes are also in the family of marked order polytopes. Therefore Theorem 1.3 yields candidates for the Feigin–Fourier–Littelmann–Vinberg polytopes in types B_n and C_n.

The paper is organized as follows. In §2 we discuss the relevant aspects of the representation theory of the simple complex Lie algebras $\mathfrak{sl}_n$. Section 3 treats marked order and chain polytopes, and gives a bijection between their lattice points. Section 4 discusses the application of the combinatorial results of §3 to the representation theoretic polytopes that interest us.

We note that the combinatorial §3 is self-contained, and may be of independent interest beyond the representation theoretic application. A possible way to read this article is to skip §2 and continue there directly.

2 Preliminaries

Consider the simple complex Lie algebra $\mathfrak{sl}_n$. Let $\mathfrak{h}$ be the Cartan subalgebra consisting of its diagonal matrices. For $i = 1,\dots,n$, let $\varepsilon_i \in \mathfrak{h}^*$ denote the projection onto the i-th diagonal component. As $\varepsilon_1 + \cdots + \varepsilon_n = 0$, the coefficient vector of an integral weight is only determined as an element of $\mathbf{Z}^n/\langle(1,\dots,1)\rangle$. We identify an integral weight with the corresponding equivalence class of coefficient vectors. If λ is a weight and we use the symbol λ in a context where it has to be interpreted as an n-tuple $\lambda = (\lambda_1,\dots,\lambda_n)$, we use the convention that a representative has been chosen implicitly. Fix simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i = 1,\dots,n-1$. The corresponding fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. Hence dominant integral weights correspond to weakly decreasing n-tuples of integers, or partitions.

Given a dominant integral weight λ, the associated Gelfand–Tsetlin [5] polytope $\mathrm{GT}(\lambda)$ is defined as follows: Consider the board given in Figure 1.

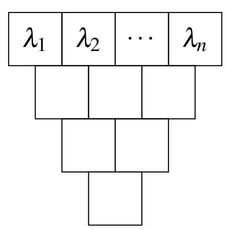

Figure 1: Board defining Gelfand–Tsetlin patterns.

Each one of the $n(n-1)/2$ empty boxes stands for a real variable. The polytope $\mathrm{GT}(\lambda) \subset \mathbf{R}^{n(n-1)/2}$ is given by the fillings of the board with real numbers with the following property: each number is less than or equal to its upper left neighbor and greater than or equal to its upper right neighbor. Note that the ambiguity in choosing an n-tuple for the weight λ amounts to an integral translation of $\mathrm{GT}(\lambda)$, and hence does not affect its number of integral points. In fact, the integral points in $\mathrm{GT}(\lambda)$ parametrize the Gelfand–Tsetlin basis of $V(\lambda)$, hence $|\mathrm{GT}(\lambda) \cap \mathbf{Z}^{n(n-1)/2}| = \dim V(\lambda)$.

Feigin, Fourier, and Littelmann [3] associate a different polytope with a dominant integral weight λ as follows: The positive roots of $\mathfrak{sl}_n$ are $\Phi_+ = \{\alpha_{i,j} \mid 0 \le i < j \le n\}$, where $\alpha_{i,j} = \varepsilon_i - \varepsilon_j$. A Dyck path is by definition a sequence $(\beta(0),\dots,\beta(k))$ in Φ_+ such that $\beta(0)$ and $\beta(k)$ are simple, and if $\beta(l) = \alpha_{i,j}$, then either $\beta(l+1)=\alpha_{i+1,j}$ or $\beta(l+1)=\alpha_{i,j+1}$. Denote the coordinates on $\mathbf{R}^{\Phi_+}$ by s_β for $\beta\in\Phi_+$. Let $\lambda=m_1\omega_1+\cdots+m_{n-1}\omega_{n-1}$. Then the polytope $\mathrm{FFLV}(\lambda)\subset\mathbf{R}^{\Phi_+}$ is given by the inequalities

$$s_\beta\geq 0$$

for all $\beta\in\Phi_+$ and

$$s_{\beta(0)}+\cdots+s_{\beta(k)}\leq m_i+\cdots+m_j$$

for all Dyck paths $(\beta(0),\ldots,\beta(k))$ such that $\beta(0)=\alpha_i$ and $\beta(k)=\alpha_j$.

For all $\alpha\in\Phi_+$, let f_α be a nonzero element of the root space $\mathfrak{g}_{-\alpha}$. Let v_λ be a highest weight vector of $V(\lambda)$. Fix any total order on Φ_+. As s ranges over the lattice points of $\mathrm{FFLV}(\lambda)$, the elements $\left(\prod_{\alpha\in\Phi_+}f_\alpha^{s_\alpha}\right)v_\lambda$ form a basis of $V(\lambda)$ [3, Th. 3.11]. Hence $|\mathrm{FFLV}(\lambda)\cap\mathbf{Z}^{\Phi_+}|=\dim V(\lambda)$.

The previous discussion shows that $|\mathrm{FFLV}(\lambda)\cap\mathbf{Z}^{\Phi_+}|=|\mathrm{GT}(\lambda)\cap\mathbf{Z}^{n(n-1)/2}|$. In the sequel, we give a combinatorial explanation and an extension of this fact.

3 Marked poset polytopes

To any finite poset P, Stanley [6] associated two polytopes in $\mathbf{R}^P$: the order polytope and the chain polytope. He showed that there is a continuous, piecewise linear bijection between them, which restricts to a bijection between their sets of integral points. In this section we construct a generalization of the order and chain polytopes, and prove the analogous result. We begin with a review of Stanley's work.

3.1 Stanley's order and chain polytopes

Let P be a finite poset. For $p,q\in P$ we say that p **covers** q, and write $p\succ q$, when $p>q$ and there is no $r\in P$ with $p>r>q$. We identify P with its **Hasse diagram**: the graph with vertex set P, having an edge going down from p to q whenever p covers q.

The **order polytope** and **chain polytope** of P are,

$$\mathscr{O}(P)=\{x\in[0,1]^P\mid x_p\leq x_q\text{ for all }p<q\},\text{ and}$$
$$\mathscr{C}(P)=\{x\in[0,1]^P\mid x_{p_1}+\cdots+x_{p_k}\leq 1\text{ for all chains }p_1<\cdots<p_k\}.$$

respectively.

Stanley proved that, even though $\mathscr{O}(P)$ and $\mathscr{C}(P)$ can have quite different combinatorial structures, they have the same Ehrhart polynomial. He did this as follows. Define the **transfer map** $\varphi:\mathbf{R}^P\to\mathbf{R}^P$ by

$$\varphi(x)_p=\begin{cases}x_p & \text{if } p \text{ is minimal,}\\ \min\{x_p-x_q\mid p\succ q\} & \text{otherwise}\end{cases}\tag{1}$$

for $x\in\mathbf{R}^P$, $p\in P$. Then:

Theorem 3.1 ([6, Theorem 3.2]). *The transfer map φ restricts to a continuous, piecewise linear bijection from $\mathscr{O}(P)$ onto $\mathscr{C}(P)$. For any $m\in\mathbf{N}$, φ restricts to a bijection from $\mathscr{O}(P)\cap\frac{1}{m}\mathbf{Z}^P$ onto $\mathscr{C}(P)\cap\frac{1}{m}\mathbf{Z}^P$.*

3.2 Marked poset polytopes

We now recall the definition of marked order and chain polytopes, and prove that they satisfy a generalization of Theorem 3.1.

An element of a poset is called **extremal** if it is maximal or minimal.

Definition 3.2. A **marked poset** (P, A, λ) consists of a finite poset P, a subset $A \subseteq P$ containing all its extremal elements, and a vector $\lambda \in \mathbf{R}^A$. We identify it with the **marked Hasse diagram**, where we label the elements $a \in A$ with λ_a in the Hasse diagram of P.

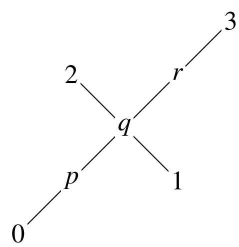

Figure 2: A marked Hasse diagram defining a partial order on the set $P = \{p,q,r\} \cup A$ with $|A| = 4$ and $\lambda = (3,2,1,0) \in \mathbf{R}^A$.

Definition 3.3. The **marked order polytope** of (P, A, λ) is

$$\begin{aligned}\mathscr{O}(P,A)_\lambda = \{x \in \mathbf{R}^{P-A} \mid\ & x_p \le x_q \text{ for } p < q,\\ & \lambda_a \le x_p \text{ for } a < p,\\ & x_p \le \lambda_a \text{ for } p < a\},\end{aligned}$$

where p and q represent elements of $P - A$, and a represents an element of A. The **marked chain polytope** of (P, A, λ) is

$$\begin{aligned}\mathscr{C}(P,A)_\lambda = \{x \in \mathbf{R}^{P-A}_{\ge 0} \mid\ & x_{p_1} + \cdots + x_{p_k} \le \lambda_b - \lambda_a\\ & \text{for } a < p_1 < \cdots < p_k < b\},\end{aligned}$$

where a, b represent elements of A, and $p_1, \ldots, p_k$ represent elements of $P - A$.

Stanley's construction is a special case of ours as follows: Given any finite poset P, add a new smallest and largest element to obtain $\tilde{P} = P \cup \{\hat{0}, \hat{1}\}$ for $\hat{0}, \hat{1} \notin P$. Let $A = \{\hat{0}, \hat{1}\}$ and $\lambda = (0,1)$. Then

$$\mathscr{O}(P) = \mathscr{O}(\tilde{P}, A)_\lambda \quad \text{and} \quad \mathscr{C}(P) = \mathscr{C}(\tilde{P}, A)_\lambda.$$

The following definitions will be needed in the proof of Theorem 3.4: The **length** of a chain $C = \{p_1 < \cdots < p_k\} \subseteq P$ is $\ell(C) = k - 1$. The **height** of $p \in P$ is the length of the longest chain ending at p. If P is graded, the height of an element is just its rank.

Theorem 3.4. *Let (P, A, λ) be a marked poset. The map $\tilde{\varphi} : \mathbf{R}^{P-A} \to \mathbf{R}^{P-A}$ defined by*

$$\tilde{\varphi}(x)_p = \min\left(\{x_p - x_q \mid p \succ q, q \notin A\} \cup \{x_p - \lambda_q \mid p \succ q, q \in A\}\right)$$

for each $p \in P - A$ restricts to a continuous, piecewise affine bijection from $\mathscr{O}(P,A)_\lambda$ onto $\mathscr{C}(P,A)_\lambda$.

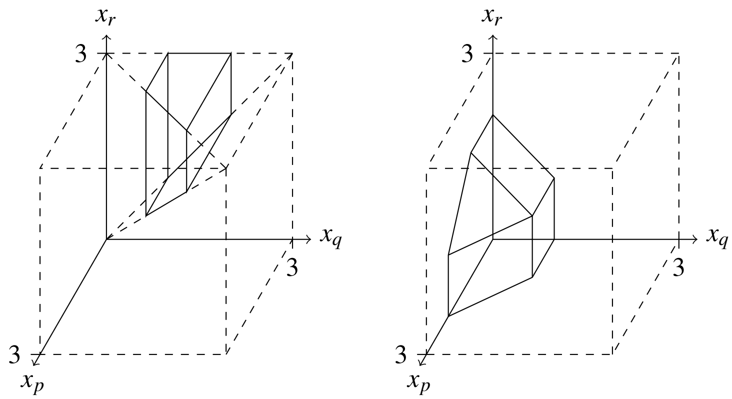

Figure 3: The marked order polytope of the marked poset in Figure 2 is given by the inequalities $0 \le x_p \le x_q \le x_r \le 3$ and $1 \le x_q \le 2$. The marked chain polytope is given by the inequalities $x_p, x_q, x_r \ge 0$, $x_p + x_q + x_r \le 3$, $x_p + x_q \le 2$, $x_q + x_r \le 2$, and $x_q \le 1$. Note that they are not combinatorially isomorphic.

The following alternative description of $\tilde{\varphi}$ may be useful. Let $\varphi : \mathbf{R}^P \to \mathbf{R}^P$ be Stanley's transfer map as defined in (1). Let $\pi : \mathbf{R}^P \to \mathbf{R}^{P-A}$ be the canonical projection which forgets the coordinates in A, and let $i : \mathbf{R}^{P-A} \to \mathbf{R}^P$ be the canonical inclusion into the fiber over $\lambda \in \mathbf{R}^A$, which adds a coordinate λ_a to each $a \in A$. Then $\tilde{\varphi} = \pi \circ \varphi \circ i$.

These maps (and some more to be defined in the proof) are illustrated in the following diagram.

$$\begin{tikzcd}
\mathbf{R}^P \arrow[r, "\varphi"] & \mathbf{R}^P \arrow[d, "\pi"] \\
\mathscr{O}(P,A)_\lambda \arrow[u, hook, "i"] \arrow[r, shift left, "\tilde{\varphi}"] & \mathscr{C}(P,A)_\lambda \arrow[l, shift left, "\tilde{\psi}"] \arrow[ul, "\psi"']
\end{tikzcd}$$

Proof. We start by showing that $\tilde{\varphi}(\mathscr{O}(P,A)_\lambda) \subseteq \mathscr{C}(P,A)_\lambda$. Let $x \in \mathscr{O}(P,A)_\lambda$ and $y = \tilde{\varphi}(x)$. Let $a, b \in A$, and $p_1, \dots, p_k \in P - A$ be such that $a < p_1 < \cdots < p_k < b$. The definition of φ implies that $y_{p_i} \le x_{p_i} - x_{p_{i-1}}$ for all $i = 2, \dots, k$ and $y_{p_1} \le x_{p_1} - \lambda_a$. Thus,

$$\begin{aligned} y_{p_1} + \cdots + y_{p_k} &\le (x_{p_1} - \lambda_a) + (x_{p_2} - x_{p_1}) + \cdots + (x_{p_k} - x_{p_{k-1}}) \\ &= x_{p_k} - \lambda_a \le \lambda_b - \lambda_a. \end{aligned}$$

Hence, $y \in \mathscr{C}(P,A)_\lambda$.

To show that $\tilde{\varphi}$ is bijective, we construct its inverse $\tilde{\psi} : \mathscr{C}(P,A) \to \mathscr{O}(P,A)$. We first define a map $\psi : \mathbf{R}^{P-A} \to \mathbf{R}^{P}$, where we define $\psi(y)_p$ recursively by going up the poset according to the rule:

$$\psi(y)_p = \begin{cases} \lambda_p & \text{if } p \in A, \\ y_p + \max\{\psi(y)_q \mid p \succ q\} & \text{if } p \notin A. \end{cases}$$

Since all elements of height 0 are in A, $\psi(y)$ is well-defined. We define $\tilde{\psi} = \pi \circ \psi$ by applying ψ and then forgetting the A-coordinates. We will prove that, when restricted to $\mathscr{C}(P,A)_\lambda$, $\tilde{\psi}$ is the inverse of $\tilde{\varphi}$.

First we show that $\tilde{\psi} \circ \tilde{\varphi}$ is the identity on $\mathscr{O}(P,A)_\lambda$. We begin by showing that $\psi \circ \tilde{\varphi} = i$; *i.e.*, that if $x \in \mathscr{O}(P,A)_\lambda$ and $y = \tilde{\varphi}(x)$ then $i(x) = \psi(y)$. We prove $i(x)_p = \psi(y)_p$ by induction on $\mathrm{ht}(p)$. The claim certainly holds for $\mathrm{ht}(p) = 0$. Suppose that we have proved it for all elements of height at most n, and let p have height $n+1$. If $p \in A$, then

$$\psi(y)_p = \lambda_p = i(x)_p$$

by definition. Otherwise, if $p \notin A$, we have

$$\begin{aligned} \psi(y)_p &= y_p + \max\{\psi(y)_q \mid p \succ q\} \\ &= y_p + \max\{i(x)_q \mid p \succ q\}. \end{aligned}$$

by the inductive hypothesis. As

$$\begin{aligned} y_p = \tilde{\varphi}(x)_p &= \pi(\varphi(i(x)))_p = \varphi(i(x))_p \\ &= \min\{i(x)_p - i(x)_q \mid p \succ q\} \\ &= i(x)_p - \max\{i(x)_q \mid p \succ q\}, \end{aligned}$$

we conclude that $\psi(y)_p = i(x)_p$, as desired.

We have shown that $\psi \circ \tilde{\varphi} = i$. By composing with the projection which forgets the A coordinates, we obtain that $\tilde{\psi} \circ \tilde{\varphi}$ is the identity on $\mathscr{O}(P,A)_\lambda$. Hence $\tilde{\varphi}$ is injective.

To prove surjectivity, let $y \in \mathscr{C}(P,A)_\lambda$ and define $x = \tilde{\psi}(y) \in \mathbf{R}^{P-A}$. We start by showing that $x \in \mathscr{O}(P,A)_\lambda$. Let $p \in P - A$. By definition,

$$x_p = \psi(y)_p = y_p + \max\{\psi(y)_q \mid p \succ q\}$$

As $y_p \geq 0$, this implies $x_p \geq \psi(y)_q$ for all q such that $p \succ q$. If $q \in A$, this says that $x_p \geq \lambda_q$. If $q \notin A$, this says that $x_p \geq x_q$. As p is arbitrary, it follows that $x \in \mathscr{O}(P,A)_\lambda$.

Finally, we claim that $\tilde{\varphi}(x) = y$. Once again, we prove that $\tilde{\varphi}(x)_p = y_p$ for all $p \in P - A$ by induction on the height of p. For height 0 this statement is vacuous. Suppose that it holds for all elements of height at most n, and consider $p \in P - A$ with $\mathrm{ht}(p) = n+1$. Then

$$\begin{aligned} \tilde{\varphi}(x)_p &= \min\{i(x)_p - i(x)_q \mid p \succ q\} \\ &= \min\{\psi(y)_p - \psi(y)_q \mid p \succ q\} \\ &= \psi(y)_p - \max\{\psi(y)_q \mid p \succ q\} \\ &= y_p + \max\{\psi(y)_q \mid p \succ q\} - \max\{\psi(y)_q \mid p \succ q\} \\ &= y_p, \end{aligned}$$

as desired. We have shown that $\tilde{\varphi} \circ \tilde{\psi}$ is the identity on $\mathscr{C}(P,A)_\lambda$, hence $\tilde{\varphi}$ is surjective.

We conclude that $\tilde{\psi} : \mathscr{C}(P,A)_\lambda \to \mathscr{O}(P,A)_\lambda$ and $\tilde{\varphi} : \mathscr{O}(P,A)_\lambda \to \mathscr{C}(P,A)_\lambda$ are inverse functions, and therefore bijective, as we wished to show. The fact that they are continuous and piecewise affine follows directly from the definitions. □

We conclude this section with the generalization of the second part of Theorem 3.1, the compatibility of the transfer map with the integral lattice. If λ is integral, then $\mathscr{O}(P,A)_\lambda$ is clearly an integral polytope, so $|\mathscr{O}(P,A)_\lambda \cap \frac{1}{m}\mathbf{Z}^{P-A}|$ is polynomial in m.

Theorem 3.5. *Let (P,A,λ) be a marked poset with $\lambda \in \mathbf{Z}^A$. Then $\tilde{\varphi}$ restricts to a bijection between $\mathscr{O}(P,A)_\lambda \cap \frac{1}{m}\mathbf{Z}^{P-A}$ and $\mathscr{C}(P,A)_\lambda \cap \frac{1}{m}\mathbf{Z}^{P-A}$. Therefore $\mathscr{O}(P,A)_\lambda$ and $\mathscr{C}(P,A)_\lambda$ have the same Ehrhart polynomial.*

Proof. This follows immediately from the proof of Theorem 3.4, as both $\tilde{\varphi}$ and $\tilde{\psi}$ preserve integrality. □

It is worth noting that Theorem 3.5 does not hold for general $\lambda \in \mathbf{R}^A$.

4 Applications

We now show how marked poset polytopes occur "in nature" in the representation theory of semisimple Lie algebras. More concretely, marked order polytopes occur as Gelfand–Tsetlin polytopes in type A, B, and C, and marked chain polytopes occur as Feigin–Fourier–Littelmann–Vinberg polytopes in type A.

4.1 *Type* A.

Let λ be a dominant integral weight for $\mathfrak{sl}_n$. Let $\mathscr{O}(P,A)_\lambda$ and $\mathscr{C}(P,A)_\lambda$ be the marked order and chain polytopes determined by the marked Hasse diagram given in Figure 4. Note that Figure 4 is obtained from

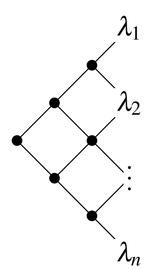

Figure 4: Marked Hasse diagram for $\mathfrak{sl}_n$.

Figure 1 by a clockwise rotation by 90°. Hence from the definitions it is immediate that $\mathrm{GT}(\lambda) = \mathscr{O}(P,A)_\lambda$. Similarly, it follows immediately from the definitions that $\mathrm{FFLV}(\lambda) = \mathscr{C}(P,A)_\lambda$. Hence the equation

$$|\mathrm{FFLV}(\lambda) \cap \mathbf{Z}^{\Phi_+}| = |\mathrm{GT}(\lambda) \cap \mathbf{Z}^{n(n-1)/2}|$$

is implied by Theorem 3.5.

It would be interesting to see whether the explicit bijection of Theorem 3.5 gives interesting information about the transition matrix between the Gelfand–Tsetlin basis and the Feigin–Fourier–Littelmann–Vinberg basis of $V(\lambda)$.

4.2 *Type* C.

Now consider the symplectic Lie algebra $\mathfrak{sp}_{2n}$. Here the role of Gelfand–Tsetlin patterns is played by the generalized Gelfand–Tsetlin patterns defined by Berenstein and Zelevinsky [1]. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sp}_{2n}$. Choose simple roots $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ such that $\alpha_i \not\perp \alpha_{i+1}$ for $i < n$ and α_n is the long root. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis of $\mathfrak{h}^*$ such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n = 2\varepsilon_n$. The corresponding fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$. This is the setting as used by Bourbaki [2]. We identify a weight λ with the n-tuple $(\lambda_1, \dots, \lambda_n)$ of its coefficients with respect to the basis $\varepsilon_1, \dots, \varepsilon_n$. Then dominant integral weights correspond to weakly decreasing n-tuples of nonnegative integers. Given a dominant integral weight λ, Berenstein and Zelevinsky define an $\mathfrak{sp}_{2n}$**-pattern** of highest weight λ to be a filling of the board in Figure 5 with nonnegative integers, such that every number is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor (if any). They show that $\dim V(\lambda)$ is the

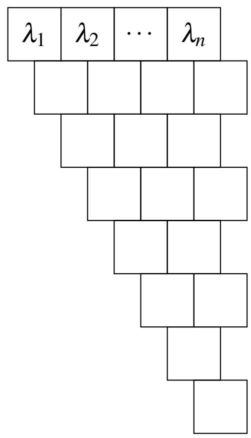

Figure 5: Board defining generalized Gelfand–Tsetlin patterns for $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n+1}$.

number of such patterns [1, Th. 4.2].

Let $\mathscr{O}(P,A)_{(\lambda,0)}$ and $\mathscr{C}(P,A)_{(\lambda,0)}$ be the marked order and chain polytopes determined by the marked Hasse diagram given in Figure 6. Note that Figure 6 is obtained from Figure 5 by a clockwise rotation by 90° and apposition of the zeroes. From the definitions it is immediate that the $\mathfrak{sp}_{2n}$-patterns of highest weight λ are the integral points in $\mathscr{O}(P,A)_{(\lambda,0)}$. This suggests the following:

Conjecture 4.1. The lattice points in $\mathscr{C}(P,A)_{(\lambda,0)}$ parametrize a PBW basis of $V(\lambda)$ for the symplectic Lie algebras, as described in §2 and in [3, Theorem 3.11].

Indeed, this conjecture is proved in an article in preparation by Feigin, Fourier, and Littelmann. [4]

4.3 *Type* B.

For the odd orthogonal Lie algebra $\mathfrak{o}_{2n+1}$, the situation is a bit more complicated. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{o}_{2n+1}$. Choose simple roots $\alpha_1, \dots, \alpha_n \in \mathfrak{h}^*$ such that $\alpha_i \not\perp \alpha_{i+1}$ for $i < n$ and α_n is the short root. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis of $\mathfrak{h}^*$ such that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $i < n$ and $\alpha_n = \varepsilon_n$. The corresponding fundamental weights are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i < n$ and $\omega_n = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_n)$. This is the setting as used by Bourbaki [2].

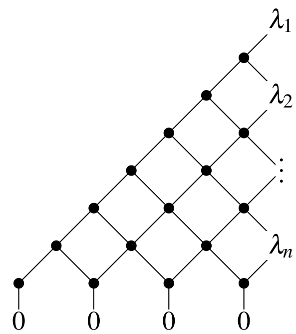

Figure 6: Marked Hasse diagram for $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_{2n+1}$.

We identify a weight λ with the n-tuple $(\lambda_1,\dots,\lambda_n)$ of its coefficients with respect to the basis $\varepsilon_1,\dots,\varepsilon_n$. Then dominant integral weights correspond to weakly decreasing n-tuples in $\frac{1}{2}\mathbf{Z}_{\geq 0}$ such that either all or none of the components are integers. Given a dominant integral weight λ, Berenstein and Zelevinsky [1] define an $\mathfrak{o}_{2n+1}$**-pattern** of highest weight λ to be a filling of the board in Figure 5 with elements of $\frac{1}{2}\mathbf{Z}_{\geq 0}$ such that every number is bounded from above by its upper left neighbor and bounded from below by its upper right neighbor (if any), and such that all numbers which possess an upper right neighbor are congruent to λ_1 modulo $\mathbf{Z}$. Let $R(\lambda)$ be the set of $\mathfrak{o}_{2n+1}$-patterns of highest weight λ.

As in type C, let $\mathscr{O}(P,A)_{(\lambda,0)}$ be the marked order polytope defined by the marked Hasse diagram in Figure 6. Then $R(\lambda)\subset\mathscr{O}(P,A)_{(\lambda,0)}$, but $R(\lambda)$ does not consist of the integral points, but of the points determined by more complicated congruence conditions. Namely, decompose

$$P-A=P'\cup P''\cup P''',$$

where P', P'', and P''' consist of all elements in P of height 1, 2, and ≥ 3, respectively, that are not contained in A. Then $R(\lambda)$ consists of all $x\in\mathscr{O}(P,A)_{(\lambda,0)}\cap(\frac{1}{2}\mathbf{Z})^{P-A}$ such that $x_p+\lambda_1\in\mathbf{Z}$ for all $p\in P''\cup P'''$. Hence $S(\lambda)=\tilde{\varphi}(R(\lambda))$ consists of all

$$y\in\mathscr{C}(P,A)_{(\lambda,0)}\cap\left((\tfrac{1}{2}\mathbf{Z})^{P'\cup P''}\times\mathbf{Z}^{P'''}\right)$$

such that

$$\max\{y_q : p\succ q\}+y_p+\lambda_1\in\mathbf{Z}$$

for all $p\in P''$. From the point of view taken in this article, $S(\lambda)$ appears to be the most natural candidate to parametrize a PBW basis of [3] in type C. Note that the elements of $S(\lambda)$ can not appear directly as exponent vectors of a PBW basis, as their components are not necessarily integral, so we are missing at least a change of coordinates in this case.

Question 4.2. *Is there a way to modify $S(\lambda)$ so that it parametrizes a PBW basis of $V(\lambda)$ for the odd orthogonal Lie algebras, as described in §2 and in [3, Theorem 3.11]?*

4.4 Type D.

The generalized Gelfand–Tsetlin polytopes [1] for the even orthogonal Lie algebras $\mathfrak{o}_{2n}$ are not marked order polytopes, so our methods do not apply here. It would be interesting to find a suitable modification of our results to this case.

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Hyperplane Arrangements and Diagonal Harmonics

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Abstract. In 2003, Haglund's bounce statistic gave the first combinatorial interpretation of the q, t-Catalan numbers and the Hilbert series of diagonal harmonics. In this paper we propose a new combinatorial interpretation in terms of the affine Weyl group of type A. In particular, we define two statistics on affine permutations; one in terms of the Shi hyperplane arrangement, and one in terms of a new arrangement — which we call the Ish arrangement. We prove that our statistics are equivalent to the area' and bounce statistics of Haglund and Loehr. In this setting, we observe that bounce is naturally expressed as a statistic on the root lattice. We extend our statistics in two directions: to "extended" Shi arrangements and to the bounded chambers of these arrangements. This leads to a (conjectural) combinatorial interpretation for all integral powers of the Bergeron-Garsia nabla operator applied to elementary symmetric functions.

Résumé. En 2003, la statistique bounce de Haglund a donné la première interprétation combinatoire de la somme des nombres q, t-Catalan et de la série de Hilbert des harmoniques diagonaux. Dans cet article nous proposons une nouvelle interprétation combinatoire à partir du groupe de Weyl affine de type A. En particulier, nous définissons deux statistiques sur les permutations affines; l'une à partir de l'arrangement d'hyperplans Shi, et l'autre à partir d'un nouvel arrangement — que nous appelons l'arrangement Ish. Nous prouvons que nos statistiques sont équivalentes aux statistiques area' et bounce de Haglund et Loehr. Dans ce contexte, nous observons que bounce s'exprime naturellement comme une statistique sur le réseau des racines. Nous prolongeons nos statistiques dans deux directions: arrangements Shi "étendus", et chambres bornées associées. Cela conduit à une interprétation (conjecturale) combinatoire pour toutes les puissances entières de l'opérateur nabla de Bergeron-Garsia appliqué aux fonctions symétriques élémentaires.

Keywords: Shi arrangement, Ish arrangement, affine permutations, diagonal harmonics, Catalan numbers, nabla operator, parking functions

1 Introduction

1.1 Diagonal Harmonics

The symmetric group $\mathfrak{S}(n)$ acts on the polynomial ring $S = \mathbb{Q}[x_1, \ldots, x_n]$ by permuting variables. Newton showed that the subring of $\mathfrak{S}(n)$-invariant polynomials is generated by the algebraically independent

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power sum polynomials: $p_k = \sum_{i=1}^n x_i^k$ for $k = 1, 2, \ldots, n$. It is known that the coinvariant ring $R = S/(p_1, \ldots, p_n)$ is a graded version of the regular representation of $\mathfrak{S}(n)$, with Hilbert series

$$\sum_{i=0}^{n} \dim R_i \, q^i = \prod_{j=1}^{n} (1 + q + q^2 + \cdots + q^j) = [n]_q!.$$

The dual ring $S^* = \mathbb{Q}[\partial/\partial x_1, \ldots, \partial/\partial x_n]$ acts on S via the pairing $(\partial/\partial x_i)x_j = \delta_{ij}$, hence the coinvariant ring is isomorphic to the quotient $S^*/(p_1^*, \ldots, p_n^*)$, where $p_k^* = \sum_{i=1}^n (\partial/\partial x_i)^k$ for $k = 1, \ldots, n$. On the other hand, this quotient is naturally isomorphic to the submodule $H \subseteq S$ annihilated by the p_k^*:

$$H = \{f \in S : p_k^* \, f = 0 \text{ for all } k\}.$$

This H is called the ring of harmonic polynomials since, in particular, p_2^* is the standard Laplacian operator on S.

Now consider the ring $DS = \mathbb{Q}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ of polynomials in two sets of commuting variables, together with the diagonal action of $\mathfrak{S}(n)$, which permutes the x variables and the y variables *simultaneously*. Weyl [31] showed that the $\mathfrak{S}(n)$-invariant subring of DS is generated by the polarized power sums: $p_{k,\ell} = \sum_{i=1}^n x_i^k y_i^\ell$ for all $k + \ell > 0$. Hence the ring of diagonal coinvariants $DR = DS/(p_{k,\ell} : k + \ell > 0)$ is naturally isomorphic to the ring of diagonal harmonic polynomials:

$$DH = \{f \in DS : \sum_{i=1}^{n} (\partial/\partial x_i)^k (\partial/\partial y_i)^\ell \, f = 0 \text{ for all } k + \ell > 0\}.$$

The diagonal action preserves the bigrading of DS by x-degree and y-degree, hence DH is a bigraded $\mathfrak{S}(n)$-module. The bigraded Hilbert series

$$\mathcal{DH}(n; q, t) := \sum_{i,j=0}^{n} \dim(DH)_{i,j} \; q^i t^j \tag{1}$$

has beautiful and remarkable properties. The study of $\mathcal{DH}(n; q, t)$ was initiated by Garsia and Haiman (see [13]) and is today an active area of research.

1.2 Some Arrangements

Let $\{e_1, e_2, \ldots, e_n\}$ be the standard basis for $\mathbb{R}^n$. Given $v \in \mathbb{R}^n$ and $k \in \mathbb{R}$, we will often use the notation "$v = k$" as shorthand for the set $\{x : (x, v) = k\} \subseteq \mathbb{R}^n$, where $(\cdot, \cdot)$ is the standard inner product. Consider the following three arrangements of hyperplanes, respectively called the Coxeter arrangement, Shi arrangement, and affine arrangement of type A_{n-1}:

$$\begin{aligned}
\mathsf{Cox}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a = 0\}, \\
\mathsf{Shi}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a \in \{0, 1\}\}, \\
\mathsf{Aff}(n) &:= \{e_i - e_j = a : 1 \le i < j \le n, \, a \in \mathbb{Z}\}.
\end{aligned}$$

Since all hyperplanes in this paper contain the line $e_1 + e_2 + \cdots + e_n$, we will typically restrict these arrangements to the $(n-1)$-dimensional space

$$\mathbb{R}_0^n := \{e_1 + e_2 + \cdots + e_n = 0\}.$$

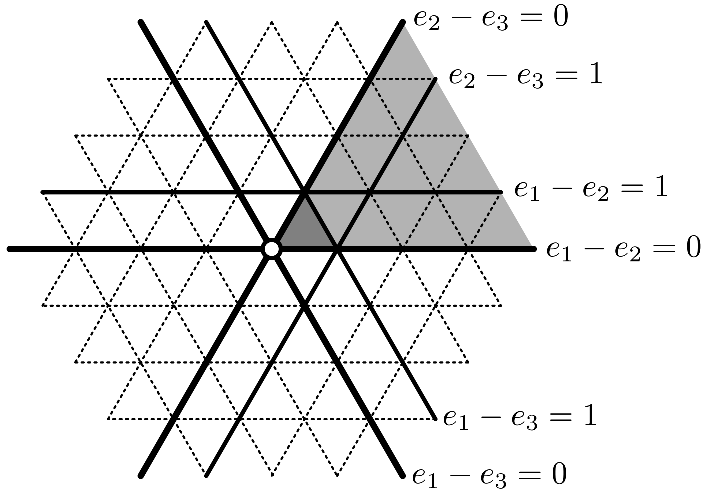

Fig. 1: Some arrangements in $\mathbb{R}^3_0$

If $\mathcal{A}$ is an arrangement in a space V then the connected components of the complement $V - \cup_{H \in \mathcal{A}} H$ are called chambers. We will refer to chambers of the Coxeter arrangement as cones; and refer to affine chambers as alcoves. Let $C_\circ$ denote the dominant cone, which satisfies the coordinate inequalities

$$e_1 > e_2 > \cdots > e_n,$$

and let $A_\circ$ denote the fundamental alcove, satisfying

$$e_1 > e_2 > \cdots > e_n > e_1 - 1.$$

Figure 1 displays the arrangements $\mathsf{Cox}(3)$, $\mathsf{Shi}(3)$, and $\mathsf{Aff}(3)$ in $\mathbb{R}^3_0$, with the dominant cone and fundamental alcove shaded. The Shi arrangement was introduced by Jian-Yi Shi (see [23, Chapter 7]) in his description of the Kazhdan-Lusztig cells for certain affine Weyl groups.

1.3 Symmetric Group

The symmetric group $\mathfrak{S}(n)$ has a faithful representation as a group of isometries of $\mathbb{R}^n_0$ generated by the set

$$S = \{s_1, s_2, \ldots, s_{n-1}\},$$

where s_i is the reflection in the hyperplane $e_i - e_{i+1} = 0$. The reflection s_i corresponds in $\mathfrak{S}(n)$ to the transposition of adjacent symbols $(i, i+1)$.

The symmetric group acts simply-transitively on the cones of the Coxeter arrangement $\mathsf{Cox}(n)$. By convention, let the dominant cone $C_\circ$ correspond to the identity permutation; then for any permutation $w \in \mathfrak{S}(n)$ the cone $wC_\circ$ satisfies

$$e_{w(1)} > e_{w(2)} > \cdots > e_{w(n)}.$$

1.4 Affine Symmetric Group

Now let s_n denote the reflection in the *affine* hyperplane $e_1 - e_n = 1$. The linear reflections $\{s_1, s_2, \ldots, s_{n-1}\}$ together with the affine reflection a_n generate the affine Weyl group of type $\tilde{A}_{n-1}$. This group acts simply-transitively on the set of alcoves, where the fundamental alcove $A_\circ$ corresponds to the identity element of the group. Note that $A_\circ$ is a (non-regular) simplex in $\mathbb{R}^n_0$ whose facets are supported by the reflecting hyperplanes of the generators $\{s_1, s_2, \ldots, s_n\}$.

Lusztig [21] introduced an affine version of the symmetric group, whose combinatorial properties were developed further by Björner and Brenti [6]. We define $\tilde{\mathfrak{S}}(n)$ as the group of infinite permutations $\tilde{w} : \mathbb{Z} \to \mathbb{Z}$ satisfying:

- $\tilde{w}(k + n) = \tilde{w}(k) + n$ for all $k \in \mathbb{Z}$,
- $\tilde{w}(1) + \tilde{w}(2) + \cdots + \tilde{w}(n) = \binom{n+1}{2}$.

The first property says that $\tilde{w}$ is periodic and the second fixes a frame of reference. The elements of $\tilde{\mathfrak{S}}(n)$ are called affine permutations, and $\tilde{\mathfrak{S}}(n)$ is the affine symmetric group. Following Björner and Brenti, we will usually express an affine permutation $\tilde{w} \in \tilde{\mathfrak{S}}(n)$ using the window notation:

$$\text{“}\tilde{w} = [\tilde{w}(1), \tilde{w}(2), \cdots, \tilde{w}(n)].\text{”}$$

For integers $i < j$ with $i \not\equiv j \pmod n$ we will write $((i, j)) : \mathbb{Z} \to \mathbb{Z}$ to denote the “affine tranposition” that swaps the elements in positions $i + kn$ and $j + kn$ for all $k \in \mathbb{Z}$. We could also write $((i, j)) = \prod_k (i + kn, j + kn)$. Lusztig proved that the correspondence $s_i \leftrightarrow ((i, i + 1))$ defines an isomorphism between the affine symmetric group and the affine Weyl group of type A. Here the affine tranposition $((i, j))$ corresponds to the reflection in the affine hyperplane

$$e_{i-n(\lceil i/n\rceil-1)} - e_{j-n(\lceil j/n\rceil-1)} = \left\lceil \frac{i}{n} \right\rceil - \left\lceil \frac{j}{n} \right\rceil. \tag{2}$$

In particular, note that the generator $s_i = ((i, i + 1))$ corresponds to $e_i - e_{i+1} = 0$ for $1 \le i \le n - 1$, and $s_n = ((n, n + 1))$ corresponds to $e_1 - e_n = 1$.

1.5 The Ish Arrangement

Finally, we introduce a new hyperplane arrangement, called the Ish arrangement. Like the Shi arrangement, the Ish arrangement begins with the $\binom{n}{2}$ linear hyperplanes of the Coxeter arrangement and then adds another $\binom{n}{2}$ affine hyperplanes:

$$\mathsf{Ish}(n) := \mathsf{Cox}(n) \cup \{e_i - e_n = a : 1 \le i \le n - 1,\ a \in \{1, \ldots, n - i\}\}.$$

Figure 2 displays the arrangements $\mathsf{Shi}(3)$ and $\mathsf{Ish}(3)$. Note that each has 16 chambers and 4 bounded chambers. There is an important reason for this: the arrangements $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ share the same *characteristic polynomial*, as we now show.

To avoid extra notation, we will use a non-standard definition of the characteristic polynomial. This definition is due to Crapo and Rota, and was applied extensively by Athanasiadis — see Stanley [29, Lecture 5] for details. Let $\mathcal{A}$ be an arrangement of *finitely many* hyperplanes in $\mathbb{R}^n$. Suppose further that

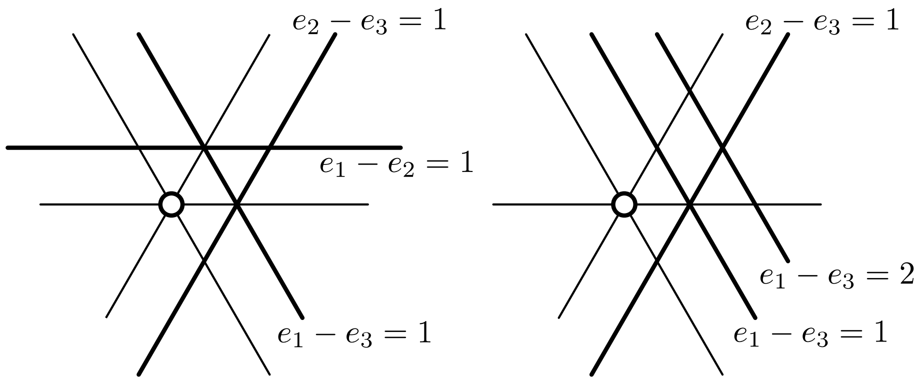

Fig. 2: The arrangements $\mathsf{Shi}(3)$ and $\mathsf{Ish}(3)$

each of these hyperplanes has an equation with integer coefficients. Then, given a (large) finite field $\mathbb{F}_q$ with q elements, we may consider the reduced arrangement $\mathcal{A}_q$ in $\mathbb{F}_q^n$. It turns out that (for all but finitely many q), the number of points of $\mathbb{F}_q^n$ **not** on any hyperplane of $\mathcal{A}_q$ is given by a polynomial in q, called the characteristic polynomial of $\mathcal{A}$:

$$\chi(\mathcal{A}, q) = \#\left(\mathbb{F}_q^n - \cup_{H \in \mathcal{A}_q} H\right) = q^n - \# \cup_{H \in \mathcal{A}_q} H.$$

The characteristic polynomial of the Shi arrangement is well known (cf. [29, Theorem 5.16]). Our new result is the following. (Proof omitted.)

Theorem 1. *The Shi arrangement and the Ish arrangement share the same characteristic polynomial, viz.*

$$\chi(\mathsf{Ish}(n), q) = q\,(q-n)^{n-1}.$$

The following is a standard result on real hyperplane arrangements. Let $\mathcal{A}$ be an arrangement in a real d-dimensional space V and suppose that the normals to $\mathcal{A}$ span a subspace $U \subseteq V$ of dimension k — called the rank of $\mathcal{A}$. If $k < d$ then $\mathcal{A}$ has no bounded chambers; its chambers that have bounded intersection with U are called relatively bounded.

Zaslavsky's Theorem (see, e.g., Theorem 2.5 of [29]). *Let $\mathcal{A}$ be a real arrangement with dimension d and rank k. Then:*

- *The number of chambers of $\mathcal{A}$ is $(-1)^d \chi(A, -1)$.*
- *The number of relatively bounded chambers of $\mathcal{A}$ is $(-1)^k \chi(A, 1)$.*

If we think of $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ as arrangements in the space $\mathbb{R}_0^n$, then $d = k = n-1$.

Corollary 1. *The arrangements $\mathsf{Shi}(n)$ and $\mathsf{Ish}(n)$ have the same number of chambers — i.e. $(n+1)^{n-1}$ — and the same number of bounded chambers — i.e. $(n-1)^{n-1}$.*

Open Problem. Find a bijective proof of the corollary.

In a recent joint paper with Rhoades [2], we have expanded on the relationship between the Shi and Ish arrangements. The paper shows that the relationship between these objects is deep and somewhat mysterious. In it we give nice combinatorial labels of the regions and show that these are equinumerous, but the problem of a bijective proof is still open.

2 Two Statistics on Shi Chambers

Now we define two statistics — called shi and ish — on the chambers of a Shi arrangement (more generally, on the elements of the group $\tilde{\mathfrak{S}}(n)$). The first statistic is well known and the second is new. Each statistic counts certain kind of "inversions" of an affine permutation. We begin by defining these.

2.1 Affine Inversions

Let w be an element of the (finite) symmetric group $\mathfrak{S}(n)$. If $w(i) > w(j)$ for indices $1 \leq i < j \leq n$ we say that the tranposition (i,j) is an inversion of w — equivalently, this means that the hyperplane $e_i - e_j = 0$ separates the cone $wC_\circ$ from the dominant cone $C_\circ$. The number of inversions of w is called its length.

In the affine symmetric group $\tilde{\mathfrak{S}}(n)$, there is again a correspondence between hyperplanes and transpositions. Recall that the affine transpositions $((i,j))$ and $((i',j'))$ coincide if $i' = i + kn$ and $j' = j + kn$ for some $k \in \mathbb{Z}$, in which case they represent the same hyperplane (2). Hence, each affine transposition has a standard representative in the set

$$\tilde{T} := \{((i,j)) : \ 1 \leq i \leq n, \ i < j, \ i \not\equiv j \mod n\} \subseteq \tilde{\mathfrak{S}}(n).$$

Given an affine permutation $\tilde{w} \in \tilde{\mathfrak{S}}(n)$ and an affine transposition $((i,j)) \in \tilde{T}$ such that $\tilde{w}(i) > \tilde{w}(j)$, we say that $((i,j))$ is an affine inversion of $\tilde{w}$ — equivalently, the hyperplane (2) separates the alcove $\tilde{w}A_\circ$ from the fundamental alcove $A_\circ$. Again, the (affine) length of $\tilde{w}$ is its number of affine inversions.

2.2 The shi statistic

Each chamber of the Shi arrangement contains a set of alcoves and among these is a unique alcove of minimum length — which we call the representing alcove of the chamber, or just a Shi alcove. This defines an injection from Shi chambers into the affine symmetric group. Figure 3 displays the representing alcoves for $\mathsf{Shi}(3)$, labeled by affine permutations. We have labeled the Shi hyperplanes with their corresponding affine transpositions,

$$\mathsf{Shi}(n) = \{((i,j)) : \quad 1 \leq i \leq n, \quad i < j < n + i\}.$$

Definition 2.1. Given a Shi chamber with representing alcove A, let $\mathsf{shi}(A)$ denote the number of Shi hyperplanes separating A from the fundamental alcove $A_\circ$. Equivalently, if $A = \tilde{w}A_\circ$ for affine permutation $\tilde{w} \in \mathfrak{S}(n)$, then $\mathsf{shi}(\tilde{w})$ is the number of affine inversions $((i,j))$ of $\tilde{w}$ satisfying $i < j < n + i$.

For example, consider the permutation $\tilde{w} = [1,5,0]$ in the figure. The inversions of $\tilde{w}$ are $((1,3)), ((2,3)), ((2,4)), ((2,6))$, and hence $\tilde{w}$ has length 4. However, only three of these — viz. $((1,3)), ((2,3)), ((2,4))$ — come from Shi hyperplanes, hence $\mathsf{shi}(\tilde{w}) = 3$.

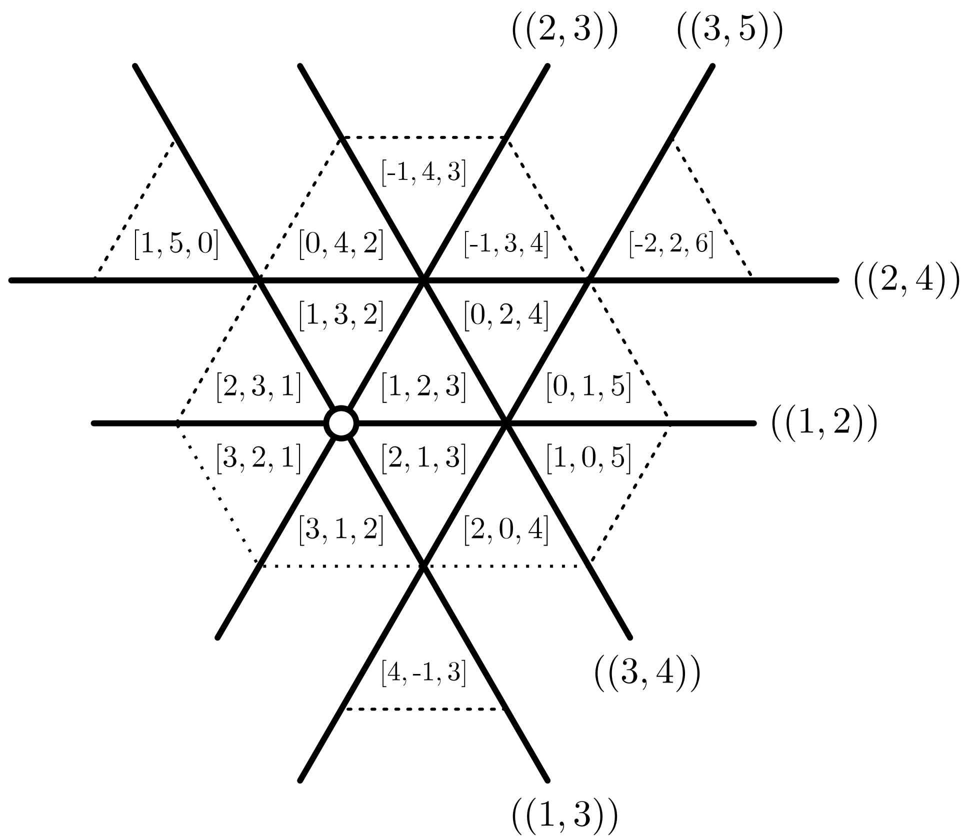

Fig. 3: Chambers of $\mathsf{Shi}(3)$ labeled by affine permutations

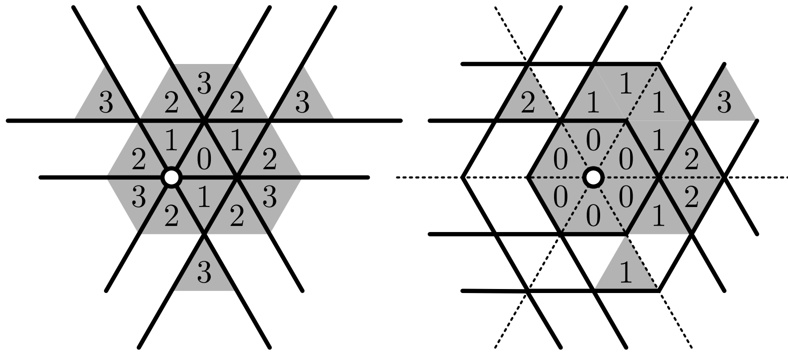

Fig. 4: The shi and ish statistics on the chambers of $\mathsf{Shi}(3)$

2.3 *The* ish *statistic*

To give a natural definition for our second statistic, we must discuss the coset space $\tilde{\mathfrak{S}}(n)/\mathfrak{S}(n)$. By abuse of notation, let $\mathfrak{S}(n)$ denote the subgroup of $\tilde{\mathfrak{S}}(n)$ generated by the subset

$$I = \{s_1, \ldots, s_{n-1}\} \subseteq \{s_1, \ldots, s_{n-1}, s_n\} = S.$$

In the language of Coxeter groups we say that $\mathfrak{S}(n)$ is a parabolic subgroup of $\tilde{\mathfrak{S}}(n)$. When $W = \tilde{\mathfrak{S}}(n)$ the standard notation for this is to write $\mathfrak{S}(n) = W_I$. Then each affine permutation $\tilde{w}$ has a canonical decomposition

$$\tilde{w} = w_I \tilde{w}^I,$$

where $w_I \in W_I$ is a finite permutation and $\tilde{w}^I \in W$ is the unique coset representative of minimum (affine) length. Combinatorially, $[\tilde{w}^I(1), \ldots, \tilde{w}^I(n)]$ is the increasing rearrangement of $[\tilde{w}(1), \ldots, \tilde{w}(n)]$ and w_I is the finite permutation needed to achieve the rearrangement. Geometrically, alcoves of the form $A = \tilde{w}^I A_\circ$ are contained in the dominant cone $C_\circ$; hence $\tilde{w} A_\circ = w_I A$ is contained in the cone $w_I C_\circ$.

We define the ish statistic in terms of minimal coset representatives.

Definition 2.2. Consider a Shi chamber with representing alcove A and suppose that $A = \tilde{w} A_\circ$. Its minimal coset representative $\tilde{w}^I A_\circ$ is an alcove in the dominant cone $C_\circ$. Let $\mathsf{ish}(A)$ denote the number of hyperplanes of the form $e_i - e_n = a$ (with $1 \leq i \leq n-1$ and $a \in \mathbb{Z}$) separating $\tilde{w}^I A_\circ$ from the fundamental alcove $A_\circ$. Equivaently, let $\mathsf{ish}(\tilde{w})$ denote the number of affine inversions of $\tilde{w}^I$ of the form $((n, j))$.

Two notes: In order to facilitate later generalization, we have defined ish in terms of *all* hyperplanes of the form $e_i - e_n = a$. In our current context, however, only the Ish hyperplanes (i.e. $a \in \{1, \ldots, n-i\}$) will contribute. We also emphasize the fact that ish **is a statistic on the (representing alcoves of) Shi chambers, not on the Ish chambers**. It seems that the chambers of the Ish arrangement are not so natural.

For example, consider the affine permutation $\tilde{w} = [\text{-}1, 4, 3]$, as shown in Figure 3. It is contained in the cone $[1, 3, 2]\, C_\circ$ and its increasing rearrangement is $[\text{-}1, 3, 4]$. Hence, it has parabolic decomposition

$$[\text{-}1, 4, 3] = \tilde{w} = w_I \tilde{w}^I = [1, 3, 2]\, [\text{-}1, 3, 4].$$

The inversions of $\tilde{w}^I = [\text{-}1, 3, 4]$ are $((2,4))$ and $((3,4))$, of which only the second is an Ish hyperplane; hence $\mathsf{ish}(\tilde{w}) = 1$. In Figure 4 we have displayed the shi and ish statistics for all chambers of $\mathsf{Shi}(3)$. (Note: to compute ish by hand, one may extend the Ish hyperplanes from the dominant cone to the other cones by reflection.) Their joint-distribution is recorded in the following table:

	shi \ ish	0	1	2	3
	0	1			
	1	2	1		
shi	2	2	3	1	
	3	1	2	2	1

2.4 Theorems and a Conjecture

We will make four assertions and then describe our state of knowledge about them (i.e. whether each is a Theorem or a Conjecture). We will use the following notation.

Recall from (1) that $\mathcal{DH}(n;q,t)$ denotes the bigraded Hilbert series of the ring of diagonal harmonic polynomials. Define

$$\mathsf{Shi}(n;q,t) := \sum_A q^{\mathsf{ish}(A)} t^{\binom{n}{2} - \mathsf{shi}(A)},$$

where the sum is taken over representing alcoves A for the chambers of the arrangement $\mathsf{Shi}(n)$. We say that an alcove is **positive** if it is contained in the dominant cone $C_\circ$ (i.e. if A is on the "positive" side of each generating hyperplane). Let $\mathsf{Shi}_+(n;q,t)$ denote the corresponding sum over positive Shi alcoves. Finally, consider the standard q-integer, q-factorial, and q-binomial coefficient:

$$[a]_q = 1 + q + \cdots + q^{a-1},$$
$$[a]_q! = [a]_q [a-1]_q \cdots [2]_q [1]_q,$$
$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[a-b]_q! [b]_q!}.$$

Assertions.

(1) $\mathsf{Shi}(n;q,t) = \mathcal{DH}(n;q,t)$, and hence is symmetric in q and t.

(2) $q^{\binom{n}{2}} \mathsf{Shi}(n;q,1/q) = [n+1]_q^{n-1}$.

(3) $\mathsf{Shi}_+(n;q,t)$ is equal to Garsia and Haiman's q,t-Catalan number, and hence is symmetric in q and t.

(4) $q^{\binom{n}{2}} \mathsf{Shi}_+(n;q,1/q) = \frac{1}{[n]_q} \begin{bmatrix} 2n \\ n-1 \end{bmatrix}_q$, the q-Catalan number.

In particular, note that $q^{\binom{n}{2}} \mathsf{Shi}_+(n;q,1/q)$ is equal to the sum of $q^{\mathsf{shi}(A) + \mathsf{ish}(A)}$ over the positive Shi alcoves A. For $n = 3$ we may compute this sum using the data in Figure 4 to obtain

$$1 + q^2 + q^3 + q^4 + q^6 = \frac{[6]_q [5]_q}{[3]_q [2]_q} = \frac{1}{[3]_q} \begin{bmatrix} 6 \\ 2 \end{bmatrix}_q,$$

which is a q-Catalan number. One may check that the other three assertions are also true in the case $n = 3$.

We will now state the main theorem of this paper, but omit its proof.

Main Theorem. *There exists a natural bijection from the* $(n+1)^{n-1}$ *chambers of the Shi arrangement to parking functions which sends our statistics* $(\mathsf{ish}, \mathsf{shi} - \binom{n}{2})$ *to the statistics* $(\mathsf{bounce}, \mathsf{area}')$ *of Haglund and Loehr [16].*

This allows us to clarify the Assertions.

Status. Each of the following depends on the Main Theorem.

(1) **Conjecture.** This is equivalent to a conjecture of Haglund and Loehr [16] (known in a different form to Haiman). No combinatorial explanation of the q, t symmetry is known.

(2) **Theorem.** This is equivalent to a theorem of Loehr [17].

(3) **Theorem.** This follows from theorems of Garsia and Haglund [9, 10]. No combinatorial explanation of the q, t symmetry is known.

(4) **Theorem.** This is equivalent to a theorem of Haglund [14], which was later proved bijectively by Loehr [18].

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The author thanks Christos Athanasiadis, Steve Griffeth, Nick Loehr, Eric Sommers, and Greg Warrington for helpful discussions. The original idea for this work was inspired by discussions with Steve Griffeth and by the paper [8] of Fishel and Vazirani.

In January 2010 we learned that Mark Haiman had a result in 2007 (unpublished) equivalent to our Main Theorem, although he approached the topic from the opposite direction — from dilations of the fundamental alcove instead of hyperplane arrangements. This should appear in a forthcoming paper.

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The Shi arrangement and the Ish arrangement

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Abstract. This paper is about two arrangements of hyperplanes. The first — the *Shi arrangement* — was introduced by Jian-Yi Shi to describe the Kazhdan-Lusztig cells in the affine Weyl group of type A. The second — the *Ish arrangement* — was recently defined by the first author who used the two arrangements together to give a new interpretation of the q,t-Catalan numbers of Garsia and Haiman. In the present paper we will define a mysterious "combinatorial symmetry" between the two arrangements and show that this symmetry preserves a great deal of information. For example, the Shi and Ish arrangements share the same characteristic polynomial, the same numbers of regions, bounded regions, dominant regions, regions with c "ceilings" and d "degrees of freedom", etc. Moreover, all of these results hold in the greater generality of "deleted" Shi and Ish arrangements corresponding to an arbitrary subgraph of the complete graph. Our proofs are based on nice combinatorial labellings of Shi and Ish regions and a new set partition-valued statistic on these regions.

Résumé. Cet article traite de deux arrangements d'hyperplans. Le premier — *arrangement Shi* — a été introduit par Jian-Yi Shi pour décrire les cellules de Kazhdan-Lusztig du groupe de Weyl affine de type A. Le deuxième — *arrangement Ish* — a été récemment défini par le premier auteur pour donner une nouvelle interprétation des nombres q,t-Catalan de Garsia et Haiman. Ici nous définissons une mystérieuse "symétrie combinatoire" entre les deux arrangements et nous montrons que cette symétrie conserve un grand nombre d'informations. Par exemple, les arrangements Shi et Ish ont le même polynôme caractéristique, le même nombre de régions, de régions bornées, de régions dominantes, de régions avec c "plafonds" et d "degrés de liberté", etc. En outre, ces résultats se généralisent aux arrangements Shi et Ish "deleted" correspondant à un sous-graphe arbitraire du graphe complet. Nos preuves reposent sur des étiquetages combinatoires des régions Shi et Ish, et sur une nouvelle statistique associée.

Keywords: hyperplane arrangement, nonnesting partition, product formula

1 Introduction

A *hyperplane arrangement* is a finite collection of affine hyperplanes in Euclidean space. Some of the nicest arrangements come from the reflecting hyperplanes of Coxeter groups. In particular, the *Coxeter arrangement of type A* (also known as the *braid arrangement*) is the arrangement in $\mathbb{R}^n$ defined by

$$\mathrm{Cox}(n) := \{x_i - x_j = 0 : 1 \leq i < j \leq n\}. \tag{1}$$

Here $\{x_1, \ldots, x_n\}$ are the standard coordinate functions on $\mathbb{R}^n$.

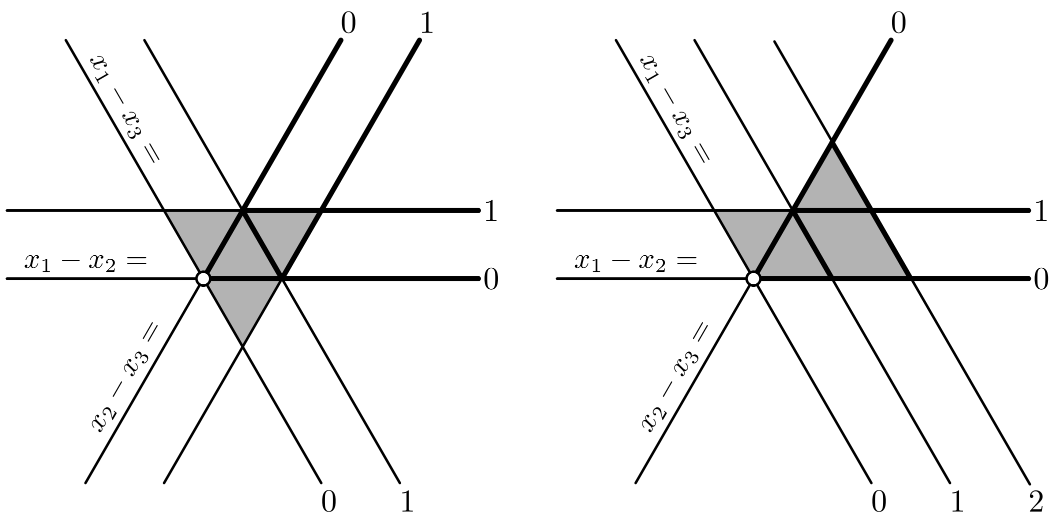


Fig. 1: The arrangements $\mathrm{Shi}(3)$ (left) and $\mathrm{Ish}(3)$ (right)

Postnikov and Stanley (9) introduced the idea of a *deformation* of the Coxeter arrangement — this is an affine arrangement each of whose hyperplanes is parallel to some hyperplane of the Coxeter arrangement. In the present paper we will study two specific deformations of the Coxeter arrangement and we will observe a deep similarity between them. The first is the *Shi arrangement* which was one of Postnikov and Stanley's motivating examples:

$$\mathrm{Shi}(n) := \mathrm{Cox}(n) \cup \{x_i - x_j = 1 : 1 \leq i < j \leq n\}. \tag{2}$$

This arrangement was defined by Jian-Yi Shi (11, Chapter 7) to study of the Kazhdan-Lusztig cellular structure of the affine Weyl group of type A. The second is the *Ish arrangement*, recently defined by the first author (1):

$$\mathrm{Ish}(n) := \mathrm{Cox}(n) \cup \{x_1 - x_j = i : 1 \leq i < j \leq n\}. \tag{3}$$

He used the Shi and Ish arrangements to give a new description of the q,t-Catalan numbers of Garsia and Haiman in terms of the affine Weyl group of type A. Figure 1 displays the arrangements $\mathrm{Shi}(3)$ and $\mathrm{Ish}(3)$. (Note that the normals to the hyperplanes of either $\mathrm{Shi}(n)$ or $\mathrm{Ish}(n)$ span the hyperplane $x_1 + x_2 + \cdots + x_n = 0$. Hence we will always draw their restrictions to this space.)

The heart of this paper is the following correspondence between Shi and Ish hyperplanes. The correspondence is natural to state but we find it geometrically mysterious. We will call this a "combinatorial symmetry":

$$x_i - x_j = 1 \longleftrightarrow x_1 - x_j = i \quad \text{for} \quad 1 \leq i < j \leq n \tag{4}$$

This symmetry allows us to define *deleted* versions of the Shi and Ish arrangements. Let $\binom{[n]}{2}$ denote the set of pairs ij satisfying $1 \leq i < j \leq n$ and consider a simple loopless graph $G \subseteq \binom{[n]}{2}$. The *deleted Shi and Ish arrangements* are defined as follows:

$$\mathrm{Shi}(G) := \mathrm{Cox}(n) \cup \{x_i - x_j = 1 : ij \in G\}, \tag{5}$$

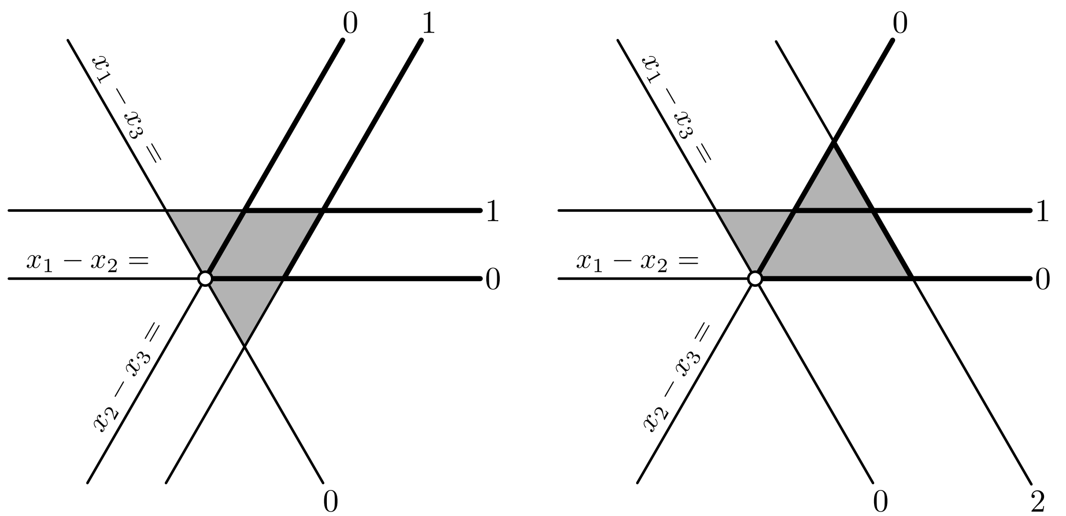


Fig. 2: The arrangements $\mathrm{Shi}(G)$ (left) and $\mathrm{Ish}(G)$ (right) corresponding to the "chain" $G = \{12, 23\} \subseteq \binom{[3]}{2}$

$$\mathrm{Ish}(G) := \mathrm{Cox}(n) \cup \{x_1 - x_j = i : ij \in G\}. \tag{6}$$

The arrangement $\mathrm{Shi}(G)$ was first considered by Athanasiadis (3). Note that $\mathrm{Shi}(G)$ (resp. $\mathrm{Ish}(G)$) interpolates between the Coxeter arrangement and the Shi (resp. Ish) arrangement. That is, if $\emptyset \in \binom{[n]}{2}$ is the "empty" graph and $K_n = \binom{[n]}{2}$ is the "complete" graph, we have that $\mathrm{Shi}(\emptyset) = \mathrm{Ish}(\emptyset) = \mathrm{Cox}(n)$, $\mathrm{Shi}(K_n) = \mathrm{Shi}(n)$, and $\mathrm{Ish}(K_n) = \mathrm{Ish}(n)$. Figure 2 displays the arrangements $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ corresponding to the "chain" $G = \{12, 23\} \subseteq \binom{[3]}{2}$.

To state our Main Theorem right away, we need a few definitions. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{R}^n$. The *intersection poset* $\mathfrak{L}(\mathcal{A})$ of $\mathcal{A}$ is the collection of all nonempty intersections of the hyperplanes in $\mathcal{A}$ partially ordered by **reverse** inclusion. The poset $\mathfrak{L}(\mathcal{A})$ has the structure of a graded meet-semilattice (13) with unique minimal element given by the 'empty intersection' $\mathbb{R}^n$. Let $\mu : \mathfrak{L}(\mathcal{A}) \times \mathfrak{L}(\mathcal{A}) \to \mathbb{Z}$ be the Möbius function of $\mathfrak{L}(\mathcal{A})$ (see (12)). The *characteristic polynomial* $\chi_{\mathcal{A}}(p) \in \mathbb{Z}[p]$ of $\mathcal{A}$ is the polynomial

$$\chi_{\mathcal{A}}(p) := \sum_{X \in \mathfrak{L}(\mathcal{A})} \mu(\mathbb{R}^n, X) p^{\dim(X)}. \tag{7}$$

The characteristic polynomial of $\mathcal{A}$ determines the Hilbert series of the Orlik-Solomon algebra of $\mathcal{A}$ and the Hilbert series of the cohomology ring of the complement of the complexification $\mathcal{A}_{\mathbb{C}} := \{\mathbb{C} \otimes_{\mathbb{R}} H : H \in \mathcal{A}\}$ of $\mathcal{A}$ in $\mathbb{C}^n$ (where cohomology is computed with coefficients in $\mathbb{C}$) (8).

If $\mathcal{A}$ is a hyperplane arrangement in $\mathbb{R}^n$, the connected components of the complement $\mathbb{R}^n \smallsetminus \bigcup_{H \in \mathcal{A}} H$ are called the *regions* of $\mathcal{A}$. If R is a region of $\mathcal{A}$, the *recession cone* of R is

$$\mathrm{Rec}(R) := \{v \in \mathbb{R}^n \,:\, R + v \subseteq R\}. \tag{8}$$

Since R is convex, $\mathrm{Rec}(R)$ is closed under nonnegative linear combinations. The dimension of the cone $\mathrm{Rec}(R)$ is called the number of *degrees of freedom* of the region R. The region R is bounded if and only if it has zero degrees of freedom.

Let R be a region of the arrangement $\mathcal{A}$ in $\mathbb{R}^n$. The hyperplanes in $\mathcal{A}$ decompose the topological closure $\bar{R}$ into *faces* of various dimensions. A hyperplane $H \in \mathcal{A}$ is called a *wall* of R if H is the affine span of a codimension-one face of R. A wall H of R is called a *ceiling* of R if H is **not** a linear hyperplane and if H does **not** separate R from the origin.

Main Theorem. *Let $G \subseteq \binom{[n]}{2}$ be a graph on n vertices and let c and d be nonnegative integers. The hyperplane arrangements $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ have the following objects in common:*

- *the characteristic polynomial;*
- *the number of regions with c ceilings and d degrees of freedom;*
- *the number of dominant regions with c ceilings.*

Proof. These are Theorems 3.1, 5.2, and 5.3, respectively. □

For example, here are the joint distributions of ceilings (c) and degrees of freedom (d) for the arrangements in Figures 1 and 2, respectively.

$$\begin{array}{cc|ccc} & & & d & \\ & & 1 & 2 & 3 \\ \hline & 0 & & & 6 \\ c & 1 & 3 & 6 & \\ & 2 & 1 & & \end{array} \qquad\qquad \begin{array}{cc|ccc} & & & d & \\ & & 1 & 2 & 3 \\ \hline & 0 & & & 6 \\ c & 1 & 2 & 4 & \\ & 2 & 1 & & \end{array}$$

We find it mysterious that the correspondence $x_i - x_j = 1 \leftrightarrow x_1 - x_j = i$ preserves so much information. However, it does not preserve everything. It can be shown using Figure 1 that $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ do not have the same intersection poset in general. It can also be shown that $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ do not have the same Tutte polynomial in general and that the Orlik-Solomon algebras of these arrangements are not in general graded-isomorphic (although by our Main Theorem these algebras do have the same Hilbert series). The authors are interested in a more compact statement of the Main Theorem as well as a more conceptual understanding of what is preserved and what is not preserved by the Shi/Ish duality.

The remainder of the paper is structured as follows. In **Section 2** we review and introduce some notation related to set partitions. In **Section 3** we prove that the arrangements $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ have the same characteristic polynomial and give an explicit expression for this polynomial. This expression is an alternating sum involving G-analogs of the Stirling numbers; it appears to be new even for the deleted Shi arrangements. In **Section 4** we introduce a nice combinatorial labeling of the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ and explain how this labeling can be used to read off the number of ceilings and number of degrees of freedom of a region. In **Section 5** we introduce a new set partition-valued statistic called "ceiling partition" on the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ and show that the ceiling partition refines the 'number of ceilings' statistic. We use the ceiling partitions to prove refined versions of Parts 2 and 3 of the Main Theorem.

2 Background on Set Partitions

If π is a set partition of $[n]$, the *arc diagram* of π is the graph on the vertex set $[n]$ whose edges are exactly the pairs ij with $i < j$ such that i and j are blockmates in π and there does not exist k with $i < k < j$ such that i, j, and k are blockmates in π. Figure 3 shows the arc diagram

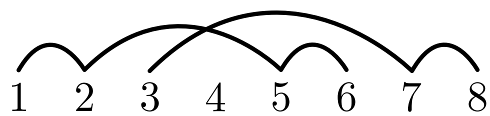


Fig. 3: A partition of $[8]$ with type $(1, 0, 1, 1, 0, 0, 0, 0)$

of the partition $\{\{1, 2, 5, 6\}, \{3, 7, 8\}, \{4\}\}$ of $[8]$. The *type* of a partition π of $[n]$ is the sequence $(r_1, \ldots, r_n) \in \mathbb{N}^n$, where r_i is the number of blocks of π of size i. The type of the partition of $[8]$ in Figure 3 is $(1, 0, 1, 1, 0, 0, 0, 0)$.

For $1 \leq k \leq n$, recall that the *Stirling number (of the second kind)* $\mathrm{Stir}(n, k)$ counts the number of partitions of $[n]$ with exactly k blocks. In order to study the deleted Shi and Ish arrangements we will find it convenient to introduce a 'G-analog' of the Stirling numbers. For a graph $G \subseteq \binom{[n]}{2}$, call a partition π of $[n]$ *G-deleted* if every edge in the arc diagram of π is also an edge in G. Denote by $\mathrm{Stir}(G, k)$ the number of G-deleted partitions of $[n]$ with exactly k blocks. For example, if $G = K_n$ is the complete graph we have that $\mathrm{Stir}(K_n, k) = \mathrm{Stir}(n, k)$ and if $G = \emptyset$ is the graph with no edges we have $\mathrm{Stir}(\emptyset, k) = \delta_{k,n}$.

A partition π of $[n]$ is called *nonnesting* if there do not exist numbers $a < b < c < d$ such that both ad and bc are arcs in the arc diagram of π. The partition in Figure 3 is not nonnesting because the arcs 37 and 56 nest. The partition $\{\{1, 3\}, \{2, 4\}\}$ of $[4]$ is nonnesting.

For $1 \leq d \leq n$, a partition π of $[n]$ is said to have d *connected components* if $d - 1$ is the maximal length of a sequence $1 < i_1 < i_2 < \ldots < i_{d-1} \leq n$ such that π refines the set partition

$$\{\{1, 2, \ldots, i_1\}, \{i_1 + 1, i_1 + 2, \ldots, i_2\}, \ldots, \{i_{d-1}, i_{d-1} + 1, \ldots, n\}\}. \tag{9}$$

(Some authors refer to connected components as "blocks" - we reserve this term for the elements of a set partition.)

3 Characteristic Polynomials

Our first result is that the characteristic polynomials of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ are equal for any graph $G \subseteq \binom{[n]}{2}$. We give an explicit formula for this common characteristic polynomial as an alternating sum involving the G-deleted Stirling numbers $\mathrm{Stir}(G, k)$.

Theorem 3.1. *Let $G \subseteq \binom{[n]}{2}$ be a graph on n vertices. The characteristic polynomials of the deleted Shi and Ish arrangement are given by:*

$$\chi_{\mathrm{Shi}(G)}(p) = \chi_{\mathrm{Ish}(G)}(p) = p \sum_{k=0}^{n-1} (-1)^k \mathrm{Stir}(G, n-k)(p-k-1)(p-k-2)\cdots(p-n+1). \tag{10}$$

Proof. (Sketch.) Since the hyperplanes in $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ have defining equations over $\mathbb{Z}$, we can apply the finite fields method of Crapo and Rota (5) to compute these characteristic polynomials. Given a sufficiently large prime p, this amounts to showing that the number of points in the complements in $\mathbb{F}_p^n$ of the reductions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ modulo p are both counted by the alternating sum in Equation 10 (here $\mathbb{F}_p$ denotes the field with p elements). We use the Principle of Inclusion-Exclusion (12, Chapter 2) to perform this enumeration. □

Headley (6) showed that the characteristic polynomial of the 'full' Shi arrangement $\mathrm{Shi}(n)$ is given by $\chi_{\mathrm{Shi}(n)}(p) = p(p-n)^{n-1}$. Athanasiadis (4) later showed that when the graph G avoids certain induced subgraphs, the arrangement $\mathrm{Shi}(G)$ is (inductively) free and the characteristic polynomial $\chi_{\mathrm{Shi}(G)}(p)$ factors as a product $p\prod_{i=1}^{n}(p-c_i)$, where the numbers $c_i \in \mathbb{N}$ can be read off from the graph G. The authors are not aware of an expression for the characteristic polynomial of $\mathrm{Shi}(G)$ for general graphs G in previous literature.

Corollary 3.2. *Let $G \subset \binom{[n]}{2}$ be a graph on n vertices. The arrangements $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ have the same number of regions.*

Proof. Combine Theorem 3.1 and Zaslavsky's Theorem (14). □

No bijective proof of Corollary 3.2 is known.

4 Labelling the Regions

In this section we will describe how to label the regions of the Shi and Ish arrangements $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$. These labels will be called *Shi ceiling diagrams* and *Ish ceiling diagrams* and will be designed to keep track of the ceilings and degrees of freedom of these regions. (Something like "Shi floor diagrams" appeared earlier in the work of Athanasiadis and Linusson (2).)

4.1 Shi ceiling diagrams

We denote by C the *dominant cone* in the Coxeter arrangement $\mathrm{Cox}(n)$ defined by the coordinate inequalities $x_1 > x_2 > \ldots > x_n$. The action of the symmetric group $\mathfrak{S}(n)$ on $\mathbb{R}^n$ by coordinate permutation induces a simply transitive action of $\mathfrak{S}(n)$ on the regions of $\mathrm{Cox}(n)$, so that every region of $\mathrm{Cox}(n)$ can be uniquely written as wC for some $w \in \mathfrak{S}(n)$.

Let $G \subseteq \binom{[n]}{2}$ be a fixed graph on n vertices. For $w \in \mathfrak{S}(n)$, we define a poset $\Phi^+(G,w)$ (i) as follows. As a set, $\Phi^+(G,w)$ consists of the following affine hyperplanes in $\mathrm{Shi}(G)$:

$$\Phi^+(G,w) := \{x_{w(i)} - x_{w(j)} = 1 \,:\, i<j, w(i)<w(j), w(i)w(j) \in G\}. \tag{11}$$

On can see that the hyperplanes in $\Phi^+(G,w)$ are precisely the hyperplanes in $\mathrm{Shi}(G)$ which intersect the cone wC. The partial order on $\Phi^+(G)$ is defined by

$$(x_{w(i')} - x_{w(j')} = 1) \preceq (x_{w(i)} - x_{w(j)} = 1) \tag{12}$$

if

$$w(i) \le w(i') < w(j') \le w(j). \tag{13}$$

Given $w \in \mathfrak{S}(n)$, we will label the regions of $\mathrm{Shi}(G)$ contained in wC with order ideals in the poset $\Phi^+(G,w)$.

Theorem 4.1. *There is a bijection between regions of $\mathrm{Shi}(G)$ which are contained in the cone wC and order ideals (down-closed sets) in the poset $\Phi^+(G,w)$. This bijection is given by sending a region R contained in wC to the set of hyperplanes in $\Phi^+(G,w)$ which do not separate R from the origin. The* **maximal** *elements of this ideal are the ceilings of R.*

(i) The notation $\Phi^+(G,w)$ is due to the fact that when $G = K_n$ is the complete graph and $w \in \mathfrak{S}(n)$ is the identity permutation, the poset $\Phi^+(G,w)$ is isomorphic to the type A_{n-1} positive root poset.

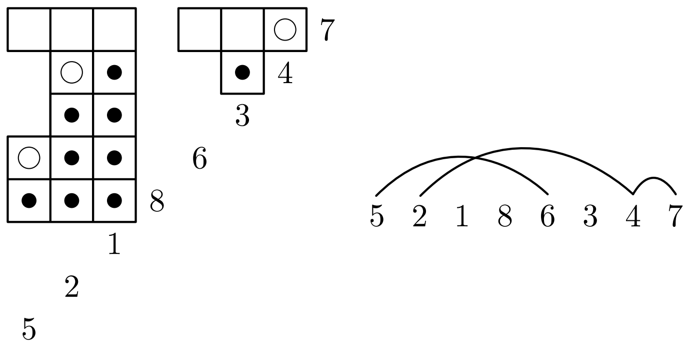


Fig. 4: An order ideal (left) and a Shi ceiling diagram (right)

Proof. Omitted. □

For example, let $n = 8$, $G = K_8$, and $w = 51286347$. The poset $\Phi^+(G, w)$ is shown on the left of Figure 4, with partial order increasing up and to the left. The elements of this poset which contain circles form an order ideal in this poset. The hollow circles are the maximal elements of this order ideal. The ceilings of the corresponding region of $\mathrm{Shi}(8)$ are $x_5 - x_8 = 1, x_1 - x_6 = 1$, and $x_3 - x_7 = 1$.

Given a region R of $\mathrm{Shi}(G)$, define the *Shi ceiling diagram* of R as follows. Let $w \in \mathfrak{S}(n)$ be the unique permutation such that $R \subseteq wC$. Write the one-line notation $w(1)w(2)\dots w(n)$ of w on a line. For every maximal element $x_{w(i)} - x_{w(j)} = 1$ of the order ideal in $\Phi^+(G, w)$ corresponding to R as in Theorem 4.1, draw an arc connecting $w(i)$ and $w(j)$. Theorem 4.1 guarantees that the partition π of $[n]$ generated by $i \sim j$ if $w(i)$ and $w(j)$ are connected by an arc is nonnesting. The pair (w, π) is the Shi ceiling diagram of $[n]$. The Shi ceiling diagram of the region of $\mathrm{Shi}(8)$ from the previous paragraph is shown on the right of Figure 4. The corresponding pair (w, π) is $(51286347, \{\{1, 4\}, \{2, 5\}, \{3\}, \{6, 8\}, \{7\}\})$. Figure 5 shows the arrangement $\mathrm{Shi}(3)$ with its regions labeled by their Shi ceiling diagrams.

Shi ceiling diagrams label the regions of $\mathrm{Shi}(G)$. They also can be used to read of the degrees of freedom of a region.

Lemma 4.2. *The set of Shi ceiling diagrams of the regions of* $\mathrm{Shi}(G)$ *is exactly the set of pairs* (w, π) *where* $w \in \mathfrak{S}(n)$ *and* π *is a nonnesting partition of* $[n]$ *satisfying:*

- *if* $i < j$ *is an arc in the arc diagram of* π*, then* $w(i) < w(j)$*,*
- *if* $i < j$ *is an arc in the arc diagram of* π*, then* $w(i)w(j) \in G$*.*

Moreover, if (w, π) *is a pair of a permutation* $w \in \mathfrak{S}(n)$ *and a nonnesting partition* π *of* $[n]$ *satisfying the two conditions above, then there exists a unique region* R *of* $\mathrm{Shi}(G)$ *with Shi ceiling diagram* (w, π)*. The number of ceilings of* R *is the number of arcs in the arc diagram of* π*. The number of degrees of freedom of* R *is the number of connected components of* π*.*

Proof. Omitted. □

For example, the region of $\mathrm{Shi}(8)$ whose Shi ceiling diagram is shown in Figure 4 has two degrees of freedom.

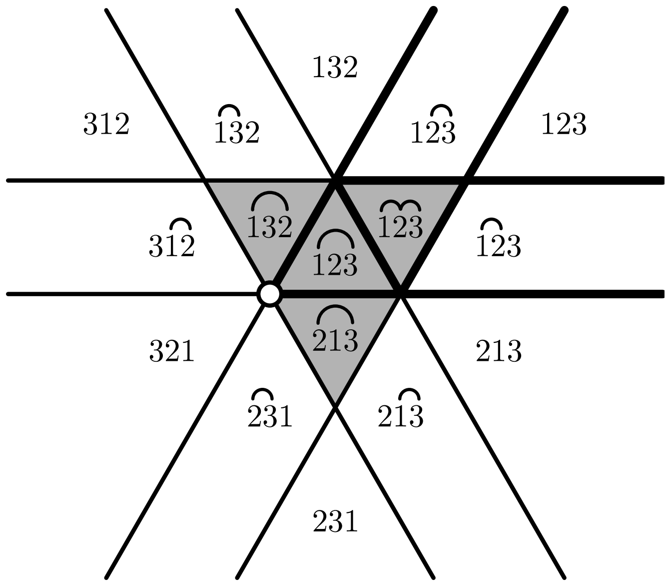


Fig. 5: The regions of $\mathrm{Shi}(3)$ and their Shi ceiling diagrams.

4.2 Ish ceiling diagrams

In order to compare the Shi and Ish arrangements, we will introduce an Ish-analog of the Shi ceiling diagrams. Fix a graph $G \subseteq \binom{[n]}{2}$.

Given a permutation $w \in \mathfrak{S}(n)$, we define a poset $\Psi^+(G,w)$ as follows. As a set, $\Psi^+(G,w)$ is the following collection of affine hyperplanes in $\mathrm{Ish}(G)$:

$$\Psi^+(G,w) := \{x_1 - x_j = i \,:\, i < j \in G, w^{-1}(i) < w^{-1}(j)\}. \tag{14}$$

It can be shown that $\Psi^+(G,w)$ is exactly the set of hyperplanes in $\mathrm{Ish}(G)$ which intersect the region wC of $\mathrm{Cox}(n)$. The partial order on $\Psi^+(G,w)$ is generated by

$$(x_1 - x_j = i) \prec (x_1 - x_{j'} = i') \tag{15}$$

if $j < j'$ or $i < i'$. In analogy with the case of the Shi arrangement, regions of $\mathrm{Ish}(G)$ which are contained in the cone wC are in a natural bijection with order *filters* in $\Psi^+(G,w)$.

Theorem 4.3. *There is a bijection between regions of* $\mathrm{Ish}(G)$ *which are contained in the cone* wC *and order filters (up-closed sets) in the poset* $\Psi^+(G,w)$*. This bijection is given by sending a region* R *to the set of hyperplanes in* $\Psi^+(G,w)$ *which do not separate* R *from the origin. The* **minimal** *elements of the ideal corresponding to* R *are the ceilings of* R*.*

Proof. Omitted. □

It is convenient to express Theorem 4.3 with a picture. Given $w \in \mathfrak{S}(n)$, we draw $w(1), w(2), \ldots, w(n)$ on a line. For each j to the right of 1, we draw $j-1$ boxes above the symbol j. If we identify the ith

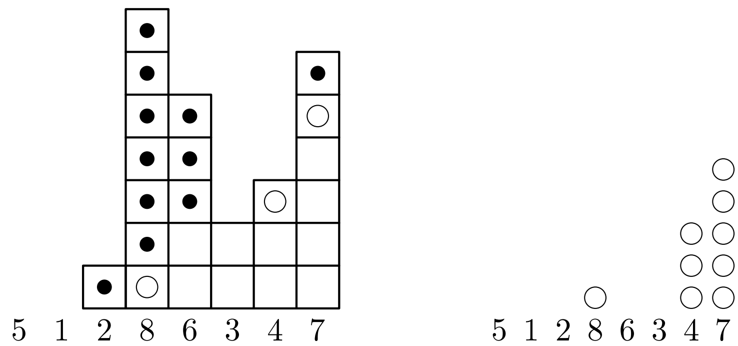


Fig. 6: An order filter (left) and an Ish ceiling diagram (right)

box above j with the hyperplane $x_1 - x_j = i$ and erase the boxes which correspond to hyperplanes not in $\mathrm{Ish}(G)$, we obtain the poset $\Psi^+(G, w)$; the partial order increases up and to the left. Given a region R of $\mathrm{Ish}(G)$ contained in wC, we can draw its order filter in $\Psi^+(G, w)$ using this pictorial representation.

For example, let $n = 8$, $G = K_8$, and $w = 51286347$. Figure 6 shows the poset $\Psi^+(G, w)$, with partial order increasing up and to the left. The boxes in this poset which contain circles form an order filter; the minimal elements in this order filter are the hollow circles. The ceilings of the corresponding region R are $x_1 - x_8 = 1, x_1 - x_4 = 3$, and $x_1 - x_7 = 5$.

The pictorial representation of order filters in the posets $\Psi^+(G, w)$ can be used to define Ish-analogs of Shi ceiling diagrams. Given a region R of $\mathrm{Ish}(G)$, the *Ish ceiling diagram* of R is the pair (w, ϵ) defined as follows. We let $w \in \mathfrak{S}(n)$ be the unique permutation such that R is contained in the region wC of $\mathrm{Cox}(n)$. We draw the poset $\Psi^+(G, w)$ and the order filter corresponding to R as in the previous paragraph. We let $\epsilon = \epsilon_1 \dots \epsilon_n$ be the sequence of nonnegative integers defined by setting $\epsilon_j = 0$ if there is no minimal element of the order filter of R above $w(j)$ in this representation and by setting $\epsilon_j = i$ if there is a minimal element in the order filter of R above $w(j)$ and this minimal element is $x_1 - x_{w(j)} = i$. We (w, ϵ) visually by drawing $w(1), \dots, w(n)$ on a line and placing ϵ_i circles above $w(i)$ for all i.

The right of Figure 6 shows the Ish ceiling diagram corresponding to the order ideal on the left of Figure 6. The corresponding pair (w, ϵ) is $(51286347, 00010035)$. Figure 7 shows the regions of $\mathrm{Ish}(3)$ labeled with their Ish ceiling diagrams.

The Ish ceiling diagrams label the regions of $\mathrm{Ish}(G)$ and can be used to read off the degrees of freedom of a region.

Lemma 4.4. *The set of Ish ceiling diagrams of the regions of* $\mathrm{Ish}(G)$ *is exactly the set of pairs* (w, ϵ) *where* $w \in \mathfrak{S}(n)$ *and* $\epsilon = \epsilon_1 \dots \epsilon_n$ *is a sequence of nonnegative integers satisfying:*

- $\epsilon_i < w(i)$ *for all* i*,*
- *if* $\epsilon_i > 0$*, then* $w^{-1}(1) < i$*,*
- *if* $\epsilon_i > 0$*, then* $\epsilon_i < w(i) \in G$*, and*
- *if* $i < j$ *and* $\epsilon_i, \epsilon_j > 0$*, then* $\epsilon_i < \epsilon_j$*.*

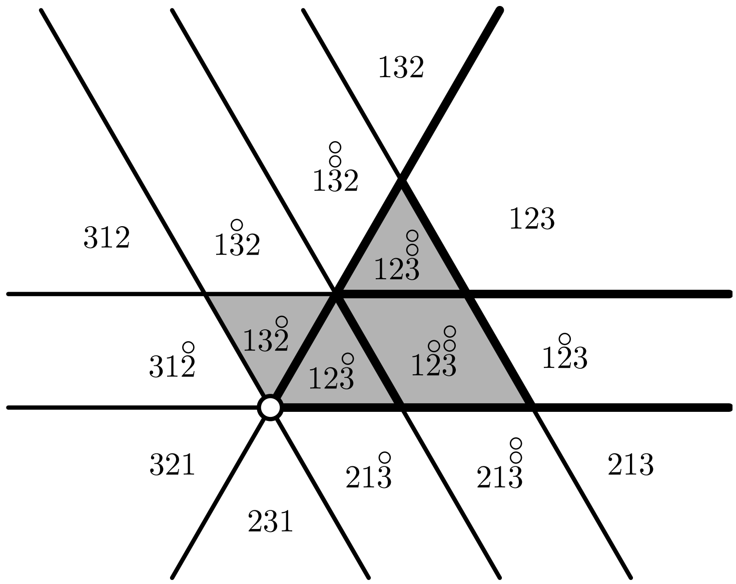


Fig. 7: The arrangement $\mathrm{Ish}(3)$ labeled with Ish ceiling diagrams

Given a pair (w, ϵ) *satisfying the four properties listed above, there exists a unique region* R *of* $\mathrm{Ish}(G)$ *with Ish ceiling diagram* (w, ϵ)*. The number of ceilings of* R *is the number of positive entries of* ϵ*. The number of degrees of freedom of* R *is* $n + w^{-1}(1) - j$*, where* $j = \max\{i : \epsilon_i > 0\}$ *if* ϵ *is not the zero sequence and* $j = w^{-1}(1)$ *if* ϵ *is the zero sequence.*

Proof. Omitted. □

For example, the region R of $\mathrm{Ish}(8)$ whose Ish ceiling diagram is shown in Figure 6 has $8 + 2 - 8 = 2$ degrees of freedom.

5 Counting the regions

In this section we use Shi and Ish ceiling diagrams to obtain our equidistribution results regarding the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$. To do this, we will introduce a new set partition-valued statistic on the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ called "ceiling partition". Two regions with the same ceiling partition will also have the same number of ceilings. We will prove that the bistatistic (ceiling partition, degrees of freedom) has the same joint distribution on the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ and compute this joint distribution explicitly. This will imply that the joint distribution of (number of ceilings, degrees of freedom) is the same on the regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$. We will also give an explicit bijection between the dominant regions of $\mathrm{Shi}(G)$ and $\mathrm{Ish}(G)$ which preserves ceiling partitions (thus preserving number of ceilings).

Definition 5.1. 1. Let R be a region of $\mathrm{Shi}(G)$. The *ceiling partition* of R is the set partition τ of $[n]$ generated by $i \sim j$ if $x_i - x_j = 1$ is a hyperplane in $\mathrm{Shi}(G)$ and a ceiling of R.

2. Let R be a region of $\mathrm{Ish}(G)$. The *ceiling partition* of R is the set partition τ of $[n]$ generated by $i \sim j$ if $x_1 - x_j = i$ is a hyperplane in $\mathrm{Ish}(G)$ and a ceiling of R.

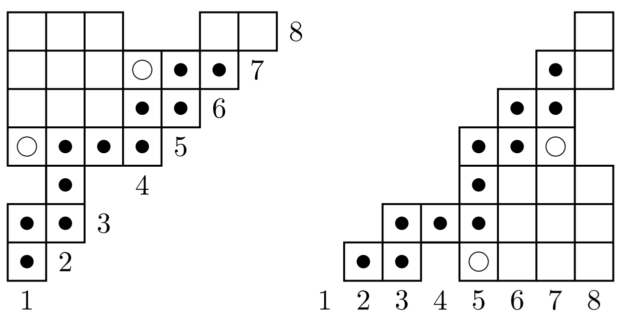
The type of a region R of $\mathrm{Shi}(G)$ or $\mathrm{Ish}(G)$ is a G-deleted partition of $[n]$. We remark that **the ceiling partition of a region of** $\mathrm{Shi}(G)$ **or** $\mathrm{Ish}(G)$ **is neither noncrossing nor nonnesting in general**.

Lemma 5.1. *Let $G \subseteq \binom{[n]}{2}$ be a graph on n vertices, let R be a region of $\mathrm{Shi}(G)$ or of $\mathrm{Ish}(G)$, and let τ be the ceiling partition of R. If τ has k blocks, then R has $n-k$ ceilings.*

Proof. Omitted. □

Theorem 5.2. *Let $G \subseteq \binom{[n]}{2}$ be a graph on n vertices and let τ be a G-deleted partition of $[n]$. If τ is not nonnesting, there are no dominant regions of $\mathrm{Shi}(G)$ or of $\mathrm{Ish}(G)$ with ceiling partition τ. If τ is nonnesting, there exists a unique dominant region of $\mathrm{Shi}(G)$ with ceiling partition τ and a unique dominant region of $\mathrm{Ish}(G)$ with ceiling partition τ.*

Proof. This is essentially a picture proof. For the identity permutation $w = \mathbf{1}$ we observe that the posets $\Phi^+(G,\mathbf{1})$ and $\Psi^+(G,\mathbf{1})$ look exactly the same, except that one is reflected in a line of slope 1. For example, here are the posets corresponding to the graph $G = \binom{[8]}{2} - \{14, 34, 48, 58\}$; Shi on the left, Ish on the right:



This reflection is an order-reversing bijection between $\Phi^+(G,\mathbf{1})$ and $\Psi^+(G,\mathbf{1})$. Hence it induces a bijection between **ideals in** $\Phi^+(G,\mathbf{1})$ **with** c **maximal elements** (dominant $\mathrm{Shi}(G)$-regions with c ceilings) and **filters in** $\Psi^+(G,\mathbf{1})$ **with** c **minimal elements** (dominant $\mathrm{Ish}(G)$-regions with c ceilings). This bijection preserves ceiling partitions. □

The bijection given in Theorem 5.2 does not extend to regions outside the dominant cone because the posets $\Phi^+(G,w)$ and $\Psi^+(G,w)$ look very different in general for permutations w other than 1. This bijection does **not** preserve degrees of freedom - indeed, it cannot as a glance at Figure 1 shows that for $n=3$ and $G = K_3$ the arrangement $\mathrm{Shi}(3)$ has two dominant regions with one degree of freedom while the arrangement $\mathrm{Ish}(3)$ has three dominant regions with one degree of freedom.

Theorem 5.3. *Let $G \subseteq \binom{[n]}{2}$ be a graph on n vertices, let $1 \le d \le n$, and let τ be a G-deleted partition of $[n]$ with k blocks.*

1. The number of regions of $\mathrm{Shi}(G)$ or of $\mathrm{Ish}(G)$ with ceiling partition τ is

$$\frac{n!}{(n-k+1)!}. \tag{16}$$

2. The number of regions of $\mathrm{Shi}(G)$ *or of* $\mathrm{Ish}(G)$ *with ceiling partition* τ *and with* d *degrees of freedom is*

$$\frac{d(n-d-1)!(k-1)!}{(n-k-1)!(k-d)!}. \tag{17}$$

Proof. (Sketch.) Use Lemmas 4.2 and 4.4 to count Shi and Ish ceiling diagrams which label regions with the desired properties. In the case of the Ish arrangement, both parts are routine counting arguments. In the case of the Shi arrangement, Part 1 requires a product formula due to Kreweras (7) which counts nonnesting partitions of $[n]$ with fixed type and Part 2 requires a product formula due to the second author (10) which counts nonnesting partitions of $[n]$ with fixed type and a fixed number of connected components. □

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Tree-like tableaux

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Abstract. In this work we introduce and study *tree-like tableaux*, which are certain fillings of Ferrers diagrams in simple bijection with permutation tableaux and alternative tableaux. We exhibit an elementary insertion procedure on our tableaux which gives a clear proof that tableaux of size n are counted by $n!$, and which moreover respects most of the well-known statistics studied originally on alternative and permutation tableaux. Our insertion procedure allows to define in particular two simple new bijections between tree-like tableaux and permutations: the first one is conceived specifically to respect the generalized pattern 2-31, while the second one respects the underlying tree of a tree-like tableau.

Résumé. Dans ce travail nous introduisons et étudions les *tableaux boisés*, qui sont certains remplissages de diagrammes de Ferrers en bijection simple avec les tableaux de permutation et les tableaux alternatifs. Nous décrivons une procédure d'insertion élémentaire sur nos tableaux qui donne une preuve limpide que les tableaux de taille n sont comptés par $n!$, et qui de plus respecte la plupart des statistiques standard sur les tableaux de permutation et tableaux alternatifs. Notre procédure d'insertion permet en particulier de définir deux nouvelles bijections simples entre tableaux et permutations: la première est conçue spécifiquement pour respecter le motif généralisé 2-31 sur les permutations, tandis que la deuxième respecte l'arbre binaire sous-jacent à un tableau boisé.

Keywords: tree-like tableaux, permutation tableaux, alternative tableaux, permutations, binary trees

Introduction

Permutation tableaux and alternative tableaux are equivalent combinatorial objects that have been the focus of intense research in recent years. Originally introduced by Postnikov [Pos06], they were soon studied by numerous combinatorialists [Bur07, CN09, SW07, Wil05, Nad09, Vie08]. They also popped up surprisingly in order to get a combinatorial understanding of the equilibrium state of the PASEP model from statistical mechanics: this is the seminal work of Corteel and Williams, see [CW07a, CW07b, CW10].

In this work we introduce and study tree-like tableaux (cf. Definition 1.1), which are objects in simple bijection with alternative tableaux. Indeed, our results have immediate reformulations in terms of alternative/permutation tableaux (see Proposition 1.2). We chose to focus on these new tableaux for one main reason: they exhibit a natural tree structure (giving them their name: cf. Figure 2, right) more clearly than the alternative tableaux, and Section 4 stresses this particular view of considering the tableaux. As is mentioned in this last section, the present work originated in fact in the study of trees.

The main result of this work is Theorem 2.2: *There is a simple bijective correspondence* $insertpoint$ *between (1) tree-like tableaux of size* n *together with an integer* $i \in \{1, \ldots, n+1\}$ *and (2) tree-like* *tableaux of size size* $n+1$. A variation for symmetric tableaux $insertpoint^*$ is also defined and shares similar properties, see Theorem 2.3. We prove that $insertpoint$ and $insertpoint^*$ carry various statistics of tableaux in a straightforward manner: we obtain thus new easy proofs of formulas enumerating tableaux and symmetric tableaux, as well as information on the number of crossings (cf. Section 1.2).

An immediate corollary of Theorem 2.2 is that tree-like tableaux of size n are enumerated by $n!$ (and Theorem 2.2 implies that symmetric tableaux of size $2n+1$ are enumerated by $2^n n!$). Several bijections between tableaux and permutations appeared already in the literature; the ones that seem essentially distinct are [SW07] and the two bijections from [CN09]. All of them give automatically a correspondence as in Theorem 2.2, but none of them is as elementary as $insertpoint$. Conversely, it is clear that $insertpoint$ allows to define various bijections between permutations and tableaux. We will focus on two of them here: the first one sends crossings to occurrences of the generalized pattern 2-31, while the second one preserves the binary trees naturally attached to permutations and tree-like tableaux.

Let us give a brief outline of this abstract. Section 1 introduces numerous definitions and notations, and most notably the tree-like tableaux which are the central focus of this work. Section 2 is the core section of this article: we introduce our main tool, the insertion $insertpoint$, and prove that it gives a 1-to-$(n+1)$ correspondence between tableaux of size n and $n+1$. We use it to give elementary proofs of refined enumeration formulas for tableaux as well as symmetric tableaux, for which we use a modified insertion $insertpoint^*$. Section 3 is centered around the enumeration of crossings; in particular, we use $insertpoint$ to define a bijection between tableaux and permutations which sends crossings to occurrences of the pattern 2-31. In Section 4 we focus on the tree structure associated to a tree-like tableau, and give in particular another bijection between tableaux and permutations. We conclude in Section 5 by listing some possible extensions of the work presented here.

1 Definitions and Notation

1.1 Ferrers diagrams, permutation and trees

Ferrers diagrams: A Ferrers diagram F is a left aligned finite set of unit cells in $\mathbb{Z}^2$, in decreasing number from top to bottom, considered up to translation: see Figure 1, left. The *half-perimeter* of F is the sum of its number of rows plus its number of columns; it is also equal to the number of *boundary edges*, which are the edges found on the Southeast border of the diagram. We will also consider *boundary cells*, which are the cells of F with no other cells to their Southeast.

There is a natural Southwest to Northeast order on boundary edges, as well as on boundary cells. Moreover, by considering the Southeast corner of boundary cells, these corners are naturally intertwined with boundary edges: we will thus speak of a boundary cell being Southwest or Northeast of a boundary edge. Two cells are *adjacent* if they share an edge.

Given two Ferrers diagrams $F_1 \subseteq F_2$, we say that the set of cells $S = F_2 - F_1$ (set-theoretic difference) is a *ribbon* if it is connected (with respect to adjacency) and contains no 2×2 square. In this case we say that S can be added to F_1, or that it can be removed from F_2. Note that a removable ribbon from F is equivalently a connected set S of boundary cells of F, such that the Southwest-most cell of S has no cell of F below it, and the Northeast-most cell of S has no cell of F to its right.

Row/Column insertion: Let F be a Ferrers diagram and e one of its boundary edges. If e is at the end of a row r, we define the insertion of a column at e to be the addition of a cell to r and all rows above it; symmetrically, if e is at the end of a column denoted by c, we can insert a row at e by adding a cell to

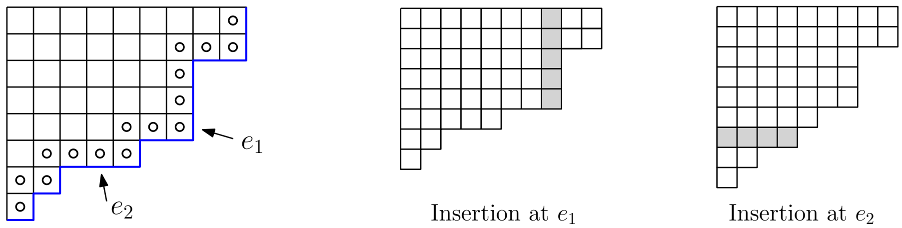

Fig. 1: A Ferrers diagram of half perimeter 17 with its highlighted boundary cells and edges (left), and examples of insertions at the boundary edges e_1 and e_2.

c and all columns to its left; see Figure 1, where the shaded cells of the figure are the added cells of the column or row.

Permutations and trees: We consider *permutations* σ of $\{1,\ldots,n\}$, which are bijections from $\{1,\ldots,n\}$ to itself, and are counted by $n!$. We will represent permutations as words $\sigma_1\ldots\sigma_n$ of length n where $\sigma_i=\sigma(i)$. A *descent* is an index $i<n$ such that $\sigma_i>\sigma_{i+1}$. An occurrence of the pattern 2-31 in σ is a pair (i,j) of two indices such that $1\leq i<j<n$ and $\sigma_{j+1}<\sigma_i<\sigma_j$.

A *planar binary tree* is a rooted tree such that each vertex has either two ordered children or no child; vertices with no child are called *leaves*, those of degree 2 are called *nodes*. The *size* of a tree is its number of nodes; see Figure 2 (right) for an example of tree of size 8.

1.2 Tree-like tableaux

We can now define the main object of this work:

Definition 1.1 (Tree-like tableau) *A* tree-like tableau *is a Ferrers diagram where each cell contains either* 0 *or* 1 *points (called respectively empty cell or pointed cell), with the following constraints:*

(1) the top left cell of the diagram possesses a point, called the root point*;*

(2) for every non-root pointed cell c, there exists either a pointed cell above c *in the same column, or a pointed cell to its left in the same row, but not both;*

(3) every column and every row possesses at least one pointed cell.

An example is shown on the left of Figure 2. Note that Condition (2) associates to each non-root point a unique other point; if we draw an edge between these two points, as well as an edge from every boundary edge to the closest point in its row or column, we create a binary tree, with nodes and leaves corresponding respectively to pointed cells and boundary edges. This is pictured on Figure 2, and explains the name given to our tableaux; we will come back to this tree structure with more detail in Section 4.

Let T be a tree-like tableau. If the diagram of T has half-perimeter $n+1$, then T is easily seen to have exactly n points: we let n be the *size* of T, and we denote by $\mathcal{T}_n$ the set of tree-like tableaux of size n. A *crossing* of T is an empty cell of T with a point both above it and to its left; we let $cr(T)$ be the number of crossings of T. The *top points* (respectively *left points*) of T are the non-root points appearing in the first row (resp. the first column) of its diagram.

Alternative tableaux and permutation tableaux: Tree-like tableaux are closely related to alternative tableaux [Nad09, Vie08] as follows: given a tree-like tableau, change every non-root point p to an arrow

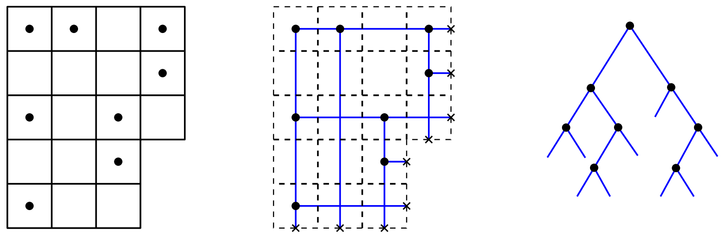

Fig. 2: A tree-like tableau (left) and the associated tree (right).

which is oriented left (respectively up) if there is no point left of p (resp. above p). This transforms the tableau into a *packed* alternative tableau [Nad09, Section 2.1.2]. To obtain an alternative tableau, one simply deletes the first row and first column (empty rows and columns may then occur). We have more precisely the following correspondences, where we refer to [Nad09] for definitions on alternative and permutation tableaux:

Proposition 1.2 *Let n, i, j, k, ℓ be nonnegative integers. There exist bijections between:*
(1) Tree-like tableaux of half-perimeter $n+1$, with i left points, j top points, k rows, ℓ crossings.
(2) Permutation tableaux of half-perimeter n with $i+1$ unrestricted rows, j top ones, k rows, ℓ superfluous ones.
(3) Alternative tableaux of half-perimeter $n-1$ with i free rows, j free columns, $k-1$ rows, ℓ free cells.

2 The fundamental insertion

This section is the core of this work. We describe a new way of inserting points in tree-like tableaux, shedding new light on numerous enumerative results on those tableaux.

2.1 Main Result

A key definition is the following one, which introduces a distinguished point in a tableau:

Definition 2.1 (Special point) *Let T be a tree-like tableau. The* special point *of T is the Northeast-most point among those that occur at the bottom of a column.*

This is well-defined since the bottom row of T necessarily has a pointed cell (Definition 1.1, (3)), which is then at the bottom of a column.

Definition of $insertpoint$: Let T be a tableau of size n and e be one of its boundary edges. We define first a tableau T' of size $n+1$ as follows: we insert a row or column at e, and mark with a point (the *new point*) the last cell of the row/column that was inserted. Let T' be the tableau obtained by this operation. Then we distinguish two cases:

(1) If e is to the Northeast of the special point of T, then we simply define $insertpoint(T, e) := T'$;

(2) Otherwise, we add a ribbon[(i)] of empty cells starting just to the right of the new point of T' and ending just below the special point of T. The result is tableau T'', and define $insertpoint(T, e) := T''$.

[(i)] In the special case where e is the lower edge of the special cell of T, define simply $T'' = T'$.

The result is a tree-like tableau of size $n+1$, since all three conditions of Definition 1.1 are clearly satisfied. Examples of the two cases of $insertpoint$ are given on Figure 3. Cells from the inserted rows/columns are shaded, while those from added ribbons are marked with a cross.

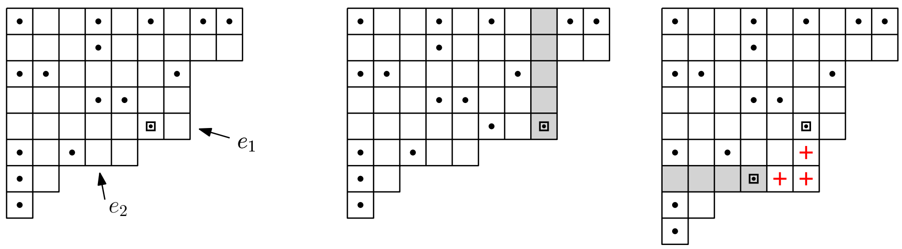

Tableau T (1) $T_1 = insertpoint(T, e_1)$ (2) $T_2 = insertpoint(T, e_2)$

Fig. 3: The two cases in the definition of $insertpoint$.

Theorem 2.2 *For any $n \geq 1$ the insertion procedure $insertpoint$ is a bijection between:*
(A) The set of pairs (T, e) where $T \in \mathcal{T}_n$ and e is one of the $n+1$ boundary edges of T, and
(B) The set $\mathcal{T}_{n+1}$.

Proof: We first define a function $removepoint$; we will then prove that it is the desired inverse of $insertpoint$. Let T be a tableau of size $n+1$ and consider the cell c of its special point. In case there is a cell adjacent to the right of c then follow the boundary cells to the Northeast of c and let c' be the first cell encountered which has a point; we then remove the ribbon of empty cells comprised between c and c'. This leaves indeed a Ferrers diagram since c' is not the bottom cell of its column. Coming back to the general case, delete now the row or column which contains c but no other points: let T_1 be the resulting tableau, and e be the boundary edge of T_1 which is adjacent to c in T. We define $removepoint(T) := (T_1, e)$; T_1 has clearly size n and e is one of its boundary edges, and we claim $removepoint$ is the desired inverse to the function $insertpoint$.

Let us now prove that $insertpoint \circ removepoint$ is the identity on $\mathcal{T}_{n+1}$. Let $T \in \mathcal{T}_{n+1}$, let c be its special cell, and let $(T_1, e) := removepoint(T)$.

- Suppose first that there is no cell in the diagram just to the right of c, that is, c lies at the end of a row. In this case the special point of T_1 must be to the Southwest of e, therefore no ribbon will be added in $insertpoint(T_1, e)$ and this last tableau is thus clearly T.

- Now suppose there is a cell just to the right of c: in this case the cell c' in the definition of $removepoint$ contains the special point of T_1, since the removal of the ribbon will turn c' into a bottom cell of a column. Now e will be to the left of c' in T_1, and so the application of $insertpoint$ will add the removed ribbon: in this case also $insertpoint(T_1, e) = T$.

Conversely, let $T' = insertpoint(T, e)$ for $T \in \mathcal{T}_n$, and we wish to show that $removepoint(T') = (T, e)$. The fundamental remark is that *the new point p added from T to T' is the special point of T'*. The proof is as follows: if the insertion added no ribbon, then p is inserted to the Northeast of the special point of T, and thus becomes the special point of T'. If the insertion added a ribbon, then no column in T' between p and the special point of T has a bottom point because of this added ribbon, and thus once again p is the special point of T'. From this remark, it is immediate that $removepoint(T') = (T, e)$, and the proof is then complete. □

Since $|\mathcal{T}_1| = 1$ we have the immediate corollary: *for* $n \geq 1$ *one has* $|\mathcal{T}_n| = n!$. We will give more precise enumerative results in Section 2.3.

So we have an elementary proof that tableaux of size n are equinumerous with permutations of length n. In fact, a multitude of bijections can be deduced from $insertpoint$; we will describe two such bijections in Sections 3.2 and 4.2.

2.2 Symmetric tableaux

In this section we consider *symmetric tableaux*, i.e. tree-like tableaux which are invariant with respect to reflection through the main diagonal of their diagram. These are in bijection with symmetric alternative tableaux from [Nad09, Section 3.5], and "type B permutation tableaux" from [LW08]. The size of such a tableau is necessarily odd, and those of size $2n+1$ (denoted by $\mathcal{T}^{sym}_{2n+1}$) are counted by $2^n n!$ (see [LW08, Nad09]); we will give a simple proof of this thanks to a modified insertion procedure.

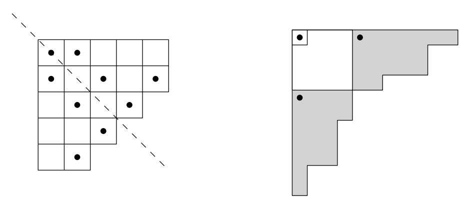

Fig. 4: A symmetric tableau, and the embedding of tree-like tableaux in symmetric tableaux.

Note that given a tree-like tableau T of size n, one can associate to it a symmetric tableau as follows: if T has k columns, then add on top of it a $k \times k$ square of cells, where only the top left cell is pointed; then add to the right of the square the reflected tableau $T*$: see Figure 4, right. In this way we embed naturally $\mathcal{T}_n$ in $\mathcal{T}^{sym}_{2n+1}$.

We now define a modified point insertion $insertpoint^*$ for symmetric tableaux. First let us call $*$-*special point* of a symmetric tableau the point at the bottom of a column which is Northeast-most *among those that are Southwest of the diagonal*.

Modified insertion: Let $T \in \mathcal{T}^{sym}_{2n+1}$ and (e, ε) be a pair consisting of a boundary edge e Southwest of the diagonal (there are $n+1$ such edges) and $\varepsilon \in \{+1, -1\}$. Insert first a row/column at e with a point at the end, and insert also the symmetric column/row, to get a tableau T'. There are then three cases:

(1) If $\varepsilon = +1$ and e is Northeast of the $*$-special point, simply define $insertpoint^*(T, e, +1) := T'$.

(2) If $\varepsilon = +1$ and e is Southwest of the $*$-special point, add a ribbon to T' between the new point (Southwest of the diagonal) and the $*$-special point of T below the diagonal; add also the symmetric ribbon. If T'' is the resulting tableau, then define $insertpoint^*(T, e, +1) := T''$.

(3) If $\varepsilon = -1$, then add a ribbon in T' between the two new points, and the resulting tableau is by definition $insertpoint^*(T, e, -1)$.

Examples of all 3 cases are given on Figure 5.

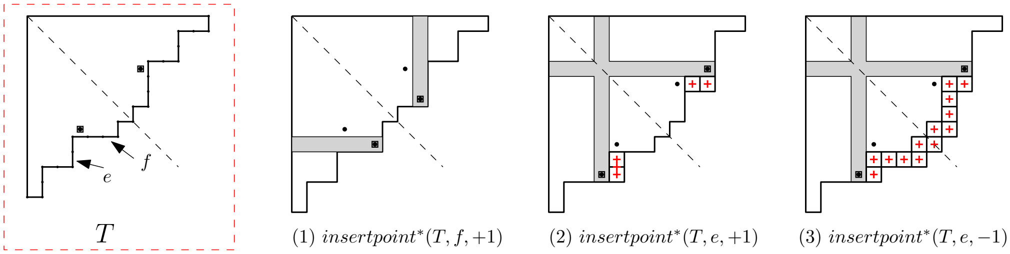

Fig. 5: The three cases in the definition of $insertpoint^*$.

Theorem 2.3 *The modified point insertion* $insertpoint^*$ *is a bijection between the set of triplets* (T, e, ε) *as above and* $\mathcal{T}^{sym}_{2n+3}$.

Consequently, $|\mathcal{T}^{sym}_{2n+3}| = 2(n+1)|\mathcal{T}^{sym}_{2n+1}|$ *and thus* $|\mathcal{T}^{sym}_{2n+1}| = 2^n n!$ *for* $n \geq 0$.

Proof: The proof is similar to the one of Theorem 2.2 and is omitted. The main ingredient is that the new point added Southwest of the diagonal is the $*$-special point of the resulting tableau, which allows one to invert the insertion procedure. $\square$

It is easy to check that when only $\varepsilon = +1$ is chosen during insertions, then the symmetric tableaux obtained in this case are precisely those corresponding to the embedding of usual tree-like tableaux into symmetric ones defined above. Indeed $insertpoint^*$ involves then only cases (1) and (2) which correspond to the usual insertion $insertpoint$.

2.3 Enumerative consequences

We now show how our insertion procedures give elementary proofs of some enumerative results on tableaux. Let $T_n(x, y)$ be the polynomial

$$T_n(x, y) = \sum_{T \in \mathcal{T}_n} x^{\mathrm{left}(T)} y^{\mathrm{top}(T)},$$

where $\mathrm{left}(T)$ and $\mathrm{top}(T)$ are respectively the number of left points and top points in T. When we insert a point in a tableau T of size n, then we add a left (respectively right) point to it if the Southwest-most edge (resp. Northeast-most edges) is picked, while for other boundary edges the number of top and left points remains the same. This gives immediately the recurrence relation $T_{n+1}(x, y) = (x + y + n - 1)T_n(x, y)$ which together with $T_1(x, y) = 1$ gives:

$$T_n(x, y) = (x + y)(x + y + 1) \cdots (x + y + n - 2). \tag{1}$$

This formula was proved in [CN09] and then bijectively in [Nad09]; the proof above is arguably the simplest one (and is bijective).

We can also give a generalization of Formula (1) to symmetric tableaux [LW08, CK10]. Following [CK10, Section 5] – reformulated in terms of tree-like tableaux – we let

$$T^{sym}_{2n+1}(x, y, z) = \sum_{T \in \mathcal{T}^{sym}_{2n+1}} x^{\mathrm{left}(T)} y^{\mathrm{top}^*(T)} z^{\mathrm{diag}(T)},$$

where $\mathrm{diag}(T)$ is the number of crossings among the diagonal cells; for $\mathrm{top}^*(T)$, consider the northmost non-root point p in the first column, then the number of points on the row of p is by definition $top^*(T)$. Let $T' = insertpoint^*(T, e, \varepsilon)$ be as in Theorem 2.3. One has $diag(T') = diag(T) + 1$ when $\varepsilon = -1$, and $\mathrm{left}(T') = \mathrm{left}(T) + 1$ when e is the Southwest-most edge. Furthermore, if the row r considered in the definition of $\mathrm{top}^*(T)$ has its boundary edge e' Southwest of the diagonal, then the insertion at $e = e'$ increases $top^*(T)$ by one; otherwise, the column c symmetric to r ends below the diagonal at a boundary edge e'', and then the insertion at $e = e''$ increases $top^*(T)$ by one. For all other choices of e we have $\mathrm{top}^*(T') = \mathrm{top}(T)$. Therefore we obtain the recurrence formula $T^{sym}_{2n+3}(x, y, z) = (1 + z)(x + y + n - 1)T^{sym}_{2n+1}(x, y, z)$, from which it follows:

$$T^{sym}_{2n+1}(x, y, z) = (1 + z)^n(x + y)(x + y + 1) \cdots (x + y + n - 2). \tag{2}$$

Note that this proof is much simpler than any of the two proofs given in [CK10].

3 Crossings and Permutations

We study here the interplay between crossings in a tableau and the insertion $insertpoint$.

3.1 Enumeration of crossings

The starting point is the following:

Lemma 3.1 *Let $T \in \mathcal{T}_n$. The crossings of T are the cells of the ribbons added in its* insertion history, *i.e in the $n - 1$ applications of $insertpoint$ necessary to construct T starting from the tableau of size* 1.

Proof: In an application of $insertpoint$, the empty cells from a ribbon correspond clearly to crossings, while the empty cells from the row or column insertions are not crossings. Furthermore $insertpoint$ does not modify crossings in the original tableau, which achieves the proof. □

Let T be a tableau of size n. From the Southwest to the Northeast, we label its boundary edges $e_0(T), \ldots, e_n(T)$ and its boundary cells $b_0(T), \ldots, b_{n-1}(T)$. We have the following proposition whose easy proof is omitted:

Proposition 3.2 *Let $T \in \mathcal{T}_n, i \in \{0, \ldots, n\}$, and consider $T' = insertpoint(T, e_i)$. Then the special cell of T' is $b_i(T')$. Moreover, if $b_k(T)$ is the special cell of T (where $0 \le k \le n - 1$), then we have $cr(T') = cr(T)$ if $k \le i$, while $cr(T') = cr(T) + (k - i)$ if $k > i$.*

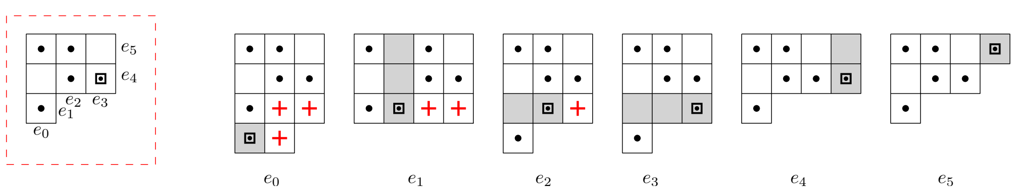

Fig. 6: A tableau of size 5 and all 6 possible point insertions.

This is illustrated on Figure 6, for which $k = 3$. A first consequence of the proposition is that given $k \in \{0, \ldots, n-1\}$, there are $n!/n = (n-1)!$ tableaux T in $\mathcal{T}_n$ where the special cell of T is $b_k(T)$. A second consequence is that given such a tableau T, the total number of ribbon cells added when constructing all tableaux $T' = insertpoint(T, e_i)$ (for $i = 0, \ldots, n$) is simply $1 + 2 + \ldots + k = \binom{k+1}{2}$, and thus the total number of crossings in these $n+1$ tableaux is $(n+1)cr(T) + \binom{k+1}{2}$. Let $Cr_n = \sum_T cr(T)$ where T runs through all tree-like tableaux of size n; the previous reasoning shows that for $n \geq 1$:

$$Cr_{n+1} - (n+1)Cr_n = (n-1)! \times \sum_{k=0}^{n-1} \binom{k+1}{2} = (n-1)! \binom{n+1}{3}.$$

If we let $X_n = Cr_n/n!$, we obtain simply $X_{n+1} - X_n = (n-1)/6$, from which we get:

Proposition 3.3 *The total number of crossings in $\mathcal{T}_n$ is given by $Cr_n = n! \times (n-1)(n-2)/12$.*

This can also be stated as: given the uniform distribution on $\mathcal{T}_n$, the expectation of $cr(T)$ is given by $(n-1)(n-2)/12$. This was proved first in [CH07, Theorem 1] by a lengthy computation, which relied on the recursive construction of (permutation) tableaux obtained by adding the leftmost column.

3.2 A first bijection with permutations

As mentioned at the end of Section 2.1, it is immediate to construct bijections from $\mathcal{T}_n$ to permutations using $insertpoint$. We will here define one with the goal of sending crossings of tableaux to occurrences of the pattern 2-31 in a permutation.

First, a tableau T of size n is naturally encoded by a list of integers $a(T) = (a_1(T), \ldots, a_n(T))$ satisfying $0 \leq a_i(T) \leq i-1$. This is done as follows: let $T_1, T_2, T_3, \ldots, T_n = T$ be the tableaux from the insertion history of T . For $i \in \{2, \ldots, n\}$, we define $a_i(T)$ as the index j such that $insertpoint(T_{i-1}, e_j) = T_i$, using the labeling of boundary edges defined before Proposition 3.2.

This proposition tells us then that $cr(T) = \displaystyle\sum_{i=1}^{n-1} max(a_i(T) - a_{i+1}(T), 0)$.

Now consider a word composed simply of n letters X, which we will replace one by one by integers. For $i = n, n-1, \ldots, 1$ in succession, we do the following: among the i remaining X's, pick the one which has $a_i(T)$ other X's to its left in the word, and replace it by the value i. For instance, if $a(T) = (0, 1, 0, 3, 1)$, then the successive words are $X5XXX$, $X5XX4$, $35XX4$, $35X24$, and finally 35124. In general, if τ is the permutation obtained after this transformation, we define $\Phi_1(T) := \tau^{-1}$.

Theorem 3.4 *Φ_1 is a bijection from $\mathcal{T}_n$ to permutations of length n. If $\sigma = \Phi_1(T)$, then $cr(T)$ is equal to the number of occurrences of 2-31 in σ.*

Proof: We will just give a sketch. First, it is clear that the construction is bijective. Now let $\Phi_1(T) = \tau^{-1}$ as above. Then note that in τ, the value $i+1$ appears left of i if and only if $a_i(T) \geq a_{i+1}(T)$, and that in this case there are $a_i(T) - a_{i+1}(T)$ values smaller than i occurring between the positions occupied in τ by i and $i+1$. This means that $a_i(T) \geq a_{i+1}(T)$ if and only if i is a descent in $\Phi_1(T)(= \tau^{-1})$, and in this case this descent stands for the '3' in $a_i(T) - a_{i+1}(T)$ occurrences of the pattern 2-31. The theorem follows then from the formula above expressing $cr(T)$ in terms of $a(T)$. $\square$

This bijection is much simpler than bijection II from [CN09], which was designed specifically to preserve the equivalent pattern 31-2.

4 Trees and Tree-like tableaux

4.1 Trees from permutations and tableaux

From permutations to binary trees: We define an *increasing tree of size* n to be a binary tree of size n where the n nodes are labeled by all integers in $\{1, \ldots, n\}$ in such a way that the labels increase along the path from the root to any node. There is a well-known bijection with permutations: given an increasing tree T, traverse its vertices in *inorder*, which means recursively traverse the left subtree, then visit the root, then traverse the right subtree. By recording node labels in the order in which they are visited, one obtains the wanted permutation: see Figure 7 (left). If σ is a permutation with associated increasing tree $inctree(\sigma)$, then we define $tree(\sigma)$ as the binary tree obtained by forgetting the labels in $inctree(T)$.
From tree-like tableaux to binary trees: We described this correspondence after Definition 1.1. It can also be obtained graphically by drawing two lines from every point of T, one down and one to the right, and stopping them at the boundary. We let $tree(T)$ be the binary tree thus constructed, see Figure 7 (right). Note that there is a natural identification between boundary edges of T and leaves of $tree(T)$.

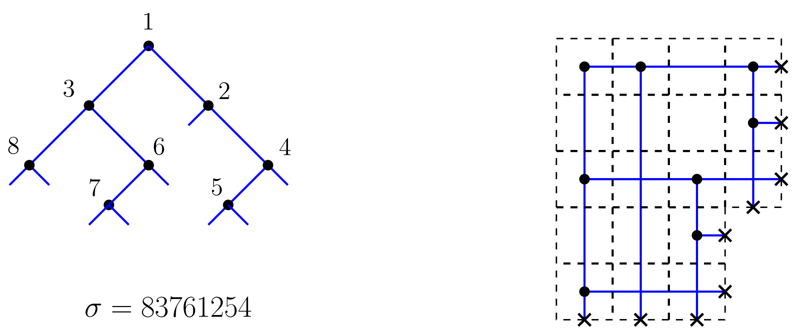

Fig. 7: The same binary tree arising from a permutation (left) and a tableau (right).

4.2 The tree preserving bijection

Using $insertpoint$, we now define a bijection Φ_2 between permutations and tree-like tableaux which preserves the binary trees attached to the objects. For this we proceed by induction on n.

Let σ be a permutation of size $n+1$, and τ be the permutation of size n obtained by deleting $n+1$ in σ. By induction hypothesis, the tableau $T := \Phi_2(\tau)$ is well defined and satisfies $tree(T) = tree(\tau)$. Define L to be the leaf of $inctree(\tau)$ appearing in the inorder traversal at the position occupied by $n+1$ in σ: then $inctree(\sigma)$ is obtained by replacing L by a node labeled $n+1$ with two leaves. Now L corresponds naturally to a boundary edge e_L in $\Phi_2(\tau)$, and we define $\Phi_2(\sigma) = insertpoint(\Phi_2(\tau), e_L)$.

Theorem 4.1 *Given $n \geq 1$, the function Φ_2 is a bijection between permutations of length n and tree-like tableaux of size n, satisfying $tree(\sigma) = tree(\Phi_2(\sigma))$.*

This is a simple consequence of the properties of $insertpoint$. The permutation σ and the tree-like tableau T from Figure 7 satisfy $\Phi_2(\sigma) = T$.

Remark: The tree structure attached to tableaux is not new: first Burstein [Bur07] defined it on so-called *bare tableaux*, which are essentially our tree-like tableaux minus a column. Then this tree structure was also studied by the third author in some detail [Nad09, Section 4]. The main difference is that, although the (unlabeled) tree structure is essentially the same, the labeling is quite different: here we have a quite simple bijection with increasing trees, while the labelings from the two aforementioned references involve some complicated increasing/decreasing conditions. The root of such complication can be traced to the fact that the boundary edges in [Bur07, Nad09] were labeled independently of the structure of the tree, while here we use the tree to determine the labeling.

4.3 Unambiguous trees

As mentioned in the introduction, this work had its origin in problems about trees, and not tableaux; we will here briefly describe such a problem. Suppose we draw the nodes of a plane binary tree as points in the center of unit cells of $\mathbb{Z}^2$, where the children of a node are drawn below and to the right of this node (as in the trees $tree(T)$ attached to a tree-like tableau T); we allow edges to cross outside of nodes. Let us call the drawing unambiguous if, when one deletes the edges of the tree, it is then possible to reconstruct them uniquely: one sees that this comes down essentially to condition (2) in Definition 1.1.

We are led to the following definition: *an unambiguous tree is a tree-like tableau T such each corner of the diagram of T contains a point* (a corner is a cell with no cell adjacent below it or to its right). The connection is as follows: given an unambiguous drawing, consider the smallest Ferrers diagram that contains it: if this tableau contains no empty row or column (corresponding to a *compact* drawing), then we obtain exactly an unambiguous tree. We proved exact enumeration formulas for the number of unambiguous tree corresponding to a given binary tree, or the total number of unambiguous trees, which will appear in a future work.

5 Further results and questions

In this work we described a very simple insertion procedure $insertpoint$ which can be seen as a 1-to-$(n+1)$ correspondence between the sets $\mathcal{T}_n$ and $\mathcal{T}_{n+1}$. We proved that from this simple seed one could produce automatically most of the enumerative results known on tableaux, as well as design bijections to permutations with various properties. Other enumeration results can also be proved with the same techniques: enumeration of tableaux according to the number of rows (this gives Eulerian numbers [Wil05]), or the total number of cells. Another set of results which was not stated here for lack of space is the generalization of the constructions of Sections 3 and 4 to symmetric tableaux.

A further question would be to revisit the work of Corteel and Williams on the PASEP model from statistical mechanics (see [CW10, CW07b, CW07a]), which involves objects related to alternative tableaux. In particular, do their (weighted) staircase tableaux have recursive decompositions similar to those given here for tree-like tableaux ?

Finally, the insertion procedure makes it very easy to generate a tree-like tableau uniformly at random among those of size n. In fact (by using adapted structures for our objects), it is possible to generate such a tableau with space and memory in $O(n^2)$. A related point is the asymptotic behavior of tree-like tableaux. For instance, is there a “limit shape” to our objects, that is a certain curve to which the boundary of tableaux (rescaled by a factor $1/n$) converge with high probability? A heuristic approach and early computer experiments tend to show that such a limit shape exists indeed, and is an arc of parabola; this is work in progress.

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The # product in combinatorial Hopf algebras

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Abstract. We show that the # product of binary trees introduced by Aval and Viennot (2008) is in fact defined at the level of the free associative algebra, and can be extended to most of the classical combinatorial Hopf algebras.

Résumé. Nous montrons que le produit # introduit par Aval et Viennot (2008) est défini au niveau de l'algèbre associative libre, et peut être étendu à la plupart des algèbres de Hopf combinatoires classiques.

Keywords: combinatorial Hopf algebras, # product, binary trees, permutations, Young tableaux

1 Introduction

There is a well-known Hopf algebra structure, due to Loday and Ronco [8], on the set of planar binary trees. Using a new description of the product of this algebra, (denoted here by $\mathbf{PBT}$) in terms of Catalan alternative tableaux, Aval and Viennot [1] introduced a new product, denoted by $\#$, which is compatible with the original graduation shifted by 1. Since then, Chapoton [2] has given a functorial interpretation of this operation.

Most classical combinatorial Hopf algebras, including $\mathbf{PBT}$, admit a realization in terms of special families of noncommutative polynomials. We shall see that on the realization, the # product has a simple interpretation. It can in fact be defined at the level of words over the auxiliary alphabet. Then, it preserves in particular the algebras based on parking functions ($\mathbf{PQSym}$), packed words ($\mathbf{WQSym}$), permutations ($\mathbf{FQSym}$), planar binary trees ($\mathbf{PBT}$), plane trees (the free tridendriform algebra $\mathfrak{TD}$), segmented compositions (the free cubical algebra $\mathfrak{TC}$), Young tableaux ($\mathbf{FSym}$), and integer compositions ($\mathbf{Sym}$). All definitions not recalled here can be found, *e.g.*, in [11; 12; 13].

In this extended abstract, most results are presented without proof.

2 A semigroup of paths

Let A be an alphabet. Words over A can be regarded as encoding paths in a complete graph with a loop on each vertex, vertices being labelled by A.

Composition of paths, denoted by $\#$, endows the set $\Sigma(A) = A^+ \cup \{0\}$ with the structure of a semigroup:

$$ua\#bv = \begin{cases} uav & \text{if } b = a, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

For example, $baaca\#adb = baacadb$ and $ab\#cd = 0$. Thus, the $\#$ product maps $A^n \times A^m$ to A^{m+n-1}. It is graded w.r.t. the path length (*i.e.*, the number of edges in the path).

We have the following obvious compatibilities with the concatenation product: $(uv)\#w = u \cdot (v\#w)$ and $(u\#v) \cdot w = u\#(vw)$.

Let d_k be the linear operator on the free associative algebra $\mathbb{K}\langle A\rangle$ (over some field $\mathbb{K}$) defined by

$$d_k(w) = \begin{cases} uav & \text{if } w = uaav \text{ for some } a, \text{ with } |u| = k-1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Then, for u of length k, $u\#v = d_k(uv)$.

3 Application to combinatorial Hopf algebras

The notion of a *combinatorial Hopf algebra* is a heuristic one, referring to rich algebraic structures arising naturally on the linear spans of various families of combinatorial objects. These spaces are generally endowed with several products and coproducts, and are in particular graded connected bialgebras, hence Hopf algebras.

The most prominent combinatorial Hopf algebras can be realized in terms of ordinary noncommutative polynomials over an auxiliary alphabet A. This means that their products, which are described by combinatorial algorithms, can be interpreted as describing the ordinary product of certain bases of polynomials in an underlying totally ordered alphabet $A = \{a_1 < a_2 < \dots\}$.

We shall see that all these realizations are stable under the $\#$ product. In the case of $\mathbf{PBT}$ (planar binary trees), we recover the result of Aval and Viennot [1]. In this case, the $\#$-product has been interpreted by Chapoton [2] in representation theoretical terms.

We shall start with the most natural algebra, $\mathbf{FQSym}$, based on permutations. It contains as subalgebras $\mathbf{PBT}$ (planar binary trees or the Loday-Ronco algebra, the free dendriform algebra on one generator), $\mathbf{FSym}$ (free symmetric functions, based on standard Young tableaux), and $\mathbf{Sym}$ (noncommutative symmetric functions).

It is itself a subalgebra of $\mathbf{WQSym}$, based on packed words (or set compositions), in which the role of $\mathbf{PBT}$ is played by the free dendriform trialgebra on one generator $\mathfrak{TD}$ (based on Schröder trees), the free cubical trialgebra $\mathfrak{TC}$ (segmented compositions).

Finally, all of these algebras can be embedded in $\mathbf{PQSym}$, based on parking functions.

Note that although all our algebras are actually Hopf algebras, the Hopf structure does not play any role in this paper.

4 Free quasi-symmetric functions: $\mathbf{FQSym}$ and its subalgebras

4.1 Free quasi-symmetric functions

4.1.1 The operation d_k on $\mathbf{FQSym}$

Recall that the alphabet A is *totally ordered*. Thus, we can associate to any word over A a permutation $\sigma = \mathrm{std}(w)$, the *standardized word* $\mathrm{std}(w)$ of w, obtained by iteratively scanning w from left to right, and labelling $1, 2, \dots$ the occurrences of its smallest letter, then numbering the occurrences of the next one, and so on. For example, $\mathrm{std}(365182122) = 687193245$.

For a permutation σ, define

$$\mathbf{G}_\sigma = \sum_{\mathrm{std}(w)=\sigma} w\,. \tag{3}$$

We shall need the following easy property of the standardization:

Lemma 4.1 *Let* $= u_1u_2\cdots u_n$ *be a word over* A, *and* $\sigma = \sigma_1\sigma_2\ldots\sigma_n = \mathrm{std}(u)$. *Then, for any factor of* u: $\mathrm{std}(u_iu_{i+1}\cdots u_j) = \mathrm{std}(\sigma_i\sigma_{i+1}\cdots\sigma_j)$.

This implies that $\mathbf{FQSym}$ is stable under the d_k:

$$d_k(\mathbf{G}_\sigma) = \begin{cases} \mathbf{G}_{\mathrm{std}(\sigma_1\cdots\sigma_{k-1}\sigma_{k+1}\cdots\sigma_n)} & \text{if } \sigma_{k+1} = \sigma_k + 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We shall make use of the dual basis of the $\mathbf{G}_\sigma$ when dealing with subalgebras of $\mathbf{FQSym}$. In the dual basis $\mathbf{F}_\sigma$ defined by $\mathbf{F}_\sigma := \mathbf{G}_{\sigma^{-1}}$, the formula is

$$d_k(\mathbf{F}_\sigma) = \begin{cases} \mathbf{F}_{\mathrm{std}(\sigma_1\cdots k\widehat{k+1}\cdots\sigma_n)} & \text{if } \sigma \text{ has a factor } k\,k{+}1, \\ 0 & \text{otherwise,} \end{cases} \tag{5}$$

where $\widehat{a}$ means that a is removed.

4.1.2 Algebraic structure

The $\mathbf{G}_\sigma$ span a subalgebra of the free associative algebra, denoted by $\mathbf{FQSym}$. The product is given by

$$\mathbf{G}_\alpha\mathbf{G}_\beta = \sum_{\gamma=uv,\ \mathrm{std}(u)=\alpha,\ \mathrm{std}(v)=\beta} \mathbf{G}_\gamma\,. \tag{6}$$

The set of permutations occuring in the r.h.s. is called the convolution of α and β, and denoted by $\alpha * \beta$.

Hence,

$$\mathbf{G}_\sigma\#\mathbf{G}_\tau = d_k(\mathbf{G}_\sigma\mathbf{G}_\tau) = \sum_{\nu\in\sigma\#\tau} \mathbf{G}_\nu, \tag{7}$$

where $\sigma\#\tau = \{\nu|\,|\nu| = k+l-1, \mathrm{std}(\nu_1\ldots\nu_k) = \sigma; \mathrm{std}(\nu_k\ldots\nu_{k+l-1}) = \tau\}$.

Indeed, $\mathbf{G}_\sigma\#\mathbf{G}_\tau$ is the sum of all words of the form $w = uxv$, with $\mathrm{std}(ux) = \sigma$ and $\mathrm{std}(xv) = \tau$. For example, $\mathbf{G}_{132}\#\mathbf{G}_{231} = \mathbf{G}_{14352} + \mathbf{G}_{15342} + \mathbf{G}_{24351} + \mathbf{G}_{25341}$.

4.1.3 Multiplicative bases

The multiplicative bases $\mathbf{S}^\sigma$ and $\mathbf{E}^\sigma$ of $\mathbf{FQSym}$ are defined by (see [4])

$$\mathbf{S}^\sigma = \sum_{\tau\le\sigma} \mathbf{G}_\tau \qquad \text{and} \qquad \mathbf{E}^\sigma = \sum_{\tau\ge\sigma} \mathbf{G}_\tau\,, \tag{8}$$

where $\le$ denotes the left weak order.

For $\alpha \in \mathfrak{S}_k$ and $\beta \in \mathfrak{S}_l$, define $\alpha \uparrow \beta \in \mathfrak{S}_{k+l-1}$ as the output of the following algorithm:

- scan the letters of α from left to right and write: $\alpha_i + \beta_1 - 1$ if $\alpha_i \le \alpha_k$ or $\alpha_i + \max(\beta) - 1$ if $\alpha_i > \alpha_k$,

- scan the letters of β starting from the second one and write: β_i if $\beta_i < \beta_1$ or $\beta_i + \alpha_k - 1$ if $\beta_i \geq \beta_1$.

Similarly, define $\alpha \downarrow \beta$ by:

- scan the letters of α and write: α_i if $\alpha_i < \alpha_k$ or $\alpha_i + \beta_1 - 1$ if $\alpha_i \geq \alpha_k$,
- scan the letters of β starting from the second one and write: $\beta_i + \alpha_k - 1$ if $\beta_i \leq \beta_1$ or $\beta_i + \max(\alpha) - 1$ if $\beta_i > \beta_1$.

For example, $3412 \uparrow 35124 = 78346125$ and $3412 \downarrow 35124 = 56148237$.

Theorem 4.2 *The permutations appearing in a #-product* $\mathbf{G}_\alpha \# \mathbf{G}_\beta$ *is an interval of the left weak order:*

$$\mathbf{G}_\alpha \# \mathbf{G}_\beta = \sum_{\gamma \in [\alpha \downarrow \beta, \alpha \uparrow \beta]} \mathbf{G}_\gamma \,. \tag{9}$$

Using only either the lower bound or the upper bound, one obtains:

Corollary 4.3 *The bases* $\mathbf{S}^\sigma$ *and* $\mathbf{E}^\sigma$ *are multiplicative for the #-product:*

$$\mathbf{S}^\alpha \# \mathbf{S}^\beta = \mathbf{S}^{\alpha \uparrow \beta} \qquad \textit{and} \qquad \mathbf{E}^\alpha \# \mathbf{E}^\beta = \mathbf{E}^{\alpha \downarrow \beta} \,. \tag{10}$$

For example, $\mathbf{S}^{3412} \# \mathbf{S}^{35124} = \mathbf{S}^{78346125}$ and $\mathbf{E}^{3412} \# \mathbf{E}^{35124} = \mathbf{E}^{56148237}$.

4.1.4 Freeness

The above description of the # product in the $\mathbf{S}$ basis implies the following result:

Theorem 4.4 *For the # product,* $\mathbf{FQSym}$ *is free on either* $\mathbf{S}^\alpha$ *or* $\mathbf{G}_\alpha$ *where* α *runs over* non-secable permutations, *that is, permutations of size* $n \geq 2$ *such that any prefix* $\alpha_1 \ldots \alpha_k$ *of size* $2 \leq k < n$ *is not, up to order, the union of an interval with maximal value* σ_k *and another interval either empty or with maximal value* n.

The generating series of the number of non-secable permutations (by shifted degree $d'(\sigma) = n - 1$ for $\sigma \in \mathfrak{S}_n$) is Sequence A077607 of [15]

$$\mathrm{NI}(t) := 2\,t + 2\,t^2 + 8\,t^3 + 44\,t^4 + 296\,t^5 + 2312\,t^6 + \ldots \tag{11}$$

or equivalently $1/(1 - \mathrm{NI}(t)) = \sum_{n \geq 1} n! t^{n-1}$.

4.2 Young tableaux: $\mathbf{FSym}$

The algebra $\mathbf{FSym}$ of free symmetric functions [3] is the subalgebra of $\mathbf{FQSym}$ spanned by the coplactic classes

$$\mathbf{S}_t = \sum_{Q(w)=t} w = \sum_{P(\sigma)=t} \mathbf{F}_\sigma \tag{12}$$

where (P, Q) are the P-symbol and Q-symbol defined by the Robinson-Schensted correspondence. This algebra is isomorphic to the algebra of tableaux defined by Poirier and Reutenauer [14]. We shall denote by $\mathrm{STab}(n)$ the standard tableaux of size n.

Note that $S_{\begin{smallmatrix}3&4\\1&2\end{smallmatrix}} = \mathbf{G}_{2413} + \mathbf{G}_{3412}$, so that $d_1(S_{\begin{smallmatrix}3&4\\1&2\end{smallmatrix}}) = \mathbf{G}_{312}$, which does not belong to $\mathbf{FSym}$. Hence $\mathbf{FSym}$ is not stable under the d_k. However, we have:

Theorem 4.5 $\mathbf{FSym}$ *is stable under the #-product.*

For example,

$$\mathbf{S}_{\ytableaushort{2,13}} \,\#\, \mathbf{S}_{\ytableaushort{3,2,1}} = \mathbf{S}_{\ytableaushort{5,4,2,13}} + \mathbf{S}_{\ytableaushort{5,24,13}} \tag{13}$$

$$\mathbf{S}_{\ytableaushort{3,12}} \,\#\, \mathbf{S}_{\ytableaushort{3,2,1}} = \mathbf{S}_{\ytableaushort{5,4,3,12}} \tag{14}$$

$$\mathbf{S}_{\ytableaushort{3,2,1}} \,\#\, \mathbf{S}_{\ytableaushort{2,13}} = \mathbf{S}_{\ytableaushort{4,3,2,15}} \tag{15}$$

$$\mathbf{S}_{\ytableaushort{3,2,1}} \,\#\, \mathbf{S}_{\ytableaushort{3,12}} = \mathbf{S}_{\ytableaushort{3,25,14}} + \mathbf{S}_{\ytableaushort{5,3,2,14}} \tag{16}$$

Note that those products do not have same number of terms, so that there is no natural definition of what would be the # product on the usual (commutative) symmetric functions.

For T an injective tableau and S a subset of its entries, let us denote by $T_{|S}$ the (sub-)tableau consisting of the restriction of T to its entries in S. For T, T' two skew tableaux, we denote their plactic equivalence (as for words) by $T \equiv T'$, that is we can obtain T' from T by playing *Jeu de Taquin*. The # product in $\mathbf{FSym}$ is given by the following simple combinatorial rule.

Proposition 4.6 *Let T_1 and T_2 be two standard tableaux of sizes k and ℓ. Then*

$$\mathbf{S}_{T_1} \,\#\, \mathbf{S}_{T_2} = \sum \mathbf{S}_T \tag{17}$$

where T runs over standard tableaux of size $k+\ell-1$ such that: $T_{|\{1,\dots,k\}} = T_1$ and $T_{|\{k,\dots,k+l-1\}} \equiv T_2$.

With this description, it is easy to compute by hand

$$\mathbf{S}_{\ytableaushort{4,123}} \,\#\, \mathbf{S}_{\ytableaushort{3,12}} = \mathbf{S}_{\ytableaushort{6,45,123}} + \mathbf{S}_{\ytableaushort{6,4,1235}} + \mathbf{S}_{\ytableaushort{46,1235}} \,. \tag{18}$$

4.3 Planar binary trees: $\mathbf{PBT}$

4.3.1 Algebraic structure

Recall that the natural basis of $\mathbf{PBT}$ can be defined by

$$\mathbf{P}_T = \sum_{\mathcal{T}(\sigma)=T} \mathbf{G}_\sigma \tag{19}$$

where $\mathcal{T}(\sigma)$ is the shape of the decreasing tree of σ.

Proposition 4.7 *The image of a tree by d_k is either 0 or a single tree:*

$$d_k(\mathbf{P}_T) = \begin{cases} \mathbf{P}_{T'}, \\ 0, \end{cases} \tag{20}$$

according to whether k is the left child of $k+1$ in the unique standard binary search tree of shape T (equivalently if the k-th vertex in the infix reading of T has no right child), in which case T' is obtained from T by contracting this edge, the result being 0 otherwise.

By the above result, any product $\mathbf{P}_{T'}\#\mathbf{P}_{T''}$ is in $\mathbf{PBT}$. We just need to select those linear extensions which are not annihilated by d_k. Since $d_k(\mathbf{F}_\sigma)$ is nonzero iff σ has (as a word) a factor $k\,k{+}1$, the image under d_k of the surviving linear extensions are precisely those of the poset obtained by identifying the rightmost node of T' with the leftmost node of T''. Thus, $\#$ is indeed the Aval-Viennot product.

4.3.2 Multiplicative bases

The multiplicative basis of initial intervals [5] (corresponding to the projective elements of [2]) is a subset of the $\mathbf{S}$ basis of $\mathbf{FQSym}$:

$$H_T = \mathbf{S}^\tau \tag{21}$$

where τ is the maximal element of the sylvester class T [5]. These maximal elements are the 132-avoiding permutations. Hence, they are preserved by the $\#$ operation, so that we recover Chapoton's result: the $\#$ product of two projective elements is a projective element. One can also apply the argument the other way round: since one easily checks that the $\#$ product of two permutations avoiding the pattern 132 also avoids this pattern, it is a simple proof that $\mathbf{PBT}$ is stable under $\#$.

As in the case of $\mathbf{FQSym}$, the fact that the $\mathbf{S}$ basis is still multiplicative for the $\#$ product implies that the product in the $\mathbf{P}$ basis is an interval in the Tamari order.

4.4 Noncommutative symmetric functions: $\mathbf{Sym}$

4.4.1 Algebraic structure

Recall that $\mathbf{Sym}$ is freely generated by the noncommutative complete functions

$$S_n(A) = \sum_{i_1 \le i_2 \le \ldots \le i_n} a_{i_1} a_{i_2} \cdots a_{i_n} = \mathbf{G}_{12\cdots n} \tag{22}$$

Here, we have obviously $S_n \# S_m = S_{n+m-1}$. This implies, for $l(I) = r$, $I = I' i_r$ and $J = j_1 J''$,

$$S^I \# S^J = S^{I' \cdot (i_r + j_1 - 1) \cdot J''} \qquad \text{and similarly} \qquad R_I \# R_J = R_{I' \cdot (i_r + j_1 - 1) \cdot J''}\,, \tag{23}$$

where S^I is the basis of products of noncommutative complete symmetric functions and R_I are noncommutative ribbon Schur functions. For example, $R_{1512} \# R_{43} = R_{15153}$.

Clearly, as a $\#$-algebra, $\mathbf{Sym}^+$ is the free graded associative algebra $\mathbb{K}\langle x, y\rangle$ over the two generators $x = S_2 = R_2$ and $y = \Lambda_2 = R_{11}$ of degree 1, the neutral element being $S_1 = R_1 = \Lambda_1$.

Now, define for any composition $I = (i_0, \ldots, i_r)$, the binary word: $b(I) = 0^{i_0 - 1} 1 (0^{i_1 - 1}) 1 \ldots (0^{i_r - 1})$. On the binary coding of a composition I, one can read an expression of R_I, S^I, and Λ^I in terms of

#-products of the generators x, y: replace the concatenation product by the #-product, replace 0 by respectively x, x, or y, and 1 by respectively y, $x+y$, or $x+y$, so that

$$R_I := (x^{i_0-1})^{\#}\#y\#(x^{i_1-1})^{\#}\#y\#\dots\#(x^{i_r-1})^{\#}, \tag{24}$$

$$S^I := (x^{i_0-1})^{\#}\#(x+y)\#(x^{i_1-1})^{\#}\#(x+y)\#\dots\#(x^{i_r-1})^{\#}, \tag{25}$$

and

$$\Lambda^I := (y^{i_0-1})^{\#}\#(x+y)\#(y^{i_1-1})^{\#}\#(x+y)\#\dots\#(y^{i_r-1})^{\#}. \tag{26}$$

Note that in particular, the maps sending S^I either to R_I or Λ^I are algebra automorphisms. This property will extend to a Hopf algebra automorphism with the natural coproduct.

4.4.2 Coproduct

In this case, we have a natural coproduct: the one for which x and y are primitive:

$$\nabla S_2 = S_2 \otimes S_1 + S_1 \otimes S_2, \qquad \text{and} \qquad \nabla\Lambda_2 = \Lambda_2 \otimes \Lambda_1 + \Lambda_1 \otimes \Lambda_2, \tag{27}$$

and, the neutral element S_1 is grouplike: $\nabla S_1 = S_1 \otimes S_1$. Then,

$$\nabla S_n = \sum_{i=1}^{n} \binom{n-1}{i-1} S_i \otimes S_{n+1-i}. \tag{28}$$

The coproduct of generic S^I, Λ^I, and R_I all are the same: since x and y are primitive, $x+y$ is also primitive, so that, *e.g.*,

$$\nabla R_I = \sum_{w,w' \mid w \sqcup\!\sqcup w' = b(I)} C_{w,w'}^{b(I)} R_J \otimes R_K, \tag{29}$$

where J (resp. K) are the compositions whose binary words are w (resp. w'), and $C_{w,w'}^{b(I)}$ is the coefficient of $b(I)$ in $w \sqcup\!\sqcup w'$. Another way of presenting this coproduct is as follows: given $b(I)$, choose for each element if it appears on the left or on the right of the coproduct (hence giving $2^{|I|-1}$ terms) and compute the corresponding products of x and y.

Hence, the maps sending S^I either to R_I or Λ^I are Hopf algebra automorphisms.

4.4.3 Duality: quasi-symmetric functions under #

Since $\mathbf{Sym}$ is isomorphic to the Hopf algebra $\mathbb{K}\langle x, y\rangle$ on two primitive generators x and y, its dual is the shuffle algebra on two generators whose coproduct is given by deconcatenation.

Since all three bases S, R, and Λ behave in the same way for the Hopf structure, the same holds for their dual bases, so that the bases M_I, F_I, and the forgotten basis of $QSym$ have the same product and coproduct formulas. In the basis F_I, this is

$$F_I \# F_J = \sum_{w \in b(I) \sqcup\!\sqcup b(J)} F_K, \tag{30}$$

where K is the composition such that $b(K) = w$.

For example, $F_3 \# F_{12} = 3\,F_{14} + 2\,F_{23} + F_{32}$, since $xx \sqcup\!\sqcup yx = 3\,yxxx + 2\,xyxx + xxyx$.

Note that since the product is a shuffle on words in x and y, all elements in a product $F_I F_J$ have same length, which is $l(I) + l(J) - 1$.

5 Word quasi-symmetric functions: $\mathbf{WQSym}$ and its subalgebras

5.1 Word quasi-symmetric functions

5.1.1 Algebraic structure

Word quasi-symmetric functions are the invariants of the quasi-symmetrizing action of the symmetric group (in the limit of an infinite alphabet), see, *e.g.*, [12].

The *packed word* $u = \mathrm{pack}\,(w)$ associated with a word w in the free monoid A^* is obtained by the following process. If $b_1 < b_2 < \ldots < b_r$ are the letters occuring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. For example, $\mathrm{pack}\,(469818941) = 235414521$. A word u is said to be *packed* if $\mathrm{pack}\,(u) = u$. Such words can be interpreted as set compositions, or as faces of the permutohedreon, and are sometimes called pseudo-permutations [6].

As in the case of permutations, we have:

Lemma 5.1 *Let $u = u_1u_2\cdots u_n$ be a word over A, and $v = v_1v_2\ldots v_n = \mathrm{pack}\,(u)$. Then, for any factor of u,* $\mathrm{pack}\,(u_iu_{i+1}\cdots u_j) = \mathrm{pack}\,(v_iv_{i+1}\cdots v_j)\,.$

The natural basis of $\mathbf{WQSym}$, which lifts the quasi monomial basis of $QSym$, is labelled by packed words. It is defined by

$$\mathbf{M}_u = \sum_{\mathrm{pack}\,(w)=u} w\,. \tag{31}$$

Note that $\mathbf{WQSym}$ is stable under the operators d_k. We have

$$d_k(\mathbf{M}_w) = \begin{cases} \mathbf{M}_{w_1\cdots w_{k-1}w_{k+1}\cdots w_n} & \text{if } w_k = w_{k+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{32}$$

so that $\mathbf{WQSym}$ is also stable under $\#$. Since in this basis the ordinary product is given by

$$\mathbf{M}_u\mathbf{M}_v = \sum_{w=u'v';\ \mathrm{pack}\,(u')=u,\ \mathrm{pack}\,(v')=v} \mathbf{M}_w\,, \tag{33}$$

we have

$$\mathbf{M}_u\#\mathbf{M}_v = d_k(\mathbf{M}_u\mathbf{M}_v) = \sum_{w\in u\#v} \mathbf{M}_w, \tag{34}$$

where $u\#v = \{w \,|\, |w| = k+l-1, \mathrm{pack}\,(w_1\ldots w_k) = u; \mathrm{pack}\,(w_k\ldots w_{k+l-1}) = v\}$.

For example, $\mathbf{M}_{121}\#\mathbf{M}_{12} = \mathbf{M}_{1212} + \mathbf{M}_{1213} + \mathbf{M}_{1312}$.

5.1.2 Multiplicative bases

Recall that there exists an order on packed words generalizing the left weak order : it is the pseudo-permutohedron order. This order has a definition in terms of inversions (see [6]) similar to the definition of the left weak order. The *generalized inversion set* of a given packed word w is the union of the set of pairs (i,j) such that $i<j$ and $w_i > w_j$ with coefficient one (full inversions), and the set of pairs (i,j) such that $i<j$ and $w_i = w_j$ with coefficient one half (half-inversions).

One then says that two words u and v satisfy $u < v$ for the pseudo-permutohedron order iff the coefficient of any pair (i,j) in u is smaller than or equal to the same coefficient in v.

Note that the definition of $u \uparrow v$ and $u \downarrow v$ (see the section about $\mathbf{FQSym}$) does not require u and v to be permutations. One then has

Theorem 5.2 *The words appearing in the product* $\mathbf{M}_u \# \mathbf{M}_v$ *is an interval of the pseudo-permutohedron order:*

$$\mathbf{M}_u \# \mathbf{M}_v = \sum_{w \in [u \downarrow v, u \uparrow v]} \mathbf{M}_w \,. \tag{35}$$

The multiplicative bases $\mathbf{S}^u$ and $\mathbf{E}^u$ of $\mathbf{WQSym}$ are defined in [12] by

$$\mathbf{S}^u = \sum_{v \leq u} \mathbf{M}_v \qquad \text{and} \qquad \mathbf{E}^u = \sum_{v \geq u} \mathbf{M}_v \,, \tag{36}$$

where $\leq$ is the pseudo-permutohedron order.

Proposition 5.3 *The* $\mathbf{S}$ *and* $\mathbf{E}$*-bases are multiplicative for the* $\#$*-product:*

$$\mathbf{S}^u \# \mathbf{S}^v = \mathbf{S}^{u \uparrow v} \qquad \textit{and} \qquad \mathbf{E}^u \# \mathbf{E}^v = \mathbf{E}^{u \downarrow v}. \tag{37}$$

5.1.3 Freeness

As in the case of $\mathbf{FQSym}$, we can describe a set of free generators in the $\mathbf{S}$ basis for $\mathbf{WQSym}$.

We shall say that a packed word u of size n is *secable* if there exists a prefix $u_1 \dots u_k$ of size $2 \leq k < n$ such that: $\{u_1, \dots, u_k\} \cap \{u_k, \dots, u_n\} = \{u_k\}$ and the set $\{u_1, \dots, u_k\}$ is, up to order the union of an interval with maximal value u_k and another interval either empty or with maximal value the maximal entry of the whole word u.

Conversely, a packed word of size at least 2 which is not secable will be called *non-secable*.

Theorem 5.4 *For the* $\#$ *product,* $\mathbf{WQSym}$ *is free on the* $\mathbf{S}^u$ *or* $\mathbf{M}_u$ *where* u *runs over non-secable packed words.*

If a packed word u is weighted by $t^{|u|-1}$, the generating series $\mathrm{PW}(t)$ of (unrestricted) packed words corresponds to Sequence A000670 of [15]:

$$\mathrm{PW}(t) = 1 + 3\,t + 13\,t^2 + 75\,t^3 + 541\,t^4 + 4683\,t^5 + 47293\,t^6 + \dots \tag{38}$$

The generating series $NSPW$ of non-secable packed words is related to PW by the relation $\mathrm{PW}(t) = 1/(1 - NSPW(t))$ which enables us to compute $NSPW$:

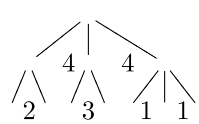
$$\mathrm{NSPW}(t) = 3\,t + 4\,t^2 + 24\,t^3 + 192\,t + 1872\,t^5 + 21168\,t^6 + \dots \tag{39}$$

5.2 The free tridendriform algebra $\mathfrak{TD}$

The realization of the free dendriform trialgebra given in [11] involves the following construction. With any word w of length n, associate a plane tree $\mathcal{T}(w)$ with $n+1$ leaves, as follows: if $m = \max(w)$ and if w has exactly $k-1$ occurences of m, write

$$w = v_1\, m\, v_2 \cdots v_{k-1}\, m\, v_k \,, \tag{40}$$

where the v_i may be empty. Then, $\mathcal{T}(w)$ is the tree obtained by grafting the subtrees $\mathcal{T}(v_1), , \dots, \mathcal{T}(v_k)$ (in this order) on a common root, with the initial condition $\mathcal{T}(\epsilon) = \emptyset$ for the empty word. For example, the tree associated with 243411 is



(41)

We shall call *sectors* the zones containing numbers and say that a sector is to the left of another sector if its number is to the left of the other one, so that the reading of all sectors from left to right of any $\mathcal{T}(w)$ gives back w.

Now define a polynomial by

$$\mathcal{M}_T := \sum_{\mathcal{T}(w)=T} \mathcal{M}_w \,. \tag{42}$$

Then, exactly as in the case of $\mathbf{PBT}$, we have

Theorem 5.5

$$d_k(\mathcal{M}_T) = \begin{cases} \mathcal{M}_{T'}, \\ 0, \end{cases} \tag{43}$$

depending on whether the k-th and $k+1$-th sectors are grafted on the same vertex or not. In the nonzero case, T' is obtained from T by gluing the k-th and $k+1$-th sectors.

Corollary 5.6 *The operation $\#$ is internal on $\mathfrak{TD}$.*

5.3 The free cubical algebra $\mathfrak{TC}$

Define a *segmented composition* as a finite sequence of integers, separated by vertical bars or commas, *e.g.*, $(2,1\,|\,2\,|\,1,2)$. We shall associate an ordinary composition with a segmented composition by replacing the vertical bars by commas. Recall that the descents of an ordinary composition are the positions of the ones in the associated binary word. In the same way, there is a natural bijection between segmented compositions of sum n and sequences of length $n-1$ over three symbols $<,=,>$: start with a segmented composition $\mathbf{I}$. If i is not a descent of the underlying composition of $\mathbf{I}$, write $<$; otherwise, if i corresponds to a comma, write $=$; if i corresponds to a bar, write $>$.

Now, with each word w of length n, associate a segmented composition $S(w)$, defined as the sequence $s_1,\dots,s_{n-1}$ where s_i is the comparison sign between w_i and w_{i+1}.

For example, given $w = 1615116244543$, one gets the sequence $<><>=<><=<>>$ and the segmented composition $(2|2|1,2|2,2|1|1)$.

Given a segmented composition $\mathbf{I}$, define: $\mathcal{M}_\mathbf{I} = \sum_{S(T)=\mathbf{I}} \mathcal{M}_T$.

It has been shown in [12] that the $\mathcal{M}_\mathbf{I}$ generate a Hopf subalgebra of $\mathfrak{TD}$ and that their product is given by

$$\mathcal{M}_{\mathbf{I}'}\mathcal{M}_{\mathbf{I}''} = \mathcal{M}_{\mathbf{I}'.\mathbf{I}''} + \mathcal{M}_{\mathbf{I}',\mathbf{I}''} + \mathcal{M}_{\mathbf{I}'|\mathbf{I}''}. \tag{44}$$

where $\mathbf{I}'.\mathbf{I}''$ is obtained by gluing the last part of $\mathbf{I}'$ with the first part of $\mathbf{I}''$.

As before, it is easy to see that

Theorem 5.7

$$d_k(\mathcal{M}_\mathbf{I}) = \begin{cases} \mathcal{M}_{\mathbf{I}'}, \\ 0, \end{cases} \tag{45}$$

depending on whether k is not or is a descent of the underlying composition of $\mathbf{I}$. In the nonzero case, $\mathbf{I}'$ is obtained from $\mathbf{I}$ by decreasing the entry that corresponds to the entry containing the k-th cell in the corresponding composition, that is, if $\mathbf{I} = (i_1,\dots,i_\ell)$ where the i are separated by commas or vertical bars, decreasing i_n where n is the smallest integer such that $i_1+\cdots+i_n > k$.

For the same reason, the following result is also true: $\mathcal{M}_{\mathbf{I}'}\#\mathcal{M}_{\mathbf{I}''} = \mathcal{M}_{\mathbf{I}'.'\mathbf{I}''}$, where $\mathbf{I}'.'\mathbf{I}''$ amounts to glue together the last part of $\mathbf{I}'$ with the first part of $\mathbf{I}''$ minus one, leaving the other parts unchanged.

6 Parking quasi-symmetric functions: $\mathbf{PQSym}$

A *parking function* on $[n] = \{1, 2, \ldots, n\}$ is a word $\mathbf{a} = a_1 a_2 \cdots a_n$ of length n on $[n]$ whose non-decreasing rearrangement $\mathbf{a}^{\uparrow} = a'_1 a'_2 \cdots a'_n$ satisfies $a'_i \le i$ for all i. We shall denote by PF the set of parking functions.

For a word w over a totally ordered alphabet in which each element has a successor, one can define [13] a notion of *parkized word* $\mathrm{park}(w)$, a parking function which reduces to $\mathrm{std}(w)$ when w is a word without repeated letters.

For $w = w_1 w_2 \cdots w_n$ on $\{1, 2, \ldots\}$, we set

$$d(w) := \min\{i | \#\{w_j \le i\} < i\} . \tag{46}$$

If $d(w) = n + 1$, then w is a parking function and the algorithm terminates, returning w. Otherwise, let w' be the word obtained by decrementing all the elements of w greater than $d(w)$. Then $\mathrm{park}(w) := \mathrm{park}(w')$. Since w' is smaller than w in the lexicographic order, the algorithm terminates and always returns a parking function.

For example, let $w = (3, 5, 1, 1, 11, 8, 8, 2)$. Then $d(w) = 6$ and the word $w' = (3, 5, 1, 1, 10, 7, 7, 2)$. Then $d(w') = 6$ and $w'' = (3, 5, 1, 1, 9, 6, 6, 2)$. Finally, $d(w'') = 8$ and $w''' = (3, 5, 1, 1, 8, 6, 6, 2)$, that is a parking function. Thus, $\mathrm{park}(w) = (3, 5, 1, 1, 8, 6, 6, 2)$.

Lemma 6.1 *Let $u = u_1 u_2 \cdots u_n$ be a word over A, and $\mathbf{c} = c_1 c_2 \ldots c_n = \mathrm{park}(u)$. Then, for any factor of u:* $\mathrm{park}(u_i u_{i+1} \cdots u_j) = \mathrm{park}(c_i c_{i+1} \cdots c_j)$.

Recall from [13], that with a parking function $\mathbf{a}$, one associates the polynomial

$$\mathbf{G_a} = \sum_{\mathrm{park}(w)=\mathbf{a}} w . \tag{47}$$

These polynomials form a basis of a subalgebra[(i)] $\mathbf{PQSym}$ of the free associative algebra over A. In this basis, the product is given by

$$\mathbf{G_a} \mathbf{G_b} = \sum_{\mathbf{c}=uv;\ \mathrm{park}(u)=\mathbf{a},\, \mathrm{park}(v)=\mathbf{b}} \mathbf{G_c} . \tag{48}$$

Thus,

$$\mathbf{G_a} \# \mathbf{G_b} = \sum_{\mathbf{c} \in \mathbf{a} \# \mathbf{b}} \mathbf{G_c}, \tag{49}$$

where $\mathbf{a} \# \mathbf{b} = \{\mathbf{c} |\, |\mathbf{c}| = k + l - 1, \mathrm{park}(\mathbf{c}_1 \ldots \mathbf{c}_k) = \mathbf{a}, \mathrm{park}(\mathbf{c}_k \ldots \mathbf{c}_{k+l-1}) = \mathbf{b}\}$. Indeed, $\mathbf{G_a} \# \mathbf{G_b}$ is the sum of all words of the form $w = uxv$, with $\mathrm{park}(ux) = \mathbf{a}$ and $\mathrm{park}(xv) = \mathbf{b}$.

Note that $\mathbf{PQSym}$ is not stable under the operators d_k. For example, $d_1(\mathbf{G}_{112})$ is not in $\mathbf{PQSym}$. However, let d'_k be the linear operator defined by

$$d'_k(\mathbf{G_c}) = \begin{cases} \mathbf{G}_{\mathbf{c}_1 \ldots \mathbf{c}_{k-1} \mathbf{c}_{k+1} \ldots \mathbf{c}_n} & \text{if } \mathbf{c}_k = \mathbf{c}_{k+1} \text{ and } \mathbf{c}_1 \ldots \mathbf{c}_{k-1} \mathbf{c}_{k+1} \ldots \mathbf{c}_n \in \mathrm{PF}, \\ 0 & \text{otherwise.} \end{cases} \tag{50}$$

(i) Strictly speaking, this subalgebra is rather $\mathbf{PQSym}^*$, the graded dual of the Hopf algebra $\mathbf{PQSym}$, but both are actually isomorphic.

Proposition 6.2 *Then, if* $\mathbf{a}$ *is of length* k*, one has:* $\mathbf{G_a}\#\mathbf{G_b} = d'_k(\mathbf{G_a}\mathbf{G_b})$.

For example, $\mathbf{G}_{121}\#\mathbf{G}_{1141} = \mathbf{G}_{121161} + \mathbf{G}_{121151} + \mathbf{G}_{121141}$.

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Powers of the Vandermonde determinant, Schur functions, and the dimension game

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Abstract. Since every even power of the Vandermonde determinant is a symmetric polynomial, we want to understand its decomposition in terms of the basis of Schur functions. We investigate several combinatorial properties of the coefficients in the decomposition. In particular, I will give a recursive approach for computing the coefficient of the Schur function s_μ in the decomposition of an even power of the Vandermonde determinant in $n+1$ variables in terms of the coefficient of the Schur function s_λ in the decomposition of the same even power of the Vandermonde determinant in n variables if the Young diagram of μ is obtained from the Young diagram of λ by adding a tetris type shape to the top or to the left.

Résumé. Comme toute puissance paire du déterminant de Vandermonde est un polynôme symétrique, nous voulons comprendre sa décomposition dans la base des fonctions de Schur. Nous allons étudier plusieurs propriétés combinatoires des coefficients de la décomposition. En particulier, nous allons donner une approche récursive pour le calcul du coefficient de la fonction de Schur s_μ dans la décomposition d'une puissance paire du déterminant de Vandermonde en $n+1$ variables, en fonction du coefficient de la fonction de Schur s_λ dans la décomposition de la même puissance paire du déterminant de Vandermonde en n variables, lorsque le diagramme de Young de μ est obtenu à partir du diagramme de Young de λ par l'addition d'une forme de type tetris vers le haut ou vers la gauche.

Keywords: Schur functions, Vandermonde determinant, Young diagrams, symmetric functions, quantum Hall effect

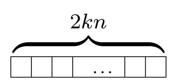
1 Introduction

Because Vandermonde determinants are so prevalent in mathematics I will not attempt to list here their many applications. In the theory of formal power series they are best known for the part they play in the classical definition of Schur functions. Since each even power of the Vandermonde determinant is a symmetric function, it is natural to ask for its decomposition in terms of the basis for the ring of symmetric functions given by Schur functions. This decomposition has been studied extensively (see [2], [3], [7], and the references therein) in connection with its usefulness in the understanding of the (fractional) quantum Hall effect. In particular, the coefficients in the decomposition correspond precisely to the coefficients in the decomposition of the Laughlin wave function as a linear combination of (normalized) Slater determinantal wave functions. The calculation of the coefficients in the decomposition becomes computationally expensive as the size of the determinant increases. Several algorithms for the expansion of the square of the Vandermonde determinant in terms of Schur functions are available (see, for example [7]). However, a combinatorial interpretation for the coefficient of a given Schur function is still unknown. Recently, Boussicault, Luque and Tollu [1] provided a purely numerical algorithm for computing the coefficient of a given Schur function in the decomposition without computing the other coefficients. The algorithm uses hyperdeterminants and their Laplace expansion. It was used by the authors to compute coefficients in the decomposition of even powers of Vandermonde determinants of size up to 11. For determinants of large size, the algorithm becomes computationally too expensive for practical purposes. In this article we present recursive combinatorial properties of some of the coefficients on the decomposition.

1.1 Statement of results

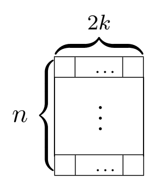
We denote by a_{δ_n} the Vandermonde determinant $a_{\delta_n} = \det(x_i^{n-j})_{i,j=1}^n = \prod_{1\le i<j\le n}(x_i - x_j)$. For fixed positive integers n and k, suppose λ is a partition of $kn(n-1)$ and μ is a partition of $kn(n+1)$. We prove the following results relating $\langle a^{2k}_{\delta_{n+1}}, s_\mu\rangle$ and $\langle a^{2k}_{\delta_n}, s_\lambda\rangle$ when the diagram of μ is obtained by adding a certain configuration of boxes, called a *tetris type shape*, to the top or to the left of the diagram of λ.

1. (Theorem 4.1) If the Young diagram of μ is obtained by adding the tetris type shape of size $2kn$



   to the top of the Young diagram of λ, then $\langle a^{2k}_{\delta_{n+1}}, s_\mu\rangle = \langle a^{2k}_{\delta_n}, s_\lambda\rangle$.

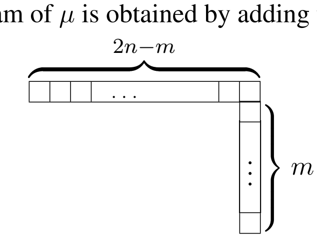
2. (Corollary 4.3) If the Young diagram of μ is obtained by adding the tetris type shape of size $2kn$



   to the left of the Young diagram of λ, then $\langle a^{2k}_{\delta_{n+1}}, s_\mu\rangle = \langle a^{2k}_{\delta_n}, s_\lambda\rangle$.

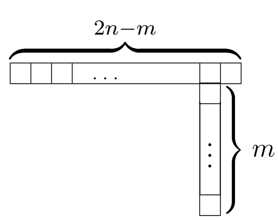
   For the remaining results let $k = 1$.

3. (Theorem 4.6) If the Young diagram of μ is obtained by adding the tetris type shape of size $2n$



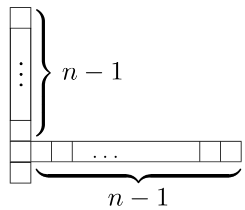
   to the top of the Young diagram of λ, then $\langle a^{2}_{\delta_{n+1}}, s_\mu\rangle = (-1)^m(2m+1)\langle a^{2}_{\delta_n}, s_\lambda\rangle$.

4. (Conjecture 4.8) If the Young diagram of μ is obtained by adding the tetris type shape of size $2n$



to the top of theYoung diagram of λ, then $\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^m (m+1) \langle a^2_{\delta_n}, s_\lambda \rangle$.

5. (Theorem 4.12) If the Young diagram of μ is obtained by adding the tetris type shape of size $2n$



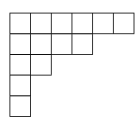
to the left of the Young diagram of λ, then $\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^n 3n \langle a^2_{\delta_n}, s_\lambda \rangle$.

We also prove several corollaries of the results above.

2 Notation and basic facts

We first introduce some notation and basic facts about Vandermonde determinants related to this problem. For details on partitions and Schur functions we refer the reader to [6, Chapter 7].

Let n be a non-negative integer. A *partition* of n is a weakly decreasing sequence of non-negative integers, $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_\ell)$, such that $|\lambda| := \sum \lambda_i = n$. We write $\lambda \vdash n$ to mean λ is a partition of n. The nonzero integers λ_i are called the *parts* of λ. We identify a partition with its *Young diagram*, i.e. the array of left-justified squares (boxes) with λ_1 boxes in the first row, λ_2 boxes in the second row, and so on. The rows are arranged in matrix form from top to bottom. By the box in position (i, j) we mean the box in the i-th row and j-th column of λ. The *length* of λ, $\ell(\lambda)$, is the number of rows in the Young diagram or the number of non-zero parts of λ. For example,



is the Young diagram for $\lambda = (6, 4, 2, 1, 1)$, with $\ell(\lambda) = 5$ and $|\lambda| = 14$.

We write $\lambda = \langle 1^{m_1}, 2^{m_2} \ldots \rangle$ to mean that λ has m_i parts equal to i.

Given a weak composition $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$ of length n, we write x^α for $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of length at most n and $\delta = \delta_n = (n-1, n-2, \ldots, 2, 1, 0)$, then the skew symmetric function $a_{\lambda+\delta}$ is defined as

$$a_{\lambda+\delta} = \det(x_i^{\lambda_j + n - j})_{i,j=1}^n. \tag{1}$$

If $\lambda = \emptyset$,

$$a_\delta = \det(x_i^{n-j})_{i,j=1}^n = \prod_{1 \le i < j \le n} (x_i - x_j) \tag{2}$$

is the Vandermonde determinant. We have [6, Theorem 7.15.1]

$$a_{\lambda+\delta}/a_\delta = s_\lambda(x_1, \ldots, x_n), \tag{3}$$

where $s_\lambda(x_1, \dots, x_n)$ is the Schur function of shape λ in variables $x_1, \dots, x_n$. Moreover, for any homogeneous symmetric function f of degree n, we have [6, Corollary 7.15.2]

$$\langle f, s_\lambda \rangle_n = [x^{\lambda+\delta}] a_\delta f, \tag{4}$$

i.e., the coefficient of $x^{\lambda+\delta}$ in $a_\delta f$. In particular, if $f = a_\delta^{2k}$, then

$$\langle a_\delta^{2k}, s_\lambda \rangle = [x^{\lambda+\delta}] a_\delta^{2k+1}. \tag{5}$$

We will often write c_λ for $\langle a_{\delta_n}^{2k}, s_\lambda \rangle$.

The goal of this work is to investigate several combinatorial properties of the numbers (1).

We note that $|\delta| = n(n-1)/2$ and a_δ is a homogeneous polynomial of degree $n(n-1)/2$. If $\langle a_\delta^{2k}, s_\lambda \rangle \neq 0$, then $|\lambda| = kn(n-1)$, $n-1 \leq \ell(\lambda) \leq n$, $k(n-1) \leq \lambda_1 \leq kn(n-1)$ and $\lambda_n \leq k(n-1)$. Moreover, if $\lambda_n = k(n-1)$, then $\lambda = (k(n-1))^n$.

Whenever it is necessary to emphasize the dimension, we write δ_n for δ and a_{δ_n} for a_δ.

By $\bar{a}_{\delta_n}$ we mean a_{δ_n} with x_i replaced by x_{i+1} for each $i = 1, 2, \dots n$. Thus,

$$\bar{a}_{\delta_n} = \prod_{2 \leq i < j \leq n+1} (x_i - x_j). \tag{6}$$

Given a weak composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ of n of length n, we denote by c_α the coefficient of x^α in $a_{\delta_n}^{2k+1}$. If ξ is a permutation of $\{1, 2, \dots, n\}$, and $\xi(\alpha)$ is the weak composition $(\alpha_{\xi(1)}, \alpha_{\xi(2)}, \dots, \alpha_{\xi(n)})$, then $c_\alpha = sgn(\xi) c_{\xi(\alpha)}$.

To simplify the notation, we write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ for a partition with $\ell(\lambda) \leq n$ by setting $\lambda_j = 0$ if $j > \ell(\lambda)$.

3 The box-complement of a partition

Definition: Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{n-1}, \lambda_n)$ be a partition of $kn(n-1)$ with $\ell(\lambda) \leq n$. The *box-complement* of λ is the partition of $kn(n-1)$ given by

$$\lambda_{bc} := (2k(n-1) - \lambda_n, 2k(n-1) - \lambda_{n-1}, \dots, 2k(n-1) - \lambda_2, 2k(n-1) - \lambda_1). \tag{7}$$

Thus, λ_{bc} is obtained from λ in the following way. Place the Young diagram of λ in the upper left corner of a box with n rows each of length $2k(n-1)$. If we remove the Young diagram of λ and rotate the remaining shape by $180°$, we obtain the Young diagram of λ_{bc}.

Example: Let $k = 1$, $n = 4$ and $\lambda = (5, 3, 2, 2)$. Then $\lambda_{bc} = (4, 4, 3, 2)$. The Young diagram of λ is shown on the left of the 4×6 box. The remaining squares of the box are marked with X. They form the diagram of λ_{bc} rotated by $180°$.

					X
			X	X	X
		X	X	X	X
		X	X	X	X

Lemma 3.1 (Box-complement lemma) *With the notation above we have*

$$\langle a_\delta^{2k}, s_\lambda\rangle = \langle a_\delta^{2k}, s_{\lambda_{bc}}\rangle. \tag{8}$$

For a proof in the case $k = 1$, see [2, Section 6] where the box-complement partition is referred to as the reversed partition. We prove the lemma for general k by elementary means, using induction on n. In [2], Dunne also explains the physical meaning of the box-complement lemma.

4 The dimension game

The goal of this section is to establish a relationship between $\langle a_{\delta_{n+1}}^{2k}, s_\mu\rangle$ and $\langle a_{\delta_n}^{2k}, s_\lambda\rangle$ when there is a relationship between $\lambda \vdash kn(n-1)$ and $\mu \vdash kn(n+1)$ of the type described in the introduction.

Theorem 4.1 *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of $kn(n-1)$ with $\ell(\lambda) \le n$ and μ is the partition of $kn(n+1)$ given by $\mu = (2kn, \lambda_1, \lambda_2, \ldots, \lambda_n)$, then*

$$\langle a_{\delta_{n+1}}^{2k}, s_\mu\rangle = \langle a_{\delta_n}^{2k}, s_\lambda\rangle. \tag{9}$$

Thus, adding the tetris type shape

$$\overbrace{\begin{array}{|c|c|c|c|c|c|}\hline & & & \cdots & & \\ \hline\end{array}}^{2kn}$$

to the top of the diagram for λ does not change the coefficient.

Proof: The proof follows by induction from

$$a_{\delta_{n+1}}^{2k+1} = \prod_{i=2}^{n+1}(x_1 - x_i)^{2k+1} \cdot \bar{a}_{\delta_n}^{2k+1}. \tag{10}$$

$\square$

Remark: The theorem is also true if λ is just a weak composition of $kn(n-1)$ (with no more than one part equal to 0).

Note: The result of the theorem for $k = 1$ is also noted in (23a) of [7].

Corollary 4.2 *If $\lambda = (2k(n-1), 2k(n-2), \ldots, 4k, 2k, 0) = 2k\delta_n$, then $\langle a_{\delta_n}^{2k}, s_\lambda\rangle = 1$.*

Using Theorem 4.1 and Lemma 3.1, we obtain the following corollary.

Corollary 4.3 *If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition of $kn(n-1)$ with $\ell(\lambda) \le n$, and μ is the partition of $kn(n+1)$ given by $\mu = \lambda + \langle (2k)^n\rangle = (\lambda_1 + 2k, \lambda_2 + 2k, \ldots, \lambda_n + 2k)$, then*

$$\langle a_{\delta_{n+1}}^{2k}, s_\mu\rangle = \langle a_{\delta_n}^{2k}, s_\lambda\rangle. \tag{11}$$

Thus adding the tetris type shape

$$n\left\{\overbrace{\begin{array}{|c|c|c|}\hline & \cdots & \\ \hline \multicolumn{3}{|c|}{\vdots} \\ \hline & \cdots & \\ \hline\end{array}}^{2k}\right.$$

to the left of the diagram of λ does not change the coefficient.

Note: For $k = 1$ this is (23b) of [7].

For the remainder of the article we set $k = 1$.

Lemma 4.4 *Suppose* $\lambda \vdash n(n-1)$ *with* $n - 1 \leq \ell(\lambda) \leq n$ *and* $\langle a^2_{\delta_n}, s_\lambda \rangle \neq 0$. *If* $\lambda_n = \lambda_{n-1} = \ldots = \lambda_{n-i} = s$, *then* $i \leq s$, i.e., *the maximum number of rows of size* s *at the bottom of the diagram is* $s + 1$.

Proof: We examine the coefficient of $x^{\lambda} + \delta_n$ in $a^3_{\delta_n}$. Suppose $i = s + 1$. Then

$$x^{\lambda+\delta_n} = (x_1^{\lambda_1+n-1} \cdots x_{n-s}^{\lambda_{n-s}+s}) \cdot M_s, \tag{12}$$

where the monomial $M_s = x^{2s+1}_{n-s-1} \cdots x^{s+2}_{n-2} x^{s+1}_{n-1} x^s_n$ has degree $\dfrac{(3s+1)(s+2)}{2}$.

On the other hand,

$$\prod_{j=n-s}^{n} (x_{n-s-1} - x_j)^3 \cdot \prod_{j=n-s+1}^{n} (x_{n-s} - x_j)^3 \cdots \prod_{j=n-1}^{n} (x_{n-2} - x_j)^3 \cdot (x_{n-1} - x_n)^3 \tag{13}$$

contributes powers of $x_{n-s-1}, x_{n-s}, \ldots, x_n$ to all monomials in $a^3_{\delta_n}$. However, each monomial in the product (13) has degree $\dfrac{(3s+3)(s+2)}{2}$. Therefore, $i \leq s$.

$\square$

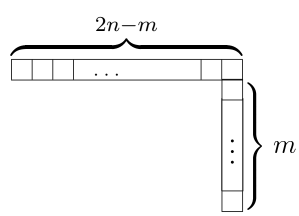
Reformulating the previous lemma in terms of the box-complement of the partition λ, we obtain the following corollary.

Corollary 4.5 *Suppose* $\lambda \vdash n(n-1)$ *with* $n - 1 \leq \ell(\lambda) \leq n$ *and* $\langle a^2_{\delta_n}, s_\lambda \rangle \neq 0$. *If* $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 2n - m - 1$, *then* $\lambda_{m+1} < 2n - m - 1$.

Theorem 4.6 *Let* $1 \leq m \leq n$. *Let* $\lambda \vdash n(n-1)$ *with* $n - 1 \leq \ell(\lambda) \leq n$ *and parts* $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 2n - m - 1$. *Let* $\mu \vdash n(n+1)$ *with parts* $\mu_1 = \mu_2 = \ldots = \mu_{m+1} = 2n - m$ *and (if* $m < n$*)* $\mu_j = \lambda_{j-1}$ *for* $j = m + 2, \ldots, n + 1$. *Then,*

$$\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^m (2m+1) \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{14}$$

Thus, adding the tetris type shape



to the top of the diagram of λ changes the coefficient by a multiple of $(-1)^m(2m+1)$.

Proof: The proof relies on tricky but elementary linear algebra and reduces to [6, Exercise 7.37(b)] which shows that

$$\langle a^2_{\delta_m}, s_{\langle (m-1)^m \rangle} \rangle = (-1)^{\binom{m}{2}} \cdot 1 \cdot 3 \cdots (2m-1). \tag{15}$$

□

Note: If $k = 1$, Theorem 4.1 fits into the pattern of Theorem 4.6 for $m = 0$.

Exercise 7.37(c) of [6] follows as an easy corollary of Theorems 4.6 and 4.1.

Corollary 4.7 *If $\lambda = ((n+i-1)^{n-i}, (i-1)^i)$, $1 \le i \le n$, then*

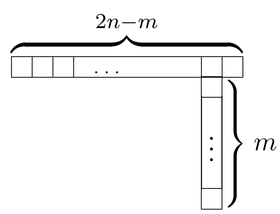
$$\langle a_{\delta_n}, s_\lambda \rangle = (-1)^{\frac{1}{2}(n-1)(n-2i)}[1 \cdot 3 \cdots (2i-1)] \cdot [1 \cdot 3 \cdots (2(n-i)-1)]. \tag{16}$$

We state conjecturally a similar combinatorial recursive property.

Conjecture 4.8 *Let $1 \le m \le n-1$. Let $\lambda \vdash n(n-1)$ with $n-1 \le \ell(\lambda) \le n$ and parts $\lambda_1 = \lambda_2 = \ldots = \lambda_m = 2n-m-2$. Let $\mu \vdash n(n+1)$ with parts $\mu_1 = 2n-m$, $\mu_2 = \ldots = \mu_{m+1} = 2n-m-1$ and (if $m < n$) $\mu_j = \lambda_{j-1}$ for $j = m+2, \ldots, n+1$. Then,*

$$\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^m (m+1) \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{17}$$

Thus adding the tetris type shape



to the top of the diagram of λ changes the coefficient by a multiple of $(-1)^m(m+1)$.

The statement of the conjecture will be proved if we can show the following.

Let $l = (l_1, l_2, \ldots, l_m, l_{m+1} = 0)$ be a partition of m of length at most $m+1$ (thus, at least one part is zero). Let α be a permutation of $\{1, 2, \ldots, m\}$ and let β be a permutation of $\{1, 2, \ldots, m, m+1\}$. Denote by $\mathcal{C}(l, \alpha)$ the coefficient of

$$x_1^{2m-3+l_{\alpha(1)}} x_2^{2m-4+l_{\alpha(2)}} \cdots x_m^{m-2+l_{\alpha(m)}} = x^{\langle (m-2)^m \rangle + \alpha(l^*) + \delta_m} \tag{18}$$

in $a^3_{\delta_m}$ (here, $l^* = (l_1, l_2, \ldots, l_m)$) and denote by $\overline{\mathcal{C}}(l, \beta)$ the coefficient of

$$x_1^{2m+l_{\beta(1)}} x_2^{2m-2+l_{\beta(2)}} x_3^{2m-3+l_{\beta(3)}} \cdots x_m^{m+l_{\beta(m)}} x_{m+1}^{m-1+l_{\beta(m+1)}} = x^{\langle m, (m-1)^m \rangle + \beta(l) + \delta_{m+1}} \tag{19}$$

in $a^3_{\delta_{m+1}}$.

We need to show that for each partition $l = (l_1, l_2, \ldots l_m)$ of m of length at most m,

$$\sum_{\beta \in S_{m+1}} \overline{\mathcal{C}}(l, \beta) = (-1)^m (m+1) \sum_{\alpha \in S_m} \mathcal{C}(l, \alpha). \tag{20}$$

The particular cases of Conjecture 4.8 when $m = 1$ or $m = n-1$ can be proved directly.

Proposition 4.9 (Case $m = 1$ of Conjecture 4.8) *Let* $\lambda = (2n-3, \lambda_2, \ldots, \lambda_n) \vdash n(n-1)$ *and* $\mu = (2n-1, 2n-2, \lambda_2, \ldots, \lambda_n) \vdash n(n+1)$. *Then*

$$\langle a^2_{\delta_{n+1}}, s_\mu \rangle = -2 \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{21}$$

Proof: The proof is elementary and relies on a case analysis of the contribution of $(x_1 - x_2)^3$ to the coefficient of $x^{\mu+\delta_{n+1}}$ in $a^3_{\delta_{n+1}}$.

□

Using Proposition 4.9 and Theorem 4.1 we can prove by induction the following observation that Dunne finds remarkable (see [2, Section 6]). Starting with the partition λ of Corollary 4.2, $\lambda = (2(n-1), 2(n-2), \ldots, 4, 2, 0)$, if we remove the last box from the jth row of the Young diagram of λ and add it to the end of the $(j+k)$th row, the coefficient changes to $(-1)^k \cdot 3 \cdot 2^{k-1}$.

Corollary 4.10 *Fix an integer* j *with* $1 \leq j \leq n-1$ *and let* k *be a fixed integer such that* $j+1 \leq k \leq n$. *If* $\nu \vdash n(n-1)$ *is given by*

$$(2(n-1), \ldots, 2(n-j+1), 2(n-j)-1, 2(n-j-1), \ldots, 2(n-k+1), 2(n-k)+1, 2(n-k-1), \ldots, 2, 0), \tag{22}$$

then

$$\langle a_{\delta_n}, s_\nu \rangle = (-1)^{k-j} \cdot 3 \cdot 2^{k-j-1}. \tag{23}$$

Proposition 4.11 (Case $m = n-1$ of Conjecture 4.8) *Let* $\lambda = \langle (n-1)^n \rangle \vdash n(n-1)$ *and* $\mu = \langle n+1, n^n \rangle \vdash n(n+1)$. *Then*

$$\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^{n-1} n \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{24}$$

Proof: We first introduce some definitions following [6, Chapter 7]. Denote by f_λ the number of standard Young tableaux (SYT) of shape λ. Given a Young diagram λ and a square $u = (i, j)$ of λ, let $h(u)$ denote the hook length of u, the number of squares directly to the right or directly below u, including u itself once. Thus

$$h(u) = \lambda_i + \lambda'_j - i - j + 1.$$

We also define the content $c(u)$ of λ at $u = (i, j)$ by

$$c(u) = j - i.$$

If λ is a partition of t, the *hook-length formula* [6, Corollary 7.21.6] gives

$$f_\lambda = \frac{t!}{\prod_{u \in \lambda} h(u)}. \tag{25}$$

If λ is a partition of $n(n-1)$ and $\lambda = \eta + \langle (n-2)^n \rangle$ (thus η is a partition of n), then, by [6, Exercise 7.37.d]

$$\langle a^2_{\delta_n}, s_\lambda \rangle = (-1)^{\binom{n}{2}} f_\eta \prod_{s \in \eta} (1 - 2c(s)). \tag{26}$$

As noted in [4],

$$f_\lambda = \sum_{\nu \in \lambda \setminus 1} f_\nu, \tag{27}$$

where $\lambda \setminus 1$ is the set of partitions obtained from λ by removing a corner. (This formula follows directly from the construction of standard Young tableaux.)

For the partitions λ and μ in the statement of the proposition, we have

$$\lambda = \langle (n-1)^n \rangle = \langle 1^n \rangle + \langle (n-2)^n \rangle \tag{28}$$

and

$$\mu = \langle 2, 1^{n-1} \rangle + \langle (n-1)^n \rangle. \tag{29}$$

Then, by (26), we have

$$\langle a^2_{\delta_n}, s_\lambda \rangle = (-1)^{\binom{n}{2}} f_{\langle 1^n \rangle} \prod_{s \in \langle 1^n \rangle} (1 - 2c(s)) = (-1)^{\binom{n}{2}} 1 \cdot 3 \cdot 5 \cdots (2n-1) \tag{30}$$

and

$$\langle a^2_{\delta_n}, s_\mu \rangle = (-1)^{\binom{n+1}{2}} f_{\langle 2, 1^{n-1} \rangle} \prod_{s \in \langle 2, 1^{n-1} \rangle} (1 - 2c(s)) = \tag{31}$$

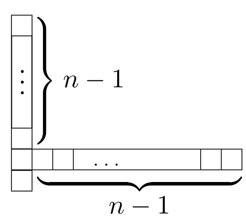
$$(-1)^{\binom{n+1}{2}} \frac{(n+1)!}{(n+1)(n-1)!} (-1) \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1) = (-1)^{n-1} n \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{32}$$

□

Theorem 4.12 *Let* $\lambda \vdash n(n-1)$ *with* $\ell(\lambda) = n-1$ *and* $\lambda_{n-1} \geq n-1$ *and let* $\mu \vdash n(n+1)$ *be given by* $\mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_{n-1} + 1, n, 1)$. *Then,*

$$\langle a^2_{\delta_{n+1}}, s_\mu \rangle = (-1)^n 3n \langle a^2_{\delta_n}, s_\lambda \rangle. \tag{33}$$

Thus adding the tetris type shape



to the left of the diagram of λ changes the coefficient by a multiple of $(-1)^n 3n$.

Proof: If $\lambda \vdash n(n-1)$ with $\ell(\lambda) = n-1$ is as in the statement of the theorem, then λ_{n-1} equals n or $n-1$.

Case I: $\lambda_{n-1} = n$. Then $\lambda = \langle n^{n-1} \rangle$ and, after repeatedly applying Theorem 4.6, we have

$$c_\lambda := \langle a^2_{\delta_n}, s_\lambda \rangle = (-1)^{\binom{n-1}{2}} 1 \cdot 3 \cdot 5 \cdots (2n-3).$$

In this case $\mu = \langle (n+1)^{n-1}, n, 1\rangle$ and the calculation of $c_\mu = \langle a_{d_{n+1}}, s_\mu\rangle$ relies on Proposition 4.11.

Case II: $\lambda_{n-1} = n-1$. Thus, $\lambda = \langle (n-1)^{n-1}\rangle + \nu$, where ν is a partition of $n-1$. The last part of ν can only be 0 or 1. If $\nu_{n-1} = 1$, then we are in Case I. Therefore, we assume $\nu_{n-1} = 0$.

Using Corollary 4.3, we have

$$\langle a_{\delta_n}^2, s_\lambda\rangle = \langle a_{\delta_{n-1}}^2, s_{\lambda/\langle 2^{n-1}\rangle}\rangle.$$

Since $\lambda/\langle 2^{n-1}\rangle = \langle (n-3)^{n-1}\rangle + \nu$ is a partition of $(n-1)(n-2)$, we can use (26) to obtain

$$\langle a_{\delta_n}^2, s_\lambda\rangle = (-1)^{\binom{n-1}{2}} f_\nu \prod_{s\in\nu}(1-2c(s)). \tag{34}$$

We can also use (25) to write

$$\langle a_{\delta_n}^2, s_\lambda\rangle = (-1)^{\binom{n-1}{2}} \frac{(n-1)!}{\prod_{u\in\nu} h(u)} \prod_{s\in\nu}(1-2c(s)). \tag{35}$$

Now let us consider $\mu = (n+\nu_1, n+\nu_2, \ldots, n+\nu_{n-1}, n, 1)$. We have

$$x^{\mu+\delta_{n+1}} = x_1^{2n+\nu_1} x_2^{2n+\nu_2-1} \cdots x_i^{2n-\nu_i-i+1} \cdots x_{n-2}^{n+3+\nu_{n-2}} x_{n-1}^{n+2+\nu_{n-1}} x_n^{n+1} x_{n+1}.$$

We write $a_{\delta_{n+1}}^3$ as

$$a_{\delta_{n+1}}^3 = a_{\delta_n}^3 \prod_{i=1}^{n} (x_i - x_{n+1})^3.$$

For each $i = 1, 2, \ldots, n$, the product $\prod_{i=1}^{n}(x_i - x_{n+1})^3$ contributes

$$-3x_i^2 x_{n+1} x_1^3 x_2^3 \cdots x_{i-1}^3 x_{i+1}^3 \cdots x_n^3$$

and $a_{\delta_n}^3$ contributes

$$x_1^{2n+\nu_1-3} x_2^{2n+\nu_2-4} x_3^{2n+\nu_3-5} \cdots x_{i-1}^{2n+\nu_{i-1}-i-1} x_i^{2n-\nu_i-i-1} x_{i+1}^{2n+\nu_{i+1}-i-3} \cdots x_{n-1}^{n+\nu_{n-1}-1} x_n^{n-2}$$

with multiplicity c_i to forming $x^{\mu+\delta_{n+1}}$.

If $\nu_{i-1} = \nu_i$, then $c_i = 0$. Moreover, $c_n = 0$ since we assumed that $\nu_{n-1} = 0$.

For each $i = 1, 2, \ldots, n$, such that $\nu_{i-1} > \nu_i$, $c_i = \langle a_{\delta_n}^2, s_{\eta_i}\rangle$, where

$$\eta_i = (n+\nu_1-2, n+\nu_2-2, n+\nu_3-2, \ldots, n+\nu_{i-1}-2, n+\nu_i-1, n+\nu_{i+1}-2, \ldots, n+\nu_{n-1}-2, n-2). \tag{36}$$

Thus

$$\eta_i = \langle (n-2)^n\rangle + \tilde{\nu}_i, \tag{37}$$

where $\tilde{\nu}_i$ is the partition of n obtained from ν by adding a box at the end of the ith row, *i.e.*,

$$\tilde{\nu}_i = (\nu_1, \nu_2, \ldots, \nu_{i-1}, \nu_i + 1, \nu_{i+1}, \ldots, \nu_{n-1}). \tag{38}$$

To find c_i we can now use (26). We have

$$c_i = (-1)^{\binom{n}{2}} f_{\tilde{\nu}_i} \prod_{s \in \tilde{\nu}_i} (1 - 2c(s)), \tag{39}$$

or, using (25),

$$c_i = (-1)^{\binom{n}{2}} \frac{n!}{\prod_{u \in \tilde{\nu}_i} h(u)} \prod_{s \in \tilde{\nu}_i} (1 - 2c(s)). \tag{40}$$

Now let us compare $\prod_{s \in \nu}(1 - 2c(s))$ and $\prod_{s \in \tilde{\nu}_i}(1 - 2c(s))$. We have

$$\prod_{s \in \tilde{\nu}_i}(1 - 2c(s)) = (1 - 2c(i, \nu_i + 1)) \prod_{s \in \nu}(1 - 2c(s)) = (2i - 2\nu_i - 1) \prod_{s \in \nu}(1 - 2c(s)). \tag{41}$$

Thus, using (34) and (39), in order to prove the theorem, we need to show that

$$n f_\nu = \sum_{\substack{i=1 \\ \nu_{i-1} > \nu_i}}^{n} f_{\tilde{\nu}_i}(2i - 2\nu_i - 1). \tag{42}$$

Note that the terms for $i = 1$ and $i = \ell(\nu) + 1$ are always included in the sum.

The Statement (42) can be proved by induction using the commutativity of the operations ”removal of one box” and ”addition of one box”. It also follows from [5].

□

5 Open problems

1. Ideally, one would be able to find other rules involving different tetris type shapes.

2. In [3], the authors define *admissible partitions* and conjecture that they determine the Schur functions with non-zero coefficients in the decomposition of the $(2k)$th power of the Vandermonde determinant. However, the conjecture fails and counterexamples have been provided by [7]. It remains an open problem to find a non-vanishing criterion for these coefficients.

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The Murnaghan–Nakayama rule for k-Schur functions

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Abstract. We prove a Murnaghan–Nakayama rule for k-Schur functions of Lapointe and Morse. That is, we give an explicit formula for the expansion of the product of a power sum symmetric function and a k-Schur function in terms of k-Schur functions. This is proved using the noncommutative k-Schur functions in terms of the nilCoxeter algebra introduced by Lam and the affine analogue of noncommutative symmetric functions of Fomin and Greene.

Résumé. Nous prouvons une règle de Murnaghan-Nakayama pour les fonctions de k-Schur de Lapointe et Morse, c'est-à-dire que nous donnons une formule explicite pour le développement du produit d'une fonction symétrique "somme de puissances" et d'une fonction de k-Schur en termes de fonctions k-Schur. Ceci est prouvé en utilisant les fonctions non commutatives k-Schur en termes d'algèbre nilCoxeter introduite par Lam et l'analogue affine des fonctions symétriques non commutatives de Fomin et Greene.

Keywords: Murnaghan–Nayakama rule, symmetric functions, noncommutative symmetric functions, k-Schur functions

1 Introduction

The Murnaghan–Nakayama rule [11, 14, 15] is a combinatorial formula for the characters $\chi_\lambda(\mu)$ of the symmetric group in terms of ribbon tableaux. Under the Frobenius characteristic map, there exists an analogous statement on the level of symmetric functions, which follows directly from the formula

$$p_r s_\lambda = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} s_\mu. \tag{1}$$

Here p_r is the r-th power sum symmetric function, s_λ is the Schur function labeled by partition λ, and the sum is over all partitions $\lambda \subseteq \mu$ for which μ/λ is a border strip of size r. Recall that a border strip is a connected skew shape without any 2×2 squares. The height $\mathrm{ht}(\mu/\lambda)$ of a border strip μ/λ is one less than the number of rows.

In [4], Fomin and Greene develop the theory of Schur functions in noncommuting variables. In particular, they derive a noncommutative version of the Murnaghan–Nakayama rule [4, Theorem 1.3] for the nilCoxeter algebra (or more generally the local plactic algebra)

$$\mathbf{p}_r \mathbf{s}_\lambda = \sum_w (-1)^{\mathrm{asc}(w)} w \mathbf{s}_\lambda \ , \tag{2}$$

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where w is a hook word of length r. Here $\mathbf{p}_r$ and $\mathbf{s}_\lambda$ are the noncommutative analogues of the power sum symmetric function and the Schur function (introduced in Section 2). The word w is a hook word if $w = b_l b_{l-1} \dots b_1 a_1 a_2 \dots a_m$ where

$$b_l > b_{l-1} > \cdots > b_1 > a_1 \le a_2 \le \cdots \le a_m \tag{3}$$

and $\operatorname{asc}(w) = m-1$ is the number of ascents in w. Actually, by [4, Theorem 5.1] it can further be assumed that the support of w is an interval.

In this paper, we derive a (noncommutative) Murnaghan–Nakayama rule for the k-Schur functions of Lapointe and Morse [10]. k-Schur functions form a basis for the ring $\Lambda_{(k)} = \mathbb{Z}[h_1, \dots, h_k]$ spanned by the first k complete homogeneous symmetric functions h_r, which is a subring of the ring of symmetric functions Λ. Lapointe and Morse [10] gave a formula for a homogeneous symmetric function h_r times a k-Schur function (at $t=1$) as

$$h_r s_\lambda^{(k)} = \sum_{\mu \in \mathcal{P}^{(k)}} s_\mu^{(k)}, \tag{4}$$

where the sum is over all k-bounded partitions $\mu \in \mathcal{P}^{(k)}$ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip. Here $\lambda^{(k)}$ denotes the k-conjugate of λ. Equation (4) is a simple analogue of the Pieri rule for usual Schur functions, called the k-Pieri rule. This formula can in fact be taken as the definition of k-Schur functions from which many of their properties can be derived. Conjecturally, the k-Pieri definition of the k-Schur functions is equivalent to the original definition by Lapointe, Lascoux, and Morse [6] in terms of atoms.

Lam [5] defined a noncommutative version of the k-Schur functions in the affine nilCoxeter algebra as the dual of the affine Stanley symmetric functions

$$F_w(X) = \sum_{a=(a_1,\dots,a_t)} \langle \mathbf{h}_{a_t}(u)\mathbf{h}_{a_{t-1}}(u)\cdots\mathbf{h}_{a_1}(u)\cdot 1, w\rangle \; x_1^{a_1}\cdots x_t^{a_t}, \tag{5}$$

where the sum is over all compositions of the length of w satisfying $a_i \in [0,k]$. Here

$$\mathbf{h}_r(u) = \sum_A u_A^{\mathrm{dec}}$$

are the analogues of homogeneous symmetric functions in noncommutative variables, where the sum is over all r-subsets A of $[0,k]$ and u_A^{dec} is the product of the generators of the affine nilCoxeter algebra in cyclically decreasing order with indices appearing in A. We denote the noncommutative analogue of $\Lambda_{(k)}$ by $\mathbf{\Lambda}_{(k)}$ as the subalgebra of the affine nilCoxeter algebra generated by these analogues of homogeneous symmetric functions. See Section 2.3 for further details.

Denote by $\mathbf{s}_\lambda^{(k)}$ the noncommutative k-Schur function labeled by the k-bounded partition λ and $\mathbf{p}_r$ the noncommutative power sum symmetric function in the affine nilCoxeter algebra. There is a natural bijection from k-bounded partitions λ to $(k+1)$-cores, denoted $\operatorname{core}_{k+1}(\lambda)$ (see Section 2.1). We define a vertical domino in a skew-partition to be a pair of cells in the diagram, with one sitting directly above the other. For the skew of two k-bounded partitions $\lambda \subseteq \mu$ we define the height as

$$\operatorname{ht}(\mu/\lambda) = \text{number of vertical dominos in } \mu/\lambda \,. \tag{6}$$

For ribbons, that is skew shapes without any 2×2 squares, the definition of height can be restated as the number of occupied rows minus the number of connected components. Notice that is compatible with the usual definition of the height of a border strip.

All notation and definitions regarding our main Definition 1.1 and Theorem 1.2 are given in Section 2 below.

Definition 1.1 *The skew of two k-bounded partitions, μ/λ, is called a k-*ribbon of size r *if μ and λ satisfy the following properties:*

(0) (containment condition) $\lambda \subseteq \mu$ and $\lambda^{(k)} \subseteq \mu^{(k)}$;

(1) (size condition) $|\mu/\lambda| = r$;

(2) (ribbon condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is a ribbon;

(3) (connectedness condition) $\mathrm{core}_{k+1}(\mu)/\mathrm{core}_{k+1}(\lambda)$ is k-connected (see Definition 2.3);

(4) (height statistics condition) $\mathrm{ht}(\mu/\lambda) + \mathrm{ht}(\mu^{(k)}/\lambda^{(k)}) = r - 1$.

Our main result is the following theorem.

Theorem 1.2 *For $1 \leq r \leq k$ and λ a k-bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_\mu (-1)^{\mathrm{ht}(\mu/\lambda)} \mathbf{s}_\mu^{(k)},$$

where the sum is over all k-bounded partitions μ such that μ/λ is a k-ribbon of size r.

Let λ, ν be k-bounded partitions of the same size and ℓ the length of ν. A k-ribbon tableau of shape λ and type ν is a filling, T, of the cells of λ with the labels $\{1, 2, \ldots, \ell\}$ which satisfies the following conditions for all $1 \leq i \leq \ell$:

(i) the shape of the restriction of T to the cells labeled $1, \ldots, i$ is a partition, and

(ii) the skew shape r_i, which is the restriction of T to the cells labeled i, is a k-ribbon of size ν_i.

We also define

$$\chi_{\lambda,\nu}^{(k)} = \sum_T \left(\prod_{i=1}^{\ell} (-1)^{\mathrm{ht}(r_i)} \right) ,$$

where the sum is over all k-ribbon tableaux T of shape λ and type ν.

Iterating Theorem 1.2 gives the following corollary. We remark that this formula may also be considered as a definition of the k-Schur functions.

Corollary 1.3 *For ν a k-bounded partition, we have*

$$\mathbf{p}_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda,\nu}^{(k)} \, \mathbf{s}_\lambda^{(k)}.$$

In Section 2 we will see that there is a ring isomorphism

$$\iota : \mathbf{\Lambda}_{(k)} \to \Lambda_{(k)}$$

sending the noncommutative symmetric functions to their symmetric function counterpart. This leads us to the following corollary.

Corollary 1.4 *Theorem 1.2 and Corollary 1.3 also hold when replacing $\mathbf{p}_r$ by the power sum symmetric function p_r, and $\mathbf{s}_\lambda^{(k)}$ by the k-Schur function $s_\lambda^{(k)}$.*

Dual k-Schur functions $\mathfrak{S}_\lambda^{(k)}$ indexed by k-bounded partitions λ form a basis of the quotient space $\Lambda^{(k)} = \Lambda/\langle p_r \mid r > k\rangle = \Lambda/\langle m_\lambda \mid \lambda_1 > k\rangle$ (they correspond to the affine Stanley symmetric functions indexed by Grassmannian elements). The Hall inner product $\langle\cdot,\cdot\rangle : \Lambda\times\Lambda \to \mathbb{Q}$ defined by $\langle h_\lambda, m_\mu\rangle = \langle s_\lambda, s_\mu\rangle = \delta_{\lambda,\mu}$, can be restricted to $\langle\cdot,\cdot\rangle : \Lambda_{(k)} \times \Lambda^{(k)} \to \mathbb{Q}$, so that $s_\lambda^{(k)}$ and $\mathfrak{S}_\mu^{(k)}$ form dual bases $\langle s_\lambda^{(k)}, \mathfrak{S}_\mu^{(k)}\rangle = \delta_{\lambda,\mu}$. Let z_λ be the size of the centralizer of any permutation of cycle type λ. Then $\langle p_\lambda, p_\mu\rangle = z_\lambda\delta_{\lambda,\mu}$.

Corollary 1.5 *For ν a k-bounded partition, we have*

$$\mathfrak{S}_\nu^{(k)} = \sum_{\lambda\in\mathcal{P}^{(k)}} \frac{1}{z_\lambda}\chi_{\nu,\lambda}^{(k)}\, p_\lambda\ .$$

Since the product of two k-bounded power symmetric functions is again a k-bounded power symmetric function, the expansion of the dual k-Schur functions in terms of p_λ of Corollary 1.5 is better suited for multiplication than the expansion in terms of monomial symmetric functions. The product of two k-bounded monomial symmetric functions is a sum of monomial symmetric functions which are not necessarily k-bounded.

The paper is organized as follows. In Section 2 we introduce all notation and definitions. In particular, we define the various noncommutative symmetric functions. In Section 3 we prove Theorem 3.1, which is the analogue of Theorem 1.2 formulated in terms of the nilCoxeter algebra. We conclude in Section 4 with some related open questions.

A long version of this paper containing a proof that Theorems 1.2 and 3.1 are equivalent is available as a preprint [1].

Acknowledgments

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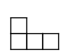
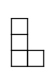
2 Notation

In this section we give all necessary definitions.

2.1 Partitions and cores

A sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is a partition if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0$. We say that ℓ is the length of λ and $|\lambda| = \lambda_1 + \cdots + \lambda_\ell$ is its size. A partition λ is k-bounded if $\lambda_1 \leq k$. We denote by $\mathcal{P}^{(k)}$ the set of all k-bounded partitions.

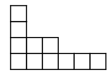
One may represent a partition λ by its partition diagram, which contains λ_i boxes in row i. The conjugate λ^t corresponds to the diagram with rows and columns interchanged. We use French convention and label rows in decreasing order from bottom to top. For example

 and 

correspond to the partition $(3, 1)$ and its conjugate $(2, 1, 1)$, respectively.

For two partitions λ and μ whose diagrams are contained, that is $\lambda \subseteq \mu$, we denote by μ/λ the skew partition consisting of the boxes in μ not contained in λ. A ribbon is a skew shape which does not contain any 2×2 squares. An r-border strip is a connected ribbon with r boxes.

A partition λ is an r-core if no r-border strip can be removed from λ and still results in a partition. For example



(7)

is a 4-core. We denote the set of all r-cores by $\mathcal{C}_r$.

For a cell $c = (i, j) \in \lambda$ in row i and column j we define its hook length to be the number of cells in row i of λ to the right of c plus the number of cells in column j of λ weakly above c (including c). An alternative definition of an r-core is a partition without any cells of hook length equal to a multiple of r [13, Ch. 1, Ex. 8]. The content of cell $c = (i, j)$ is given by $j - i \pmod r$.

There exists a bijection [9]

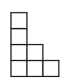
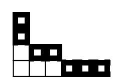
$$\mathrm{core}_{k+1} : \mathcal{P}^{(k)} \to \mathcal{C}_{k+1} \tag{8}$$

from k-bounded partitions to $(k+1)$-cores defined as follows. Let $\lambda \in \mathcal{P}^{(k)}$ considered as a set of cells. Starting from the smallest row, check whether there are any cells of hook length greater than k. If so, slide the row and all those in the rows below to the right by the minimal amount so that none of cells in that row have a hook length greater than k. Then continue the procedure with the rows below. The positions of the cells define a skew partition and the outer partition is a $(k+1)$-core.

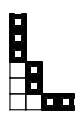
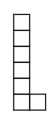
The inverse map $\mathrm{core}_{k+1}^{-1} : \mathcal{C}_{k+1} \to \mathcal{P}^{(k)}$ is slightly easier to compute. The partition $\mathrm{core}_{k+1}^{-1}(\kappa)$ is of the same length as the $(k+1)$-core κ and the i^{th} entry of the partition is the number of cells in the the i^{th} row of κ which have a hook smaller or equal to k.

Let $\lambda \in \mathcal{P}^{(k)}$. Then the k-conjugate $\lambda^{(k)}$ of λ is defined as $\mathrm{core}_{k+1}^{-1}(\mathrm{core}_{k+1}(\lambda)^t)$.

Example 2.1 *For $k = 3$, take $\lambda = (3, 2, 1, 1) \in \mathcal{P}^{(k)}$ so that*

$\mathrm{core}_4 :$  $\mapsto$ 

which is the 4-core in (7) (where we have drawn the original boxes of λ in bold). To obtain the k-conjugate $\lambda^{(3)}$ of λ we calculate

$\mathrm{core}_4^{-1} :$  $\mapsto$  .

2.2 Affine nilCoxeter algebra

The affine nilCoxeter algebra $\mathcal{A}_k$ is the algebra over $\mathbb{Z}$ generated by $u_0, u_1, \ldots, u_k$ satisfying

$$\begin{aligned} &u_i^2 = 0 && \text{for } i \in [0, k], \\ &u_i u_{i+1} u_i = u_{i+1} u_i u_{i+1} && \text{for } i \in [0, k], \\ &u_i u_j = u_j u_i && \text{for } i, j \in [0, k] \text{ such that } |i - j| \geq 2, \end{aligned} \tag{9}$$

where all indices are taken modulo $k + 1$. We view the indices $i \in [0, k]$ as living on a circle, with node i being adjacent to nodes $i - 1$ and $i + 1$ (modulo $k + 1$). As with Coxeter groups, we have a notion of reduced words of elements $u \in \mathcal{A}_k$ as the shortest expressions in the generators. If $u = u_{i_1} \cdots u_{i_m}$ is a reduced expression, we call $\{i_1, \ldots, i_m\}$ the support of u denoted $\mathrm{supp}(u)$ (which is independent of the reduced word and only depends on u itself). Also, $i_1 \ldots i_m$ is the corresponding reduced word and $\mathrm{len}(u) = m$ is the length of u.

A word w in the letters $[0, k]$ is cyclically decreasing (resp. increasing) if the length of w is at most k, every letter appears at most once, and if $i, i-1 \in w$ then i occurs before (resp. after) $i-1$. Note that since u_i and u_j commute if i is not adjacent to j, all cyclically decreasing (resp. increasing) words w with the same support give rise to the same affine nilCoxeter group element $\prod_{i \in w} u_i$. For a proper subset $A \subsetneq [0, k]$ we define $u_A^{\mathrm{dec}} \in \mathcal{A}_k$ (resp. $u_A^{\mathrm{inc}} \in \mathcal{A}_k$) to be the element corresponding to cyclically decreasing (resp. increasing) words with support A.

Example 2.2 *Take $k = 6$ and $A = \{0, 2, 3, 4, 6\}$. Then $u_A^{\mathrm{dec}} = (u_0 u_6)(u_4 u_3 u_2) = (u_4 u_3 u_2)(u_0 u_6)$ and $u_A^{\mathrm{inc}} = (u_6 u_0)(u_2 u_3 u_4) = (u_2 u_3 u_4)(u_6 u_0)$.*

If $u \in \mathcal{A}_k$ is supported on a proper subset S of $[0, k]$, then we specify a canonical interval I_S which contains the subset S. Identify the smallest element a (from the numbers 0 through k with the integer order) which does not appear in S. Then the canonical cyclic interval which we choose orders the elements

$$a + 1 < a + 2 < \cdots < k < 0 < 1 < \cdots < a - 1,$$

(where we identify k and -1 when necessary).

Definition 2.3 *An element $u \in \mathcal{A}_k$ (resp. word w) is k-connected if its support S is an interval in I_S.*

Example 2.4 *For $k = 6$, the word $w = 0605$ is k-connected, whereas $w = 06052$ is not.*

Suppose $u \in \mathcal{A}_k$ has support $S \subsetneq [0, k]$. We say that u corresponds to a hook word if it has a reduced word w of the form of Equation (3) with respect to the canonical order I_S. In this case we denote by $\mathrm{asc}(u)$ or $\mathrm{asc}(w)$ the number of ascents $\mathrm{asc}_{I_S}(w)$ in the canonical order.

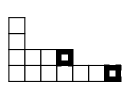
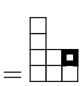
Example 2.5 *Take $u = u_3 u_2 u_6 u_0 u_4 \in \mathcal{A}_6$. In this case $S = \{0, 2, 3, 4, 6\}$ and I_S is given by $2 < 3 < 4 < 5 < 6 < 0$. The word $w = (3)(2460)$ is a hook word with respect to I_S and $\mathrm{asc}(u) = 3$.*

The generators u_i in the nilCoxeter algebra $\mathcal{A}_k$ act on a $(k+1)$-core $\nu \in \mathcal{C}_{k+1}$ by

$$u_i \cdot \nu = \begin{cases} \nu \text{ with all corner cells of content } i \text{ added if they exist,} \\ 0 \text{ otherwise.} \end{cases} \tag{10}$$

This action is extended to the rest of the algebra $\mathcal{A}_k$ and can be shown to be consistent with the relations of the generators. Under the bijection core_{k+1}^{-1} to k-bounded partitions only the topmost box added to diagram survives. The action of u_i on a k-bounded partition λ under core_{k+1} is denoted $u_i \cdot \lambda$.

Example 2.6 *Taking $\nu = \mathrm{core}_4(\lambda)$ from Example 2.1 we obtain*

$u_2 \cdot \nu =$  *and* $\mathrm{core}_4^{-1}(u_2 \cdot \nu) =$ 

where the boxes added by u_2 of content 2 are indicated in bold.

2.3 Noncommutative symmetric functions

We now give the definition of the noncommutative symmetric functions $\mathbf{e}_r$, $\mathbf{h}_r$, $\mathbf{s}_{(r-i,1^i)}$, $\mathbf{p}_r$, and $\mathbf{s}^{(k)}_\lambda$ in terms of the affine nilCoxeter algebra.

Following Lam [5], for $r = 1, \ldots, k$, we define the noncommutative homogeneous symmetric functions

$$\mathbf{h}_r = \sum_{A \in \binom{[0,k]}{r}} u_A^{\mathrm{dec}} ,$$

where u_A^{dec} is a cyclically decreasing element with support A as defined in Section 2.2. We take as a defining relation for the elements $\mathbf{e}_r$ the equation $\sum_{i=0}^{r} (-1)^i \mathbf{e}_{r-i}\mathbf{h}_i = 0$. It can be shown [5, Proposition 16] that then

$$\mathbf{e}_r = \sum_{A \in \binom{[0,k]}{r}} u_A^{\mathrm{inc}} ,$$

where u_A^{inc} is a cyclically increasing element with support A. More generally, the hook Schur functions for $r \le k$ are given by

$$\mathbf{s}_{(r-i,1^i)} = \mathbf{h}_{r-i}\mathbf{e}_i - \mathbf{h}_{r-i+1}\mathbf{e}_{i-1} + \cdots + (-1)^i \mathbf{h}_r$$

and we will demonstrate in Corollary 3.5 (below) that these elements may also be expressed as a sum over certain words.

The noncommutative power sum symmetric functions for $1 \le r \le k$ are defined through the analogue of a classical identity with ribbon Schur functions

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i,1^i)}.$$

Lam [5, Proposition 8] proved that, even though the variables u_i do not commute, the elements $\mathbf{h}_r$ for $1 \le r \le k$ commute and consequently, so do the other elements $\mathbf{e}_r$, $\mathbf{p}_r$, $\mathbf{s}_{(r-i,1^i)}$ we have defined in terms of the $\mathbf{h}_r$. We define $\mathbf{\Lambda}_{(k)} = \mathbb{Z}[\mathbf{h}_1, \ldots, \mathbf{h}_k]$ to be the noncommutative analogue of $\Lambda_{(k)} = \mathbb{Z}[h_1, \ldots, h_k]$.

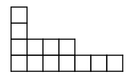
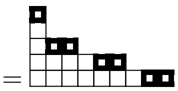
We define the noncommutative k-Schur functions $\mathbf{s}^{(k)}_\lambda$ by the noncommutative analogue of the k-Pieri rule (4). Let us denote by $\mathcal{H}^{(k)}_r$ the set of all pairs (μ, λ) of k-bounded partitions μ, λ such that μ/λ is a horizontal r-strip and $\mu^{(k)}/\lambda^{(k)}$ is a vertical r-strip (which describes the summation in the k-Pieri rule). Then for a k-bounded partition λ we require that

$$\mathbf{h}_r \mathbf{s}^{(k)}_\lambda = \sum_{\mu : (\mu,\lambda) \in \mathcal{H}^{(k)}_r} \mathbf{s}^{(k)}_\mu. \tag{11}$$

This definition can be used to expand the $\mathbf{h}_\mu$ elements in terms of the elements $\mathbf{s}^{(k)}_\lambda$. The transition matrix is described by the number of k-tableaux of given shape and weight (see [9]). Since this matrix is unitriangular, this system of relations can be inverted over the integers and hence $\{\mathbf{s}^{(k)}_\lambda \mid \lambda \in \mathcal{P}^{(k)}\}$ forms a basis of $\mathbf{\Lambda}_{(k)}$.

As shown in [9, 7], for $1 \le r \le k$, we have if $(\mu, \lambda) \in \mathcal{H}^{(k)}_r$, then there is a cyclically decreasing element $u \in \mathcal{A}_k$ of length r such that $\mu = u \cdot \lambda$. Moreover, if $u \in \mathcal{A}_k$ is cyclically decreasing and $\mu = u \cdot \lambda \ne 0$, then $(\mu, \lambda) \in \mathcal{H}^{(k)}_r$.

Example 2.7 *Take* $\lambda = (3,3,1,1) \in \mathcal{P}^{(3)}$ *and* $u = u_0 u_3$. *Then*

$\mathrm{core}_4(\lambda) =$  *and* $u \cdot \mathrm{core}_4(\lambda) =$ 

so that $((3,3,2,1,1),(3,3,1,1)) \in \mathcal{H}_2^{(3)}$.

Hence, we may rewrite (11) as

$$\mathbf{h}_r \mathbf{s}_\lambda^{(k)} = \sum_{\mu:(\mu,\lambda)\in\mathcal{H}_r^{(k)}} \mathbf{s}_\mu^{(k)} = \sum_{A\in\binom{[0,k]}{r}} \mathbf{s}^{(k)}_{u_A^{\mathrm{dec}}\cdot\lambda}\,,$$

where we assume $\mathbf{s}^{(k)}_{u_A^{\mathrm{dec}}\cdot\lambda} = 0$ if $u_A^{\mathrm{dec}} \cdot \lambda = 0$. The elements $\mathbf{h}_r = \sum_{A\in\binom{[0,k]}{r}} u_A^{\mathrm{dec}}$ generate $\mathbf{\Lambda}_{(k)}$, and therefore more generally for any element $\mathbf{f} = \sum_u c_u u \in \mathbf{\Lambda}_{(k)}$ with $u \in \mathcal{A}_k$ and $c_u \in \mathbb{Z}$

$$\mathbf{f}\ \mathbf{s}_\lambda^{(k)} = \sum_u c_u \mathbf{s}^{(k)}_{u\cdot\lambda}\,. \tag{12}$$

Since all of the noncommutative symmetric functions in this section commute and satisfy the same defining relations as their commutative counterparts, there is a ring isomorphism

$$\iota : \mathbf{\Lambda}_{(k)} \to \Lambda_{(k)}$$

sending $\mathbf{h}_r \mapsto h_r$, $\mathbf{e}_r \mapsto e_r$, $\mathbf{p}_r \mapsto p_r$, $\mathbf{s}_\lambda^{(k)} \mapsto s_\lambda^{(k)}$.

3 Main result: Murnaghan–Nakayama rule in terms of words

We now restate Theorem 1.2 in terms of the action of words. This result is proved in the remainder of this section.

Theorem 3.1 *For* $1 \le r \le k$ *and* λ *a* k*-bounded partition, we have*

$$\mathbf{p}_r \mathbf{s}_\lambda^{(k)} = \sum_{(w,\mu)} (-1)^{\mathrm{asc}(w)} \mathbf{s}_\mu^{(k)}, \tag{13}$$

where the sum is over all pairs (w,μ) *of reduced words* w *in the affine nilCoxeter algebra* $\mathcal{A}_k$ *and* k*-bounded partitions* μ *satisfying*

(1′) (size condition) $\mathrm{len}(w) = r$;

(2′) (ribbon condition) w *is a hook word;*

(3′) (connectedness condition) w *is* k*-connected;*

(4′) (weak order condition) $\mu = w \cdot \lambda$.

The proof of Theorem 3.1 essentially amounts to computing an expression for $\mathbf{p}_r$ in terms of words. Since all words involved will be of length $\le k$, there will be a canonical order on the support as introduced in Section 2.2. The statistic $\mathrm{asc}(w)$, and the property of being a hook word, will always be in terms of this canonical ordering.

Lemma 3.2 *For $0 \leq i \leq r \leq k$,*

$$\mathbf{h}_{r-i}\mathbf{e}_i = \sum_w w \,, \tag{14}$$

where the sum is over all words w satisfying $(1')$, $(2')$ with respect to the canonical order, and $\mathrm{asc}(w) \in \{i-1, i\}$.

Proof: $\mathbf{h}_{r-i}$ is the sum over all cyclically decreasing nilCoxeter group elements of length $r-i$ and $\mathbf{e}_i$ is the sum over all cyclically increasing nilCoxeter group elements of length i. Hence

$$\mathbf{h}_{r-i}\mathbf{e}_i = \sum_{\substack{(u,v) \\ u \text{ cycl. dec., } |u|=r-i \\ v \text{ cycl. inc., } |v|=i}} uv.$$

Rearrange each u and v so that they together form a hook with respect to the canonical order associated to the set $\mathrm{supp}(u) \cup \mathrm{supp}(v)$. Either the last letter of u is smaller than the first letter of v, in which case the total ascent is i, or the last letter of u is bigger than the first letter in v, in which case the total number of ascents is $i-1$. This yields a bijection between hook words in the canonical order and pairs appearing in this sum with the number of ascents in $\{i, i-1\}$. In the corner case $i = 0$ (resp. $i = r$) the number of ascents can only be 0 (resp. $r-1$ due to the fact that the words are of length r). □

Example 3.3 *Take $k = 8$, $u = (u_1u_0u_8)(u_5u_4)$ and $v = (u_2u_3)(u_0)$, so that $i = 3$ and $r = 8$. In this case the canonical order is $7 < 8 < 0 < 1 < 2 < 3 < 4 < 5$ and we would write uv as $uv = [(u_5u_4)(u_1u_0u_8)][(u_0)(u_2u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i = 3$ ascents. If on the other hand $u = (u_1u_0)(u_5u_4)$ and $v = (u_2u_3)(u_8u_0)$, so that $i = 4$ and $r = 8$, then we would write $uv = [(u_5u_4)(u_1u_0)][(u_8u_0)(u_2u_3)]$, giving rise to the word $w = (5410)(8023)$ with $i - 1 = 3$ ascents.*

Remark 3.4 *Note that there may be multiplicities in (14) with respect to affine nilCoxeter group elements because there may be several hook words with the same number of ascents that are equivalent to the same affine nilCoxeter element. For example, $(4)(20)$ and $(0)(24)$ are two different hook words with exactly one ascent with respect to the interval $I_{\{0,2,4\}} = \{2 < 4 < 0\}$. Of course, they both correspond to the same affine nilCoxeter element since all letters in the word commute. The element with $u = u_2$ and $v = u_4u_0$ would give rise to the hook word $w = (240)$ with 2 ascents.*

We can use this lemma to get an expression for hook Schur functions.

Corollary 3.5 *For $0 \leq i \leq r \leq k$, the hook Schur function is*

$$\mathbf{s}_{(r-i,1^i)} = \sum_w w \,,$$

where the sum is over all words w satisfying $(1')$, $(2')$ with respect to the canoncial order, and $\mathrm{asc}(w) = i$.

Proof: From our definition of the noncommutative Schur functions indexed by a hook partition, it follows that

$$\mathbf{s}_{(r-i,1^i)} = \mathbf{h}_{r-i}\mathbf{e}_i - \mathbf{h}_{r-i+1}\mathbf{e}_{i-1} + \cdots + (-1)^i\mathbf{h}_r.$$

Hence by Lemma 3.2 the only words which do not appear in two terms with opposite signs are those that have $\mathrm{asc}(w) = i$, which implies the corollary. □

Example 3.6 *Let $k = 3$. Then for $r = 3$ and $i = 1$ we have*

$$\begin{aligned}\mathbf{s}_{2,1} = {} & u_1u_0u_1 + u_2u_1u_2 + u_3u_2u_3 + u_0u_3u_0 \\ & + u_1u_3u_0 + u_1u_0u_2 + u_2u_0u_1 + u_2u_1u_3 + u_3u_1u_2 + u_3u_2u_0 + u_0u_2u_3 + u_0u_3u_1.\end{aligned}$$

We can now write an expression for $\mathbf{p}_r$ by using the definition.

Corollary 3.7 *For $1 \le r \le k$,*

$$\mathbf{p}_r = \sum_w (-1)^{\mathrm{asc}(w)} w,$$

where the sum is over all words w satisfying $(1')$ and $(2')$ in the canonical order.

Proof: This follows immediately from the definition

$$\mathbf{p}_r = \sum_{i=0}^{r-1} (-1)^i \mathbf{s}_{(r-i,1^i)} \ .$$

□

In fact, we may restrict our attention to those words in the sum also satisfying $(3')$ because it is possible to show that those not satisfying $(3')$ will cancel.

Lemma 3.8 *For $r \le k$,*

$$\mathbf{p}_r = \sum_w (-1)^{\mathrm{asc}(w)} w,$$

where the sum is over all words w satisfying $(1')$, $(2')$, and $(3')$.

Proof: Since each canonical interval can be viewed as an interval of the finite nilCoxeter group, the sign-reversing involution described before [4, Theorem 5.1] still holds and there is a sign-reversing involution on the terms which do not satisfy $(3')$. Hence it suffices to sum only over terms which are connected cyclic intervals. □

Example 3.9 *Let $k = 3$. Then*

$$\mathbf{p}_2 = u_1u_0 + u_2u_1 + u_3u_2 + u_0u_3 - (u_1u_2 + u_2u_3 + u_3u_0 + u_0u_1).$$

Theorem 3.1 now follows from the action of words on $\mathbf{s}_\lambda^{(k)}$ given by Equation (12).

4 Outlook

By Corollaries 1.3, 1.4 and 1.5, the Murnaghan-Nakayama rule proved in this paper gives the expansion of the power sum symmetric functions in terms of the k-Schur functions $s_\lambda^{(k)} \in \Lambda_{(k)}$ and the expansion of the dual k-Schur functions $\mathfrak{S}_\lambda^{(k)} \in \Lambda^{(k)}$ in terms of the power sums:

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \chi_{\lambda,\nu}^{(k)} \, s_\lambda^{(k)} \qquad \text{and} \qquad \mathfrak{S}_\nu^{(k)} = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \chi_{\nu,\lambda}^{(k)} \, p_\lambda \ .$$

Unlike in the symmetric function case, where the Schur functions $s_\lambda \in \Lambda$ are self-dual, there should be a dual version of the Murnaghan-Nakayama rule of this paper, namely a combinatorial formula for the coefficients $\tilde{\chi}^{(k)}_{\lambda,\nu}$ in the expansion of the power sum symmetric functions in terms of the dual k-Schur functions

$$p_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \tilde{\chi}^{(k)}_{\lambda,\nu} \, \mathfrak{S}^{(k)}_\lambda$$

or, equivalently by the same arguments as in the proof of Corollary 1.5,

$$s^{(k)}_\nu = \sum_{\lambda \in \mathcal{P}^{(k)}} \frac{1}{z_\lambda} \tilde{\chi}^{(k)}_{\nu,\lambda} \, p_\lambda \, .$$

Since the $s^{(k)}_\nu$ are known to be Schur-positive symmetric functions [8], they correspond to representations of the symmetric group under the Frobenius characteristic map. Furthermore, the characters of these representations are given by the $\tilde{\chi}^{(k)}_{\nu,\lambda}$. An explicit description of such representations is an interesting open problem, which has been studied by Li-Chung Chen and Mark Haiman [2]. In the most generality they conjecture a representation theoretical model for the k-Schur functions with a parameter t which keeps track of the degree grading; the $\tilde{\chi}^{(k)}_{\nu,\lambda}$ described above should give the characters of these representations without regard to degree.

Computer evidence suggests that the ribbon condition (2) of Definition 1.1 might be superfluous because it is implied by the other conditions of the definition. This was checked for $k, r \leq 11$ and for all $|\lambda| = n \leq 12$ and $|\mu| = n + r$.

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On the enumeration of column-convex permutominoes

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Abstract. We study the enumeration of *column-convex permutominoes*, i.e. column-convex polyominoes defined by a pair of permutations. We provide a direct recursive construction for the column-convex permutominoes of a given size, based on the application of the ECO method and generating trees, which leads to a functional equation. Then we obtain some upper and lower bounds for the number of column-convex permutominoes, and conjecture its asymptotic behavior using numerical analysis.

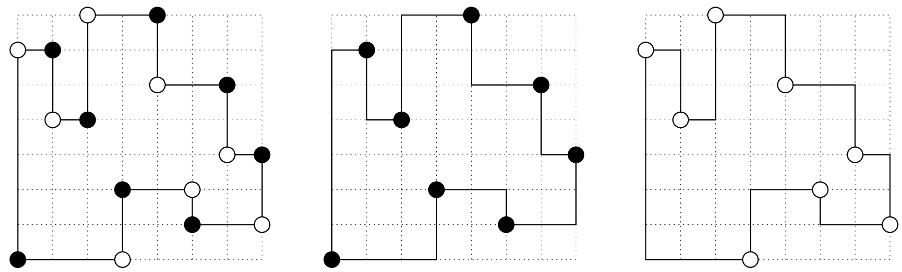
Résumé. Nous étudions l'énumeration des *permutominos verticalement convexes*, c.à.d. les polyominos verticalement convexes définis par un couple de permutations. Nous donnons une construction recursive directe pour ces permutominos de taille fixée, basée sur une application de la méthode ECO et les arbres de génération, qui nous amène à une équat ion fonctionelle. Ensuite nous obtenons des bornes superieures et inférieures pour le nombre de ces permutominos convexes et nous conjecturons leur comportement asymptotique à l'aide d'analyses numériques.

Keywords: polyominoes, permutations, generating functions

1 Introduction

In the plane $\mathbb{Z} \times \mathbb{Z}$ a *cell* is a unit square, and a *polyomino* is a finite connected union of cells having no cut point. Polyominoes are defined up to translations. A *column* (*row*) of a polyomino is the intersection between the polyomino and an infinite strip of cells lying on a vertical (horizontal) line. A polyomino is said to be *column-convex* (*row-convex*) when its intersection with any vertical (horizontal) line of cells in the square lattice is connected, and *convex* when it is both column and row-convex. For the main definitions of these objects we refer to [6].

Let P be a polyomino without "holes", i.e. a polyomino whose boundary is a single loop, and having n rows and n columns, $n \geq 1$; we assume without loss of generality that the south-west corner of its minimal bounding rectangle is placed at $(1, 1)$. We say that P is a *permutomino* if for each abscissa (ordinate) between 1 and n there is exactly one vertical (horizontal) bond in the boundary of P with that coordinate, and n is called the *size* of the permutomino. A permutomino can be equivalently defined by two permutations of length $n + 1$, denoted π_1 and π_2, as depicted in Fig. 1. For more detailed definitions of permutations on permutominoes we refer to [4, 9, 11].



$\pi_1 = (1, 7, 5, 3, 8, 2, 6, 4)$ $\pi_2 = (7, 5, 8, 1, 6, 3, 4, 2)$

Fig. 1: A permutomino and the two associated permutations.

Permutominoes were introduced in [15], and then considered by F. Incitti while studying the problem of determining the $\widetilde{R}$-polynomials associated with a pair (π_1, π_2) of permutations [13]. In recent years, a particular class of permutominoes, namely convex permutominoes and their associated permutations, have been widely studied [4, 9, 11]. The main enumerative results known about convex permutominoes are the following:

i. the number of *parallelogram permutominoes* of size n is equal to the n-th *Catalan number*,

$$C_n = \frac{1}{n+1}\binom{2n}{n};$$

ii. the number of *directed convex permutominoes* of size n is equal to $\binom{2n-1}{n}$;

iii. the number of *convex permutominoes* of size n is:

$$2\,(n+3)\,4^{n-2} - \frac{n}{2}\binom{2n}{n} \qquad n \geq 1. \tag{1}$$

We point out that formula (1) was proved using analytical techniques independently in [5, 9], and a bijective proof of (1) was given in [8].

We also recall from [4] that permutations defining convex permutominoes are strictly related with the so-called *square permutations* (or *convex permutations*), recently considered by several authors [1, 10, 14].

Our aim is to deal with the enumeration of column-convex permutominoes. We determine a direct recursive construction for the column-convex permutominoes of a given size, based on the application of the ECO method [3] and generating trees [2], which leads to a functional equation. However we are not able to solve the equation so as to obtain the generating function of column-convex permutominoes. We are only able to obtain some upper and lower bounds for the number of column-convex permutominoes, and conjecture its asymptotic behavior using numerical analysis.

2 Generation of column-convex permutominoes

The ECO method [3] is a method for the enumeration and the recursive construction of classes of combinatorial objects. It is substantially based on an operator ϑ, which constructs each object of a given size in a unique way, starting from those of immediately lower size. The recursive construction determined by ϑ can be suitably described by a *generating tree* [2], which then leads to a functional equation satisfied by the generating function of the class.

In order to define the ECO construction for column-convex permutominoes, we classify the corners of their boundary, using *reentrant* and *salient points*. So let us briefly recall the definition of these objects.

Let P be a polyomino. Starting from the lowest among the leftmost points of P, and moving in a clockwise direction, the boundary of P can be encoded as a word of a four letter alphabet, {N, E, S, W}, where N (resp. E, S, W) represents a *north* (resp. *east*, *south*, *west*) unit step. Any occurrence of a sequence NE, ES, SW, or WN in the word encoding P defines a *salient point* of P, while any occurrence of a sequence EN, SE, WS, or NW defines a *reentrant point* of P (see for instance, Figure 2).

Reentrant and salient points were considered for instance in [7], and it was proved that in any polyomino the difference between the number of salient and reentrant points is equal to 4. Moreover, we observe that the set of reentrant points of a convex permutomino of size n defines a permutation matrix of size $n-1$. This property it is not true for column-convex permutominoes. However it is easy to prove that in a column-convex permutomino of size n we have exactly one reentrant point for each abscissa between 1 and $n-1$.

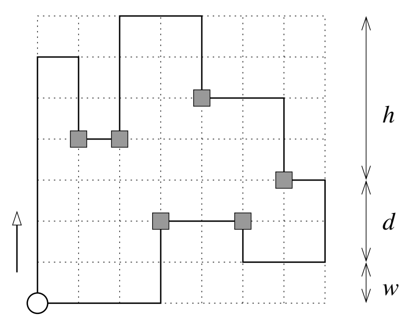


Fig. 2: In the picture, the coding of the boundary of the permutomino starts from the circled point, and the reentrant points are those denoted by squares.

Let $\mathcal{C}_n$ be the set of column-convex permutominoes of size n, and let $P \in \mathcal{C}_n$; the number of cells in the rightmost column of P is called the *degree* of P, and will be denoted by $d(P)$. Let the level of a cell in P be the ordinate of its upper edge. Then we denote by $w(P)$ the level of the lowest cell of P minus 1 and by $h(P)$ the level of the uppermost cell of P (see Fig. 2); clearly $h(P) + d(P) + w(P) = n$. To a column-convex permutomino P we assign the label $(h(P), d(P), w(P))$ ((h, d, w), for brevity).

Now we define an ECO operator $\vartheta : \mathcal{C}_n \to 2^{\mathcal{C}_{n+1}}$ which defines a recursive construction of all the column-convex permutominoes of size $n+1$ in a unique way from those of size n. The operator ϑ acts on a column-convex permutomino performing some local expansions on the cells of its rightmost column.

The operator ϑ consists in four operations denoted by α, β, γ, and δ, and below we give a detailed description of each of them:

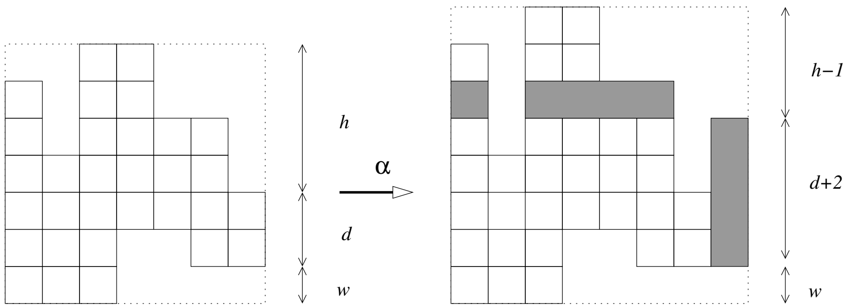


Fig. 3: Operation α performed on a column-convex permutomino, adding a new column of length $d+2$. The added cells have been highlighted.

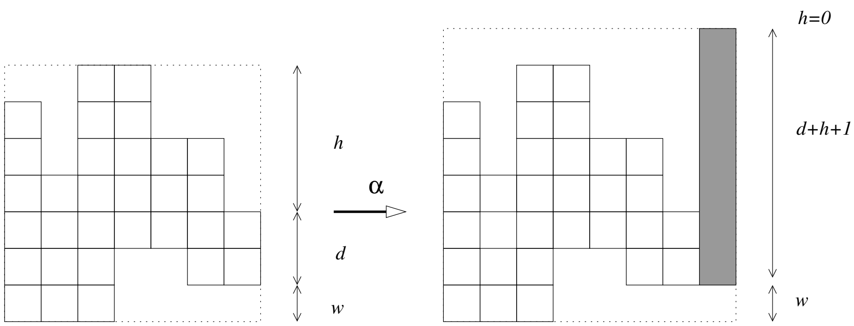


Fig. 4: Operation α performed on a column-convex permutomino, adding a new column of length $d+h+1$.

(α) for each i such that $d+1 \leq i \leq d+h+1$, operation α adds a column of length i next to the rightmost column of the permutomino, starting from the bottom, and adds a cell to the top of each cell at level $w+i$, in order to maintain the property of being permutomino, as in Figure 3. When $i = d+h+1$, no other cell is added, according to Figure 4.

It is clear that the resulting polyomino is a column-convex permutomino of size $n+1$, and that the rightmost reentrant point in such a new permutomino is always of type EN.

(β) is performed on each cell of the rightmost column. So let q_i be the ith cell of such a column, from bottom to top, with $1 \leq i \leq d$. Operation β adds a cell to the top of each cell of the row containing q_i, and a new column made of i cells on the right of q_i, as illustrated in Figure 5.

It is clear that, for any i, the obtained polyomino is a column-convex permutomino of size $n+1$, and its rightmost reentrant point is of type SE.

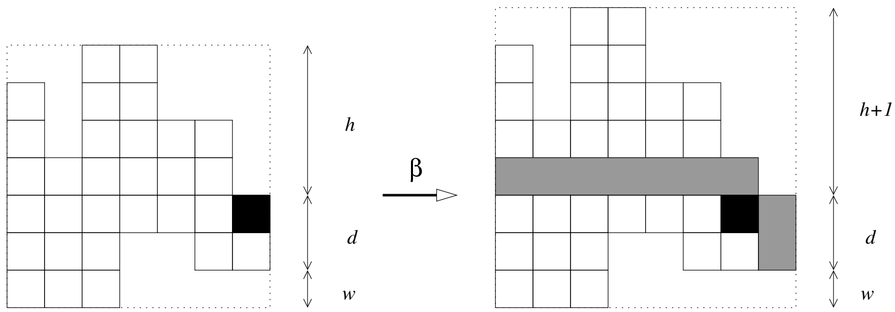


Fig. 5: Operation β performed on the highlighted cell of a column-convex polyomino. The added cells have been highlighted.

(γ) is performed on each cell of the rightmost column. So let q_i be the ith cell of such column, numbered from bottom to top, with $1 \leq i \leq d$. Operation γ adds a cell on the top of each cell of the row containing q_i, and a new column made of $d-i+1$ cells on the right of q_i, as illustrated in Figure 6.

It is clear that, for any i, the obtained polyomino is a column-convex permutomino of size $n+1$, and its rightmost reentrant point is of type WS.

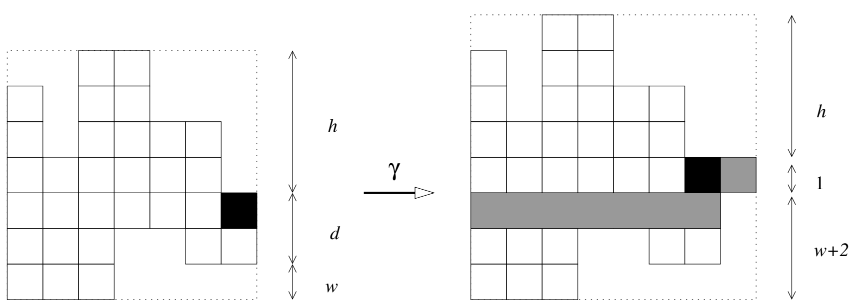


Fig. 6: Operation γ performed on the highlighted cell of a column-convex polyomino. The added cells have been highlighted.

(δ) for each i such that $1 \leq i \leq w+1$, operation δ adds a column of length $d+i$ next to the rightmost column of the permutomino, starting from the top, and adds a cell to the bottom of each cell at level

$w - i + 1$. When $i = w + 1$, no other cell is added, see Figure 7.

It is clear that the resulting polyomino is a column-convex permutomino of size $n + 1$, and that the rightmost reentrant point in such new permutomino is always of type NW.

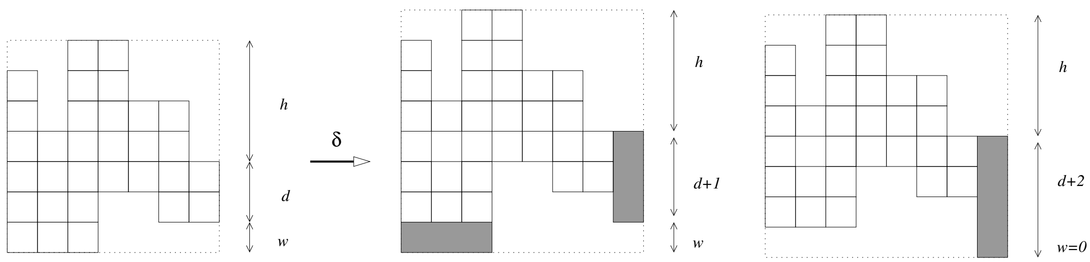


Fig. 7: The two column-convex permutominoes produced through the application of δ to the permutomino on the left.

The proof of the following statement is then straightforward.

Theorem 2.1 *Any column-convex permutomino of size $n \geq 2$ is uniquely obtained through the application of the operator ϑ to a convex permutomino of size $n - 1$.*

Formally, the application of ϑ to a column-convex permutomino with a generic label (h, d, w), produces $2d + h + w + 2$ column-convex permutominoes according to the following succession rule:

$$\begin{array}{llll} (h,d,w) & \stackrel{\alpha}{\rightsquigarrow} & (h-i+1, d+i, w) & 1 \leq i \leq h+1 \\ & \stackrel{\beta}{\rightsquigarrow} & (d+h-i+1, i, w) & 1 \leq i \leq d \\ & \stackrel{\gamma}{\rightsquigarrow} & (h, i, d+w-i+1) & 1 \leq i \leq d \\ & \stackrel{\delta}{\rightsquigarrow} & (h, d+i, w-i+1) & 1 \leq i \leq w+1. \end{array}$$

These have root $(0, 1, 0)$, which is the label of the one cell permutomino.

3 Enumeration of column-convex permutominoes

The previous succession rule can be suitably represented by means of a *generating tree*, which is a rooted tree where the objects at level n are the labels of the column-convex permutominoes of size n. Thus, the root is $(0, 1, 0)$, and the sons of a generic label (h, d, w) at level n are given by the succession rule production. The number f_n of column-convex permutominoes of size n is then given by the number of objects at level n of the generating tree.

Let $\mathcal{C}$ denote the set of all column-convex permutominoes. Our aim is to determine the generating function:

$$\begin{aligned} F(x,y,z) &= \sum_{P\in\mathcal{C}} x^{h(P)} y^{d(P)} z^{w(P)} \\ &= y + xy + yz + 2y^2 + 3x^2y + 2xyz + 3z^2y + 4xy^2 + 4zy^2 + 6y^3 + \ldots \end{aligned}$$

where, in particular, $F(y,y,y)$ is the generating function of column-convex permutominoes according to size. Using the succession rule, and standard methods, we can write

$$\begin{aligned} F(x,y,z) \;=\; & y + \sum_{P\in\mathcal{C}} x^h y^{d+1} z^w + \ldots + x^0 y^{d+h+1} z^w + \sum_{P\in\mathcal{C}} x^{d+h} y^1 z^w + \ldots + x^{h+1} y^d z^w \\ & + \sum_{P\in\mathcal{C}} x^h y z^{d+w} + \ldots + x^h y^d z^{w+1} + \sum_{P\in\mathcal{C}} x^h y^{d+1} z^w + \ldots + x^h y^{d+w+1} z^0, \end{aligned}$$

and then, after some calculation, we obtain the following functional equation

$$F(x,y,z) \,=\, y + \frac{yz}{z-y}F(x,z,z) - \frac{y^2}{z-y}F(x,y,y) + \frac{xy}{x-y}F(x,x,z) - \frac{y^2}{x-y}F(y,y,z). \tag{2}$$

From this equation we are able to compute the first terms of sequence f_n:

$1, 4, 22, 152, 1262, 12232, 135544, 1690080, 23417928, 356958816, 5936071344, 106944112320, 2074955738160, 43135041684288\ldots$

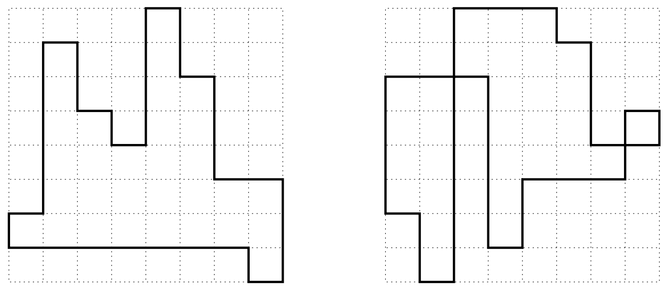
Indeed, we have obtained the first 200 coefficients, and report on their analysis in Section 3.3. We have not been able to solve the equation, or to find a closed formula for the number of column-convex permutominoes. In the following sections we add some combinatorial considerations, and give upper and lower bounds. We remark that from the ECO construction it easily follows that $F(x,y,z) = F(z,y,x)$; this fact does not help us, however, in solving (2).

3.1 Directed column-convex permutominoes

A simple lower bound is given by the number of directed column-convex permutominoes. Let us define the ECO operator ϑ' as the operator which performs the operations α, β, and γ defined before, but not operation δ. Starting from the one cell permutomino, the operator ϑ' generates all column-convex permutominoes which do not contain any WS reentrant point; that is to say, the column-convex permutominoes which are *north-west directed* (briefly, directed, see Fig. 8 (a)). Let $G(x,y,z)$ be the generating function of directed column-convex permutominoes. This time, using the operator ϑ', and the associated succession rule, we can obtain the following functional equation for $g(y) = G(y,y,y)$:

$$g(y) \,=\, 2y\,g(y) + y^2 g'(y) + y\,,$$

and then we readily find that the number of directed column-convex permutominoes of size n is $\frac{(n+1)!}{2}$. We also have a simple bijective proof of this fact.



(a) (b)

Fig. 8: (a) A directed column-convex permutomino; (b) A column-convex permutominide.

3.2 Column-convex permutominides

To provide an upper bound, we need to introduce another combinatorial object, the *column-convex permutominide*. Without going further into formal definitions, a column-convex permutominide of size n is substantially a polyomino whose boundary is allowed to cross itself, while the columns remain connected, and with exactly one edge for every abscissa and exactly one edge for every ordinate between 1 and n, see Fig. 8 (a). These objects have been treated and enumerated, according to size, in [8].

To a column-convex permutominide P of size n we associate a *side* permutation $\pi(P)$ of length $n+1$, as follows: Among the two horizontal edges of P starting from the left side of the minimal bounding square there is only one with length 1, and its ordinate is denoted $\pi(1)$. Now, removing this edge, there is one horizontal edge for each abscissa between 1 and n, so let $\pi(i+1)$ be the ordinate of the edge with abscissa i. For instance, the side permutation associated with the column-convex permutominide depicted in Fig. 9 (b) is $(3, 7, 1, 9, 2, 4, 8, 5, 6)$. The reader can easily observe that the permutation $\pi(P)$ does not uniquely determine P; in fact, we have the following:

Proposition 1 *Given a permutation σ of size $n+1$, there are 2^{n-2} column-convex permutominides of size n having σ as side permutation.*

Proof: Let us fix a set $\Gamma \subseteq \{4, \ldots, n+1\}$. We want to prove that σ and Γ uniquely determine a column-convex permutominide. Let us consider the points $(1, \sigma(1))$, and $(i, \sigma(i))$, for $i > 1$. We will write $\sigma(i)$ to mean the point $(i, \sigma(i))$. Now we join $\sigma(1)$ with $\sigma(3)$ and, from left to right, all the points $\sigma(i)$, with $i \in \Gamma$, until we reach the right side of the minimal bounding square, see Fig. 9 (a). Now the permutominide of size n is uniquely determined, see Fig. 9 (b). $\square$

The following statement then readily follows:

Proposition 2 *The number of column-convex permutominides of size n is $2^{n-2}(n+1)!$*

This number is an upper bound on the number of column-convex permutominoes. Using combinatorial arguments, we may slightly refine this upper bound. For instance, if P is a permutomino then $\pi(2)$ must not be between $\pi(1)$ and $\pi(3)$.

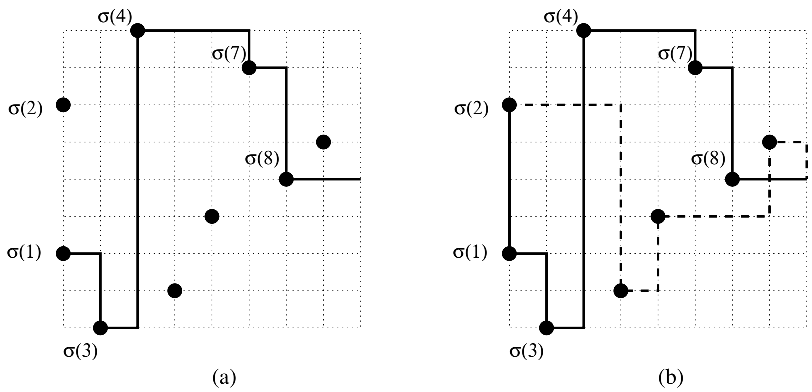


Fig. 9: (a) The path obtained connecting $\sigma(1)$ with the points belonging to the set $\Gamma = \{4, 7, 8\}$; (b) The column-convex permutominide determined from σ and Γ.

3.3 Final remarks

Our bounds allow us to write that for sufficiently large values of n we have:

$$(n+1)! < f_n < (n+1)!\, 2^n\,.$$

The upper and lower bounds suggest that an exponential generating function is likely to be appropriate, and so we applied the simplest possible analysis. Using the first 200 terms of the sequence $\{f_n\}$, we simply calculated the sequence $\tilde{f}_n = f_n/(n+1)!$, and then calculated the ratio of successive terms of this new sequence, viz. $\tilde{f}_n/\tilde{f}_{n-1}$. We found this sequence converges very rapidly to a constant, which we find to be

$$h = 1.38593327599819425386062181488 2515\ldots.$$

This suggests that

$$f_n \sim k\,(n+1)!\, h^n\,, \tag{3}$$

where we estimated the value of k by the simple expedient of constructing the sequence $f_n/(n+1)!/h^n$. Again we obtain a rapidly convergent sequence, and estimate that

$$k = 0.3419111315217955078839250169 8973\ldots.$$

Detailed study of the rate of convergence suggests that it is exponential. Exponential convergence is one manifestation of a second singularity beyond the radius of convergence. We investigated this possibility by using differential approximants [12] to study the sequence $\{\tilde{f}_n\}$. This study shows a dominant singularity – a simple pole – at $1/h \approx 0.72153572588$, plus a second singularity at 1. The nature of the second singularity is less clear. It appears to be a confluent singularity. Its presence and nature indicates that the generating function, or indeed the exponential generating function, while having simple dominant asymptotics, has some subdominant terms that are more subtle.

We have attempted to study this sub-dominant singularity as follows: Given that we know the position of the dominant singularity, and its amplitude, very precisely, we can largely subtract it, and investigate the remaining series. So we formed the sequence $d_n = \tilde{f}_n - k \times h^n$, using the values of k and h given above. Assuming that the sub-dominant singularity of the generating function for $\tilde{f}_n$ is precisely at 1, as our above analysis suggests, we then expect $d_n \sim e \times n^g$, where e is a constant. If that is the case, it follows that we can construct a sequence of estimators for g from the sequence $d_n/d_{n-1} = 1 + g/n + \mathrm{o}(1/n)$, so the sequence $n(d_n/d_{n-1} - 1)$ should provide a sequence of estimators of g. In this way we find $g \approx -1.272$ where the last digit is in doubt. Indeed, we cannot really exclude that the correct value of this exponent is a simple rational fraction, such as $-5/4$, or it may be something bizarre, such as $-\log(\sqrt{2}-1)/\log(2) = 1.2715\ldots$. Then e can be estimated by forming the sequence d_n/n^g, and in this way we estimate $e \approx 0.6625$. Putting together the pieces of our analysis, we conjecture that

$$\tilde{f}_n \sim k \times h^n + e \times n^g.$$

It is natural to investigate the possibility that h is an algebraic number. We have been unable to find a simple polynomial of degree less than 25 with h as a root which suggests that it is not a simple algebraic number. Further investigation using interactive solvers that seek representations in terms of a variety of transcendental constants and their powers, as well as logarithms and Dirichlet functions has also been unsuccessful.

To solve equation (2) or to give a proof of (3) are open problems.

The form for f_n given by (3) suggests that formulating the problem in terms of the exponential generating function may prove useful. If we write

$$\begin{aligned} E(t,x,y,z) &= \sum_{P \in \mathcal{C}} \frac{t^{h(P)+d(P)+w(P)} x^{h(P)} y^{d(P)} z^{w(P)}}{(h(P)+d(P)+w(P))!} \\ &= ty + \frac{t^2}{2}(xy + yz + 2y^2) + \frac{t^3}{6}(3x^2y + 2xyz + 3z^2y + 4xy^2 + 4zy^2 + 6y^3) + \ldots \end{aligned}$$

then by the same method used for (2), we obtain

$$\frac{\partial}{\partial t} E(t,x,y,z) = y + \frac{yz}{z-y} E(t,x,z,z) - \frac{y^2}{z-y} E(t,x,y,y) + \frac{xy}{x-y} E(t,x,x,z) - \frac{y^2}{x-y} E(t,y,y,z). \tag{4}$$

Unfortunately this equation seems no more tractable than (2).

It may be the case that there exists a different method for constructing column-convex permutominoes. One possibility involves adding the horizontal bonds, one column at a time, from left to right (column-convexity implies that there are precisely two horizontal bonds in each column). If the bonds in column i are at heights a_i and b_i, with $a_i > b_i$, for columns $i = 1 \ldots k$, then the bonds in column $k+1$ must satisfy either

- $a_{k+1} = a_k$ and $b_{k+1} \neq a_1, \ldots, a_k, b_1, \ldots, b_k$; or
- $b_{k+1} = b_k$ and $a_{k+1} \neq a_1, \ldots, a_k, b_1, \ldots, b_k$.

While we have not been able to obtain an enumeration via this method, we note that the corresponding version for column-convex permutominides uses the slightly relaxed rules

- $a_{k+1} = a_k$ or b_k and $b_{k+1} \neq a_1, \ldots, a_k, b_1, \ldots, b_k$; or
- $b_{k+1} = a_k$ or b_k and $a_{k+1} \neq a_1, \ldots, a_k, b_1, \ldots, b_k$.

In this case an enumeration readily follows, and agrees with the result of Proposition 2.

From another perspective, we remark that column-convex permutominoes are an interesting class of permutominoes, since they show some features that make them different from the other classes of permutominoes previously studied.

In fact, we have seen that the generating functions of the classes of previously considered permutominoes are of the same nature (i.e. rational or algebraic) as the corresponding class of polyominoes (enumerated according to the semi-perimeter). So, for instance, the classes of convex, directed-convex, parallelogram permutominoes have algebraic generating functions, while the classes of stack, centered permutominoes have rational generating functions. Using the previously stated result, we are led to think that, while the class of column-convex polyominoes is an algebraic class [6], the class of column-convex permutominoes has a transcendental generating function. The reason for this distinction probably arises from the fact that, in passing to column-convex permutominoes the relation:

$$\text{size of the permutomino } P \ = \ 2 \ \text{ semi-perimeter of the permutomino } P\,,$$

holding for convex permutominoes, now fails. Thus, it would also be interesting to enumerate column-convex permutominoes according to the semi-perimeter, but in this case the ECO construction we have given is not helpful.

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Primitive orthogonal idempotents for R-trivial monoids

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Abstract. We construct a recursive formula for a complete system of primitive orthogonal idempotents for any R-trivial monoid. This uses the newly proved equivalence between the notions of R-trivial monoid and weakly ordered monoid.

Résumé. Nous construisons une formule récursive pour un système complet d'idempotents orthogonaux primitifs pour tout monoïde R-trivial. Nous employons une nouvelle équivalence entre les notions de monoïde R-trivial et de monoïde faiblement ordonné.

Keywords: monoids, primitive orthogonal idempotents, 0-Hecke algebras, left regular bands

1 Introduction

Recently, Denton ([6], [7]) gave a formula for a complete system of primitive orthogonal idempotents for the 0-*Hecke algebra* of type A, the first since the question was posed by Norton [9] in 1979. A complete system of primitive orthogonal idempotents for *left regular bands* was found by Brown [5] and Saliola [12]. Finding such collections is an important problem in representation theory because they decompose an algebra into projective indecomposable modules: if $\{e_J\}_{J\in\mathfrak{J}}$ is such a collection for a finite dimensional algebra A, then $A = \oplus_{J\in\mathfrak{J}} Ae_J$ for indecomposable modules Ae_J. They also allow for the explicit computation of the quiver, the Cartan invariants, and the Wedderburn decomposition of the algebra (see [4], [2]). For example, in [8], Denton, Hivert, Schilling, and Thiéry use a construction of a system of idempotents for any J-trivial monoid M to derive combinatorially the Cartan matrix and quiver of M.

Schocker [13] constructed a class of monoids, called *weakly ordered monoids*, to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras. We realize this goal here.

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A key step in being able to do so is recognizing that the notions of weakly ordered monoid and *R-trivial monoid* are one and the same. This was first pointed out to us by Nicolas M. Thiéry [17] after an intense discussion between the authors and Denton, Hivert, Schilling, and Thiéry. In Section 2, we fill out an outline of a proof provided by Steinberg [16], who independently made this same observation. In Section 3, we use this equivalence to construct a complete system of primitive orthogonal idempotents.

2 Weakly ordered monoids and R-trivial monoids

Given any monoid T, that is, a set with an associative multiplication and an identity element, we define a preorder $\leq$ as follows. Given $u, v \in T$, write $u \leq v$ if there exists $w \in T$ such that $uw = v$. We write $u < v$ if $u \leq v$ but $u \neq v$. Unless stated otherwise, the monoids throughout the paper are endowed with this "weak" preorder. (In the semigroup theory literature, the *dual* of this preorder is known as *Green's R-preorder*.)

Definition 2.1 *A finite monoid W is said to be a* **weakly ordered monoid** *if there is a finite upper semi-lattice $(\mathcal{L}, \preceq)$ together with two maps $C, D : W \to \mathcal{L}$ satisfying the following axioms.*

1. *C is a surjection of monoids.*
2. *If $u, v \in W$ are such that $uv \leq u$, then $C(v) \preceq D(u)$.*
3. *If $u, v \in W$ are such that $C(v) \preceq D(u)$, then $uv = u$.*

Remark 2.2 *This notion was introduced by Schocker [13] to generalize 0-Hecke monoids and left regular bands, with the broader aim of finding a complete system of orthogonal idempotents for the corresponding monoid algebras. In his paper, he actually calls these weakly ordered semigroups. However our understanding is that monoids include an identity element and semigroups do not. So throughout the paper we call these weakly ordered monoids.*

Definition 2.3 *A monoid S is* ***R*-trivial** *if, for all $x, y \in S$, $xS = yS$ implies $x = y$. It is easy to see that a monoid S is R-trivial if and only if the preorder $\leq$ defined above is a partial order.*

We restrict our discussion to *finite* R-trivial monoids.

Example 2.4 A monoid W is called a **left regular band** if $x^2 = x$ and $xyx = xy$ for all $x, y \in W$. Left-regular bands are R-trivial. Indeed, if $xW = yW$, then there exist $u, v \in W$ such that $xu = y$ and $x = yv$. But then, since $uv = uvu$,

$$x = yv = xuv = xuvu = yvu = xu = y.$$

Finitely generated left regular bands are also weakly ordered monoids, see Shocker [13], e.g. 2.4 and Brown [5], Appendix B.

Example 2.5 Let G be a Coxeter group with simple generators $\{s_i : i \in I\}$ and relations:

- $s_i^2 = 1$,
- $\underbrace{s_i s_j s_i s_j \cdots}_{m_{ij}} = \underbrace{s_j s_i s_j s_i \cdots}_{m_{ij}}$ for positive integers m_{ij}.

Then the **0-Hecke monoid** $H^G(0)$ has generators $\{T_i : i \in I\}$ and relations:

- $T_i^2 = T_i$,
- $\underbrace{T_iT_jT_iT_j\cdots}_{m_{ij}} = \underbrace{T_jT_iT_jT_i\cdots}_{m_{ij}}$ for positive integers m_{ij}.

Of particular interest is the case when G is the symmetric group $\mathfrak{S}_n$. Norton [9] gave a decomposition of the monoid algebra $\mathbb{C}H^{\mathfrak{S}_n}(0)$ into left ideals and classified its irreducible representations. She raised the question of constructing a complete system of orthogonal idempotents for the algebra. Denton [6] gave the first construction of a set of orthogonal idempotents for $\mathbb{C}H^{\mathfrak{S}_n}(0)$.

The weakly ordered monoid $H^{\mathfrak{S}_n}(0)$ has maps C and D onto the lattice of subsets of $\{1, \ldots, n-1\}$. The map C is the *content set* of an element: $C(T_{i_1}T_{i_2}\cdots T_{i_k}) = \{i_1, i_2, \ldots, i_k\}$. The map D is the subset of right descents of an element: $D(x) = \{i \in \{1, \ldots, n-1\} : xT_i = x\}$. Note that the preorder for this monoid coincides with the weak order on the elements of the Coxeter group.

Example 2.6 Let S be the monoid with identity generated by the following matrices.

$$g_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad g_2 := \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $S = \{1,\ g_1,\ g_2,\ g_1g_2,\ g_2g_1\}$ and S is both an R-trivial monoid and a weakly ordered monoid. For example, we can take $\mathcal{L}$ be to be usual lattice of subsets of $\{1, 2\}$, with $C : S \to \mathcal{L}$ given by

$$C(1) = \emptyset,\ C(g_1) = \{1\},\ C(g_2) = \{2\},\ C(g_1g_2) = C(g_2g_1) = \{1, 2\},$$

and $D : S \to \mathcal{L}$ given by

$$D(1) = \emptyset,\ D(g_1) = \{1\},\ D(g_2) = D(g_1g_2) = \{2\},\ D(g_2g_1) = \{1, 2\}.$$

The monoid S, however, is neither a left regular band, since g_1g_2 is not idempotent, nor isomorphic to the 0-Hecke monoid $H^{\mathfrak{S}_3}(0)$ on two generators, since the latter has six elements.

The fact that the above examples are both weakly ordered and R-trivial is no coincidence: the purpose of this section is to show that these two notions are equivalent.

Remark 2.7 *A weakly ordered monoid is an R-trivial monoid. Indeed, if W is a weakly ordered monoid, then Lemma* 2.1 *in [13] shows that the defining conditions of a weakly ordered monoid imply that the preorder on W is a partial order (see Definition 2.3).*

We will show that any finite R-trivial monoid S is a weakly ordered monoid using an argument outlined by Steinberg [16]. We must establish the existence of an upper semi-lattice $\mathcal{L}$ and two maps C and D from S to $\mathcal{L}$ that satisfy the conditions of Definition 2.1. We gather here the definitions of $\mathcal{L}$, C and D:

1. $\mathcal{L}$ is the set of left ideals Se generated by idempotents e in S, ordered by reverse inclusion;
2. $C : S \to \mathcal{L}$ is defined as $C(x) = Sx^\omega$, where x^ω is the idempotent power of x;

3. $D : S \to \mathcal{L}$ is defined as $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S : us = u\}$ (with respect to the preorder $\leq$).

The remainder of this section is dedicated to showing that these objects are well-defined and that they satisfy the conditions of Definition 2.1. We begin by recalling some classical results from the semigroup literature. The following is [10, Proposition 6.1].

Lemma 2.8 *If S is a finite semigroup, then for each $x \in S$, there exists a positive integer $\omega = \omega(x)$ such that x^ω is idempotent, i.e. $(x^\omega)^2 = x^\omega$. Furthermore, if S is R-trivial, then we also have $x^\omega x = x^\omega$.*

Proof: Consider the elements $x, x^2, x^3, \ldots$. Since S is finite, there exists positive integers i and p such that $x^{i+p} = x^i$. Then $x^{k+p} = x^k$ for all $k \geq i$, so if we take $\omega = ip$, then $(x^\omega)^2 = x^{\omega+ip} = x^\omega$.
If S is R-trivial, then $x^\omega \leq x^\omega x \leq x^\omega x^\omega = x^\omega$, and so $x^\omega x = x^\omega$. □

Remark 2.9 *In what follows, if $x \in \mathbb{C}S$ and there exists an N such that $x^{N+1} = x^N$, we sometimes abuse notation by writing x^ω in place of x^N.*

We are now ready to construct a lattice corresponding to the R-trivial monoid S. Define

$$\mathcal{L} := \{Se \;:\; e \in S \text{ such that } e^2 = e\}.$$

That is, $\mathcal{L}$ is the set of left ideals generated by the idempotents of S. Define a partial order on $\mathcal{L}$ by

$$Se \preceq Sf \iff Se \supseteq Sf.$$

Proposition 2.10 *If e, f are idempotents in S, then $S(ef)^\omega$ is the least upper bound of Se and Sf in $\mathcal{L}$.*

Remark 2.11 *A fully detailed and elementary proof of this result for R-trivial monoids can be found in [3], although the motivated reader can deduce this from the above results and definitions. This is a special case of more general results in the semigroup theory literature. For example, it follows by restricting a result of Schützenberger to R-trivial monoids [14]. For a detailed discussion within the context of the representation theory of finite monoids, see [1] and [8].*

As a result, we may define the join of two elements Se and Sf in $\mathcal{L}$ by

$$Se \vee Sf = S(ef)^\omega.$$

That is, $\mathcal{L}$ is an upper semilattice with respect to this join operation. This observation proves the following.

Proposition 2.12 *The map $C : S \to \mathcal{L}$ defined by $C(x) = Sx^\omega$ is a surjective monoid morphism.*

Here is an alternate and useful characterization of $C(x)$.

Proposition 2.13 $C(x) = \{a \in S \;:\; ax = a\}$ *for all* $x \in S$.

Proof: Take an arbitrary element in $C(x) = Sx^\omega$, say tx^ω. Since $\big(tx^\omega\big)x = t\big(x^\omega x\big) = tx^\omega$ by Lemma 2.8, we see that $tx^\omega \in \{a \in S \;:\; ax = a\}$. On the other hand, take $b \in \{a \in S \;:\; ax = a\}$. Then

$$bx^\omega = (bx)x^{\omega-1} = bx^{\omega-1} = (bx)x^{\omega-2} = bx^{\omega-2} = \cdots = bx = b.$$

Therefore, $b \in Sx^{\omega}$. $\square$

We now define the map $D : S \to \mathcal{L}$. Given $u \in S$, let $D(u) = C(e)$, where e is a maximal element in the set $\{s \in S \,:\, us = u\}$. To check that D is well-defined, let e and f be two distinct maximal elements in $\{s \in S \,:\, us = u\}$. Since $e \leq ef$ and $u(ef) = (ue)f = uf = u$, by the maximality of e, $e = ef$. Similarly, since $f \leq fe$ and $u(fe) = u$, the maximality of f implies $f = fe$. Then, by Proposition 2.12,

$$C(e) = C(ef) = C(e) \vee C(f) = C(f) \vee C(e) = C(fe) = C(f).$$

Note that the maximality of e and $ue^2 = u$ also implies that $e = e^2$, that is, e is idempotent.

The next proposition shows that the maps C and D interact in precisely the manner given in conditions 2 and 3 in Definition 2.1. The following lemma will help us prove this proposition.

Lemma 2.14 *Let $x, y \in S$. If $x \leq y$, then $C(x) \preceq C(y)$.*

Proof: If $s \in C(y)$, then $sy = s$. Since $x \leq y$, there exists $t \in S$ such that $y = xt$. So $sxt = s$, implying $sx \leq s$. That is, $s \in C(x)$. Hence $C(y) \subseteq C(x)$, or $C(x) \preceq C(y)$ since $s \leq sx$ and S is R-trivial. $\square$

Proposition 2.15 *Let $u, v \in S$. (i) If $uv \leq u$, then $C(v) \preceq D(u)$. (ii) If $C(v) \preceq D(u)$, then $uv = u$.*

Proof: (i) Since $u \leq uv$, $u = uv$. Hence v lies in the set $\{s \in S \,:\, us = u\}$. Let e be a maximal element in this set such that $v \leq e$. Then, by Lemma 2.14, $C(v) \preceq C(e) = D(u)$.

(ii) By definition, $D(u) = C(e)$, where e is a maximal element of $\{s \in S \,:\, us = u\}$. So if $C(v) \preceq D(u)$, then $C(v) \preceq C(e)$. Hence $C(e) \subseteq C(v)$. Since $ue = u$, u lies in $C(e)$. So u is also a member of $C(v)$; that is, $uv = u$. $\square$

Propositions 2.12 and 2.15 tell us that an R-trivial monoid is a weakly ordered monoid. Combining this with Corollary 2.7, we have the following result.

Theorem 2.16 *A monoid W is a weakly ordered monoid if and only if it is an R-trivial monoid.*

3 Constructing idempotents

We begin this section with a small technical lemma about R-trivial monoids. The proof is rather trivial, but we use it often enough in proofs to justify stating it at the onset.

Lemma 3.1 *Suppose W is an R-trivial monoid. If $x, y, z \in W$ are such that $xyz = x$, then $xy = x$.*
Consequently, if $x, y_1, y_2, \ldots, y_m \in W$ are such that $xy_1 \cdots y_m = x$, then $xy_i = x$ for all $1 \leq i \leq m$.

Proof: If $xyz = x$ then $xyW = xW$. Therefore $xy = x$ by the definition of W being R-trivial. The second statement immediately follows from the first. $\square$

Definition 3.2 *Let A be a finite dimensional algebra with identity* 1. *We say that a set of nonzero elements $\Lambda = \{e_J : J \in \mathcal{I}\}$ of A is a* **complete system of primitive orthogonal idempotents for** *A if:*

1. *each e_J is* idempotent*: that is, $e_J^2 = e_J$ for all $J \in \mathcal{I}$;*
2. *the e_J are pairwise* orthogonal*: that is, $e_J e_K = 0$ for $J, K \in \mathcal{I}$ with $J \neq K$;*

3. *each e_J is* primitive *(meaning that it cannot be further decomposed into orthogonal idempotents): if $e_J = x + y$ with x and y orthogonal idempotents in A, then $x = 0$ or $y = 0$;*

4. *$\{e_J : J \in \mathcal{I}\}$ is* complete *(meaning that the elements sum to the identity): $\sum_{J\in\mathcal{I}} e_J = 1$.*

Remark 3.3 *If Λ is a* maximal *set of nonzero elements satisfying conditions 1 and 2, then Λ is a complete system of primitive orthogonal idempotents (that is, 3 and 4 also hold). Indeed, e_J is primitive, for if e_J could be written as $x + y$, then we could replace e_J in Λ with x and y, contradicting the maximality of Λ. To see 4, we just note that if $\sum_K e_K \neq 1$, then $1 - \sum_K e_K$ is idempotent and orthogonal to all other e_K. Combining this element with Λ would again contradict the maximality of Λ.*

Let W denote a weakly ordered monoid with C and D being the associated "content" and "descent" maps from W to an upper semi-lattice $\mathcal{L}$. We let $\mathcal{G}$ denote a set of generators of W. The main goal of this paper is to build a method for finding a complete system of orthogonal idempotents for the monoid algebra $\mathbb{C}W$. In particular, this solves the problem posed by Norton about the 0-Hecke algebra for the symmetric group.

For each $J \in \mathcal{L}$, we define a **Norton element** $A_J T_J$. Let us begin by defining T_J:

$$T_J = \Big(\prod_{\substack{g\in\mathcal{G}\\ C(g)\preceq J}} g^\omega \Big)^\omega \in W.$$

Remark 3.4 *A different ordering of the set $\mathcal{G}$ of generators may produce different T_J's; so we fix an (arbitrarily chosen) order.*

We now define the A_J in the Norton element $A_J T_J$. First we let

$$B_J = \prod_{\substack{g\in\mathcal{G}\\ C(g)\not\preceq J}} (1 - g^\omega) \in \mathbb{C}W.$$

In the spirit of Lemma 2.8, we would like to raise B_J to a sufficiently high power so that it is idempotent. However, B_J is not an element of the monoid W, so $(B_J)^\omega$ may not be well defined. The following lemma and corollary resolve this problem.

Definition 3.5 *Given $x = \sum_{w\in W} c_w w \in \mathbb{C}W$, the* ***coefficient*** *of w in x is c_w. We say w is a* ***term*** *of x if the coefficient of w in x is nonzero.*

Lemma 3.6 *Let $b \in W$ and suppose $bx^\omega = b$ for some $x \in \mathcal{G}$ with $C(x) \not\preceq J$. If c is a term of bB_J, then $c > b$.*

Proof: Let $\mathcal{D} = \{x^\omega : x \in \mathcal{G}, C(x) \not\preceq J, bx^\omega = b\}$. By assumption $\mathcal{D}$ is not empty. Let $g_1, g_2, \ldots, g_m$ be the generators which appear in the definition of B_J. Then

$$B_J = \sum_{i_1<i_2<\cdots<i_k} (-1)^k g_{i_1}^\omega g_{i_2}^\omega \cdots g_{i_k}^\omega.$$

It follows from Lemma 3.1 that the coefficient of b in bB_J is counting the terms in B_J where each of $g_{i_1}, \ldots, g_{i_k}$ come from $\mathcal{D}$, weighted with sign $(-1)^k$. If $|\mathcal{D}| = m \geq 1$ then this is $1 - m + \binom{m}{2} - \binom{m}{3} + \cdots + (-1)^m = 0$. Therefore $c \neq b$. The statement now follows from the definition of order, as every term c of bB_J must be of the form $c = bz$ for some term z appearing in B_J, and hence $c \geq b$. $\square$

Lemma 3.7 *For every $J \in \mathcal{L}$, there exists an integer N such that $y^\omega B_J^N = 0$ for all $y \in \mathcal{G}$ with $C(y) \not\preceq J$.*

Proof: Let $N = \ell + 1$, where ℓ is the length of the longest chain of elements in the poset $(W, \leq)$.

Suppose $y^\omega B_J^N \neq 0$. Let c_N be a term of B_J^N. Then c_N is a term of $c_{N-1} B_J$ for some term c_{N-1} in $y^\omega B_J^{N-1}$. Since $y^\omega y^\omega = y^\omega$, Lemma 3.6 implies that y^ω is not a term of $y^\omega B_J^k$ for any $k \geq 1$, so that $c_{N-1} = y^\omega g_1^\omega \cdots g_m^\omega$ for some $m \geq 1$ and $g_i \in \mathcal{G}$ with $C(g_i) \not\preceq J$. In particular, $c_{N-1} g_m^\omega = c_{N-1}$, and so, again by Lemma 3.6, $c_N > c_{N-1}$. Repeated application of this argument produces a decreasing chain

$$c_N > c_{N-1} > c_{N-2} > \cdots > c_1$$

of elements in W, contradicting the fact that the length of the longest chain of elements in $(W, \leq)$ is ℓ. □

Corollary 3.8 *For every $J \in \mathcal{L}$ there exists an N such that $B_J^{N+1} = B_J^N$.*

Proof: By Lemma 3.7, $(B_J - 1)B_J^N = 0$ for a sufficiently large N since every element of $B_J - 1$ is of the form αy^ω where $\alpha \in \mathbb{C}$, $y \in \mathcal{G}$ and $C(y) \not\preceq J$. □

This now allows us to define $A_J = B_J^\omega$.

Lemma 3.9 *Let $J \in \mathcal{L}$. Then:*

1. *$T_J x = T_J$ for all x such that $C(x) \preceq J$;*
2. *$y^\omega A_J = 0$ for all y such that $C(y) \not\preceq J$ and $y \in \mathcal{G}$.*

Proof: Since $J = C(T_J)$, $C(x) \preceq J$ implies $C(x) \supseteq C(T_J)$. We also know that $T_J \in C(T_J)$ because T_J is idempotent. So $T_J \in C(x)$, that is, $T_J x = T_J$.

The second part follows from Lemma 3.7 since $A = B^N$. □

Remark 3.10 *Although T_J and A_J are idempotents individually, their product, the Norton element z_J, need not be. For example, take the 0-Hecke algebra $H_6(0)$ corresponding to the symmetric group $\mathfrak{S}_6$. Let J be the subset $\{1, 4, 5\}$ of $\{1, 2, 3, 4, 5\}$. Then $T_J = T_1 T_4 T_5 T_4$, $A_J = (1 - T_2)(1 - T_3)(1 - T_2)$ and z_J is their product. No power of z_J is idempotent.*

Lemma 3.11 *The coefficient of T_J in $z_J = A_J T_J$ is 1. All other terms y in z_J have $C(y) \succ J$.*

Proof: The coefficient of the identity element 1 in A_J is 1. Each term of $A_J T_J$ is of the form $a T_J$ for a term a of A_J. If $a \neq 1$, then $C(a) \not\preceq J$ so $C(aT_J) = C(a) \vee C(T_J) \succ C(T_J) = J$. Hence the coefficient of T_J in $A_J T_J$ is 1 and all other terms have content greater than J. □

Lemma 3.12 *If $J \not\preceq K$ then $z_J z_K = 0$.*

Proof: Since $J \not\preceq K$, there exists a $g \in \mathcal{G}$ with $C(g) \preceq J$ but $C(g) \not\preceq K$. Then, using Lemma 3.9 (1) and Lemma 3.9 (2), $z_J z_K = A_J T_J A_K T_K = A_J (T_J g^\omega) A_K T_K = A_J T_J (g^\omega A_K) T_K = 0$. □

Lemma 3.13 *For all $J \in \mathcal{L}$, there exists an N such that $(1 - z_J)^N z_J^2 = 0$.*

Proof (Outline): The proof is somewhat involved, so we only include an outline of the main argument here. A complete and detailed proof can be found in [3]. To simplify the notation, we temporarily set $T = T_J$, $A = A_J$ and $z = AT$. Fisrt note that $(1-z)^k z^2 = A(T(1-A)T)^k AT$. The idea is to argue that $(T(1-A)T)^N A = 0$ for N larger than the length of the largest chain in $(W, \leq)$.

Let $\mathcal{A}$ be the set of terms in $1 - A$. Every term of $(T(1-A)T)^N$ is of the form $Ta_1Ta_2T \cdots a_NT$ with $a_i \in \mathcal{A}$. If we write $x_i = Ta_1Ta_2T \cdots a_iT$, then in the R-order we have $x_1 \leq x_2 \leq \cdots \leq x_N$. For some i we must have $x_i = x_{i+1}$, so by Lemma 3.1, $x_i = x_i a_{i+1}$. This implies that $x_i(1-A)T = x_i a_{i+1}(1-A)T = x_iT = x_i$, from which it follows that $x_iA = 0$. □

Definition 3.14 *Let $J \in \mathcal{L}$. Let*

$$P_J := \sum_{n,m \geq 0} (1-z_J)^{n+m} z_J^2 = \sum_{k \geq 0} (k+1)(1-z_J)^k z_J^2.$$

(In Remark 3.20 we establish a summation-free formula for P_J.)

Remark 3.15 *Lemma 3.13 shows there are only finitely many terms in the summation of P_J. Therefore P_J is a well-defined element of $\mathbb{C}W$ for each $J \in \mathcal{L}$.*

Remark 3.16 *A monoid S is called J-trivial if $SxS = SyS$ implies $x = y$ for all $x, y \in S$. When S is J-trivial it suffices to define*

$$P_K = \sum_{n \geq 0} (1-z_K)^n z_K.$$

Lemma 3.17 *The coefficient of T_J in P_J is 1 and all other terms y of P_J have $C(y) \succ J$.*

Proof: If $n + m > 0$ then, using that T_J is idempotent,

$$A_JT_JA_JT_J(1 - A_JT_J)^{n+m} = A_JT_JA_J(T_J - T_JA_JT_J)^{n+m}.$$

Each term x in $(T_J - T_JA_JT_J)^{n+m}$ has $C(x) \succ J$, so no T_J appears in $z_J^2(1-z_J)^{n+m}$. The coefficient of T_J in z_J is 1, by Lemma 3.11. Hence T_J appears in $z_J^2(1-z_J)^0$ with coefficient 1. By Lemma 3.11, since all of the terms $y \neq T_J$ of z_J have $C(y) \succ J$ and P_J is a polynomial in z_J, all other terms w of P_J must have $C(w) \succ J$. □

Remark 3.18 *As polynomials in x we have for any nonnegative integer N:*

$$x \sum_{n=0}^{N} (1-x)^n = 1 - (1-x)^{N+1}.$$

Proposition 3.19 *For each $J \in \mathcal{L}$, the element P_J is idempotent.*

Proof: Let $J \in \mathcal{L}$ be fixed and let N be such that $(1 - z_J)^N z_J^2 = 0$. Let us temporarily denote z_J by z. We can use Lemma 3.18 to rewrite P_J as

$$\begin{aligned}
P_J &= \sum_{n,m\geq 0} z^2(1-z)^{n+m} = \sum_{n=0}^{N}\sum_{m=0}^{N-n} z^2(1-z)^{n+m} \\
&= \sum_{n=0}^{N}(1-z)^n \left(z^2 \sum_{m=0}^{N-n}(1-z)^m \right) = \sum_{n=0}^{N}(1-z)^n \left(z - z(1-z)^{N-n+1} \right) \\
&= z\left(\sum_{n=0}^{N}(1-z)^n\right) - (N+1)z(1-z)^{N+1} = 1-(1-z)^{N+1} - (N+1)z(1-z)^{N+1}.
\end{aligned}$$

This implies that $z^2 P_J = z^2$ since $z^2(1-z)^{N+1} = 0$, and so

$$P_J^2 = \left(\sum_{n=0}^{N}\sum_{m=0}^{N-n}(1-z)^{n+m}z^2\right) P_J = \sum_{n=0}^{N}\sum_{m=0}^{N-n}(1-z)^{n+m}z^2 = P_J.$$

□

Remark 3.20 *As shown in the calculation above, one could define P_J as*

$$P_J = 1 - (1 + (N+1)z_J)(1-z_J)^{N+1},$$

where N is the length of the longest chain in the monoid. For a J-trivial monoid, it suffices to take $P_J = 1 - (1 - z_J)^{N+1}$.

Lemma 3.21 *For all $J, K \in \mathcal{L}$, with $J \not\preceq K$, $P_J P_K = 0$.*

Proof: Follows from Lemma 3.12 and the fact that P_J is a polynomial in z_J with no constant term. □

Definition 3.22 *For each $J \in \mathcal{L}$, let*

$$e_J := P_J \left(1 - \sum_{K \succ J} e_K\right).$$

Lemma 3.23 *T_J occurs in e_J with coefficient 1. All other terms y of e_J have $C(y) \succ J$. In particular, $e_J \neq 0$.*

Proof: We proceed by induction. If J is maximal, then $e_J = P_J$, so the statement is implied by Lemma 3.17.

Now suppose the statement is true for all $M \succ J$. Then $e_J = P_J(1 - \sum_{M \succ J} e_M)$. By induction, all terms x of e_M have $C(x) \succeq M \succ J$. So terms y from $P_J e_M$ have $C(y) \succeq M \succ J$. The only other terms are those from P_J, for which the statement was proved in Lemma 3.17. □

Lemma 3.24 $e_K P_J = 0$ *for $K \not\preceq J$.*

Proof: The proof is by a downward induction on the semilattice. If K is maximal, then $e_K = P_K$, so by Lemma 3.21, $e_K P_J = P_K P_J = 0$.

Now suppose that for every $L \succ K$, $e_L P_J = 0$ for $L \not\preceq J$, and we will show that $e_K P_J = 0$ for $K \not\preceq J$. We expand $e_K P_J$:

$$e_K P_J = P_K \left(1 - \sum_{L \succ K} e_L\right) P_J = P_K P_J - \sum_{L \succ K} P_K e_L P_J.$$

Since $K \not\preceq J$, we have $P_K P_J = 0$ by Lemma 3.21, and $e_L P_J = 0$ by induction, since $L \succ K$ and $K \not\preceq J$ implies $L \not\preceq J$. □

Corollary 3.25 *e_J is idempotent.*

Proof: We expand $e_J e_J$:

$$e_J e_J = P_J \left(1 - \sum_{M \succ J} e_M\right) P_J \left(1 - \sum_{M \succ J} e_M\right) = P_J \left(P_J - \sum_{M \succ J} e_M P_J\right) \left(1 - \sum_{M \succ J} e_M\right)$$

$$\stackrel{(1)}{=} P_J^2 \left(1 - \sum_{M \succ J} e_M\right) \stackrel{(2)}{=} P_J \left(1 - \sum_{M \succ J} e_M\right) = e_J,$$

where (1) follows from Lemma 3.24, and (2) follows from Lemma 3.19. □

Lemma 3.26 *$e_J e_K = 0$ for $J \neq K$.*

Proof: The proof is by downward induction on the lattice $\mathcal{L}$. For a maximal element $M \in \mathcal{L}$, $e_M = P_M$, so $e_M e_K = P_M P_K (1 - \sum e_L) = 0$ by Lemma 3.21. Now suppose that for all $M \succ J$, $e_M e_K = 0$ for $M \neq K$ and we will show that $e_J e_K = 0$ for $J \neq K$. We expand $e_J e_K$:

$$e_J e_K = P_J (1 - \sum_{L \succ J} e_L) e_K = P_J (e_K - \sum_{L \succ J} e_L e_K) \qquad (1)$$

If $K \not\succ J$, then $\sum_{L \succ J} e_L e_K = 0$ by our induction hypothesis, so $P_J (e_K - \sum_{L \succ J} e_L e_K) = P_J e_K = P_J P_K (1 - \sum_{M \succ K} e_M) = 0$ by Lemma 3.21.

If $K \succ J$, then $\sum_{L \succ J} e_L e_K = e_K$ since e_K is idempotent and $e_L e_K = 0$ for $L \neq K$ by the inductive hypothesis. Therefore $e_K - \sum_{L \succ J} e_L e_K = 0$ and hence the right hand side of (1) is zero. □

Theorem 3.27 *The set $\{e_J : J \in \mathcal{L}\}$ is a complete collection of orthogonal idempotents for $\mathbb{C}W$.*

Proof: From [13], we know that the maximal number of such idempotents is the cardinality of $\mathcal{L}$. The rest of the claim is just Lemma 3.23, Corollary 3.25 and Lemma 3.26. □

Appendix: An example

We show by example how to use the above construction to create orthogonal idempotents for the free left regular band on two generators.

Idempotents for the free left regular band on two generators

Let S be the left regular band freely generated by two elements a, b. Then $S = \{1, a, b, ab, ba\}$. All elements of S are idempotent. Also $aba = ab$ and $bab = ba$. The lattice $\mathcal{L}$ has four elements: $\emptyset := S, \mathfrak{a} := Sa, \mathfrak{b} := Sb$ and $\mathfrak{ab} := Sab = Sba$, where $\emptyset \prec \mathfrak{a} \prec \mathfrak{ab}$ and $\emptyset \prec \mathfrak{b} \prec \mathfrak{ab}$, but $\mathfrak{a}$ and $\mathfrak{b}$ have no relation. We begin by computing the elements P_J.

$J = \emptyset$: Neither of the generators satisfies $C(g) \preceq J$, so $T_\emptyset = 1 \in S$. $B_\emptyset = (1-a)(1-b)$. Also

$$\begin{aligned} B_\emptyset^2 &= (1-a)(1-b)(1-a)(1-b) = (1-a-b+ab)(1-a)(1-b) \\ &= (1-a-b+ab)(1-b) = (1-a-b+ab) = B_\emptyset. \end{aligned}$$

Therefore $A_\emptyset = B_\emptyset = 1-a-b+ab$, so $z_\emptyset = 1-a-b+ab$ is idempotent and

$$P_\emptyset = 1-a-b+ab.$$

$J = \mathfrak{a}$: Then $C(a) \preceq \mathfrak{a}$ and $C(b) \not\preceq \mathfrak{a}$, so $T_\mathfrak{a} = a$ and $B_\mathfrak{a} = 1-b = A_\mathfrak{a}$ since $1-b$ is idempotent. Therefore $z_\mathfrak{a} = (1-b)a = a-ba$. $z_\mathfrak{a}^2 = a-ab$ and one can check that $z_\mathfrak{a}^3 = z_\mathfrak{a}^2$, so

$$P_\mathfrak{a} = z_\mathfrak{a}^2(1+(1-z_\mathfrak{a})+(1-z_\mathfrak{a})^2+\dots) = z_\mathfrak{a}^2 = a-ab.$$

One can check that $P_\mathfrak{a}$ is idempotent.

$J = \mathfrak{b}$: Similarly,

$$P_\mathfrak{b} = b-ba.$$

$J = \mathfrak{ab}$: $C(a), C(b) \preceq \mathfrak{ab}$, so $T_\mathfrak{ab} = ab$ and $A_\mathfrak{ab} = 1$. $z_\mathfrak{ab} = ab$ is idempotent, so

$$P_\mathfrak{ab} = ab.$$

We can now compute the idempotents e_J. Since $\mathfrak{ab}$ is maximal,

$$e_\mathfrak{ab} = ab.$$

Since $P_\mathfrak{a} e_\mathfrak{ab} = (a-ab)ab = ab-ab = 0$,

$$e_\mathfrak{a} = P_\mathfrak{a}(1-e_\mathfrak{ab}) = P_\mathfrak{a} = a-ab$$

and similarly,

$$e_\mathfrak{b} = b-ba.$$

Finally, note that $P_\emptyset e_\mathfrak{a} = (1-a-b+ab)(a-ab) = 0$ and similarly $P_\emptyset e_\mathfrak{b} = 0$, so that

$$e_\emptyset = P_\emptyset(1-e_\mathfrak{a}-e_\mathfrak{b}-e_\mathfrak{ab}) = P_\emptyset - P_\emptyset e_\mathfrak{ab} = 1-a-b+ab-ab+ba = 1-a-b+ba.$$

One can check that $\{e_\emptyset, e_\mathfrak{a}, e_\mathfrak{b}, e_\mathfrak{ab}\}$ is a collection of mutually orthogonal idempotents.

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Deformed diagonal harmonic polynomials for complex reflection groups

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Abstract. We introduce deformations of the space of (multi-diagonal) harmonic polynomials for any finite complex reflection group of the form $W = G(m, p, n)$, and give supporting evidence that this space seems to always be isomorphic, as a graded W-module, to the undeformed version.

Résumé. Nous introduisons une déformation de l'espace des polynômes harmoniques (multi-diagonaux) pour tout groupe de réflexions complexes de la forme $W = G(m, p, n)$, et soutenons l'hypothèse que cet espace est toujours isomorphe, en tant que W-module gradué, à l'espace d'origine.

Keywords: diagonal harmonic polynomials, complex reflection group, rational Steenrod algebra, deformations

1 Introduction

The aim of this work is to give support to an extension and a generalization of the main conjecture of [HT04], to the diagonal case as well as to the context of finite complex reflection groups. This is stated explicitly in the new Conjecture 1.2 below, after a few words concerning notations and a description of the overall context.

Let X denote a $\ell \times n$ matrix of variables

$$X := (\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n),$$

with each of the columns $\mathbf{x}_j = (x_{ij})_{1\leq i\leq \ell}$ containing ℓ variables. For any fixed i (a row of X), we say that the variables $x_{i1}, x_{i2}, \ldots, x_{in}$ form a *set of variables* (the i^{th} set), and thus X consists in ℓ sets of n variables. For $\mathbf{d} \in \mathbb{N}^\ell$, we set

$$|\mathbf{d}| := d_1 + d_2 + \cdots + d_\ell \qquad \text{and} \qquad \mathbf{d}! := d_1!d_2!\cdots d_\ell!,$$

and write $\mathbf{x}_j^{\mathbf{d}}$ for the column monomial of *degree* $\mathbf{x}_j^{\mathbf{d}}$:

$$\mathbf{x}_j^{\mathbf{d}} := \prod_{i=1}^{\ell} x_{ij}^{d_i} \ .$$

The ground field $\mathbb{K}$ is assumed to be of characteristic zero and, whenever needed, to contain roots of unity and/or a parameter q; typically, $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{C}(q)$, although algebraic or transcendental extensions of $\mathbb{Q}$ are better suited for some of the computer calculations. The parameter q is called formal if it is transcendental over $\mathbb{Q}$.

Let W be a complex reflection group of rank n. Elements of W may be thought of as $n \times n$ matrices with complex entries. The *diagonal action* of W on a polynomial $f(X)$ is defined, for $w \in W$, by:

$$w \cdot f(X) := f(X\, w), \tag{1.1}$$

where $X\,w$ stands for matrix multiplication. In other words, W acts in a similar "diagonal" manner on each set of variables in X. A polynomial is *diagonally W-invariant* if

$$w \cdot f(X) = f(X), \qquad \text{for all} \quad w \in W\, .$$

We denote by $\mathcal{I}_W^{(\ell)}$ the ideal generated by constant-term-free diagonally W-invariant polynomials.

For each of the variables $x_{ij} \in X$, there is an associated partial derivation denoted here by $\partial_{x_{ij}}$ or ∂_{ij} for short. For a polynomial $f(X)$, let $f(\partial_X)$ stand for the differential operator obtained by replacing variable in X by the corresponding derivation in ∂_X. The space $\mathcal{H}W^{(\ell)}$ of *diagonally W-harmonic polynomials* (or *harmonic polynomials* for short) is then defined as the set of the polynomials $g(X)$ that satisfy all of the linear partial differential equations

$$f(\partial_X)(g(X)) = 0, \qquad \text{for} \quad f(X) \in \mathcal{I}_W^{(\ell)}\, . \tag{1.2}$$

In the following, we first restrict ourselves to the complex reflection groups $W = G(m,n)$, for $m,n \in \mathbb{N}$, and then extend our discussion to the subgroups $G(m,p,n)$. Recall that the *generalized symmetric group* $G(m,n)$ may be constructed as the group of $n \times n$ matrices having exactly one non zero entry in each row and each column, whose value is a m-th root of unity. Since the cases $\ell = 1$ and $W = \mathfrak{S}_n$ have been extensively considered previously (see [Ber09]), we write for short $\mathcal{H}W = \mathcal{H}W^{(1)}$, $\mathcal{H}_n^{(\ell)} = \mathcal{H}\mathfrak{S}_n^{(\ell)}$, and $\mathcal{H}_n = \mathcal{H}\mathfrak{S}_n$.

The ring $\mathbb{K}[X]^W$ of diagonally W-invariant polynomials for $W = G(m,n)$ is generated by *polarized* powersums, this is to say the polynomials

$$P_{\mathbf{d}} = \sum_{j=1}^{n} \mathbf{x}_j^{\mathbf{d}}\,,$$

for $|\mathbf{d}| = m\,k$, with $1 \le k \le n$. Let us write $D_{\mathbf{d}}$ for the operator $P_{\mathbf{d}}(\partial_X)$:

$$D_{\mathbf{d}} = \sum_{j=1}^{n} \boldsymbol{\partial}_j^{\mathbf{d}}\,,$$

where

$$\boldsymbol{\partial}_j^{\mathbf{d}} := \partial_{1j}^{d_1}\partial_{2j}^{d_2}\cdots\partial_{\ell j}^{d_\ell}\,.$$

Then, the space $\mathcal{H}W^{(\ell)}$ is the intersection of the kernels of all the operators $D_{\mathbf{d}}$, for $|\mathbf{d}| = m\,k$, with $1 \le k \le n$. The space $\mathcal{H}W^{(\ell)}$ is graded by (multi-)degree, and thus decomposes as a direct sum

$$\mathcal{H}W^{(\ell)} = \bigoplus_{\mathbf{d}\in\mathbb{N}^\ell} \mathcal{H}W^{(\ell)}_{\mathbf{d}}\,,$$

of its homogeneous components of degree $\mathbf{d}$. Recall that $f(X)$ is homogeneous of degree $\mathbf{d}$, if and only if we have

$$f(\mathbf{t}\,X) = \mathbf{t}^{\mathbf{d}}\,f(X)\,,$$

where $\mathbf{t}\,X$ stands for the multiplication of the matrix X by the diagonal matrix

$$\begin{pmatrix} t_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & t_\ell \end{pmatrix}.$$

The *Hilbert series of the space* $\mathcal{H}W^{(\ell)}$ is defined as

$$\mathcal{H}W^{(\ell)}(\mathbf{t}) := \sum_{\mathbf{d}\in\mathbb{N}^\ell} \dim(\mathcal{H}W^{(\ell)}_{\mathbf{d}})\,\mathbf{t}^{\mathbf{d}}\,.$$

It is well known that, for $\ell = 1$, the graded space $\mathcal{H}W$ is isomorphic, as a W-module, to the graded regular representation of W. In particular, its Hilbert series is given by the formula

$$\mathcal{H}W(t) = \prod_{k=1}^{n} \frac{t^{k\,m}-1}{t-1}\,.$$

Our specific story starts with a q-deformation of the polarized powersums:

$$P_{q,\mathbf{d}} := \sum_{j=1}^{n} \mathbf{x}_j^{\mathbf{d}}\,(1 + q\,(x_{1j}\partial_{1j} + \ldots + x_{nj}\partial_{nj})). \tag{1.3}$$

and the corresponding q-deformation of the operators $D_{\mathbf{d}} = P_{\mathbf{d}}(\partial_X)$ above:

$$D_{q,\mathbf{d}} := \sum_{j=1}^{n} (1 + q\,(x_{1j}\partial_{1j} + \ldots + x_{nj}\partial_{nj}))\,\boldsymbol{\partial}_j^{\mathbf{d}}. \tag{1.4}$$

An homogeneous polynomial $f(X)$ is said to be *diagonally* W, q*-harmonic* (or q-harmonic for short) if

$$D_{q,\mathbf{d}}\,f(X) = 0, \tag{1.5}$$

for all $\mathbf{d} \in \mathbb{N}^\ell$ such that $|\mathbf{d}|$ is divisible by m. The W-module of all q-harmonic polynomials is henceforth denoted by $\mathcal{H}W_q^{(\ell)}$, and thus by $\mathcal{H}_{n,q}^{(\ell)}$ when $W = \mathfrak{S}_n$. Our aim here is to discuss for which q the following assertion holds.

Assertion 1.1 *$\mathcal{H}W_q^{(\ell)}$ is isomorphic (as a graded W-module) to $\mathcal{H}W^{(\ell)}$.*

Conjecture 1.2 *Let $W = G(m,n)$, $\ell \in \mathbb{N}$. Then, Assertion 1.1 holds for q a formal parameter. In particular $\mathcal{H}W_q^{(\ell)}(\mathbf{t}) = \mathcal{H}W^{(\ell)}(\mathbf{t})$.*

This conjecture is an extension of the main conjecture of [HT04] (case $W = \mathfrak{S}_n$, $\ell = 1$), which itself is a q-analogue of a conjecture of Wood [Woo98, Woo01] on the "hit polynomials" for the *rational Steenrod algebra* $\mathcal{S} := \mathbb{K}[P_{1,d} \mid d \geq 1]$. Beside the extensive computer exploration and results reported on in [HT04] for $W = \mathfrak{S}_n$ and $\ell = 1$, our supporting evidence for this conjecture, includes the following results:

(1) Applying a classical specialization argument at $q = 0$ (see e.g. [HT04]), gives $\dim \mathcal{H}W_q^{(\ell)} \leq \dim \mathcal{H}W^{(\ell)}$ (also homogeneous component by homogeneous component). Furthermore, equality holds if and only if Conjecture 1.2 does.

(2) Conjecture 1.2 holds for all groups $G(m,2)$ for $\ell = 1$ (see Section 5), as well as for all ℓ when $m \leq 5$. For example, with $W = G(3,2)$, we get

$$\begin{aligned}\mathcal{H}W_q^{(\ell)}(\mathbf{t}) \quad = \quad & 1 + 2\,h_1(\mathbf{t}) + 2\,h_2(\mathbf{t}) + h_1^2(\mathbf{t}) + h_3(\mathbf{t}) + 2\,h_2(\mathbf{t})h_1(\mathbf{t}) \\ & +2\,h_4(\mathbf{t}) + h_2^2(\mathbf{t}) + 3\,h_5(\mathbf{t}) + 2\,h_6(\mathbf{t}) + h_7(\mathbf{t}).\end{aligned}$$

(3) Conjecture 1.2 holds for all ℓ, in the case $W = \mathfrak{S}_n = G(1,n)$ for $n \leq 4$. For example, we get the Hilbert series

$$\mathcal{H}_{3,q}^{(\ell)}(\mathbf{t}) = 1 + 2\,h_1(\mathbf{t}) + h_{11}(\mathbf{t}) + h_2(\mathbf{t}) + h_3(\mathbf{t})\,.$$

(4) There seems to be an analogue of Conjecture 1.2 for the subgroups $G(m,p,n)$ of $G(m,n)$ (see Section 2); in particular, Conjecture 1.2 holds for $n = 2$ (including the dihedral groups $G(m,m,2)$) when $\ell = 1$ (see Section 5), and for any ℓ for small values of m, p, n.

Another interesting feature of the space $\mathcal{H}W_q^{(\ell)}$, is that it may be characterized as the intersection of the kernels of a much smaller family of operators than the set

$$\{D_{q,\mathbf{d}} \mid |\mathbf{d}| = m\,k, \quad 1 \leq k \leq n\}. \tag{1.6}$$

Indeed, a straightforward calculation shows that the usual Lie-bracket relation between the generators of the rational Steenrod algebra generalize naturally:

$$[D_{q,\mathbf{d}}, D_{q,\mathbf{d}'}] = q(|\mathbf{d}| - |\mathbf{d}'|)\, D_{q,\mathbf{d}+\mathbf{d}'}. \tag{1.7}$$

An efficient way to setup this calculation is to let both sides act on the generating function for all monomials, namely the formal series

$$\exp(Z.X) = \sum_{\mathbf{d} \in \mathbb{N}^{\ell \times n}} \mathbf{x}^{\mathbf{d}} \frac{\mathbf{z}^{\mathbf{d}}}{\mathbf{d}!}\,,$$

where Z stands for a matrix of variables just as X does, and $Z.X := \sum_{ij} z_{ij}x_{ij}$. It follows from (1.7) that a polynomial is in the kernel of $D_{q,\mathbf{d}+\mathbf{d}'}$, whenever it lies in the kernels of both $D_{q,\mathbf{d}}$ and $D_{q,\mathbf{d}'}$. From this, we can immediately deduce that

Proposition 1.3 *The space of q-harmonic polynomials for $W = G(m,n)$ can be obtained as*

$$\mathcal{H}W_q^{(\ell)} = \bigcap_{|\mathbf{d}|=m \text{ or } 2\,m} \mathrm{Ker}(D_{q,\mathbf{d}})\,.$$

For example, when $\ell = 1$, and as already noted in [HT04] in the case $W = \mathfrak{S}_n$, the space $\mathcal{H}W_q$ is defined by just two linear differential equations:

$$\mathcal{H}W_q = \mathrm{Ker}(D_{q,m}) \cap \mathrm{Ker}(D_{q,2\,m})\,.$$

Similarly, when $\ell = 2$, the space $\mathcal{H}W_q^{(2)}$ is the intersection of the kernels of only five operators:

$$D_{q,(1,0)}, \qquad D_{q,(2,0)}, \qquad D_{q,(1,1)}, \qquad D_{q,(0,1)}, \quad \text{and} \quad D_{q,(0,2)}\,.$$

This is striking because, assuming that Conjecture 1.2 holds, a direct calculation of this joint kernel and a specialization at $q = 0$ would yield back the famous space $\mathcal{H}_n^{(2)}$ of diagonally harmonic polynomials. Yet, even if the mysterious structure of $\mathcal{H}_n^{(2)}$ has been extensively studied in the past 20 years (see [Hai03]), no nice Gröbner basis for the ideal $\mathcal{I}_W^{(2)}$ is known, even for $W = \mathfrak{S}_n$.

It is also noteworthy that systematic variations on the main conjecture in [HT04] have been extensively studied in [BGW10]. In an upcoming work, we plan to describe how these variations may be adapted to the context of the reflection groups considered here, including the diagonal point of view. In particular, since there is a close tie (*loc. sit.*) between the case $\ell = 1$, with $W = \mathfrak{S}_n$, and Wood's conjecture (stated in [Woo98] or [Woo01]), we also plan to analyze how to generalize it to our new expanded context.

2 Deformed harmonic polynomials for $G(m,p,n)$

This section presents work in progress toward generalizing the construction of q-harmonic polynomials, and Conjecture 1.2, to all finite complex reflection groups. For simplicity, we restrict ourselves to a single set of variables: namely $\ell = 1$. However, computer calculations suggest that the extension to the diagonal case is straightforward.

Recall that all but a small number of finite complex reflection groups are part of an infinite family of natural subgroups of the generalized symmetric groups which we consider now. For $m, n \in \mathbb{N}$, let p be a divisor of m. Then, the complex reflection group $G(m,p,n)$ is defined as:

$$G(m,p,n) := \{g \in G(m,n) \mid \det g^{m/p} = 1\}\,.$$

In particular, setting $p = 1$, we get back $G(m, n) = G(m, 1, n)$. Recall, for example, that the classical dihedral groups correspond to the family $G(m, m, 2)$.

The invariant ring for $W = G(m, p, n)$ is obtained by adjoining $e_n^{m/p}$ to the invariant ring of $G(m, n)$, with $e_n = e_n(\mathbf{x}) := x_1 \cdots x_n$ standing for the product of the variables. It is well known that the invariant ring $\mathbb{K}[\mathbf{x}]^W$ may then be described as the graded free commutative algebra:

$$\mathbb{K}[\mathbf{x}]^W = \mathbb{K}[p_m, \ldots, p_{(m-1)n}, e_n^{m/p}]\,,$$

with $\deg(p_k) = k$ and $\deg(e_n^{m/p}) = nm/p$. Notice the necessary suppression of the generator p_{mn} for this presentation to be free.

The choice of a canonical q-analogue of $e_n^{m/p}$ does not appear to be straightforward. Indeed, and as far as we know (see the discussion in [HT04, Section 7.1]), there is no natural analogue of the elementary symmetric polynomial inside the rational Steenrod algebra $\mathcal{S}_q$. Besides, experienced gained in [BGW10] suggests that (generically) any choice of q-analogue would give an isomorphic space. We therefore take the simplest option, which is to not deform $e_n^{m/p}$ at all. Hence, for $W = G(m, p, n)$, we define the *q-deformed rational W-Steenrod algebra* as

$$\mathcal{S}_q^W := \mathbb{K}[P_{q,m}, \ldots, P_{q,mn}, e_n^{m/p}].$$

Accordingly, we obtain the graded space $\mathcal{H}W_q$ of the q-harmonic polynomials for $W = G(m, p, n)$, just as previously. Specifically, writing ε for the operator $e_n(\partial_X)$, $\mathcal{H}W_q$ is the intersection of $\mathcal{H}G(m, n)_q$ and $\operatorname{Ker}(\varepsilon^{m/p})$. A natural question here is to ask whether Conjecture 1.2 holds for $G(m, p, n)$. We will show in Section 5 that it does for $n = 2$ and $\ell = 1$.

A first step to confirm the choice of $e_n^{m/p}$ would be to prove the following conjecture.

Conjecture 2.1 *The q-harmonic polynomials for $G(m, 1, n)$, as defined above, coincide with those for $G(m, n)$.*

An equivalent but more concrete condition is that no q-harmonic polynomial for $G(m, n)$ shall contain a monomial divisible by e_n^m. This property holds for $n = 2$, and all q-harmonic polynomials for $\mathfrak{S}_n$ calculated in [HT04].

In fact, we expect $\mathcal{H}G(m, n)_q$ to decompose as a direct sum of m layers $L_0(q) \oplus \cdots \oplus L_{m-1}(q)$ such that all the elements of $L_k(q)$ are divisible by e_n^k but not by e_n^{k+1}, as in Figure 5. Furthermore, ε (depicted by the gray down arrows in this figure) would be an isomorphism from $L_k(q)$ to $L_{k-1}(q/(1 + q))$, the change of q being due to the equation

$$e_n(\partial_X) D_{q,k} = (1 + q) D_{\frac{q}{1+q},k}\, e_n(\partial_X)\,. \tag{2.1}$$

In that case, the q-harmonic polynomials for $G(m, p, n)$ would be given by

$$\mathcal{H}G(m, p, n)_q = L_0(q) \oplus \cdots \oplus L_{m/p-1}(q)\,. \tag{2.2}$$

For example, in Figure 5, the q-harmonic polynomials for the dihedral group $G(4,1,2)$ are given by $\mathcal{H}G(4,1,2)_q = L_0(q)$. Similarly, $\mathcal{H}G(4,2,2)_q = L_0(q) \oplus L_1(q)$. We expect that, in general, the space $\mathcal{H}G(m,n)_q$ consists of m copies of $\mathcal{H}G(m,n,1)_q$, that may be constructed by "lifting back" through ε.

This further suggests that the lack of operators commuting appropriately with the action of the rational Steenrod algebra, as reported in [HT04, Section 7] can be circumvented by allowing q to change during the commutation. For example, for $m = \ell = 1$, the harmonic polynomials are usually constructed from the Vandermonde determinant by iterative applications of operators ∂_i; it would be worth finding analogues of those operators which would construct new q-harmonic polynomials from q'-harmonic polynomials for some other q'.

3 Inflating q-harmonic polynomials from $\mathfrak{S}_n$ to $G(m,n)$ and $G(m,m,n)$

In this section, we do some preliminary steps in the following direction.

Problem 3.1 *Assume that a basis of the q-harmonic polynomials for $\mathfrak{S}_n$ is given. Is it possible to construct from it a basis of the q-harmonic polynomials for $G(m,n)$? for $G(m,m,n)$?*

Beware that the analogous problem for constructing diagonally q-harmonic polynomials from q-harmonic polynomials is already hard at $q = 0$.

To start with, any q-harmonic polynomial for $W = \mathfrak{S}_n$ remains q-harmonic for $G(m,n)$. It also remains q-harmonic for $G(m,p,n)$ as soon as Conjecture 2.1 holds for $\mathfrak{S}_n$. We now construct more q-harmonic polynomials by inflating those of $\mathfrak{S}_n$. To this end, we consider the inflation algebra morphism and its analogue (which is just a linear morphism) on the dual basis:

$$\phi_r : \begin{cases} \mathbb{K}[X] & \hookrightarrow \mathbb{K}[X^r] \\ \mathbf{x}^{\mathbf{d}} & \mapsto \mathbf{x}^{r\mathbf{d}} \end{cases} \qquad \overline{\phi}_r : \begin{cases} \mathbb{K}[X] & \hookrightarrow \mathbb{K}[X^r] \\ \mathbf{x}^{(\mathbf{d})} & \mapsto \mathbf{x}^{(r\mathbf{d})} \end{cases}, \tag{3.1}$$

where, by a slight notational abuse, X^r stands for the matrix of the r-th powers of the variables and $\mathbf{x}^{(\mathbf{d})} := \frac{1}{\mathbf{d}!}\mathbf{x}^{\mathbf{d}}$.

Proposition 3.2 *Let $W = G(m,n)$ and r be a divisor of m. Then, the morphism $\overline{\phi}_r$ restricts to a graded $\mathfrak{S}_n$-module embedding (resp. isomorphism if $r = m$) from $\mathcal{H}\mathfrak{S}_{nq}$ to $\mathcal{H}W_{q/m} \cap \mathbb{K}[X^r]$, up to an appropriate r-scaling of the grading. The statement extends to any $G(m,p,n)$ as soon as Conjecture 2.1 holds for $\mathfrak{S}_n$.*

The first step toward this proposition is to define an appropriate inflation on the q-rational Steenrod algebra. Note that the operators $P_{q,k}$ live inside the subalgebra $\mathbb{K}[X \cdot \partial_X, X]$ of the Weyl algebra, where $X \cdot \partial_X$ denotes the matrix of the Euler operators $x\partial_x$ for $x \in X$. The only non-trivial brackets in this algebra are $[x\partial_x, x] = x$, for $x \in X$, from which it follows that

$[x\partial_x, x^k] = kx^k$. Similarly, the operators $D_{q,k}$ live inside the subalgebra $\mathbb{K}[XPX, \partial_X]$, with analogous relations.

Remark 3.3 *Fix $r \in \mathbb{N}$. Then, the two mappings*

$$x\partial_x \mapsto 1/r(x\partial_x), \quad x \mapsto x^r, \text{ for } x \in X \qquad \text{and} \qquad x\partial_x \mapsto 1/r(x\partial_x), \quad \partial_x \mapsto \partial_x^r, \text{ for } x \in X$$

extend respectively to algebra isomorphisms

$$\Phi_r : \mathbb{K}[X \cdot \partial_X, X] \hookrightarrow \mathbb{K}[X \cdot \partial_X, X^m] \qquad \text{and} \qquad \overline{\Phi}_r : \mathbb{K}[X \cdot \partial_X, \partial_X] \hookrightarrow \mathbb{K}[X \cdot \partial_X, \partial_X^m].$$

Furthermore, those isomorphisms are compatible with the action on inflated polynomials: for $f \in \mathbb{K}[X]$ and F in $\mathbb{K}[X \cdot \partial_X, X]$ (resp. in $\mathbb{K}[X \cdot \partial_X, \partial_X]$), we have

$$\Phi_r(F)(\phi_r(f)) = \phi_r(F(f)) \qquad \text{and} \qquad \overline{\Phi}_r(F)(\overline{\phi}_r(f)) = \overline{\phi}_r(F(f)). \tag{3.2}$$

Using this remark, a straightforward calculation shows that:

$$\Phi_r(P_{q,\mathbf{d}}) = P_{q/r,r\mathbf{d}} \qquad \text{and} \qquad \overline{\Phi}_r(D_{q,\mathbf{d}}) = D_{q/r,r\mathbf{d}}. \tag{3.3}$$

This readily implies that Φ_m restricts to an isomorphism from the q-rational Steenrod algebra for $\mathfrak{S}_n$ to that for $G(m,n)$. Computer exploration suggests that the Gröbner basis for the right ideal generated by the Steenrod algebra for $G(m,n)$ is simply obtained by inflating that for $\mathfrak{S}_n$. This possibly opens the door for controlling the leading monomials of "q-hit polynomials" for $G(m,n)$ from those for $\mathfrak{S}_n$.

Returning to our main goal, we now have all the ingredients to prove Proposition 3.2.
Proof of Proposition 3.2: Let $f \in \mathbb{K}[X]$. Then, using Equation (3.2),

$$D_{q/r,\,r\mathbf{d}}(\phi(f)) = \overline{\Phi}_r(D_{q,\mathbf{d}})(\phi(f)) = \phi(D_{q,\mathbf{d}}(f)).$$

Hence $D_{q/r,r\mathbf{d}}(\phi(f)) = 0$ if and only if $D_{q,\mathbf{d}}(f) = 0$. The statements follows. □

4 Singular values

As discussed in [HT04], Assertion 1.1 may fail for very specific values of q. In that case, q is said to be *singular*. Computer exploration (see Table A.3 of [HT04]) and the complete analysis of the case $n = 2$ suggested that the only such singular values for $W = \mathfrak{S}_n$ and $\ell = 1$ are negative rational numbers of the form $-a/b$ with $a \leq n$. In [DM10] D'Adderio and Moci refined this statement to $a \leq n \leq b$ (with $q = -a/b$ not necessarily reduced), and proved that all such values are indeed singular by constructing explicit exceptional q-harmonic polynomials.

Proposition 4.1 *Let $W = G(m,n)$, $\ell \in \mathbb{N}$, and $q = -a/b$ with $a \leq n \leq b$, for $a, b \in \mathbb{N}$. Then, q is singular.*

Proof (sketch of): Let $f(x_1, \ldots, x_n)$ be the q-harmonic polynomial for $\mathfrak{S}_n$ which was constructed in [DM10]. Then, as stated in Proposition 3.2, $f(x_1^m, \ldots, x_n^m)$ is a q/m-harmonic for W of high enough degree (as in [DM10]) to disprove the statement of Conjecture 1.2. Going from $\ell = 1$ to ℓ arbitrary is then straightforward, since the intersection of $\mathcal{H}W_q^{(\ell)}$ with the polynomial ring in the first set of variables is $\mathcal{H}W_q$. □

It is worth noting that, for $n = 2$, and $\ell = 1$ the singular values are exactly those listed in Proposition 4.1 (see Section 5). However, at this stage, we lack computer data to extend this to a conjecture for all n and ℓ.

5 Complete study for $n = 2$

In this section, we prove Conjecture 1.2 for any group $W = G(m, p, 2)$ in the case $\ell = 1$. We denote for short the two variables x and y instead of x_1 and x_2. Naturally ∂_x and ∂_y are the corresponding differential operators. We also introduce the following q-analogue of the Pockhammer symbol $(d)_k$:

$$\langle d\rangle_k := d\,(d-1)\,\cdots\,(d-k+1)\,(1+q\,(d-k)).$$

Then, for any monomial $x^\alpha y^\beta$, one has:

$$D_{q,k}(x^\alpha y^b) = \langle\alpha\rangle_k\, x^{\alpha-k} y^\beta + \langle\beta\rangle_k\, x^\alpha y^{\beta-k}\,, \tag{5.1}$$

which is well defined for any nonnegative numbers α and β, since $\langle\alpha\rangle_k = 0$ whenever $\alpha < k$.

Remark 5.1 *Let $W = G(m, m, 2)$ be the dihedral group. Then, the space $\mathcal{H}W_q$ is isomorphic to $\mathcal{H}W$, and in fact coincides with it, if and only if q is not of the form $-1/b$ with $1 \le b \le m$. In that case, it is of dimension $2m$ and a basis is given by*

$$\{1, x, x^2, x^3, \ldots, x^{m-1}, x^m - y^m, y^{m-1}, y^{m-2}, \ldots, y^2, y\}\,. \tag{5.2}$$

Otherwise, the basis of $\mathcal{H}W_q$ contains additionally the monomials x^{b+m} and y^{b+m}, or just the binomial $x^{2m} - y^{2m}$ if $b = m$.

Proof: Let $f = f(x, y)$ be an homogeneous q-harmonic polynomial in $\mathbb{K}[x, y]$. It satisfies:

$$D_{q,m}(f) = 0, \qquad D_{q,2m}(f) = 0, \qquad \textit{and} \qquad \varepsilon(f) = 0,$$

where $\varepsilon = \partial_x \partial_y$. By the last equation, f is of the form $\lambda x^d + \mu y^d$, and the two other equations rewrite as $(d)_k(\lambda x^{d-km} + \mu y^{d-km})$ for $k = 1, 2$. The statement follows. □

Proposition 5.2 *Let $W = G(m,2)$ and $\ell = 1$. Then, the space $\mathcal{H}W_q$ is isomorphic as a graded W-module to $\mathcal{H}W$ if and only if q is not of the form $-a/b$ with $1 \leqslant a \leqslant 2 \leqslant b$, and $a, b \in \mathbb{N}$. In that case, it is of dimension $2m^2$ and a basis is given by*

$$\{x^\alpha y^\beta\}_{\substack{0\leqslant a<m \\ 0\leqslant b<m}} \cup \{\langle \beta + m\rangle_m x^{\alpha+m} y^\beta - \langle \alpha + m\rangle_m x^\alpha y^{\beta+m}\}_{\substack{0\leqslant \alpha<m \\ 0\leqslant \beta<m}} . \tag{5.3}$$

Proof: As suggested by Equation (5.1), the implicit combinatorial ingredient is the length of the longest string $\ldots, x^{\alpha-m}y^{\beta+m}, x^\alpha y^\beta, x^{\alpha+m}y^{\beta-m}, \ldots$ containing any given monomial.

Obviously, whenever $\alpha < m$ and $\beta < m$, the monomial $x^\alpha y^\beta$ is killed by both operators $D_{q,m}$ and $D_{q,2\,m}$, and is therefore q-harmonic. This gives the first m^2 monomials in (5.3). Using Equation 5.1, a direct calculation shows that the remaining m^2 binomials

$$\langle \beta + m\rangle_m x^{\alpha+m} y^\beta - \langle \alpha + m\rangle_m x^\alpha y^{\beta+m}$$

for $\alpha < m$ and $\beta < m$ are also q-harmonic.

We now consider a monomial $x^{\alpha'} y^{\beta'}$ that does not appear in any of the q-harmonic polynomials of (5.3), and prove that it cannot appear in any other q-harmonic polynomial. It is straightforward that we can choose α and β such that

$$x^{\alpha'} y^{\beta'} \in \{x^{\alpha+m}y^{\beta-m}, x^\alpha y^\beta, x^{\alpha-m}y^{\beta+m}\} .$$

Let $h = c_1 x^{\alpha+m}y^{\beta-m} + c_2 x^\alpha y^\beta + c_3 x^{\alpha-m}y^{\beta+m} + \cdots$ be a q-harmonic polynomial. Then,

$$0 = D_{q,m}(h)_{|x^\alpha y^{\beta-m}} = c_1\langle \alpha+m\rangle_m + c_2\langle\beta\rangle_m .$$

Looking similarly at $D_{q,m}(h)_{|x^{\alpha-m}y^\beta}$ and $D_{q,2\,m}(h)_{|x^{\alpha-m}y^{\beta-m}}$, shows that the coefficients c_1, c_2, and c_3 must satisfy the following system of equations:

$$\begin{array}{lllllll}
\langle \alpha+m\rangle_m \; c_1 & + & \langle\beta\rangle_m \, c_2 & & & = & 0 \\
 & & \langle\alpha\rangle_m \, c_2 & + & \langle \beta+m\rangle_m \; c_3 & = & 0 \\
\langle \alpha+m\rangle_{2m} \, c_1 & & & + & \langle \beta+m\rangle_{2m} \, c_3 & = & 0
\end{array}$$

whose determinant is:

$$\frac{(\alpha+m)!(\beta+m)!}{(\alpha-m)!(\beta-m)!}(1+q\,(\alpha-m))(1+q\,(\beta-m))(2+q\,(\alpha+\beta)) .$$

Therefore $c_1 = c_2 = c_3 = 0$ whenever q is not one of the announced singular values. □

Corollary 5.3 *For $W = G(m,p,2)$ and $q \neq -a/b, 1 \leqslant a \leqslant 2 \leqslant b$, the space $\mathcal{H}W_q$ is isomorphic as a graded W-module to $\mathcal{H}W$. Its basis is obtained by considering the layers $L_0(q), \ldots, L_{m/p-1}(q)$ of the q-harmonic polynomials for $G(m,2)$.*

Proof: Select out of Equation (5.3) the elements which satisfy the extra equation $\varepsilon^{m/p}(f) = 0$. See also, in Figure 5, the vertical arrows which depict the action of ε. □

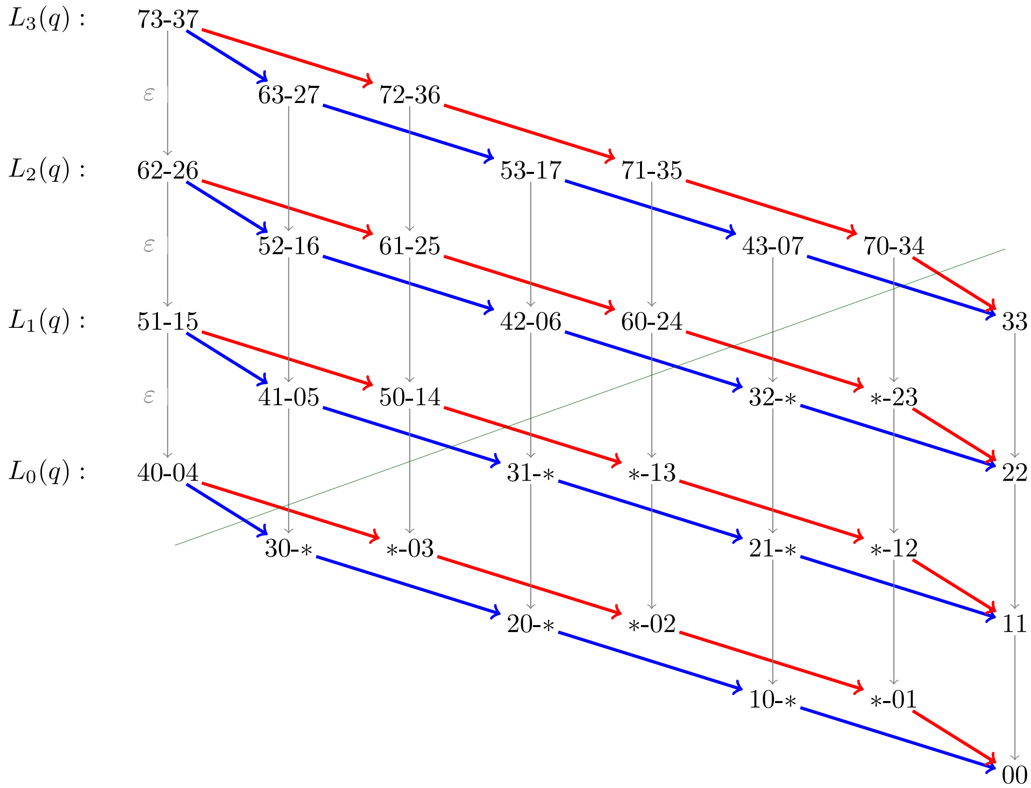

Fig. 1: Structure of q-Harmonic polynomials of $G(4,2)$. For short, the q-harmonic binomial $\langle d\rangle_4 x^\alpha y^\beta - \langle a\rangle_4 x^{\alpha'} y^{\beta'}$ is denoted "$\alpha\beta$-$\alpha'\beta'$". Similarly, the q-harmonic monomial $x^\alpha y^\beta$ is denoted "$\alpha\beta$", "$\alpha\beta$-$*$", or "$*$-$\alpha\beta$". The blue (resp. red) arrows denote the action of the would be q-analogues of the operators ∂_x and ∂_y within each layer L_i. The gray arrows denote the action of the operator $\varepsilon = e_2(\partial_X) = \partial_x \partial_y$ (recall that it changes the value of q). The green line separates the q-harmonic monomials and binomials.

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Enumerating projective reflection groups

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Abstract. Projective reflection groups have been recently defined by the second author. They include a special class of groups denoted $G(r, p, s, n)$ which contains all classical Weyl groups and more generally all the complex reflection groups of type $G(r, p, n)$. In this paper we define some statistics analogous to descent number and major index over the projective reflection groups $G(r, p, s, n)$, and we compute several generating functions concerning these parameters. Some aspects of the representation theory of $G(r, p, s, n)$, as distribution of one-dimensional characters and computation of Hilbert series of some invariant algebras, are also treated.

Résumé. Les groupes de réflexions projectifs ont été récemment définis par le deuxième auteur. Ils comprennent une classe spéciale de groupes notée $G(r, p, s, n)$, qui contient tous les groupes de Weyl classiques et plus généralement tous les groupes de réflexions complexes du type $G(r, p, n)$. Dans ce papier on définit des statistiques analogues au nombre de descentes et à l'indice majeur pour les groupes $G(r, p, s, n)$, et on calcule plusieurs fonctions génératrices. Certains aspects de la théorie des représentations de $G(r, p, s, n)$, comme la distribution des caractères linéaires et le calcul de la série de Hilbert de quelques algèbres d'invariants, sont aussi abordés.

Keywords: reflection groups, characters, permutation statistics, generating functions

1 Introduction

The study of permutation statistics, and in particular of those depending on up-down or descents patterns, is a very classical subject of study in algebraic combinatorics that goes back to the early 20th century to the work of MacMahon [Ma], and has found a new interest in the last decade after the fundamental work of Adin and Roichman [AR]. In their paper they defined a new statistic, the flag-major index that extend to the Weyl group of type B the concept of major index, which is a classical and well studied statistic over the symmetric group. This paper opened the way to several others concerning the study of statistics over classical Weyl groups, wreath products, and some types of complex reflection groups. Now we have a picture of several combinatorial and algebraic properties holding for all these families of groups.

Recently, the second author introduced a new class of groups called projective reflection groups [Ca]. Among them, there is an infinite family denoted $G(r, p, s, n)$ where r, p, s, n are integers such that $p, s|r$ and $ps|rn$. They include all complex reflection groups in fact $G(r, p, 1, n) = G(r, p, n)$. In this context is fundamental a notion of duality. Let $G = G(r, p, s, n)$, then its dual group is $G^* = G(r, s, p, n)$, where the parameters p and s are interchanged. In [Ca], it is shown that much of the theory of reflection groups can be extended to projective reflection groups, and that the combinatorics of the groups $G = G(r, p, s, n)$ is strictly related to the invariant theory of the dual group G^*.

In this paper we continue that study. We introduce two descent numbers des and fdes, and a color sum col over $G(r,p,s,n)$, that together with the flag major index fmaj [Ca] allow the extension to $G(r,p,s,n)$ of the following results.

We show that the polynomial $\sum \chi(g)q^{\mathrm{fmaj}(g)}$, where χ denotes any one-dimensional character of $G(r,p,s,n)$ and the sum is over $g \in G(r,p,s,n)$, admits a nice product formula. This generalizes a classical result of Gessel and Simion [Wa] for the symmetric group, and the main results of Adin-Gessel-Roichman [AGR] for the Coxeter group of type B, and of the first author [Bi] for the type D case.

The enumeration of permutations by number of descents and major index yields a remarkable q-analogue of a well-known identity for the Eulerian polynomials, usually called Carlitz's identity. This identity has been generalized in several ways and to several groups, see e.g. [ABR], [BC], [BB], [BZ, BZ1],[ChG],[Re]. All these extensions, can be divided into two families, depending from the used descent statistic, i.e. geometric descents or flag descents. We give a general method to compute the trivariate distribution of des (or fdes), fmaj and col over $G(r,p,s,n)$. This unifies and generalizes all previous cited results. We further exploit our method to compute the generating function of the six statistics $\mathrm{des}, \mathrm{ides}, \mathrm{fmaj}, \mathrm{ifmaj}, \mathrm{col}, \mathrm{icol}$ over the group $G(r,p,s,n)$, ($\mathrm{ides}(g)$ denotes $\mathrm{des}(g^{-1})$ and similarly for the others). From a specialization of our identity we deduce another important result, the computation of the generating function of the Hilbert series of the diagonal invariant algebras of the groups $G(r,p,s,n)$.

The definition of the above statistics depends on the particular order chosen on the colored integers numbers. By comparing our results with others in the literature we deduce that different choices of the order can or cannot give rise to same results. In some affirmative cases we provide bijective explanations of this phenomenon.

2 Notation and preliminaries

In this section we collect the notations that are used in this paper as well as the preliminary results that are needed.

We let $\mathbb{Z}$ be the set of integer numbers and $\mathbb{N}$ be the set of nonnegative integer numbers. For $a, b \in \mathbb{Z}$, with $a \leq b$ we let $[a,b] = \{a, a+1, \ldots, b\}$ and, for $n \in \mathbb{N}$ we let $[n] \stackrel{\text{def}}{=} [1,n]$. For $r \in \mathbb{N}$, we let $R_r : \mathbb{Z} \to [0, r-1]$ be the map "residue module r", where $R_r(i)$ is determined by $R_r(i) \equiv i \bmod r$. If $r \in \mathbb{N}$, $r > 0$, we denote by ζ_r the primitive r-th root of unity $\zeta_r \stackrel{\text{def}}{=} e^{\frac{2\pi i}{r}}$. As usual for $n \in \mathbb{N}$, we let $[n]_q \stackrel{\text{def}}{=} 1 + q + q^2 + \cdots + q^{n-1}$, and

$$(a;q)_n \stackrel{\text{def}}{=} \begin{cases} 1, & \text{if } n = 0; \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \geq 1. \end{cases}$$

Let $F \in \mathbb{C}[[q_1, q_2, \ldots]]$ be a formal power series with complex coefficients, and let q_i be one of its variables. We denote by $\{F\}_{q_i^p}$ the power series obtained from F by discarding all homogeneous components in the variable q_i whose degree is not divisible by p.

The main subject of this work are the complex reflection groups [Sh], or simply reflection groups, with particular attention to their combinatorial representation theory. The most important example of a complex reflection group is the group of permutations of $[n]$, known as the symmetric group, that we denote by S_n. We know by the work of Shephard-Todd [ShT] that all but a finite number of irreducible reflection groups are the groups $G(r,p,n)$, where r, p, n are positive integers with $p|r$, that we are going to describe. If A is a matrix with complex entries, we denote by $|A|$ the real matrix whose entries are the absolute values of the entries of A. The *wreath product* $G(r,n) = G(r,1,n)$ is given by all $n \times n$ matrices satisfying the following two conditions: the non-zero entries are r-th roots of unity; there is exactly one non-zero entry in every row and every column (i.e. $|A|$ is a permutation matrix). If p divides r then the *reflection group* $G(r,p,n)$ is the subgroup of $G(r,n)$ given by all matrices $A \in G(r,n)$ such that $\frac{\det A}{\det |A|}$ is a $\frac{r}{p}$-th root of unity.

A *projective reflection group* is a quotient of a reflection group by a scalar subgroup (see [Ca]). Observe that a scalar subgroup of $G(r,n)$ is necessarily a cyclic group C_s of order s, generated by $\zeta_s I$, for some $s|r$, where I denotes the identity matrix. It is also easy to characterize all possible scalar subgroups of the groups $G(r,p,n)$: in fact the scalar matrix $\zeta_s I$ belongs to $G(r,p,n)$ if and only if $s|r$ and $ps|rn$. In this case we let $G(r,p,s,n) \stackrel{\text{def}}{=} G(r,p,n)/C_s$. If $G = G(r,p,s,n)$ then the projective reflection group $G^* \stackrel{\text{def}}{=} G(r,s,p,n)$, where the roles of the parameters p and s are interchanged, is always well-defined. We say that G^* is the *dual* of G and we refer the reader to [Ca] for the main properties of this duality.

We sometimes think of an element $g \in G(r,n)$ as a *colored permutation*. If the non-zero entry in the i-th row of $g \in G(r,n)$ is $\zeta_r^{c_i}$ we let $c_i(g) \stackrel{\text{def}}{=} R_r(c_i)$ and say that $c_1(g), \ldots, c_n(g)$ are the *colors* of g. Now it is easy to see that an element $g \in G(r,n)$ is uniquely determined by the permutation $|g| \in S_n$ defined by $|g|(i) = j$ if $g_{i,j} \neq 0$, and by its colors $c_i(g)$ for all $i \in [n]$. More precisely, when we consider an element of $G(r,n)$ as a colored permutation we represent its elements either as couples $(c_1, \ldots, c_n; \sigma)$, where $\sigma = \sigma(1) \cdots \sigma(n)$ is a permutation in S_n and $(c_1, \ldots, c_n)$ is the sequence of its colors, or in *window notation* as $g = [g(1), \ldots, g(n)] = [\sigma(1)^{c_1}, \ldots, \sigma(n)^{c_n}]$. Sometimes we will call $\sigma(i)$ the *absolute value* of $g(i)$, denoted $|g(i)|$. When it is not clear from the context, we will denote c_i by $c_i(g)$. Moreover, if $c_i = 0$, it will be omitted in the window notation of g. If we let $\mathrm{col}(g) \stackrel{\text{def}}{=} c_1(g) + \cdots + c_n(g)$, then for $p|r$ we can simply express the reflection group $G(r,p,n)$ by $G(r,p,n) = \{(c_1, \ldots, c_n; \sigma) \in G(r,n) \mid \mathrm{col}(g) \equiv 0 \bmod p\}$, and for $q|r$, $pq|rn$ the projective reflection group by $G(r,p,s,n) = G(r,p,n)/C_s$, where C_s is the cyclic subgroup of $G(r,p,n)$ of order s generated by $(r/s, \ldots, r/s; id)$.

3 Descent-type statistics

In this section we recall the notions of descents, flag descents, and flag major index for the wreath products. Then we introduce analogous definitions for descents and flag descents for projective reflection groups. These two notions and the flag major index for $G(r,p,s,n)$ defined by Caselli in [Ca] will be used in the rest of the paper.

Definition 1 *In all the paper we will use the following order, called* color order

$$1^{r-1} < \ldots < n^{r-1} < \ldots < 1^1 < \ldots < n^1 < 0 < 1 < \ldots < n \tag{1}$$

For $g \in G(r,n)$ we define the *descent set* as

$$\mathrm{Des}_G(g) \stackrel{\text{def}}{=} \{i \in [0, n-1] \mid g(i) > g(i+1)\}, \tag{2}$$

where $g(0) := 0$, and denote its cardinality by $\mathrm{des}_G(g)$. A geometric interpretation of this set in terms of Coxeter-like generators and length can be given as in [Ba]. If we consider only positive descents we obtain $\mathrm{Des}_A(g) \stackrel{\mathrm{def}}{=} \mathrm{Des}_G(g) \setminus \{0\}$, and we denote by $\mathrm{des}_A(g)$ its cardinality. The *flag major index* [AR] and the *flag descent number* ([ABR],[BB]) are defined by

$$\mathrm{fmaj}(g) \stackrel{\mathrm{def}}{=} r \cdot \mathrm{maj}(g) + \mathrm{col}(g) \quad \text{and} \quad \mathrm{fdes}(g) \stackrel{\mathrm{def}}{=} r \cdot \mathrm{des}_A(g) + c_1(g), \tag{3}$$

where as usual the *major index* $\mathrm{maj}(g) = \sum_{i \in \mathrm{Des}_A(g)} i$ is the sum of all positive descents of g.

Following [Ca, §5], for $g = (c_1, \ldots, c_n; \sigma) \in G(r,p,s,n)$ we let

$$\begin{aligned} \mathrm{HDes}(g) &\stackrel{\mathrm{def}}{=} \{i \in [n-1] \mid c_i = c_{i+1}, \text{ and } \sigma(i) > \sigma(i+1)\} \\ h_i(g) &\stackrel{\mathrm{def}}{=} \#\{j \geq i \mid j \in \mathrm{HDes}(g)\} \\ k_i(g) &\stackrel{\mathrm{def}}{=} \begin{cases} R_{r/s}(c_n) & \text{if } i = n \\ k_{i+1} + R_r(c_i - c_{i+1}) & \text{if } i \in [n-1]. \end{cases} \end{aligned}$$

We call the elements in $\mathrm{HDes}(g)$ the *homogeneous descents* of g. Note that $(k_1(g), \ldots, k_n(g))$ is a partition such that $g = (R_r(k_1(g)), \ldots, R_r(k_n(g)); \sigma)$. Moreover it is characterized by the following property of minimality: if $\beta_1 \geq \ldots \geq \beta_n$ with $\beta_i \equiv c_i$ for all $i \in [1,n]$, then $\beta_i \geq k_i(g)$, for all $i \in [1,n]$.

For $g = (c_1, \ldots, c_n; \sigma) \in G(r,p,s,n)$, we let

$$\lambda_i(g) \stackrel{\mathrm{def}}{=} r \cdot h_i(g) + k_i(g). \tag{4}$$

The sequence $\lambda(g) \stackrel{\mathrm{def}}{=} (\lambda_1(g), \ldots, \lambda_n(g))$ is a partition such that $g = (\lambda(g); \sigma)$. The *flag-major index* for the projective reflection group $G(r,p,s,n)$ is defined in [Ca, §5]

$$\mathrm{fmaj}(g) \stackrel{\mathrm{def}}{=} |\lambda(g)| = \sum_{i=1}^{n} \lambda_i(g). \tag{5}$$

We define the *descent number* and the *flag descent number* of $g \in G(r,p,s,n)$ respectively by

$$\mathrm{des}(g) \stackrel{\mathrm{def}}{=} \lfloor \frac{s\lambda_1(g) + r - s}{r} \rfloor \quad \text{and} \quad \mathrm{fdes}(g) \stackrel{\mathrm{def}}{=} \lambda_1(g). \tag{6}$$

Finally, for $g \in G(r,p,s,n)$ we define the *color* of g by

$$\mathrm{col}(g) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} R_{r/s}(c_i(\tilde{g})), \tag{7}$$

where $\tilde{g}$ is any lift of g in $G(r,p,n)$.

Example 3.1 *Let* $g = (2,3,3,5,1,7,3,2; 27648153) \in G(6,2,3,8)$. *Then we have* $\mathrm{HDes}(g) = \{2,5\}$, $(h_1, \ldots, h_8) = (2,2,1,1,1,0,0,0)$ *and* $(k_1, \ldots, k_8) = (18,13,13,9,5,5,1,0)$, *and so* $\mathrm{des}(g) = 15$, $\mathrm{fdes}(g) = 6 \cdot 2 + 18 = 30$, $\mathrm{fmaj}(g) = 6 \cdot 7 + 64 = 106$, *and* $\mathrm{col}(g) = 6$.

Remark 1 *From [Ca, Lemma 5.1], it follows that for* $p = s = 1$, *the flag major index on* $G(r,p,s,n)$ *defined in (5) coincides with the flag major index of Adin and Roichman for wreath products* $G(r,n)$

defined in (3). Moreover the definition of fdes *in (6) is consistent with that in (3), and* $\mathrm{des}(g) = \mathrm{des}_G(g)$. *To see this last equality notice that*

$$\mathrm{des}(g) = \lfloor \frac{\lambda_1(g) + r - 1}{r} \rfloor = \mathrm{des}_A(g) + \lfloor \frac{c_1(g) + r - 1}{r} \rfloor = \mathrm{des}_G(g),$$

where the second equality holds since

$$\lfloor \frac{c_1(g) + r - 1}{r} \rfloor = \begin{cases} 0, & \textit{if } c_1(g) = 0; \\ 1, & \textit{otherwise}. \end{cases}$$

These equalities give motivations to the definitions of the previous statistics.

4 One-dimensional characters and flag major index

The generating function of the major index with the unique non-trivial one-dimensional characters admits a nice factorization formula over the symmetric group, as shown in Theorem 4.1 by Gessel-Simion (see [Wa]). The same happens for the other classical Weyl groups, as proved by Adin-Gessel-Roichman [AGR] for the type B case, and by the first author [Bi] for the type D case. In this section we generalize these results to all projective groups of type $G(r,p,s,n)$.

We start by recalling the classical result of Gessel-Simion for the symmetric group. As usual for $\sigma \in S_n$, we denote by $\mathrm{inv}(\sigma) \stackrel{\mathrm{def}}{=} \{(i,j) \in [n] \times [n] \mid i < j \text{ and } \sigma(i) > \sigma(j)\}$, the number of its *inversions*, and by $\mathrm{sign}(\sigma) \stackrel{\mathrm{def}}{=} (-1)^{\mathrm{inv}(\sigma)}$ its *sign*.

Theorem 4.1 (Gessel-Simion) *We have*

$$\sum_{\sigma \in S_n} \mathrm{sign}(\sigma) q^{\mathrm{maj}(\sigma)} = [1]_q [2]_{-q} [3]_q \cdots [n]_{(-1)^{n-1}q}.$$

The particular case of the Weyl groups $B_n \stackrel{\mathrm{def}}{=} G(2,n)$ is treated in [AGR]. In this paper the authors focus on the sum

$$\sum_{g \in B_n} (-1)^{\ell(g)} q^{\mathrm{fmaj}(g)},$$

where $\ell(g)$ is the Coxeter length of g with respect to a given set of simple reflections, making use of a combinatorial interpretation of $\ell(g)$ due to Brenti [BjB]. Theorem 4.2 is then achieved in this case since $(-1)^{\ell(g)} = (-1)^{\mathrm{inv}(|g|)}(-1)^{c(g)}$ for all $g \in B_n$. Although one can define an analogue of the Coxeter length for the wreath products $G(r,n)$, this does not lead to a one-dimensional character of the group, and the corresponding sum does not factorize nicely. This is why we focus on one-dimensional characters, obtaining in particular also a new proof in the case of Weyl groups of type B that does not make use of the combinatorial interpretation of the length function.

The irreducible representations of $G(r,p,s,n)$ are classified in [Ca, §6]. In particular, the one-dimensional characters of $G(r,p,s,n)$ are all of the form

$$\chi_{\epsilon,k}(g) = \epsilon^{\mathrm{inv}(|g|)} \zeta_r^{k \cdot c(g)},$$

where $\epsilon = \pm 1$, and $k \in [0, \frac{r}{p} - 1]$ with the further condition that s divides kn, and $c(g)$ is the sum of the colors of any element in $G(r,p,n)$ representing the class of g (in particular $c(g) = \mathrm{col}(g)$ if $s = 1$). Our main result is the following one.

Theorem 4.2 *Let $\chi_{\epsilon,k}$ be a one dimensional character of $G(r,p,s,n)$. Then*

$$\sum_{g\in G(r,p,s,n)} \chi_{\epsilon,k}(g) q^{\mathrm{fmaj}(g)} = \left(\prod_{j\in[n-1]} \left[\frac{jr}{p}\right]_{(\epsilon^{j-1}\zeta^k q)^p} \right) \left[\frac{nr}{ps}\right]_{(\epsilon^{n-1}\zeta^k q)^p} \left\{ [p]_{\zeta^k q}^{n-m} [p]_{\epsilon\zeta^k q}^{m} \right\}_{q^p},$$

where $m = \lfloor \frac{n}{2} \rfloor$. (We recall that $\{F(q)\}_{q^p}$ is the polynomial obtained from $F(q)$ by discarding all the homogeneous components in the variable q of degree not divisible by p.)

For $r = 2$ and $p = s = 1$ in Theorem 4.2, we obtain [AGR, Theorems 5.1, 6.1, 6.2]; for $r = s = 2$ and $p = 1$ [Bi, Theorem 4.8].

5 Carlitz's Identities

In this section we give a general method to compute the trivariate distribution of des (or fdes), fmaj and col over $G(r,p,s,n)$. This will unify and generalize all results cited in the introduction and will provide two different generalizations of Carlitz's identity for the group $G(r,p,s,n)$.

For $f = (f_1,\dots,f_n) \in \mathbb{N}^n$ let $|f| = f_1 + \cdots + f_n$ and

$$\mathbb{N}^n(p) = \{f \in \mathbb{N}^n \mid |f| \equiv 0 \bmod p\}.$$

Moreover, if $f \in \mathbb{N}^n(p)$ we let

$$\mathrm{col}(f) \stackrel{\mathrm{def}}{=} \sum_{i=1}^{n} R_{r/s}(f_i)$$

be the sum of the residues of the f_i's modulo r/s.

Lemma 5.1 *There is a bijection*

$$\mathbb{N}^n(p) \longleftrightarrow G(r,p,s,n) \times \mathcal{P}_n \times [0, s-1],$$

such that if $f \leftrightarrow (g,\lambda,h)$, then

1. $|f| = \mathrm{fmaj}(g) + r|\lambda| + hnr/s$;
2. $\mathrm{col}(f) = \mathrm{col}(g)$;
3. $\max f = \lambda_1(g) + r\lambda_1 + hr/s$;

Proof: This result is similar to several others appearing in the literature and in particular is a special case of the bijection appearing in [Ca, Theorem 8.3] and so we simply describe how the bijection is defined for the reader's convenience. If $f = (f_1,\dots,f_n) \in \mathbb{N}^n(p)$, then g is the unique element in $G(r,p,s,n)$ having a lift $\tilde{g} \in G(r,p,n)$ satisfying:

- $f_{|g(i)|} \geq f_{|g(i+1)|}$ for all $i \in [n-1]$;
- if $f_{|g(i)|} = f_{|g(i+1)|}$ then $|g(i)| < |g(i+1)|$;
- $c_i(\tilde{g}) \equiv f_{|g(i)|} \bmod r$ for all $i \in [n]$.

Letting μ be the partition obtained by reordering the entries in $(f_1, \dots, f_n)$ in non-increasing order, one can show that $\mu - \lambda(g)$ (component-wise difference) is still a partition whose parts are all congruent to the same multiple of $r/s \bmod r$ (i.e. μ is g-compatible in the notation of [Ca]). The partition $\lambda \in \mathcal{P}_n$ and the integer $h \in [0, s-1]$ are therefore uniquely determined by the requirement $\mu = \lambda(g) + r \cdot \lambda + (hr/s, \dots, hr/s)$.

The inverse map of this bijection is much simpler. Let (g, λ, h) be a triple in $G(r,p,s,n) \times \mathcal{P}_n \times [0, s-1]$, then the corresponding element $f \in \mathbb{N}^n(p)$ is given by $f_i = \lambda_{|g^{-1}(i)|}(g) + r\lambda_{|g^{-1}|(i)} + hr/s$.

All the other statements are straightforward consequences, since $\lambda_i(g) \equiv c_i(\tilde{g}) \bmod r/s$ for any lift $\tilde{g}$ of g and so we also have $\mathrm{col}(g) = \sum R_{r/s}(\lambda_i(g))$. □

We can now state the main results of this section.

Theorem 5.2

$$\left\{ \sum_{k \geq 0} t^k \left([k+1]_{q^{r/s}} + aq[k]_{q^{r/s}} \left[\frac{r}{s} - 1 \right]_{aq} \right)^n \right\}_{q^p} = \frac{\sum_{g \in G(r,p,s,n)} t^{\mathrm{des}(g)} q^{\mathrm{fmaj}(g)} a^{\mathrm{col}(g)}}{(1-t)(1-t^s q^r) \cdots (1-t^s q^{(n-1)r})(1-tq^{nr/s})}$$

Letting $a = 1$ in the previous result we easily obtain $[k+1]_{q^{r/s}} + q[k]_{q^{r/s}}[\frac{r}{s} - 1]_q = \left[\frac{r}{s}k + 1\right]_q$ and hence we obtain the following result.

Corollary 5.3 (Carlitz's identity for $G(r,p,s,n)$ with des)

$$\left\{ \sum_{k \geq 0} t^k \left[\frac{r}{s} k + 1 \right]_q^n \right\}_{q^p} = \frac{\sum_{g \in G(r,p,s,n)} t^{\mathrm{des}(g)} q^{\mathrm{fmaj}(g)}}{(1-t)(1-t^s q^r)(1-t^s q^{2r}) \cdots (1-t^s q^{(n-1)r})(1-tq^{nr/s})}.$$

The special case with $r = 2$, $p = s = 1$ of Corollary 5.3 is the main result of [ChG], and for $p = s = 1$ we obtain [ChM, Theorem 10 (iv)].

A simple modification of the same ideas lead to the generalization of other identities that use flag descents.

Theorem 5.4

$$\sum_{k \geq 0} t^k \left([Q_{r/s}(k) + 1]_{q^{r/s}} + aq[r/s - 1]_{aq} \cdot [Q_{r/s}(k)]_{q^{r/s}} + aq^{mr/s+1)}[R_{r/s}(k)]_{aq} \right)^n$$
$$= \frac{\sum_{g \in G(r,p,s,n)} t^{\mathrm{fdes}(g)} q^{\mathrm{fmaj}(g)} a^{\mathrm{col}(g)}}{(1-t)(1-t^s q^r) \cdots (1-t^s q^{(n-1)r/s}) \cdots (1-tq^{nr/s})}, \tag{8}$$

where $Q_{r/s}(k)$ is the quotient of the division of k by r/s.

By letting $a = 1$ in Equation 8, one obtain a second Carlitz's identity type, with flag-descents.

Theorem 5.5 (Carlitz's identity of $G(r,p,s,n)$ with fdes)

$$\left\{ \sum_{k \geq 0} t^k [k+1]_q^n \right\}_{q^p} = \frac{\sum_{g \in G(r,p,s,n)} t^{\mathrm{fdes}(g)} q^{\mathrm{fmaj}(g)}}{(1-t)(1-t^r q^r)(1-t^r q^{2r}) \cdots (1-t^r q^{(n-1)r})(1-t^{\frac{r}{s}} q^{\frac{nr}{s}})}.$$

For $r = 2$ and $p = s = 1$ we obtain [ABR, Theorem 4.2], for $r = s = 2$ and $p = 1$ [BC, Theorem 4.3], and for $p = 1$ [BB, Theorem 11.2].

6 Multivariate generating functions

In this section we make further use of the bijection [Ca, Theorem 8.3] to compute new multivariate distributions for the groups $G(r,p,s,n)$. We first concentrate on the case of the groups $G(r,n)$ to make the used arguments more clear to the reader. We need to state the particular case of this bijection that is needed for our purposes. The bijection can be reformulated as follows in the special case of 2-partite partitions.

We recall that a 2-*partite partition of length* n (see [GG]) is a $2\times n$ matrix with non negative integer coefficients $f=\begin{pmatrix} f_1^{(1)} & f_2^{(1)} & \dots & f_n^{(1)} \\ f_1^{(2)} & f_2^{(2)} & \dots & f_n^{(2)} \end{pmatrix}$ satisfying the following conditions:

- $f_1^{(1)} \geq f_2^{(1)} \geq \ldots \geq f_n^{(1)}$;
- If $f_i^{(1)} = f_{i+1}^{(1)}$ then $f_i^{(2)} \geq f_{i+1}^{(2)}$.

One may think of a 2-partite partition as a generic multiset of pairs of non negative integers (the columns of f) of cardinality n. If f is a 2-partite partition we denote by $f^{(1)}$ and $f^{(2)}$ the first and the second row of f respectively. We denote by $\mathcal{B}(n)$ the set of 2-partite partitions of length n and we let

$$\begin{aligned} \mathcal{B}(r,n) &\stackrel{\text{def}}{=} \{f\in\mathcal{B}(n) \mid f_i^{(1)}+f_i^{(2)} \equiv 0 \bmod r \text{ for all } i\in[n]\} \qquad \text{and} \\ \mathcal{B}(r,s,1,n) &\stackrel{\text{def}}{=} \{f\in\mathcal{B}(n) \mid \text{ there exists } l\in[s-1] \;:\; f_i^{(1)}+f_i^{(2)} \equiv lr/s \bmod r \text{ for all } i\in[n]\}. \end{aligned}$$

Note that $\mathcal{B}(1,n)=\mathcal{B}(n)$ and that $\mathcal{B}(r,n)=\mathcal{B}(r,1,1,n)$.

Proposition 6.1 *There exists a bijection between $\mathcal{B}(r,s,1,n)$ and 5-tuples (g,λ,μ,h,k), where $g\in G(r,1,s,n)$, $\lambda,\mu\in\mathcal{P}_n$ and $h,k\in[0,s-1]$. In this bijection if $f\leftrightarrow(g,\lambda,\mu,h,k)$ then*

1. $\max(f^{(1)})=\lambda_1(g)+r\lambda_1+hr/s$, *and* $\max(f^{(2)})=\lambda_1(g^{-1})+r\mu_1+kr/s$;
2. $\mathrm{col}(f^{(1)})=\mathrm{col}(g)$, *and* $\mathrm{col}(f^{(2)})=\mathrm{col}(g^{-1})$;
3. $|f^{(1)}| \stackrel{\text{def}}{=} \sum f_j^{(1)} = \mathrm{fmaj}(g)+r|\lambda|+hnr/s$, *and* $|f^{(2)}| \stackrel{\text{def}}{=} \sum f_j^{(2)} = \mathrm{fmaj}(g^{-1})+r|\mu|+knr/s$.

Proof: This is again a particular case of [Ca, Theorem 8.3]. In this case the bijection is defined as follows: if $(g,\lambda,\mu,h,k)\leftrightarrow f$ then

$$f_i^{(1)}=\lambda_i(g)+r\lambda_i+hr/s \text{ and } f_i^{(2)}=\lambda_{|g(i)|}(g^{-1})+r\mu_{|g(i)|}+kr/s,$$

and all the other statements follow immediately. □

We first state the result for the wreath products $G(r,n)$.

Theorem 6.2 *Let $r\in\mathbb{N}$. Then*

$$\sum_{k_1,k_2\geq 0} t_1^{k_1}t_2^{k_2}\Bigg(\prod_{\substack{i\in[0,rk_1],\, j\in[0,rk_2]:\\ i+j\equiv 0 \bmod r}} \frac{1}{1-ua_1^{R_r(i)}a_2^{R_r(j)}q_1^iq_2^j}\Bigg) =$$

$$=\sum_{n\geq 0}u^n\sum_{g\in G(r,n)} t_1^{\mathrm{des}(g)}t_2^{\mathrm{des}(g^{-1})}q_1^{\mathrm{fmaj}(g)},q_2^{\mathrm{fmaj}(g^{-1})}a_1^{\mathrm{col}(g)}a_2^{\mathrm{col}(g^{-1})}\prod_{j=0}^{n}\frac{1}{(1-t_1q_1^{jr})(1-t_2q_2^{jr})}.$$

The corresponding result for the the full class of projective reflection group $G(r,p,s,n)$ is the following one (recall that $p \mid n\frac{r}{s}$).

Theorem 6.3 *Let $r,p,s \in \mathbb{N}$, such that $s,p|r$. Then*

$$\left\{ \sum_{k_1,k_2 \geq 0} t_1^{k_1} t_2^{k_2} \sum_{l=0}^{s-1} \Big(\prod_{\substack{i \in [0,k_1\frac{r}{s}],\, j \in [0,k_2\frac{r}{s}]: \\ i+j \equiv l\frac{r}{s} \bmod r}} \frac{1}{1-ua_1^{R_{r/s}(i)} a_2^{R_{r/s}(j)} q_1^i q_2^j} \Big) \right\}_{q_1^p} =$$

$$= \sum_{n \geq 0} u^n \frac{\sum_{g \in G(r,p,s,n)} t_1^{\mathrm{des}(g)} t_2^{\mathrm{des}(g^{-1})} q_1^{\mathrm{fmaj}(g)} q_2^{\mathrm{fmaj}(g^{-1})} a_1^{\mathrm{col}(g)} a_2^{\mathrm{col}(g^{-1})}}{(1-t_1)(1-t_2)(1-t_1 q_1^{n\frac{r}{s}})(1-t_2 q_2^{n\frac{r}{s}}) \prod_{j=1}^{n-1} (1-t_1^s q_1^{jr})(1-t_2^s q_2^{jr})}.$$

6.1 Hilbert series of diagonal invariant algebras

A specialization of Theorem 6.3 with $a_1 = a_2 = 1$ yields the following result

$$\left\{ \sum_{l=0}^{s-1} \prod_{i+j \equiv lr/s} \frac{1}{1-uq_1^i q_2^j} \right\}_{q_1^p} = \sum_{n \geq 0} u^n \frac{\sum_{g \in G(r,p,s,n)} q_1^{\mathrm{fmaj}(g)} q_2^{\mathrm{fmaj}(g^{-1})}}{(1-q_1^{n\frac{r}{s}})(1-q_2^{n\frac{r}{s}}) \prod_{j=1}^{n-1} (1-q_1^{jr})(1-q_2^{jr})}.$$

There is a nice algebraic interpretation of the previous identity. We let $S_p[X,Y]$ be the subalgebra of the algebra of polynomials in $2n$ variables $x_1,\ldots,x_n,y_1,\ldots,y_n$ generated by 1 and the monomials whose degrees in both the x's and the y's variables are divisible by p. Then we can observe that the factor

$$\frac{1}{(1-q_1^{n\frac{r}{s}})(1-q_2^{n\frac{r}{s}}) \prod_{j=1}^{n-1} (1-q_1^{jr})(1-q_2^{jr})}$$

is the bivariate Hilbert series of the invariant algebra corresponding to the "tensorial action" of the group $G(r,s,p,n)^2 = G(r,s,p,n) \times G(r,s,p,n)$ (note the interchanging of the roles of p and s) on the ring of polynomials $S_p[X,Y]$. Therefore, by [Ca, Corollary 8.6], we can deduce that

$$\frac{\sum_{g \in G(r,p,s,n)} q_1^{\mathrm{fmaj}(g)} q_2^{\mathrm{fmaj}(g^{-1})}}{(1-q_1^{n\frac{r}{s}})(1-q_2^{n\frac{r}{s}}) \prod_{j=1}^{n-1} (1-q_1^{jr})(1-q_2^{jr})} = \mathrm{Hilb}(S_p[X,Y]^{\Delta G(r,s,p,n)}),$$

where $\Delta G(r,s,p,n)$ is the diagonal embedding of $G(r,s,p,n)$ in $G(r,s,p,n)^2$. We can therefore conclude that Theorem 6.3 provides the following interpretation for the generating function of the Hilbert series of the diagonal invariant algebras of the groups $G(r,p,s,n)$.

Corollary 6.4 *Let $G = G(r,p,s,n)$. Then we have*

$$\sum_{n \geq 0} u^n \mathrm{Hilb}(S_s[X,Y]^{\Delta G})(q_1,q_2) = \left\{ \sum_{l=0}^{p-1} \prod_{i+j \equiv lr/p} \frac{1}{1-uq_1^i q_2^j} \right\}_{q_1^s}.$$

7 Orderings

In the literature, when considering statistics based on descents and/or inversions on the groups $G(r,n)$ at least two distinct orderings have been considered on the set of colored integers. In this last section we would like to clarify some of the relationships between two of these orderings concerning the results shown in this paper.

We recall that we have been using the order defined in (1). One can consider also the order $<'$

$$n^{r-1} <' \cdots <' n^1 <' \cdots <' 1^{r-1} <' \cdots <' 1^1 <' 0 <' 1 <' \cdots <' n.$$

The order $<'$ is the "good" one to give a combinatorial interpretation of the length function in $G(r,n)$ in terms of inversions (see e.g. [Ba], [BZ1]), while the order $<$ is often used in the study of some algebraic aspects such as the invariant theory of B_n and $G(r,n)$ (see [AR], [Ca]). Here we give a motivation for these choices. So we denote $\mathrm{Des}'(g) \stackrel{\mathrm{def}}{=} \{i \in [0, n-1] \mid g(i) >' g(i+1)\}$, $\mathrm{des}'(g) \stackrel{\mathrm{def}}{=} |\mathrm{Des}'(g)|$, and $\mathrm{fmaj}'(g) = r \cdot \sum_{i \in \mathrm{Des}'(g)} i + \mathrm{col}(g)$.

If one considers the special case of Theorem 5.2 with $p = s = 1$, and relate it to [BZ, Theorem 5.1] one can deduce that the two polynomials

$$\sum_{g \in G(r,n)} t^{\mathrm{des}(g)} q^{\mathrm{fmaj}(g)} a^{\mathrm{col}(g)} \text{ and } \sum_{g \in G(r,n)} t^{\mathrm{des}'(g)} q^{\mathrm{fmaj}'(g)} a^{\mathrm{col}(g)}$$

are equal. This can also be easily proved bijectively.

Proposition 7.1 *There exists an explicit involution $\phi : G(r,n) \to G(r,n)$ such that*

$$\mathrm{des}(\phi(g)) = \mathrm{des}'(g) \textit{ and } \mathrm{col}(\phi(g)) = \mathrm{col}(g),$$

and in particular $\mathrm{fmaj}(\phi(g)) = \mathrm{fmaj}'(g)$.

Proof: For $g \in G(r,n)$ let $S(g) = \{g(1), \ldots, g(n)\}$. The set $S(g)$ is totally ordered by both $<$ and $<'$. We let $\iota : S(g) \to S(g)$ be the unique involution such that $\iota(x) < \iota(y)$ if and only if $x <' y$, for all $x, y \in S(g)$. We define $\phi(g) = [\iota(g(1)), \ldots, \iota(g(n))]$. It is clear that the map ϕ satisfies the conditions of the statement. $\square$

It follows from Proposition 7.1 that the two statistics fmaj and fmaj' are equidistributed over $G(r,n)$. Nevertheless the two polynomials $\sum_{g \in G(r,n)} \chi(g) q^{\mathrm{fmaj}(g)}$ and $\sum_{g \in G(r,n)} \chi(g) q^{\mathrm{fmaj}'(g)}$, where χ is any linear character of $G(r,n)$, do not coincide in general. So, in order to obtain the results appearing in §4 we must consider the order $<$. We should also mention that the polynomial $\sum_{g \in G(r,n)} \chi(g) q^{\mathrm{fmaj}'(g)}$ does not factor nicely at all in general.

There is a more subtle difference if one considers the following multivariate distribution

$$\sum_{g \in G(r,n)} t_1^{\mathrm{des}(g)} t_2^{\mathrm{des}(g^{-1})} q_1^{\mathrm{fmaj}(g)} q_2^{\mathrm{fmaj}(g^{-1})} a_1^{\mathrm{col}(g)} a_2^{\mathrm{col}(g^{-1})}, \tag{9}$$

and the corresponding with the order $<'$. In fact, by comparing Theorem 6.2 and [BZ1, Theorem 7.1] we deduce that these two polynomials coincide for $r = 1, 2$, and are distinct if $r > 2$. The case $r = 1$ being trivial, we can justify this coincidence for $r = 2$ with a bijective proof.

By the Robinson-Schensted correspondence for B_n (see [StW, Ca]) we have a bijection

$$g \mapsto [(P_0, P_1), (Q_0, Q_1)],$$

where (P_0, P_1) and (Q_0, Q_1) are bitableaux of the same shape. Given a tableau P let

$$\mathrm{Des}(P) = \{i \mid \text{ both } i \text{ and } i+1 \text{ belong to } P \text{ with } i \text{ strictly above } i+1\}.$$

Let $\mathrm{Neg}(g) = \{i \in [n] \mid c_i(g) = 1\}$. Let $\varphi : B_n \to B_n$ be the bijection defined by the requirement that if

$$\begin{array}{rcl} g & \mapsto & [(P_0, P_1), (Q_0, Q_1)], \quad \text{then} \\ \varphi(g) & \mapsto & [(P_0, P_1'), (Q_0, Q_1')], \end{array}$$

where T' denotes the transposed tableau. The next result gives a bijective proof of the equality between (9) and its $<'$-analogue.

Proposition 7.2 *The bijection φ satisfies the following properties:*

1. $\mathrm{Neg}(g) = \mathrm{Neg}(\varphi(g))$.
2. $\mathrm{Des}(g) = \mathrm{Des}'(\varphi(g))$.
3. $\mathrm{Des}(g^{-1}) = \mathrm{Des}'(\varphi(g)^{-1})$.

Example 7.3 *If $g = [5, -2, -1, -4, 6, -3, -7] \in B_7$, then*

$$g \mapsto \left[\left(\begin{array}{cc} 5 & 6 \end{array} , \begin{array}{ccc} 1 & 3 & 7 \\ 2 & 4 & \end{array} \right), \left(\begin{array}{cc} 1 & 5 \end{array} , \begin{array}{ccc} 2 & 4 & 7 \\ 3 & 6 & \end{array} \right) \right].$$

The element $\varphi(g)$ is the defined by

$$\varphi(g) \mapsto \left[\left(\begin{array}{cc} 5 & 6 \end{array} , \begin{array}{cc} 1 & 2 \\ 3 & 4 \\ 7 & \end{array} \right), \left(\begin{array}{cc} 1 & 5 \end{array} , \begin{array}{cc} 2 & 3 \\ 4 & 6 \\ 7 & \end{array} \right) \right].$$

One then can check that $\varphi(g) = [5, -3, -7, -1, 6, -4, -2]$. Hence $\mathrm{Neg}(g) = \mathrm{Neg}(\varphi(g)) = \{2, 3, 4, 6, 7\}$, $\mathrm{Des}(g) = \mathrm{Des}'(\varphi(g)) = \{1, 2, 5\}$, *and* $\mathrm{Des}(g^{-1}) = \mathrm{Des}'(\varphi(g)^{-1}) = \{0, 1, 3, 6\}$.

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Finite Eulerian posets which are binomial or Sheffer

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Abstract. In this paper we study finite Eulerian posets which are binomial or Sheffer. These important classes of posets are related to the theory of generating functions and to geometry. The results of this paper are organized as follows:

- We completely determine the structure of Eulerian binomial posets and, as a conclusion, we are able to classify factorial functions of Eulerian binomial posets;
- We give an almost complete classification of factorial functions of Eulerian Sheffer posets by dividing the original question into several cases;
- In most cases above, we completely determine the structure of Eulerian Sheffer posets, a result stronger than just classifying factorial functions of these Eulerian Sheffer posets.

We also study Eulerian triangular posets. This paper answers questions posed by R. Ehrenborg and M. Readdy. This research is also motivated by the work of R. Stanley about recognizing the *boolean lattice* by looking at smaller intervals.

Résumé. Nous étudions les ensembles partiellement ordonnés finis (EPO) qui sont soit binomiaux soit de type Sheffer (deux notions reliées aux séries génératrices et à la géométrie). Nos résultats sont les suivants:

- nous déterminons la structure des EPO Euleriens et binomiaux; nous classifions ainsi les fonctions factorielles de tous ces EPO;
- nous donnons une classification presque complète des fonctions factorielles des EPO Euleriens de type Sheffer;
- dans la plupart de ces cas, nous déterminons complètement la structure des EPO Euleriens et Sheffer, ce qui est plus fort que classifier leurs fonctions factorielles.

Nous étudions aussi les EPO Euleriens triangulaires. Cet article répond à des questions de R. Ehrenborg and M. Readdy. Il est aussi motivé par le travail de R. Stanley sur la reconnaissance du treillis booléen via l'étude des petits intervalles.

Keywords: Eulerian poset, binomial poset, Sheffer poset

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1 Introduction

There are many theories which unify various aspects of enumerative combinatorics and generating functions. One such successful theory introduced by Doubilet, Rota and Stanley [3] is that of binomial posets. Classically, binomial posets are infinite posets with the property that every two intervals of the same length have the same number of maximal chains. Doubilet, Rota and Stanley show this chain regularity condition gives rise to universal families of generating functions. Ehrenborg and Readdy [5] and Reiner [9] independently generalized the notion of binomial posets to a larger class of posets called Sheffer posets or upper binomial posets.

Ehrenborg and Readdy [4] gave a complete classification of the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Recall that infinite posets are those posets which contain an infinite chain. They posed the open question of characterizing the finite case. This paper deals with these questions.

A *Sheffer poset* is a graded poset such that the number of maximal chains $D(n)$ in an n-interval $[\hat{0}, y]$ depends only on $\rho(y) = n$, the rank of the element y, and the number $B(n)$ of maximal chains in an n-interval $[x, y]$, where $x \neq \hat{0}$, depends only on $\rho(x, y) = \rho(y) - \rho(x)$. The two functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function*, respectively. *Binomial posets* are a special class of Sheffer posets. A binomial poset is a graded poset such that the number of maximal chains $B(n)$ in an n-interval $[x, y]$ depends only on $\rho(x, y) = \rho(y) - \rho(x)$.

Binomial posets were previously considered in [1], [3], [8], [11] and [13]. Ehrenborg and Readdy [5] used Sheffer posets and a generalization of R-labeling to study augmented r-signed permutations. Reiner [9] used them to derive generating functions counting signed permutations by two statistics.

A graded poset P is *Eulerian* if every non-singleton interval of P satisfies the *Euler-Poincaré* relation. (See Definition 2.1.) Eulerian posets form an important class of posets as there are many geometric examples such as the face lattices of convex polytopes, and more generally, the face posets of regular CW-spheres.

As we mentioned above, Ehrenborg and Readdy in [4] classify the factorial functions of infinite Eulerian binomial posets and infinite Eulerian Sheffer posets. Since we are concerned here with finite posets, we drop the requirement that binomial, Sheffer and triangular posets have an infinite chain. This paper studies the following natural questions, as suggested by Ehrenborg and Readdy in [4].

1. Which Eulerian posets are binomial?
2. Which Eulerian posets are Sheffer?

Stanley has proved that one can recognize *boolean lattices* by looking at smaller intervals (see [7], Lemma 8). Farley and Schmidt answer a similar question for *distributive lattices* in [6]. The project of studying Eulerian binomial posets and Eulerian Sheffer posets is also motivated by their work. In many cases we use the factorial function of smaller intervals to characterize the whole poset.

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2 Background and Definition

We encourage readers to consult Chapter 3 of [12] for basic poset terminology. All the posets which are considered in this paper are finite.

We begin by recalling that a graded interval satisfies the *Euler-Poincaré relation* if it has the same number of elements of even rank as of odd rank.

Definition 2.1. 1. *A graded poset is* Eulerian *if every non-singleton interval satisfies the Euler-Poincaré relation. Equivalently, a poset P is Eulerian if its Möbius function satisfies $\mu(x, y) = (-1)^{\rho(x,y)}$ for all $x \leq y$ in P, where ρ denotes the rank function of P.*

2. *Consider a graded poset P with rank function ρ. If $\rho(x, y) = n$, then we call $[x, y]$ an n-*interval.

Definition 2.2. *A finite graded poset P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ is called a* (finite) binomial poset *if it satisfies the following condition:*

1. *For all $n \in \mathbb{N}$, $n \leq$ rank(P), any two n-intervals have the same number $B(n)$ of maximal chains. We call $B(n)$ the* factorial function *or* binomial factorial function *of the poset P.*

Definition 2.3. *A finite graded poset P with a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$ is called a* (finite) Sheffer poset *if it satisfies the following two conditions:*

1. *Any pair of n-intervals $[\hat{0}, y]$ and $[\hat{0}, v]$ have the same number $D(n)$ of maximal chains.*

2. *Any pair of n-intervals $[x, y]$ and $[u, v]$ such that $x \neq \hat{0}$ and $u \neq \hat{0}$ have the same number $B(n)$ of maximal chains.*

Let us consider a Sheffer poset P. An interval $[\hat{0}, y]$, where $y \neq \hat{0}$, is called a *Sheffer interval* whereas an interval $[x, y]$ with $x \neq \hat{0}$ is called a *binomial interval*. The functions $B(n)$ and $D(n)$ are called the *binomial factorial function* and *Sheffer factorial function* of P, respectively. Next we define $A(n)$ and $C(n)$ to be the number of coatoms in a binomial interval of length n, respectively, a Sheffer interval of length n. The functions $A(n)$ and $C(n)$ are called the *atom function* and *coatom function* of P, respectively. The number of elements of rank k in a Sheffer interval of rank n is

$$\frac{D(n)}{D(k)B(n-k)}. \tag{1}$$

Moreover, for a binomial interval $[x, y]$ of rank n in a Sheffer poset, the number of elements of rank k is equal to

$$\frac{B(n)}{B(k)B(n-k)}. \tag{2}$$

The *dual suspension* of a poset P is defined in [4] as follows.

Definition 2.4. *Let P be a poset with $\hat{0}$. We define the* dual suspension *of P, denoted $\Sigma^*(P)$, to be the poset P with two new elements a_1 and a_2. $\Sigma^*(P)$ has the following order relation: $\hat{0} <_{\Sigma^*(P)} a_i <_{\Sigma^*(P)} y$, for all $y > \hat{0}$ in P and $i = 1, 2$. That is, the elements a_1 and a_2 are inserted between $\hat{0}$ and atoms of P. Clearly if P is Eulerian then so is $\Sigma^*(P)$. Moreover, if P is a binomial poset then $\Sigma^*(P)$ is a Sheffer poset with the factorial function $D_{\Sigma^*(P)}(n) = 2B(n-1)$, for $n \geq 2$.*

Definition 2.5. *Let P be a poset with $\hat{1}$. We define the* suspension *of P, denoted by $\Sigma(P)$, to be the poset P with two new elements a_1 and a_2 adjoined with the additional order relations that $y <_{\Sigma(P)} a_i <_{\Sigma(P)} \hat{1}$, for all $y < \hat{1}$ in P and $i = 1, 2$.*

The *dual* of the poset P, denoted P^*, is defined as follows: P^* has the same set of elements as P and the following order relation: $x <_{P^*} y$ if and only if $y <_P x$.

Definition 2.6. *The* boolean lattice *B_n of rank n is the poset of subsets of $[n] = \{1, \ldots, n\}$ ordered by inclusion.*

Definition 2.7. *The* butterfly poset *T_n of rank n consists of the elements of $\{\hat{0}\} \cup (D_{n-1} \times \{1, 2\}) \cup \{\hat{1}\}$, where $D_{n-1} \times \{1, 2\}$ is the direct product of the chain of length $n-1$, denoted by D_{n-1}, and the anti-chain of rank 2, with the order relation $(k, i) \prec (k+1, j)$ for all $i, j \in \{1, 2\}$. Also $\hat{0}$ and $\hat{1}$ are the unique minimal and maximal elements of this poset, respectively. Clearly, $T_n \cong \Sigma^*(T_{n-1})$.*

3 Finite Eulerian binomial posets

In this section, we classify the structure of finite Eulerian binomial posets.

First we provide some examples of finite binomial posets. See [4] for infinite versions of Examples 3.1 and 3.2.

Example 3.1. *The boolean lattice B_n of rank n is an Eulerian binomial poset with factorial function $B(k) = k!$ and atom function $A(k) = k$, $k \leq n$. Every interval of length k of this poset is isomorphic to B_k.*

Example 3.2. *The butterfly poset T_n of rank n is an Eulerian binomial poset with factorial function $B(k) = 2^{k-1}$ for $1 \leq k \leq n$ and atom function $A(k) = 2$, for $2 \leq k \leq n$, and $A(1) = 1$.*

It is not hard to see that in any n-interval of an Eulerian binomial poset P with factorial function $B(k)$ for $1 \leq k \leq n$, the Euler-Poincaré relation is stated as follows:

$$\sum_{k=0}^{n} (-1)^k \cdot \frac{B(n)}{B(k)B(n-k)} = 0. \tag{3}$$

The following is [4, Lemma 2.6].

Lemma 3.3. *Let P be a graded poset of odd rank such that every proper interval of P is Eulerian. Then P is an Eulerian poset.*

Lemma 3.4. *Let P be an Eulerian binomial poset of rank 3. Then the poset P and its factorial function $B(n)$ satisfy the following conditions:*

(i) *$B(2) = 2$ and $B(3) = 2q$, where q is a positive integer such that $q \geq 2$.*

(ii) *There is a list of integers $q_1, \ldots, q_r$, $q_i \geq 2$, such that $P \cong \boxplus_{i=1,\ldots,r} P_{q_i}$, where P_{q_i} is the face lattice of the q_i-gon.*

This result is [4, Example 2.5].

R. Ehrenborg and M. Readdy proved the following two propositions. See [4, Lemma 2.17 and Prop. 2.15].

Proposition 3.5. *Let P be a binomial poset of rank n with factorial function $B(k) = 2^{k-1}$ for $1 \le k \le n$. Then the poset P is isomorphic to the butterfly poset T_n.*

Proposition 3.6. *Let P be a binomial poset of rank n with factorial function $B(k) = k!$ for $1 \le k \le n$. Then the poset P is isomorphic to the boolean lattice B_n of rank n.*

The following is [4, Lemma 2.12].

Lemma 3.7. *Let $P^{'}$ and P be two Eulerian binomial posets of rank $2m+2$, $m \ge 2$, having atom functions $A^{'}(n)$ and $A(n)$, respectively, which agree for $n \le 2m$. Then the following equality holds:*

$$\frac{1}{A(2m+1)}\left(1-\frac{1}{A(2m+2)}\right) = \frac{1}{A'(2m+1)}\left(1-\frac{1}{A'(2m+2)}\right). \tag{4}$$

Lemma 3.8. *Every Eulerian binomial poset P of rank 4 is isomorphic to either T_4 or B_4.*

In the following theorem we obtain the structure of Eulerian binomial posets of even rank.

Theorem 3.9. *Every Eulerian binomial poset of even rank $n = 2m \ge 4$ is isomorphic to either T_n or B_n (the butterfly poset of rank n or boolean lattice of rank n).*

Theorem 3.10. *Let P be an Eulerian binomial poset of odd rank $n = 2m + 1 \ge 5$. Then the poset P satisfies one of the following conditions:*

(i) *There is a positive integer k such that P is the k-summation of the boolean lattice of rank n. In other words, $P \cong \boxplus^k(B_n)$.*

(ii) *There is a positive integer k such that P is the k-summation of the butterfly poset of rank n. In other words, $P \cong \boxplus^k(T_n)$.*

Proof. We prove the theorem for two different cases $B(3) = 4$ and $B(3) = 6$. Lemma 3.8 implies that every interval of length 4 is isomorphic either to B_4 or T_4. Thus the factorial function $B(3)$ can only take the values 4 or 6 and therefore we are in one of these two cases.

1. $B(3) = 6$. In this case we claim that there is a positive integer k such that $P \cong \boxplus^k(B_n)$. In order to show that $P \cong \boxplus^k(B_n)$, we make the following construction. We remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. Consider one of the obtained connected components and add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to it. Denote the resulting poset by Q. We wish to show that $Q \cong B_n$. This implies that $P \cong \boxplus^k(B_n)$. It is not hard to see that Q is an Eulerian binomial poset. The posets P and Q have the same factorial functions and atom functions up to rank $2m$. Hence $B_Q(k) = B_P(k)$ and $A_Q(k) = A_P(k)$, for $1 \le k \le 2m$. Therefore, Eq. (2) implies that the number of atoms and coatoms are the same in the poset Q. Denote this number by t. Let $x_1, \dots, x_t$ and $a_1, \dots, a_t$ be an ordering of the atoms and coatoms of Q, respectively. Also, let $c_1, \dots, c_l$ be the set of elements of rank $2m - 1$ in Q. We show that $t = 2m + 1$, and this implies that $Q \cong B_{2m+1}$. For each element y of rank at least 2 in Q, let $S(y)$ be the set of atoms of Q that are below y. Set $A_i := S(a_i)$ for each element a_i of rank $2m$, $1 \le i \le t$, and also set $C_i := S(c_i)$ for each element c_i of rank $2m - 1$, $1 \le i \le l$. In order to show that $Q \cong B_n$, we prove the following.

(1) We show that $|A_i \cap A_j| = 2m-1$ for $i \neq j$.

(2) We use part (1) to show that $t = 2m+1$.

(1) We first show that $|A_i \cap A_j| = 2m-1$ for $i \neq j$. By considering the factorial functions, Theorem 3.9 implies that the intervals $[\hat{0}, a_i]$ and $[x_j, \hat{1}]$ have the same factorial functions as B_{2m} and so they are isomorphic to B_{2m} for

$1 \leq i \leq t$ and $1 \leq j \leq t$. We conclude that any interval $[\hat{0}, c_k]$ of rank $2m-1$ is isomorphic to B_{2m-1}. As a consequence, we have $|A_i| = |S(a_i)| = 2m$, $1 \leq i \leq t$ and also $|C_k| = |S(c_k)| = 2m-1$, $1 \leq k \leq l$.

If there exist i and j such that $A_i \cap A_j \neq \emptyset$, where $1 \leq i, j \leq t$, we claim that $2m-1 \leq |A_i \cap A_j| \leq 2m$. Consider an atom $x_k \in A_i \cap A_j$, $1 \leq k \leq t$. Theorem 3.9 implies that $[x_k, \hat{1}] \cong B_{2m}$. Thus, by considering properties of boolean lattices, there is an element c_h of rank $2m-2$ in this interval which is covered by a_i and a_j, $1 \leq h \leq l$. Notice that c_h is an element of rank $2m-1$ in Q. Therefore, $|C_h| = 2m-1 \leq |A_i \cap A_j| \leq |A_i| = |S(a_i)| = 2m$.

We claim that for all distinct pairs i and j, $1 \leq i, j \leq t$, we have $A_i \cap A_j \neq \emptyset$. In order to show this claim, associate the graph G_Q to the poset Q as follows: $A_1, \ldots, A_t$ are vertices of this graph, and we connect vertices A_i and A_j if and only if $A_i \cap A_j \neq \emptyset$.

We will show that G_Q is a complete graph and so $|A_i \cap A_j| \neq 0$ for all $i \neq j$. Since $Q - \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. We show that if $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are different edges of G_Q, $\{A_i, A_k\}$ is also an edge of G_Q. Since $\{A_i, A_j\}$ and $\{A_j, A_k\}$ are edges of G_Q, we have $|A_i \cap A_j| \geq 2m-1$ as well as $|A_j \cap A_k| \geq 2m-1$. On the other hand, since $|A_i| = |A_j| = |A_k| = 2m$, we conclude that $A_i \cap A_k \neq \emptyset$. Therefore $\{A_i, A_k\}$ is also an edge of G_Q. As a consequence, the connected graph G_Q is a complete graph. Thus $A_i \cap A_j \neq \emptyset$ and also $2m-1 \leq |A_i \cap A_j| \leq 2m$ for $1 \leq i, j \leq t$ and $i \neq j$.

Now, we show that $|A_i \cap A_j| = 2m-1$ for all $i \neq j$. We proceed by contradiction. Suppose this claim does not hold. Then there are different i and j such that $|A_i \cap A_j| = 2m$. We claim that in the case $|A_i \cap A_j| = 2m$, there are two elements of rank $2m-1$ in Q such that they both are covered by coatoms a_i and a_j. To show this claim, consider an atom $x_f \in A_i \cap A_j$, so we have $[x_f, \hat{1}] \cong B_{2m}$. Hence, there is a unique element c_h of rank $2m-2$ in this interval which is covered by both a_i and a_j. By induction on m, Lemma 3.4, and the property that $|C_h| \leq |A_i \cap A_j| = 2m$, we conclude that $[\hat{0}, c_h]$ is isomorphic to B_{2m-1} and so $|C_h| = 2m-1$. Therefore there is an atom $x_d \in A_i \cap A_j \setminus C_h$. Since the interval $[x_d, \hat{1}]$ is isomorphic to B_{2m}, there is an element $c_k \neq c_h$ of rank $2m-1$ which is covered by coatoms a_i and a_j .

Since $|C_h| = |S(c_h)| = |C_k| = |S(c_k)| = 2m-1$ and C_k and C_h are both subsets of $A_i \cap A_j$, we conclude that there should be an atom $x_s \in C_k \cap C_h$. Therefore the interval $[x_s, \hat{1}]$ has two elements c_k and c_h of rank $2m-2$ such that they both are covered by two elements a_i and a_j of rank $2m-1$ in the interval $[x_s, \hat{1}]$. We know $[x_s, \hat{1}] \cong B_{2m}$ and there are no two elements of rank $2m-2$ covered by two elements of rank $2m-1$ in B_{2m}. This contradicts our assumption, and so $|A_i \cap A_j| = 2m-1$ for pairs i and j of distinct elements.

In summary, we have:

(a) $|A_i| = 2m$ for $1 \leq i \leq t$,

(b) $|A_i \cap A_j| = 2m - 1$ for all $1 \leq i < j \leq t$,

(c) $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$.

As a consequence, we have $t > 2m$.

(2) Now, we show that $t = 2m + 1$. We are going to show that $t = 2m + 1$. Without loss of generality, consider the three different sets $A_1 = S(a_1), A_2 = S(a_2)$ and $A_3 = S(a_3)$ associated with the three coatoms a_1, a_2 and a_3. We know that $|A_1| = |A_2| = |A_3| = 2m$ and $|A_1 \cap A_2| = |A_2 \cap A_3| = |A_1 \cap A_3| = 2m - 1$. Without loss of generality, let us assume that $A_1 = \{x_1, x_2, \ldots, x_{2m-1}, y_1\}$ and $A_2 = \{x_1, x_2, \ldots, x_{2m-1}, y_2\}$ where $y_i \neq x_1, \ldots, x_{2m-1}$ for $i = 1, 2$. We have the following two different cases:

(a) A_3 does not contain y_1 and y_2.

(b) A_3 contains at least one of y_1 and y_2.

First we study the case, $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_3\}$ where $y_3 \notin \{y_1, y_2, x_1, \ldots, x_{2m-1}\}$. Considering the $t - 3$ other coatoms a_k, $4 \leq k \leq t$, there are different atoms y_k, $4 \leq k \leq t$, such that $y_k \notin \{y_1, y_2, y_3, x_1, \ldots, x_{2m-1}\}$ and $A_k = S(a_k) = \{x_1, x_2, \ldots, x_{2m-1}, y_k\}$. This implies that the number of atoms is $|\bigcup_{i=1}^{t} A_i| = t + 2m - 1$, which is a contradiction. So it must be the case that A_3 contains one of y_1 or y_2. In this case $|A_2 \cap A_3| = |A_1 \cap A_3| = 2m - 1$ implies that $A_3 = \{x_1, x_2, \ldots, x_{2m-1}, y_1, y_2\} \setminus \{x_j\} \subset A_1 \cup A_2$ for some x_j. Since A_3 was chosen arbitrarily, it follows that for each A_k we have $A_k \subset A_1 \cup A_2$.
Therefore,

$$\bigcup_{i=1}^{t} A_k = \{x_1, \ldots, x_{2m-1}, y_1, y_2\}. \tag{5}$$

Thus the number of coatoms in the poset Q is $t = 2m + 1$.
By Theorem 3.9, $B_Q(k) = k!$ for $1 \leq k \leq 2m$, therefore $B_Q(2m + 1) = (2m + 1)!$. By Proposition 3.6, Q is isomorphic to B_{2m+1} and so P is a union of copies of B_{2m+1} with their minimal elements and maximal elements identified. In other words, $P \cong \boxplus^k (B_{2m+1})$. It can be seen that P is binomial and Eulerian and the proof follows.

(ii) $B(3) = 4$. With the same argument as part (i), we remove $\hat{1}$ and $\hat{0}$ from P. The remaining poset is a disjoint union of connected components. We add a minimal element $\hat{0}$ and a maximal element $\hat{1}$ to each of these connected components. We show that the obtained posets are isomorphic to T_n. This implies that $P \cong \boxplus^k (T_n)$.

We construct the binomial poset Q by adding $\hat{1}$ and $\hat{0}$ to one of the connected components of $P - \{\hat{0}, \hat{1}\}$. We claim that Q is isomorphic to T_{2m+1}. Similar to part (i), let $a_1, \ldots, a_t$ and $x_1, \ldots, x_t$ denote coatoms and atoms of Q. We show that $t = 2$ which implies $Q \cong T_{2m+1}$.

Set $A_i = S(a_i)$. By Theorem 3.9, we have $|A_i| = 2$. It is easy to see that $\bigcup_{i=1}^{t} A_i = \{x_1, \ldots, x_t\}$. Define G_Q to be the graph with vertices $x_1, \ldots, x_t$ and edges $A_1, \ldots, A_t$. Since $Q \setminus \{\hat{0}, \hat{1}\}$ is connected, G_Q is also a connected graph. Since $[x_i, \hat{1}] \cong T_{2m}$, the degree of each vertex of G_Q is 2 and G_Q is the cycle of length t. Therefore if $t > 2$, we have $|A_i \cap A_j| = 1$ or 0, $1 \leq i < j \leq t$.

We claim that $t = 2$. Suppose this claim does not hold and $t > 2$. Consider an element c of rank 3 in Q. Lemma 3.4 and Theorem 3.9 imply that both intervals $[\hat{0}, c]$ and $[c, \hat{1}]$ are isomorphic to butterfly posets. Hence there are two coatoms above c, say a_k and a_l, and similarly there are two atoms below c, say x_h and x_s. Therefore, we have $A_k = A_l = \{x_h, x_s\}$. This is not possible when $t > 2$. As a consequence, $t = 2$ and all the A_i's have two elements and $|\bigcup_1^t A_i| = |\{x_1, \ldots, x_t\}| = 2 = t$.

Similar to part (i), $B_Q(k) = 2^{k-1}$ for $1 \leq k \leq 2m+1$. By Proposition 3.5, we conclude that Q is isomorphic to T_{2m+1}. Therefore, there is an integer $k > 0$ such that $P \cong \boxplus^k(T_n)$.

□

4 Finite Eulerian Sheffer Posets

In this section, we give an almost complete classification of the factorial functions and the structure of Eulerian Sheffer posets. We study Eulerian Sheffer posets of ranks $n = 3$ and 4 in Lemmas 3.4 and 4.2. By these two lemmas, we reduce the set of possible values of $B(3)$ to 4 or 6. In Section 4.1, Lemma 4.3 and Theorems 4.4, 4.9, 4.10 and 4.11 deal with Eulerian Sheffer posets with $B(3) = 6$. Finally in Section 4.2, Theorem 4.12 deals with Eulerian Sheffer posets with $B(3) = 4$.

It is clear that every binomial poset is also a Sheffer poset. Here is an other example of Sheffer posets, some of which appear in [4] and [9].

Example 4.1. *Let T be the poset with the elements $\hat{0}_1, \hat{0}_2, \hat{1}$ and the cover relations $\hat{0}_1 < \hat{1}$ and $\hat{0}_2 < \hat{1}$.*

Let T^n be the Cartesian product of n copies of the poset T. The poset $C_n = T^n \cup \{\hat{0}\}$ denotes the face lattice of an n-dimensional cube, also known as the cubical lattice*. The cubical lattice is a Sheffer poset with $B(k) = k!$ for $1 \leq k \leq n$ and $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq n+1$.*

It is not hard to see that Lemma 3.4 also characterize the structure of Eulerian Sheffer posets of rank 3. Lemma 4.2 deals with Eulerian Sheffer posets of rank 4.

Lemma 4.2. *Let poset P be an Eulerian Sheffer poset of rank 4. Then one of the following conditions hold.*

1. *$B(3) = 2b$, $D(3) = 4$, $D(4) = 4b$, where $b \geq 2$.*
2. *$B(3) = 8$, $D(3) = 3!$, $D(4) = 2^3 \cdot 3!$.*
3. *$B(3) = 10$, $D(3) = 3!$, $D(4) = 5!$.*
4. *$B(3) = 4$, $D(3) = 3!$, $D(4) = 2 \cdot 3!$.*
5. *$B(3) = 3!$, $D(3) = 3!$, $D(4) = 4!$.*
6. *$B(3) = 3!$, $D(3) = 4$, $D(4) = 2 \cdot 3!$.*
7. *$B(3) = 3!$, $D(3) = 10$, $D(4) = 5!$.*
8. *$B(3) = 3!$, $D(3) = 8$, $D(4) = 2^3 \cdot 3!$.*
9. *$B(3) = 4$, $D(3) = 2b$, $D(4) = 4b$ where $b \geq 2$.*

4.1 Characterization of the factorial functions and structure of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$.

In this subsection. we mainly consider Eulerian Sheffer posets with $B(3) = 3!$. As a consequence of Lemma 4.2, we know that Eulerian Sheffer posets of rank $n \geq 4$ with $B(3) = 3!$ have the Sheffer factorial functions $D(3) = 4, 6, 8$ and 10. Lemma 4.3 shows that for any such poset of rank $n \geq 6$, the Sheffer factorial function $D(3)$ can only take the values $4, 6$ or 8.

In Subsections 4.1.1, 4.1.2 and 4.1.3, we consider posets with $B(3) = 6$ and different cases $D(3) = 4, 6$ and 8, respectively. The question of studying the finite Eulerian Sheffer posets of rank 5 with $B(3) = 6$ and $D(3) = 10$ remains open. There is such a poset, namely the face lattice of the 120-cell with Schläfli symbol $\{5, 3, 3\}$.

Lemma 4.3. *Let P be an Eulerian Sheffer poset of rank $n \geq 6$ with $B(3) = 3!$. Then $D(3)$ can take only the values $4, 6, 8$.*

4.1.1 Characterization of the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$.

In this subsection, we study the factorial functions of Eulerian Sheffer posets of rank $n \geq 5$ for which $B(3) = 3!$ and $D(3) = 8$. Theorem 4.4 characterizes the factorial functions of such posets of even rank. However, the question of characterizing the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m + 1 \geq 5$ with $B(3) = 3!$ and $D(3) = 8$ remains open.

Theorem 4.4. *Let P be an Eulerian Sheffer poset of even rank $n = 2m + 2 \geq 4$ with $B(3) = 3!$ and $D(3) = 8$. Then P has the same factorial functions as C_n, the cubical lattice of rank n, that is, $D(k) = 2^{k-1}(k-1)!$, $1 \leq k \leq n$ and $B(k) = k!$, $1 \leq k \leq n - 1$.*

In order to prove Theorem 4.4, we establish the following two lemmas.

Lemma 4.5. *Let Q be an Eulerian Sheffer poset of odd rank $2m + 1$, $m \geq 2$, with $B(3) = 3!$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$ for $2 \leq n \leq 2m$ and $C(2m+1) \neq 4m + 1$.*

Lemma 4.5 implies the following.

Corollary 4.6. *Let P be an Eulerian Sheffer poset of rank $2m + 2$, $m \geq 2$, with $B(k) = k!$, for $1 \leq k \leq 2m$. Then the coatom function of P must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) \neq 4m + 1$ and $C(2m+2) \neq 4(2m+1)$.*

Lemma 4.7. *Let Q be an Eulerian Sheffer poset of rank $2m+2$, $m \geq 2$, with $B(k) = k!$ for $1 \leq k \leq 2m$. Then the coatom function of Q must satisfy at least one of the following inequalities: $C(n) \neq 2(n-1)$, $2 \leq n \leq 2m$, $C(2m+1) \neq 4m - 1$ and $C(2m+2) \neq \frac{4}{3}(2m+1)$.*

The following lemma can be obtained by applying the proof of Lemma 4.8 in [4].

Lemma 4.8. *Let P and $P^{'}$ be two Eulerian Sheffer posets of rank $2m + 2$, $m \geq 2$, such that their binomial factorial functions and coatom functions agree up to rank $n \leq 2m$. That is, $B(n) = B^{'}(n)$ and $C(n) = C^{'}(n)$, where $m \geq 2$. Then the following equation holds:*

$$\frac{1}{C(2m+1)}\left(1 - \frac{1}{C(2m+2)}\right) = \frac{1}{C'(2m+1)}\left(1 - \frac{1}{C'(2m+2)}\right). \tag{6}$$

of Theorem 4.4. In order to prove the theorem, we inductively show that the Eulerian Sheffer poset P and C_{2m+2}, the cubical latice of rank $2m+2$, have the same coatom functions.

Let $C(k)$ and $C^{'}(k) = 2(k-1)$ respectively be the coatom functions of the Eulerian Sheffer poset P and C_{2m+2} for $2 \leq k \leq 2m+2$. We only need to show that $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m+2$. We prove this claim by induction on m. By Lemma 4.2, an Eulerian Sheffer poset of even rank 4 with $B(3) = 3!$ and $D(3) = 8$ has the same factorial function as C_4. Therefore, $C(4) = C^{'}(4) = 6$ and the claim holds for $m = 1$. Suppose $m \geq 2$. By the induction hypothesis $C(k) = C^{'}(k) = 2(k-1)$ for $2 \leq k \leq 2m$. Set $F = C(2m+1)$ and $E = C(2m+2)$. Theorem 3.10 implies that $B(k) = k!$ for $1 \leq k \leq 2m$ and there is a positive integer α such that $B(2m+1) = \alpha(2m+1)!$. We know that $D(k) = 2^{k-1}(k-1)!$ for $1 \leq k \leq 2m$, so $D(2m+1) = F2^{2m-1}(2m-1)!$ and $D(2m+2) = EF2^{2m-1}(2m-1)!$. Since P is an Eulerian Sheffer poset, the Euler-Poincaré relation implies that

$$1 + \sum_{k=1}^{2m+2} \frac{(-1)^k D(2m+2)}{D(k)B(2m+2-k)} = 0. \tag{7}$$

By substituting the values of the factorial functions, we have

$$2 - E + \frac{EF}{2}\left[\frac{1}{2m} - \frac{1}{2m(2m+1)} + \frac{2^{2m}}{2m(2m+1)} - \frac{2^{2m}}{2\alpha m(2m+1)}\right] = 0. \tag{8}$$

Thus,

$$E\left(1 - F\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right)\right) = 2. \tag{9}$$

In case $\alpha \geq 2$, it is easy to verify that

$$\left(\frac{2\alpha m + (\alpha-1)2^{2m}}{4\alpha m(2m+1)}\right) > \frac{1}{2m}. \tag{10}$$

Since $F \geq A(2m) \geq 2m$, the left-hand side of Eq. (9) becomes negative in this case. Therefore, $\alpha = 1$ and the posets P and C_{2m+2} have the same binomial factorial functions. Since $2m+1 = A(2m+1) \leq C(2m+2) < \infty$, Lemma 4.8 implies that $4m-1 \leq C(2m+1) = F \leq 4m+1$. Since $\alpha = 1$, Eq. (9) implies that $2 - E + \frac{EF}{4m+2} = 0$. Thus E and F must satisfy one of the following cases:

(1) $F = 4m-1$ and $E = \frac{4}{3}(2m+1)$.

(2) $F = 4m$ and $E = 4m+2$.

(3) $F = 4m+1$ and $E = 4(2m+1)$.

As we have discussed in Corollary 4.6 and Lemma 4.7, the cases (1) and (3) are not possible. Case (2) occurs in the cubical lattice of rank $2m+2$, C_{2m+2}. Thus, the poset P has the same factorial functions as C_{2m+2}, as desired. $\square$

Classification of the factorial functions of Eulerian Sheffer posets of odd rank $n = 2m+1 \geq 5$ with $B(3) = 6$ and $D(3) = 8$ remains open. Let α be a positive integer and set $Q_\alpha = \boxplus^\alpha(C_{2m+1})$. It can be seen that Q_α is an Eulerian Sheffer poset and it has the following factorial functions: $D(k) = 2^{k-1}(k-1)!$

for $1 \le k \le n-1$, $D(n) = \alpha \cdot 2^{n-1}(n-1)!$ and $B(k) = k!$ for $1 \le k \le n-1$. We ask the following question:

Question: Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \ge 5$ with $B(3) = 6, D(3) = 8$. Is there a positive integer α such that the poset P has the same factorial functions as poset $Q_\alpha = \boxplus^\alpha(C_{2m+1})$, where C_{2m+1} is a cubical lattice of rank $2m+1$?

4.1.2 Characterization of the structure of Eulerian Sheffer posets of rank $n \ge 5$ for which $B(3) = 3!$, and $D(3) = 3! = 6$.

Theorem 4.9. *Let P be an Eulerian Sheffer poset of rank $n \ge 3$ with $B(3) = D(3) = 3! = 6$ for 3-intervals. P satisfies one of the following cases:*

(i) *There is an integer $k \ge 1$ such that $P \cong \boxplus^k(B_n)$, where n is odd.*

(ii) *$P \cong B_n$, where n is even.*

4.1.3 Characterization of the structure of Eulerian Sheffer posets of rank $n \ge 5$ for which $B(3) = 3!$ and $D(3) = 4$.

Let P be an Eulerian Sheffer poset of rank $n \ge 5$, with $B(3) = 3!$ and $D(3) = 4$. In this section we show that in the case $n = 2m+2$ the poset P satisfies $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$ for some integer $\alpha \ge 1$ and in the case $n = 2m+1$, $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$, for some integer $\alpha \ge 1$.

Theorem 4.10. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 \ge 4$ with $B(3) = 3!$ and $D(3) = 4$. Then $P \cong \Sigma^*(\boxplus^\alpha(B_{2m+1}))$, where $\alpha = \frac{B(2m+1)}{(2m+1)!}$ is a positive integer for $n \ge 6$ and $\alpha = 1$ for $n = 4$. Consequently the poset P has the following binomial and Sheffer factorial functions.*

(i) $B(k) = k!$ *for* $1 \le k \le 2m$, *and* $B(2m+1) = \alpha(2m+1)!$,

(ii) $D(1) = 1$, $D(k) = 2(k-1)!$ *for* $2 \le k \le 2m+1$, *and* $D(2m+2) = 2\alpha(2m+1)!$.

Theorem 4.11. *Let P be an Eulerian Sheffer poset of odd rank $n = 2m+1 \ge 5$ with $B(3) = 6$ and $D(3) = 4$. Then $P \cong \boxplus^\alpha(\Sigma^*(B_{2m}))$ for some positive integer α.*

4.2 Characterization of the structure and factorial functions of Eulerian Sheffer posets of rank $n \ge 5$ with $B(3) = 4$.

In this section, we characterize Eulerian Sheffer posets of rank $n \ge 5$ with $B(3) = 4$. Let P be an Eulerian Sheffer poset of rank $n \ge 5$ with $B(3) = 4$. It can be seen that the poset P satisfies one of the cases:

1. P has the following binomial factorial function $B(k) = 2^{k-1}$, where $1 \le k \le n-1$;
2. n is even and there is a positive integer $\alpha > 1$ such that poset P has the binomial factorial function $B(k) = 2^{k-1}$ for $1 \le k \le n-2$ and $B(n-1) = \alpha \cdot 2^{n-2}$ for some positive integer α.

As a consequence of Theorems 3.11 and 3.12 in [4], we can characterize posets in the case (i). Theorem 4.12 deals with the case (ii).

Theorem 4.12. *Let P be an Eulerian Sheffer poset of even rank $n = 2m+2 > 4$ with the binomial factorial function $B(k) = 2^{k-1}$ for $1 \le k \le 2m$, and $B(2m+1) = \alpha \cdot 2^{2m}$, where $\alpha > 1$ is a positive integer. Then $P \cong \Sigma^*(\boxplus^\alpha(T_{2m+1}))$.*

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Rational smoothness and affine Schubert varieties of type A

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Abstract. The study of Schubert varieties in G/B has led to numerous advances in algebraic combinatorics and algebraic geometry. These varieties are indexed by elements of the corresponding Weyl group, an affine Weyl group, or one of their parabolic quotients. Often times, the goal is to determine which of the algebraic and topological properties of the Schubert variety can be described in terms of the combinatorics of its corresponding Weyl group element. A celebrated example of this occurs when G/B is of type A, due to Lakshmibai and Sandhya. They showed that the smooth Schubert varieties are precisely those indexed by permutations that avoid the patterns 3412 and 4231. Our main result is a characterization of the rationally smooth Schubert varieties corresponding to affine permutations in terms of the patterns 4231 and 3412 and the twisted spiral permutations.

Résumé. L'étude des variétés de Schubert dans G/B a mené à plusieurs avancées en combinatoire algébrique. Ces variétés sont indexées soit par l'élément du groupe de Weyl correspondant, soit par un groupe de Weyl affine, soit par un de leurs quotients paraboliques. Souvent, le but est de déterminer quelles propriétés algébriques et topologiques des variétés de Schubert peuvent être décrites en termes des propriétés combinatoires des éléments du groupe de Weyl correspondant. Un exemple bien connu, dû à Lakshmibai et Sandhya, concerne le cas où G/B est de type A. Ils ont montré que les variétés de Schubert lisses sont exactement celles qui sont indexées par les permutations qui évitent les motifs 3412 et 4231. Notre résultat principal est une caractérisation des variétés de Schubert lisses et rationnelles qui correspondent à des permutations affines pour les motifs 4231 et 3412 et les permutations spirales tordues.

Keywords: pattern avoidance, affine permutations, Schubert varieties

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1 Introduction

The study of Schubert varieties and their singular loci incorporates tools from algebraic geometry, representation theory, and combinatorics. One celebrated result in this area due to Lakshmibai-Sandhya is that, in classical type A, the smooth Schubert varieties are precisely those that are indexed by permutations that avoid the patterns 4231 and 3412 [22], see also [29; 34]. A second important theorem in this area concerns a weaker notion than smoothness based on cohomology, called rational smoothness. Other generalizations of smoothness can be found in [32]. In general, smoothness implies rational smoothness, but not conversely. For classical Schubert varieties, Peterson showed smoothness is equivalent to rational

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smoothness precisely in types A, D, E. Recently, there has been a surge of research activity related to affine Schubert varieties [2; 20; 21; 23; 24; 25; 27]. It is natural to ask how properties of smoothness, rational smoothness, singular loci and tangent spaces for affine Schubert varieties relate to their classical counterparts.

This paper summarizes the results of [1], which give a criterion for detecting rationally smooth Schubert varieties in affine type A. These varieties are indexed by the set of affine permutations, denoted $\widetilde{S}_n$. Generalizing the theorem of Lakshmibai-Sandhya, it was shown that the patterns 4231 and 3412 can be interpreted as patterns for affine permutations. A permutation avoiding these two patterns will again index a rationally smooth affine Schubert variety. However, there is an infinite family of affine permutations in $\widetilde{S}_n$ which contain 3412 and yet index rationally smooth affine Schubert varieties. These varieties are related to the spiral varieties studied by Mitchell [28], see also [2]. Thus, the main result is a complete characterization of the rationally smooth Schubert varieties in G/B in type $\widetilde{A}_n$.

Theorem 1.1 [1, Theorem 1.1] *Let $w \in \widetilde{S}_n$ for $n \geq 3$. The affine Schubert variety X_w is rationally smooth if and only if one of the following hold:*

1. *w avoids the patterns 3412 and 4231,*
2. *w is a twisted spiral permutation (defined in Section 4.1).*

Note, $\widetilde{S}_2$ is the infinite dihedral group. It follows that X_w is rationally smooth for all $w \in \widetilde{S}_2$. Hence, throughout this paper we will assume $n \geq 3$ unless otherwise specified.

If a point $p \in X_w$ is not rationally smooth, then p must be singular. Hence we get a necessary condition to detect singular affine Schubert varieties.

Corollary 1.2 *Let $w \in \widetilde{S}_n$ for $n \geq 3$. If w contains either a 3412 or 4231 pattern, then X_w is singular.*

The outline of this paper is as follows. In Section 2, we briefly introduce affine permutations. In Section 3, we then define the associated geometric objects, the affine Schubert varieties. Section 4 contains an outline of the proof of Theorem 1.1, and Section 5 discusses some corollaries of Theorem 1.1 and directions for future research.

2 Affine permutations

Let $\widetilde{S}_n$ be the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ such that the following two properties hold:

1. $w(i+n) = w(i) + n$ for all $i \in \mathbb{Z}$,
2. $\sum_{i=1}^{n} w(i) = \binom{n+1}{2}$.

Here the group operation is function composition. Elements of $\widetilde{S}_n$ are called *affine permutations*. Affine permutations first appeared in [26, §3.6] and were then further studied in [30]. See [6] for more background.

We can view an affine permutation in its one-line notation as the infinite string

$$\cdots w_{-1} w_0 w_1 w_2 \cdots w_n w_{n+1} \cdots ,$$

where, for ease of notation, we write $w_i = w(i)$. An affine permutation is completely determined by its action on any window of n consecutive indices $[w_i, w_{i+1}, \ldots, w_{i+n-1}]$. In particular, it is enough to record the base window $[w_1, \ldots, w_n]$ to capture all the information about w.

As a Coxeter group, $\widetilde{S}_n$ is generated by the set of *simple reflections* $S = \{s_0, s_1, \ldots, s_{n-1}\}$, where s_i exchanges $i + kn$ and $i + 1 + kn$ for all $k \in \mathbb{Z}$ and fixes all other values. The relations amongst these generators are encoded in the following Coxeter graph.

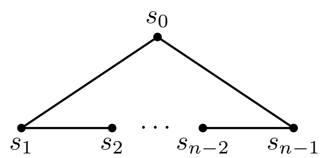

Fig. 1: Coxeter graph for $\widetilde{S}_n$.

As a group, $\widetilde{S}_n$ is also generated by the reflections t_{ij} for $i < j$ and $i \not\equiv j \,(\mathrm{mod}\ n)$. Here t_{ij} interchanges $i + kn$ and $j + kn$ for all $k \in \mathbb{Z}$ and leaves all other values fixed. Since $t_{ij} = t_{i+n,j+n}$, it suffices to work with the set

$$T = \{t_{ij} : 1 \leq i \leq n, i < j, \text{ and } i \not\equiv j \pmod{n}\}.$$

Each $w \in \widetilde{S}_n$ can be written (in infinitely many ways) as a product of elements from the generating set S. We denote the minimal length of such an expression as $\ell(w)$, and call this statistic the *length* of w. An alternative description of this statistic (which is easier to compute) is given by the following proposition.

Proposition 2.1 [6, Proposition 8.3.1] *Let $w \in \widetilde{S}_n$. Then*

$$\ell(w) = \#\{(i,j) : 1 \leq i \leq n, i < j, \text{ and } w_i > w_j\}.$$

There is a partial order on $\widetilde{S}_n$ defined as the transitive closure of the covering relations, where $u \lessdot v$ if there exists a reflection $t \in T$ with $ut = v$ and $\ell(u) = \ell(v) - 1$. This partial order is called the *Bruhat order* on $\widetilde{S}_n$. Under Bruhat order, $\widetilde{S}_n$ is a ranked poset, ranked by the length function.

Given two elements $u \leq w$ related by Bruhat order, let $[u, w] = \{v \in \widetilde{S}_n : u \leq v \leq w\}$ denote the interval between u and w. Call the interval $[\mathrm{Id}, w]$ the *order ideal* of w, where $\mathrm{Id} \in \widetilde{S}_n$ is the identity permutation. The *Poincaré polynomial* $P_w(q)$ is the rank generating function for the order ideal of w. Specifically,

$$P_w(q) = \sum_{v \leq w} q^{\ell(v)}.$$

This polynomial gets its name from the fact that $P_w(q^2)$ is the Poincaré polynomial of the cohomology ring of X_w. Note that $v \leq w$ if and only if $v^{-1} \leq w^{-1}$, so we have $P_w(q) = P_{w^{-1}}(q)$.

We use the following theorem due to Carrell-Peterson to define *rational smoothness* of a Schubert variety in terms of the combinatorics of the Bruhat order and Coxeter groups. For $x, w \in \widetilde{S}_n$, let $\mathcal{R}(x, w) = \{t \in T : x < xt \leq w\}$. For any polynomial $f(q)$ of degree n, call f *palindromic* if $f(q) = q^n f(q^{-1})$.

Theorem 2.2 [9, Theorem E] *Let W be an (affine) Weyl group (e.g., $W = \widetilde{S}_n$) and let $w \in W$. Then the following are equivalent.*

1. *The (affine) Schubert variety X_w is rationally smooth.*
2. *$P_w(q)$ is palindromic.*
3. *$\#\mathcal{R}(x, w) = \ell(w) - \ell(x)$ for all $x \leq w$.*

Our main result in Theorem 1.1 provides an efficient criterion for the three equivalent conditions in Theorem 2.2 in the case W is the affine Weyl group of type A.

3 Affine Schubert varieties

We first briefly review the classical, non-affine case. Let G be a connected, reductive algebraic group such as $GL_n(\mathbb{C})$. Let B be a Borel subgroup of G. For example, if $G = GL_n(\mathbb{C})$, we can take B to be the upper triangular matrices. The cosets G/B form the points of a *flag variety*. For each G, there is an associated finite Weyl group W. For each $w \in W$, we obtain the *Schubert variety* X_w by taking the Zariski closure of the orbit $B \cdot e_w$, where e_w represents w embedded in G. There are many good references on Schubert varieties including [8; 12; 13; 15; 16; 17; 33].

We now move to the construction of the affine Schubert varieties of type A. For an excellent reference on this construction, see [27] and the references contained therein. Let $\widetilde{G} = \mathrm{GL}_n(\mathbb{C}((t)))_0$ be the group of invertible $n \times n$ matrices whose entries are rational functions in t and whose determinant has order 0. Each affine permutation corresponds to a matrix in $\widetilde{G}$ as follows. Let $w \in \widetilde{S}_n$ and for each $1 \leq i \leq n$, write $w_i = a_i + b_i n$, where $1 \leq a_i \leq n$. Then $\{a_1, \ldots, a_n\} = \{1, \ldots, n\}$ and $\sum_{i=1}^{n} b_i = 0$ by the requirements 1 and 2 in the definition of an affine permutation. Set $e_w = (m_{i,j})_{i,j=1}^{n}$, where $m_{i,a_i} = t^{b_i}$ and all other entries are 0. Such a matrix is call an *affine permutation matrix*.

Example 3.1 *Let $w = [8, -1, 6, 3, 1, 4] \in \widetilde{S}_6$. Then $(a_1, \ldots, a_6) = (2, 5, 6, 3, 1, 4)$ and $(b_1, \ldots, b_6) = (1, -1, 0, 0, 0, 0)$. Hence*

$$e_w = \begin{bmatrix} 0 & t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The *affine flag manifold* is the space of cosets in $\widetilde{G}/\widetilde{B}$ where

$$\widetilde{B} = \{b \in \mathrm{GL}_n(\mathbb{C}[[t]]) : b|_{t=0} \text{ is upper triangular}\}.$$

Note that the entries of elements in $\widetilde{B}$ are formal power series in t. A coset $g\widetilde{B}$ in $\widetilde{G}/\widetilde{B}$ has a unique representative that is in column echelon form. The pivot points in this echelon form will correspond to an affine permutation matrix e_w. The $\widetilde{B}$-orbit of this coset will then give all matrices in $\widetilde{G}$ whose echelon forms correspond to e_w. The set $C_w = \widetilde{B} e_w \widetilde{B}/\widetilde{B}$ is called the *Schubert cell* corresponding to w.

Example 3.2 *For $w = [8, -1, 6, 3, 1, 4]$, C_w is the set of all matrices whose column echelon form is*

$$\begin{bmatrix} a+bt & t & c & d & et^{-1}+f & g \\ 0 & 0 & 0 & 0 & t^{-1} & 0 \\ h & 0 & i & j & k & 1 \\ \ell & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

Let $X_w = \overline{C}_w$ be the Zariski closure of the Schubert cell corresponding to w. We call X_w the *affine Schubert variety* corresponding to w.

4 Sketch of proof of Theorem 1.1

4.1 Factoring the Poincaré polynomial

First suppose that $w \in \widetilde{S}_n$ avoids the patterns 3412 and 4231. By Theorem 2.2, we must show that $P_w(q)$ is palindromic. In particular, each w with a palindromic Poincaré polynomial will factor using a parabolic decomposition as follows. Given a subset $J \subseteq S$ of the generators for a Coxeter group W, we can define W_J to be the subgroup of W generated by the elements of J. W_J is called the *parabolic subgroup generated by* J. Each coset in W/W_J contains a unique element of minimal length [18, Proposition 1.10]. The set of all minimal length coset representatives is denoted

$$W^J = \{w \in W | \ell(ws) > \ell(w) \text{ for all } s \in J\}$$

and hence we can identify W^J with these cosets. Similarly, let ${}^JW$ denote the minimal length left coset representatives in $W_J \backslash W$. Bruhat order on W induces partial orders on ${}^JW$ and W^J. When we wish to refer to the Poincaré polynomial of a minimal length coset representative w for the induced order on either of the quotients ${}^JW$ or W^J, we will denote it by ${}^JP_w$ or P_w^J, respectively. The *parabolic decomposition* for elements of W is given as follows.

Proposition 4.1 [6, Proposition 2.4.4,2.5.1] *For every $w \in W$ there exists a unique $u \in W_J$ and a unique $v \in {}^JW$ such that $w = u \cdot v$ and $\ell(w) = \ell(u) + \ell(v)$. Moreover, the map $w \mapsto v$ is order preserving as a map from W to the set of minimal length coset representatives.*

Let $m(w, J)$ be the unique maximal element in $[0, w] \cap W_J$ (see [4, Theorem 2.2]). Computing $m(w, J)$ is equivalent to taking the Demazure product from [19, Definition 3.1] of the subword of $s_{i_1} \cdots s_{i_p}$ consisting of all of the letters from J. Then the following theorem shows how this parabolic decomposition gives rise to a factorization of the Poincaré polynomial.

Theorem 4.2 [3, Theorem 6.4] *Suppose $w \in W$ has the parabolic decomposition $w = u \cdot v$ with $u \in W_J$ and $v \in {}^JW$. If $u = m(w, J)$, then $P_w(q) = P_u(q) \cdot {}^JP_v(q)$.*

Returning to the case where w avoids 3412 and 4231, we show that we can decompose w as $w = u \cdot v$, where $u = m(w, J)$ for some $J \subset S$. Moreover, ${}^JP_v(q)$ will be palindromic and u avoids 3412 and 4231. We then use induction on $\ell(w)$. Thus, we have outlined the proof for the following theorem.

Theorem 4.3 *Let $w \in \widetilde{S}_n$ be an affine permutation that avoids the patterns 3412 and 4231. Then $P_w(q)$ is palindromic.*

Recall that, in the non-affine case, Gasharov shows that the factors of the Poincaré polynomial for (rationally) smooth elements are all of the form $(1+q+\cdots+q^k)$ (see [14]). However, for $w \in \widetilde{S}_n$, these factors sometimes end up being the more general q-binomial coefficients instead.

4.2 Twisted spiral permutations

There is one important infinite family of affine permutations consisting of permutations that contain 3412, but are still rationally smooth. For $a, b \in \mathbb{Z}$, define

$$c(a,b) = \begin{cases} s_a s_{a+1} s_{a+2} \cdots s_{a+b-1}, & \text{if } b > 0, \\ s_a s_{a-1} s_{a-2} \cdots s_{a+b+1}, & \text{if } b < 0, \\ 1, & \text{if } b = 0, \end{cases} \tag{4.1}$$

where all of the subscripts are taken modulo n. By definition, we have $c(a,1) = c(a,-1) = s_a$, and $\ell(c(a,b)) = |b|$. For any $k \in \mathbb{Z}$ with $k \neq 0$ and any $1 \leq i \leq n$, let

$$w^{(i,k)} = w_0^{J_i} c(i, k(n-1)) \in \widetilde{S}_n, \tag{4.2}$$

where $w_0^{J_i}$ is the unique longest element in the parabolic subgroup generated by $J_i := S \backslash \{i\}$. For $k \neq 0$, we call $w^{(i,k)}$ a *twisted spiral permutation*, since its reduced expression is obtained by spiraling around the Coxeter graph for $\widetilde{S}_n$ and then twisting by $w_0^{J_i}$.

By a result of Mitchell [28], the Poincaré polynomial for $c(i, k(n-1))$ in the quotient ${}^{J_i}(\widetilde{S}_n)$ is palindromic (in fact, it is a q-binomial coefficient). Theorem 4.2 then shows that we may lift each $c(i, k(n-1))$ to a palindromic element in $\widetilde{S}_n$ by left multiplying by $w_0^{J_i}$. Hence the Poincaré polynomial for $w^{(i,k)}$ is palindromic.

4.3 When w contains 4231 or 3412

The converse to Theorem 1.1 asserts that the rationally smooth varieties found in Section 4.1 are the only ones that are rationally smooth. In the case where w contains 4231, we show that the Poincaré polynomial fails to be palindromic at degree 1. The coefficient of q in $P_w(q)$ is the number of distinct generators occurring in any reduced expression for w. The coefficient of $q^{\ell(w)-1}$ is the number of elements $v \leq w$ with $\ell(v) = \ell(w) - 1$. Hence we construct a graph whose edges represent these covering relations and argue that there are more edges than distinct generators.

The case where w contains 3412 and avoids 4231 is more complicated, in the sense that the Poincaré polynomial can fail to be palindromic at various degrees. For this, we introduce an affine version of *Bruhat pictures*, which first appeared in [5]. We use these pictures to *flatten* a pair $x < w$ as much as possible, while preserving the length difference and the size of the set $\mathcal{R}(x,w) = \{t \in T : x < xt \leq w\}$.

Define the *rank function* for $w \in \widetilde{S}_n$ by

$$r_w(p,q) = \#\{i \leq p : w_i \geq q\}.$$

Define the *difference function* for the pair $x, w \in \widetilde{S}_n$ by

$$d_{x,w}(p,q) = r_w(p,q) - r_x(p,q).$$

The difference function gives another useful characterization of Bruhat order and generalizes the Ehresmann criterion for Bruhat order [11], see also [12].

Theorem 4.4 [6, Theorem 8.3.7] *For $x, w \in \widetilde{S}_n$, $x \leq w$ if and only if $d_{x,w}(p,q) \geq 0$ for all $p, q \in \mathbb{Z}$.*

Given an affine permutation $w \in \widetilde{S}_n$, we can visualize w as $\{(i, w_i) : i \in \mathbb{Z}\} \subset \mathbb{Z}^2$. We think of each pair (i, w_i) as a point in the plane drawn in Cartesian coordinates. Let $\mathrm{pt}_w(i) = (i, w_i)$. Furthermore, when comparing two affine permutations x, w in Bruhat order using the rank difference function $d_{x,w}$ and Theorem 4.4, it is useful to visualize the nonzero entries of the function $d_{x,w}$ as a union of *shaded* rectangles in the plane. Combining both visualizations we get an *affine Bruhat picture*.

For example, take $w = [8, 3, 1, 0, 4, 5]$ and $x = wt_{1,4} = [0, 3, 1, 8, 4, 5]$, then $d_{x,w}$ will be positive in the translated shaded regions of the affine Bruhat picture for $x < w$ in Figure 2. Here the points of w are represented by dots, while the points of x are represented by x's.

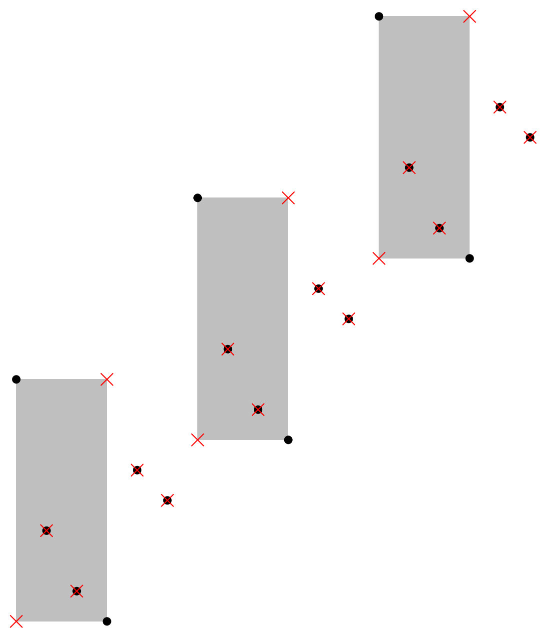

Fig. 2: $w = [8, 3, 1, 0, 5, 4]$ and $x = wt_{1,4} = [0, 3, 1, 8, 5, 4]$.

Given integers $p < q$ such that $x_p < x_q$, we will use the affine Bruhat pictures to determine if $x < xt_{p,q} \leq w$. Observe that $d_{x,xt_{p,q}}$ is positive on the periodic union of rectangles

$$\mathcal{A}_{p,q,k}(x) = [p + kn, q - 1 + kn] \times [x_p + 1 + kn, x_q + kn], \text{ for all } k \in \mathbb{Z}.$$

It is possible that these rectangles overlap in affine Bruhat pictures as in Figure 3. If a point (i, j) is contained in exactly m consecutive translates of $\mathcal{A}_{p,q,0}(x)$, then $d_{x,xt_{p,q}} = m$. Thus, we get the following criterion for determining if $t_{p,q} \in \mathcal{R}(x, w)$.

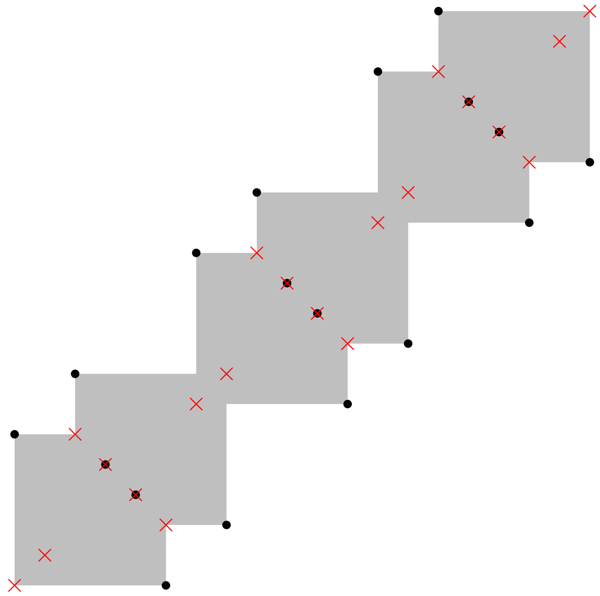

Fig. 3: $w = [6, -3, 8, 5, 4, 1]$ and $x = [1, 2, 6, 5, 4, 3]$.

Lemma 4.5 *Given affine permutations $x < w$ and integers $p < q$ such that $x_p < x_q$, then $t_{p,q} \in \mathcal{R}(x, w)$, provided $d_{x,w}(i, j) \geq m$ for every (i, j) contained in m consecutive rectangles $\mathcal{A}_{p,q,k}(x)$.*

As an example of computing $\mathcal{R}(x, w)$ using shading, let $w = [6, -3, 8, 5, 4, 1]$ and $x = [1, 2, 6, 5, 4, 3]$ (see Figure 3). Then

$$\mathcal{R}(x, w) = \{t_{12}, t_{13}, t_{14}, t_{15}, t_{16}, t_{23}, t_{24}, t_{25}, t_{26}, t_{37}, t_{38}, t_{47}, t_{48}, t_{57}, t_{58}, t_{67}, t_{68}\}.$$

If w contains 3412 but avoids 4231, and w is not a twisted spiral, we identify a specific affine permutation $x < w$ such that $\#\mathcal{R}(x, w) > \ell(\omega)$ by using the Bruhat pictures. Then, we can appeal to Theorem 2.2. The proof requires analyzing several cases which whittle down the possible Bruhat pictures until only the twisted spirals remain. As mentioned about however, the twisted spirals are rationally smooth. This completes the outline of the proof of the following theorem.

Theorem 4.6 *If $w \in \widetilde{S}_n$ contains either a 3412 or a 4231 pattern, but is not a twisted spiral permutation, then $P_w(q)$ is not palindromic.*

5 Further directions

It was shown in [10] that there are only a finite number of 3412 avoiding affine permutations in $\widetilde{S}_n$. Thus, there exists a finite number of smooth Schubert varieties indexed by $\widetilde{S}_n$. Using this fact, we have verified the following conjecture up to $n = 5$. Significant progress on this conjecture has since been made by Cheng, Crites, and Kuttler, the results of which are currently in preparation.

Conjecture 1 *Let $w \in \widetilde{S}_n$. The Schubert variety X_w is smooth if and only if w avoids 3412 and 4231.*

In fact, we can get a tighter bound on the length of any w avoiding 3412 and 4231. In the proof of Theorem 1.1, we proved that any 3412 and 4231 avoiding affine permutation can be written in the form $w = w'\sigma$, with $\ell(w) = \ell(w') + \ell(\sigma)$. Both w' and σ are elements of a proper parabolic subgroup of $\widetilde{S}_n$, and hence $\ell(w'), \ell(\sigma) \leq \binom{n}{2}$. Thus, we have the following corollary.

Corollary 5.1 (To proof of Theorem 1.1) *If $w \in \widetilde{S}_n$ avoids 3412 and 4231, then $\ell(w) \leq 2\binom{n}{2}$.*

Since the number of affine permutations of length at most $2\binom{n}{2}$ is finite, we would like to compute how many affine permutations in $\widetilde{S}_n$ avoid both 3412 and 4231. Conjecturally, this is equivalent to the number of smooth affine Schubert varieties of type $\widetilde{A}_n$. In general, not much is known about the number of affine permutations that avoid a given set of patterns, except for a characterization of when this number is finite [10]. Starting with $n = 2$, the first few terms of this sequence are $5, 31, 173, 891, 4373$, which now appear in Sloane's [31] as sequence A180635.

In [7], Björner and Ekedahl give general inequalities amongst the coefficients of the Poincaré polynomial for elements of any crystallographic Coxeter group. As mentioned in Section 4.3, we showed that when w contains 4231, the coefficient of q in $P_w(q)$ is strictly less than the coefficient of $q^{\ell(w)-1}$. Combining this fact with [7, Theorem C], proves the following corollary.

Corollary 5.2 *Let $w \in \widetilde{S}_n$ and assume w contains a 4231. Then if $P_{\mathrm{Id},w} = 1 + a_1 q + \cdots + a_d q^d$ is the Kazhdan-Lusztig polynomial indexed by Id, w, then $a_1 > 0$.*

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A tight colored Tverberg theorem for maps to manifolds (extended abstract)

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Abstract. Any continuous map of an N-dimensional simplex Δ_N with colored vertices to a d-dimensional manifold M must map r points from disjoint rainbow faces of Δ_N to the same point in M, assuming that $N \geq (r-1)(d+1)$, no r vertices of Δ_N get the same color, and our proof needs that r is a prime. A face of Δ_N is called a *rainbow face* if all vertices have different colors.

This result is an extension of our recent "new colored Tverberg theorem", the special case of $M = \mathbb{R}^d$. It is also a generalization of Volovikov's 1996 topological Tverberg theorem for maps to manifolds, which arises when all color classes have size 1 (i.e., without color constraints); for this special case Volovikov's proofs, as well as ours, work when r is a prime power.

Résumé. Étant donné un simplex Δ_N de dimension N ayant les sommets colorés, une face de Δ_N est dite *arc-en-ciel*, si tous les sommets de cette face ont des couleurs différentes. Toute fonction continue d'un simplex Δ_N de dimension N aux sommets colorés vers une variété d-dimensionnelle M doit envoyer r points provenant de faces arc-en-ciel disjointes de Δ_N au mêmes points dans M; en supposant que $N \geq (r-1)(d+1)$, un ensemble de r sommets de Δ_N doit être coloré à l'aide d'au moins deux couleurs. Notre démonstration requiert que r soit un nombre premier.

Ce résultat est une extension de notre "nouveau théorème de Tverberg coloré", le cas particulier où $M = \mathbb{R}^d$. Il est également une généralisation du théorème de Tverberg topologique de Volovikov datant de 1996, pour les fonctions vers une variété, dont les classes de couleurs sont de taille 1 (c'est-à-dire sans contraintes de couleur). Dans ce cas particulier, la démonstration de Volovikov et la nôtre fonctionnent lorsque r est une puissance d'un premier.

Keywords: equivariant algebraic topology, convex geometry, colored Tverberg problem, configuration space/test map scheme, group cohomology

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1 Introduction

More than 50 years ago, the Cambridge undergraduate Bryan Birch [5] showed that "$3N$ points in a plane" can be split into N triples that span triangles with a non-empty intersection. He also conjectured a sharp, higher-dimensional version of this, which was proved by Helge Tverberg [15] in 1964.

In a 1988 Computational Geometry paper [2], Bárány, Füredi & Lovász noted that they needed a "colored version of Tverberg's theorem". Soon after this Bárány & Larman [3] proved such a theorem for rN colored points in a plane where the number of overlapping faces r is 2 or 3. Moreover, they conjectured a general version for any higher dimension d and any number of overlaps $r \geq 2$, offering a proof by Lovász for the case $r = 2$ and any dimension d. A 1992 paper [17] by Živaljević & Vrećica obtained this in a slightly weaker version, though not with a tight bound on the number of points. The proof relied on equivariant topology and beautiful combinatorics of "chessboard complexes".

Recently we proposed a new "colored Tverberg theorem", which is tight, generalizes Tverberg's original theorem in the case of primes and gives the best known answers for the Bárány–Larman conjecture.

Theorem 1.1 (Tight colored Tverberg theorem [7]) *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$.*

Then for every continuous map $f : \Delta_N \to \mathbb{R}^d$ there are r disjoint faces $F_1, \ldots, F_r$ of Δ_N such that the vertices of each face F_i have all different colors and the images under f have a point in common: $f(F_1) \cap \ldots \cap f(F_r) \neq \emptyset$.

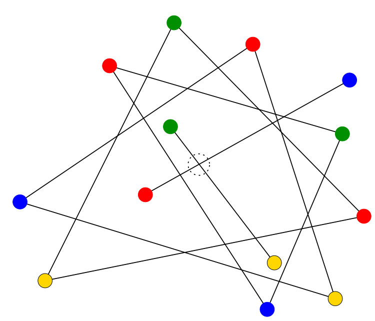

Fig. 1: Example of Theorem 1.1 for $d = 2$, $r = 5$, $N + 1 = 13$.

Here a *coloring* of the vertices of the simplex Δ_N is a partition of the vertex set into color classes, $C_1 \uplus \ldots \uplus C_m$. The condition $|C_i| \leq r-1$ implies that there are at least $d+2$ different color classes. In the following, a face whose all vertices have different colors, $|F_j \cap C_i| \leq i$ for all 1, will be called a *rainbow face*. Figure 1 shows an example for Theorem 1.1.

Theorem 1.1 is tight in the sense that it fails for maps of a simplex of smaller dimension, or if r vertices have the same color. It implies an optimal result for the Bárány–Larman conjecture in the case where $r+1$ is a prime, and an asymptotically-optimal bound in general; see [7, Corollaries 2.4, 2.5]. The special case where all vertices of Δ_N have different colors, $|C_i| = 1$, is the prime case of the topological Tverberg theorem, as proved by Bárány, Shlosman & Szűcs [4].

In this talk we present an extension of Theorem 1.1 that treats continuous maps $R \to M$ from the a subcomplex R of the N-simplex to an arbitrary d-dimensional manifold M with boundary in place of $\mathbb{R}^d$. Here, R is the *rainbow subcomplex* Δ_N, which consists of all rainbow faces.

Theorem 1.2 (Tight colored Tverberg theorem for M) *For $d \geq 1$ and a prime $r \geq 2$, set $N := (d+1)(r-1)$, and let the $N+1$ vertices of an N-dimensional simplex Δ_N be colored such that all color classes are of size at most $r-1$. Let R be the corresponding rainbow subcomplex.*

Then for every continuous map $f : R \to M$ to a d-dimensional manifold, the rainbow subcomplex R has r disjoint rainbow faces whose images under f have a point in common.

Theorem 1.2 without color constraints (that is, when all color classes are of size 1, and thus all faces are rainbow faces and $R = \Delta_N$) was previously obtained by Volovikov [16], using different methods. His proof (as well as ours in the case without color constraints) works for prime powers r.

An extension of Theorem 1.2 to a prime power that is not a prime seems out of reach at this point, even in the case $M = \mathbb{R}^d$. Similarly, for the case when r is not a prime power there currently does not seem to be a viable approach to the case without color constraints, even for $M = \mathbb{R}^d$. This is the remaining open case of the topological Tverberg conjecture [4].

Finally we remark that the restriction of the domain to a proper subcomplex of Δ_N, as given by Theorem 1.2, appears to be a non-trivial strengthening, even though any partition can use only faces in $R \subset \Delta_N$ of dimension at most $N - r + 1$. Let us give an example to illustrate that. Let $d = r = 2$ and let M be the 2-dimensional sphere. Then $N = 3$ and we give the vertices of the tetrahedra Δ_N all different colors. Since the N-dimensional face of Δ_N is never part of a Tverberg partition, we might guess that the conclusion of Theorem 1.2 should hold true also for any map $f : \partial\Delta_3 \to M$. However this is wrong: any homeomorphism f gives a counter-example!

2 Proof

In this extended abstract we only consider the case when f extends to a map $\Delta^N \to M$ on the whole simplex. If the given number of colors used to color the vertices is at least $d + 3 + \left\lfloor \frac{d}{r-1} \right\rfloor$ then the same proof will also work for non-extendable maps $f : R \to M$. Our proof of the general case of Theorem 1.2 needs some additional machinery due to Volovikov [16].

We prove Theorem 1.2 in this case in two steps:

- First, a geometric reduction lemma implies that it suffices to consider only manifolds M that are of the form $M = \widetilde{M} \times I^g$, where $I = [0, 1]$ and $\widetilde{M}$ is another manifold. More precisely we will need for the second step that
$$(r-1)\dim(M) > r \cdot \text{cohdim}(M), \tag{1}$$
where cohdim(M) is the cohomology dimension of M. This is done in Section 2.1.
- In the second step, we can assume (1) and prove Theorem 1.2 for maps $\Delta_N \to \widetilde{M}$ via the configuration space/test map scheme and Fadell–Husseini index theory, see Sections 2.2 and 2.4. The basic idea is the following: Assuming that Theorem 1.2 has a counter-example, construct an equivariant map from it. Then we show using equivariant topology that such a map cannot exist.

In the second step we rely on the computation of the Fadell–Husseini index of joins of chessboard complexes that we obtained in [8, Corollary 2.6].

2.1 A geometric reduction lemma

In the proof of Theorem 1.2 we may assume that M satisfy the above inequality (1) by using the following reduction lemma repeatedly.

Lemma 2.1 *Theorem 1.2 for parameters (d, r, M, f) can be derived from the case with parameters $(d', r', M', f') = (d+1, r, M \times I, f')$, where the continuous map f' is defined in the following.*

Proof: Suppose we have to prove the theorem for the parameters (d, r, M, f). Let $d' = d+1$, $r' = r$, and $M' = M \times I$. Then $N' := (d'+1)(r-1) = N + r - 1$. Let $v_0, \dots, v_N, v_{N+1}, \dots, v_{N'}$ denote the vertices of $\Delta_{N'}$. We regard Δ_N as the front face of $\Delta_{N'}$ with vertices $v_0, \dots, v_N$. We give the new vertices $v_{N+1}, \dots, v_{N'}$ a new color. Define a new map $f' : \Delta_{N'} \to M'$ by

$$\lambda_0 v_0 + \ldots + \lambda_{N'} v_{N'} \longmapsto (f(\lambda_0 v_0 + \ldots + \lambda_{N-1} v_{N-1} + (\lambda_N + \ldots + \lambda_{N'}) v_n), \lambda_{N+1} + \ldots + \lambda_{N'}) .$$

Suppose we can show Theorem 1.2 for the parameters (d', r', M', f'). That is, we found a Tverberg partition $F'_1, \dots, F'_r$ for these parameters. Put $F_i := F'_i \cap \Delta_N$. Since f' maps the front face Δ_N to $M \times \{0\}$ and since $\Delta_{N'}$ has only $r - 1 < r$ vertices more than Δ_N, already the F_i will intersect in $M \times \{0\}$. Hence the r faces $F_1, \dots, F_r$ form a solution for the original parameters (d, r, M, f). This reduction is sketched in Figure 2. □

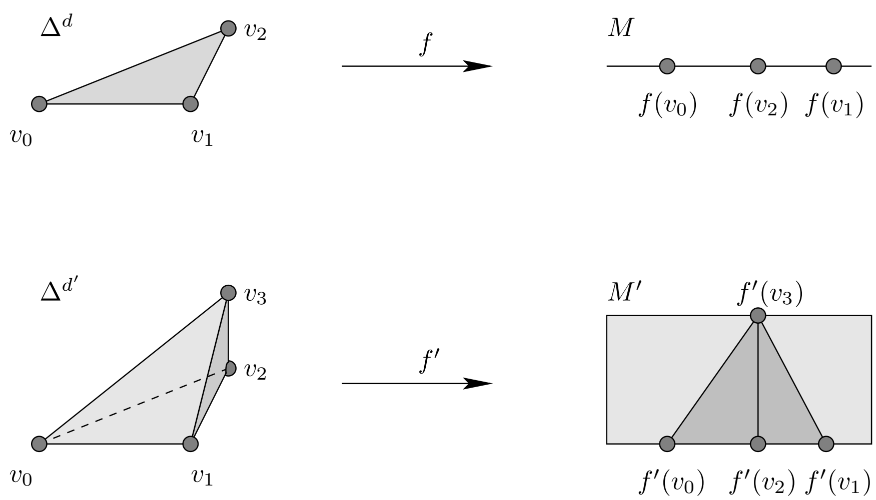

Fig. 2: Exemplary reduction in the case $d = 1$, $r = 2$, $N = 2$.

If the reduction lemma is applied $g = 1 + \left\lfloor \frac{d}{r-1} \right\rfloor$ times, the problem is reduced from the arbitrary parameters (d, r, M, f) to parameters (d'', r'', M'', f'') where $M'' = M \times I^g$. Thus M'' has vanishing cohomology in its g top dimensions. Therefore $(r-1)\dim(M'') > r \cdot \text{cohdim}(M'')$.

Having this reduction in mind, in what follows we may simply assume that the manifold M already satisfies inequality (1).

2.2 The configuration space/test map scheme

Now we reduce Theorem 1.2 to a problem in equivariant topology. Suppose we are given a continuous map

$$f : \Delta_N \longrightarrow M,$$

and a coloring of the vertex set vert$(\Delta_N) = [N+1] = C_0 \uplus \ldots \uplus C_m$ such that the color classes C_i are of size $|C_i| \leq r-1$. We want to find a colored Tverberg partition, that is, pairwise disjoint rainbow faces $F_1, \ldots, F_r$ of Δ_N, $|F_j \cap C_i| \leq 1$, whose images under f intersect.

The test map F is constructed using f in the following way. Let $f^{*r} : (\Delta_N)^{*r} \longrightarrow_{\mathbb{Z}_r} M^{*r}$ be the r-fold join of f. Since we are interested in pairwise disjoint faces $F_1, \ldots, F_r$, we restrict the domain of f^{*r} to the simplicial r-fold 2-wise deleted join of Δ_N, $(\Delta_N)^{*r}_{\Delta(2)} = [r]^{*(N+1)}$. This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint faces. (See [13, Chapter 5.5] for an introduction to these notions.) Since we are interested in colored faces F_j, we restrict the domain further to the subcomplex

$$R^{*r}_{\Delta(2)} = (C_0 * \ldots * C_m)^{*r}_{\Delta(2)} = [r]^{*|C_0|}_{\Delta(2)} * \ldots * [r]^{*|C_m|}_{\Delta(2)}.$$

This is the subcomplex of $(\Delta_N)^{*r}$ consisting of all joins $F_1 * \ldots * F_r$ of pairwise disjoint rainbow faces. The space $[r]^{*k}_{\Delta(2)}$ is known as the *chessboard complex* $\Delta_{r,k}$ [13, p. 163]. We write

$$K := (\Delta_{r,|C_0|}) * \ldots * (\Delta_{r,|C_m|}). \tag{2}$$

Hence we get a *test map*

$$F' : K \longrightarrow_{\mathbb{Z}_r} M^{*r}.$$

Let $T_{M^{*r}} := \{\sum_{i=1}^{r} \frac{1}{r} \cdot x \ : \ x \in M\}$ be the thin diagonal of M^{*r}. Its complement $M^{*r} \backslash T_{M^{*r}}$ is called the topological r-fold r-wise deleted join of M and it is denoted by $M^{*r}_{\Delta(r)}$.

The preimages $(F')^{-1}(T_{M^{*r}})$ of the thin diagonal correspond exactly to the colored Tverberg partitions. Hence the image of F' intersects the diagonal if and only if f admits a colored Tverberg partition.

Suppose that f admits *no* colored Tverberg partition, then the test map F' induces a $\mathbb{Z}_r$-equivariant map that avoids $T_{M^{*r}}$, that is,

$$F : K \longrightarrow_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}. \tag{3}$$

We will derive a contradiction to the existence of such an equivariant map using the Fadell–Husseini index theory.

2.3 The Fadell–Husseini index

In this section we review equivariant cohomology of G-spaces via the Borel construction. This will provide the right tool to prove the non-existence of the test-map (3). We refer the reader to [1, Chap. V] and [10, Chap. III] for more details.

In the following H^* denotes singular or Čech cohomology with $\mathbb{F}_r$-coefficients, where r is a prime. Let G a finite group and let EG be a contractible free G-CW complex, for example the infinite join $G*G*\cdots$, suitably topologized. The quotient $BG := EG/G$ is called the *classifying space of* G. To every G-space X we can associate the *Borel construction* $EG \times_G X := (EG \times X)/G$, which is the total space of the fibration $X \hookrightarrow EG \times_G X \xrightarrow{pr_1} BG$.

The *equivariant cohomology* of a G-space X is defined as the ordinary cohomology of the Borel construction,

$$H^*_G(X) \ := \ H^*(EG \times_G X).$$

If X is a G-space, we define the *cohomological index* of X, also called the *Fadell–Husseini index* [11], [12], to be the kernel of the map in cohomology induced by the projection from X to a point,

$$\mathrm{Ind}_G(X) \ := \ \ker\left(H^*_G(\mathrm{pt}) \xrightarrow{p^*} H^*_G(X)\right) \ \subseteq \ H^*_G(\mathrm{pt}).$$

The cohomological index is monotone in the sense that if there is a G-map $X \longrightarrow_G Y$ then

$$\mathrm{Ind}_G(X) \supseteq \mathrm{Ind}_G(Y). \tag{4}$$

If r is odd then the cohomology of $\mathbb{Z}_r$ with $\mathbb{F}_r$-coefficients as an $\mathbb{F}_r$-algebra is

$$H^*(\mathbb{Z}_r) = H^*(B\mathbb{Z}_r) \cong \mathbb{F}_r[x,y]/(y^2),$$

where $\deg(x) = 2$ and $\deg(y) = 1$. If $r = 2$ then $H^*(\mathbb{Z}_r) \cong \mathbb{F}_2[t]$, $\deg t = 1$.

The index of the configuration space K, defined in (2), was computed in [8, Corollary 2.6]:

Theorem 2.2 $\mathrm{Ind}_{\mathbb{Z}_r}(K) = H^{*\geq N+1}(B\mathbb{Z}_r)$.

Therefore in the proof of Theorem 1.2 it remains to show that $\mathrm{Ind}_{\mathbb{Z}_r}(M^{*r}_{\Delta(r)})$ contains a non-zero element in dimension less or equal to N. Indeed, the monotonicity of the index (4) then implies the non-existence of a test map (3), which in turn implies the existence of a colored Tverberg partition.

Let us remark that the index of K becomes larger with respect to inclusion than in Theorem 2.2 if just one color class C_i has more than $r-1$ elements. That is, in this case our proof of Theorem 1.2 does not work anymore. In fact, for any r and d there exist $N+1$ colored points in $\mathbb{R}^d$ such that one color class is of size r and all other color classes are singletons that admit no colored Tverberg partition.

2.4 The index of the deleted join of the manifold

In this section we prove that $\mathrm{Ind}_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}$ contains a non-zero element in degree N. Together with Theorem 2.2 we deduce that $\mathrm{Ind}_{\mathbb{Z}_r} M^{*r}_{\Delta(r)}$ is not contained in $\mathrm{Ind}_{\mathbb{Z}_r}(K)$, hence by the monotonicity of the index, the test-map (3) does not exist, which finishes the proof.

We have inclusions

$$T_{M^{*r}} \ \hookrightarrow \ \left\{\sum \lambda_i x \in M^{*r} \ : \ \lambda_i > 0, \sum \lambda_i = 1, x \in M\right\} \cong M \times \Delta^\circ_{r-1} \ \hookrightarrow \ M^{*r},$$

where Δ°_{r-1} denotes the open $(r-1)$-simplex. Since M is a smooth $\mathbb{Z}_r$-invariant manifold, $T_{M^{*r}}$ has a $\mathbb{Z}_r$-equivariant tubular neighborhood in M^{*r}; see [6, Section VI.2]. Its closure can be described as the disk bundle $D(\xi)$ of an equivariant vector bundle ξ over M. We denote its sphere bundle by $S(\xi)$. The fiber F of ξ is as a $\mathbb{Z}_r$-representation the $(d+1)$-fold sum of W_r, where $W_r = \{x \in \mathbb{R}[\mathbb{Z}_r] \ : \ x_1 + \ldots + x_r = 0\}$ is the augmentation ideal of $\mathbb{R}[\mathbb{Z}_r]$.

The representation sphere $S(F)$ is of dimension $N-1$. It is a free $\mathbb{Z}_r$-space, hence its index is

$$\mathrm{Ind}_{\mathbb{Z}_r}(S(F)) = H^{*\geq N}(B\mathbb{Z}_r). \tag{5}$$

This can be directly deduced from the Leray–Serre spectral sequence associated to the Borel construction $E\mathbb{Z}_r \times_{\mathbb{Z}_r} S(F) \to B\mathbb{Z}_r$, noting that the images of the differentials to the bottom row give precisely the index of $S(F)$. The latter can be seen from the edge-homomorphism. For background on Leray–Serre spectral sequences we refer to [14, Chapters 5, 6].

The Leray–Serre spectral sequence associated to the fibration $S(\xi) \to M$ collapses at E_2, since $N = (r-1)(d+1) \geq d+1$ and hence there is no differential between non-zero entries. Thus the map $i^* : H^{N-1}(S(\xi)) \to H^{N-1}(S(F))$ induced by inclusion is surjective.

The Mayer–Vietoris sequence associated to the triple $(D(\xi), M^{*r}_{\Delta(r)}, M^{*r})$ contains the subsequence

$$H^{N-1}(M^{*r}_{\Delta(r)}) \oplus H^{N-1}(D(\xi)) \xrightarrow{j^*+k^*} H^{N-1}(S(\xi)) \xrightarrow{\delta} H^N(M^{*r}).$$

We see that $H^N(M^{*r})$ is zero: This follows from the formula

$$\widetilde{H}^{*+(r-1)}(M^{*r}) \cong \widetilde{H}^*(M)^{\otimes r},$$

as long as $N-(r-1) > re$, where e is the cohomological dimension of M. This inequality is equivalent to $d > \frac{r}{r-1}e$, which can be assumed by applying the reduction from Section 2.1 at least $\lfloor 1 + \frac{e}{r-1} \rfloor$ times. Hence we can assume that $H^N(M^{*r}) = 0$.

Furthermore inequality (1) implies that $N-1 \geq d > \operatorname{cohdim}(M)$. Hence the term $H^{N-1}(D(\xi)) = H^{N-1}(M)$ of the sequence is zero as well.

Thus the map $j^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(\xi))$ is surjective. Therefore the composition $(j \circ i)^* : H^{N-1}(M^{*r}_{\Delta(r)}) \to H^{N-1}(S(F))$ is surjective as well. We apply the Borel construction functor $E\mathbb{Z}_r \times_{\mathbb{Z}_r} (\_) \to B\mathbb{Z}_r$ to this map and apply Leray–Serre spectral sequences; see Figure 3.

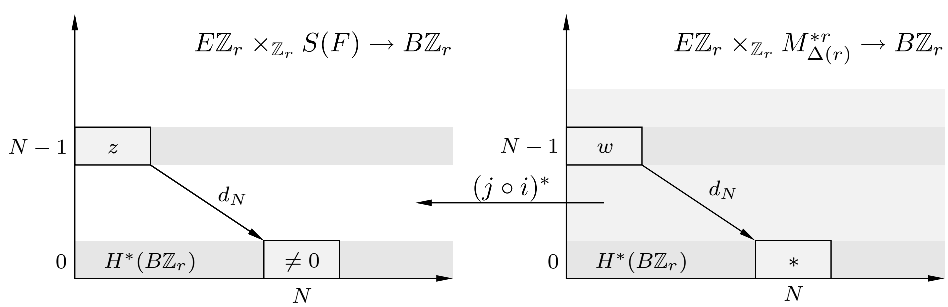

Fig. 3: We associate to the map $S(F) \xrightarrow{j \circ i} M^{*r}_{\Delta(r)}$ the Borel constructions and spectral sequences to deduce that $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N.

At the E_2-pages, the generator z of $H^{N-1}(S(F))$ has a preimage w since $(j \circ i)^*$ is surjective. At the E_N-pages $(j \circ i)^*(d_N(w)) = d_N(z)$, which is non-zero by (5). Hence $d_N(w) \neq 0$, which is an element in the kernel of the edge-homomorphism $H^*(B\mathbb{Z}_r) \to H^*_{\mathbb{Z}_r}(M^*_{\Delta(r)})$.

Therefore, the index of $M^{*r}_{\Delta(r)}$ contains a non-zero element in dimension N. This completes the proof of Theorem 1.2 if f can be extended to Δ^N. $\square$

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Shortest path poset of Bruhat intervals

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Abstract. Let $[u, v]$ be a Bruhat interval and $B(u, v)$ be its corresponding Bruhat graph. The combinatorial and topological structure of the longest u-v paths of $B(u, v)$ has been extensively studied and is well-known. Nevertheless, not much is known of the remaining paths. Here we describe combinatorial properties of the shortest u-v paths of $B(u, v)$. We also derive the non-negativity of some coefficients of the complete $\mathbf{cd}$-index of $[u, v]$.

Résumé. Soit $[u, v]$ un intervalle de Bruhat et $B(u, v)$ le graphe de Bruhat associé. La structure combinatoire et topologique des plus longs chemins de u à v dans $B(u, v)$ est bien comprise, mais on sait peu de chose des autres chemins. Nous décrivons ici les propriétés combinatoires des plus courts de chemins de u à v. Nous prouvons aussi que certains coefficients du $\mathbf{cd}$-indice complet de $[u, v]$ sont positifs.

Keywords: Bruhat interval, shortest-path poset, complete $\mathbf{cd}$-index

1 Introduction

While the paths of the Bruhat graph $B(u, v)$ of the Bruhat interval $[u, v]$ only depend on the isomorphism type of $[u, v]$ (see (Dye91)), all of the u-v paths of $B(u, v)$ are needed to compute the $\widetilde{R}$-polynomial, as well as the complete $\mathbf{cd}$-index of $[u, v]$. Unfortunately, the structure of $B(u, v)$ is not easy to understand. Thus we focus on the shortest paths of $B(u, v)$, since their combinatorial structure is more manageable. In particular, they form a Hasse diagram of a poset, which we denote by $SP(u, v)$.

The order of the paper is as follows: In Section 2 we summarize the basic properties of $SP(u, v)$, and describe their structure two specific cases: (i) if W is finite, with $u = e$ and $v = w_0^W$ (longest-length element of W) and (ii) if the number of rising chains (under a reflection order) is one. In Section 2.3 we provide an algorithm that allows us to separate the chains in $SP(u, v)$ into subposets, each of which has properties resembling properties of $[u, v]$. In Section 3 we derive consequences of the work done to the complete $\mathbf{cd}$-index.

1.1 Basic definitions

Let (W, S) be a Coxeter system, and let $T \stackrel{\text{def}}{=} T(W) = \{wsw^{-1} : s \in S, w \in W\}$ be the set of *reflections* of (W, S). The *Bruhat graph* of (W, S), denoted by $B(W, S)$ or simply $B(W)$, is the directed graph with vertex set W, and a directed edge $w_1 \to w_2$ between $w_1, w_2 \in W$ if $\ell(w_1) < \ell(w_2)$ and there exists $t \in T$ with $tw_1 = w_2$. Here ℓ denotes the *length function* of (W, S). The edges of $B(W)$ are labeled

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by reflections; for instance the edge $w_1 \to w_2$ is labeled with t. The Bruhat graph of an interval $[u, v]$, denoted by $B(u, v)$, is the subgraph of $B(W)$ obtained by only considering the elements of $[u, v]$. A *path* in the Bruhat graph $B(u, v)$, will always mean a *directed* path from u to v. As it is the custom, we will label these paths by listing the edges that are used. Furthermore, we denote the set of paths of length k in $B(u, v)$ by $B_k(u, v)$.

A *reflection order* $<_T$ is a total order of T so that $r <_T rtr <_T rtrtr <_T \ldots <_T trt <_T t$ or $t <_T trt <_T trtrt <_T \ldots <_T rtr <_T r$ for each Coxeter system $(\langle r, t\rangle, \{r, t\})$ where $r, t \in T$. Let $\Delta = (t_1, t_2, \ldots, t_k)$ be a path in $B(u, v)$, and define the *descent set* of Δ by $D(\Delta) = \{j : t_{j+1} <_T t_j\} \subset [k-1]$. If $D(\Delta) = \emptyset$, we say that Δ is rising.

Let $w(\Delta) = x_1 x_2 \cdots x_{k-1}$, where $x_i = \mathbf{a}$ if $t_i < t_{i+1}$, and $x_i = \mathbf{b}$, otherwise. In other words, set x_i to $\mathbf{a}$ if $i \notin D(\Delta)$ and to $\mathbf{b}$ if $i \in D(\Delta)$. Billera and Brenti (BB) showed that $\sum_{\Delta \in B(u,v)} w(\Delta)$ becomes a polynomial in the variables $\mathbf{c}$ and $\mathbf{d}$, where $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$. This polynomial is called the *complete* $\mathbf{cd}$*-index* of $[u, v]$, and it is denoted by $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$. Notice that the complete $\mathbf{cd}$-index of $[u, v]$ is an encoding of the distribution of the descent sets of each path Δ in the Bruhat graph of $[u, v]$, and thus seems to depend on $<_T$. However, it can be shown that this is not the case. For details on the complete $\mathbf{cd}$-index, see (BB).

As an example, consider S_3 with generators $s_1 = (1\ \ 2)$ and $s_2 = (2\ \ 3)$. Then $t_1 = s_1 <_T t_2 = s_1 s_2 s_1 <_T t_3 = s_2$ is a reflection ordering. The paths of length 3 are: $(t_1, t_2, t_3), (t_1, t_3, t_1), (t_3, t_1, t_3)$, and (t_3, t_2, t_1), that encode to $\mathbf{a^2} + \mathbf{ab} + \mathbf{ba} + \mathbf{b^2} = \mathbf{c^2}$. There is one path of length 1, namely t_2, which encodes simply to 1. So $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d}) = \mathbf{c^2} + 1$.

Given a monomial $m \in \mathbb{Z}\langle \mathbf{c}, \mathbf{d}\rangle$, we denote the coefficient of m in $\widetilde{\psi}_{u,v}(\mathbf{c}, \mathbf{d})$ by $[m]_{u,v}$. Notice that $[\mathbf{c}^n]_{u,v}$ is the number of rising paths in $B_{n+1}(u, v)$.

2 Shortest path poset

We begin with some basic properties of $SP(u, v)$.

Proposition 2.1 *Let $[u, v]$ be a Bruhat interval, then the undirected edges of the shortest paths of $B(u, v)$ form the Hasse diagram of a poset.*

We point out that in general the edges of paths in $B_k(u, v)$ need not form a Hasse diagram of a poset. Indeed, it is possible to have elements $u \leq x_0 < x_1 < x_2 < x_3 \leq v$ so that $x_0 \to x_1 \to x_2 \to x_3$ and $x_0 \to x_3$ are all in $B(u, v)$.

We call the poset of Proposition 2.1 , the *shortest path poset* of $[u, v]$, and we denote it by $SP(u, v)$. Furthermore, the edges of the Hasse diagram of $SP(u, v)$ inherit the labels of the corresponding edges in $B(u, v)$. In particular, we say that a maximal chain C in $SP(u, v)$ is *rising* if the path corresponding to C in $B(u, v)$ is rising.

Proposition 2.2 *$SP(u, v)$ is a graded poset, and for $x \in SP(u, v)$, the rank of x is the length of the shortest u-x path in $B(u, x)$.*

To illustrate the definition consider B_2 and $SP(e, \underline{1}\,\underline{2})$ as depicted in Figure 1. Notice that the rank of $SP(e, \underline{1}\,\underline{2})$ is 2, the length of the shortest paths in $B(B_2)$.

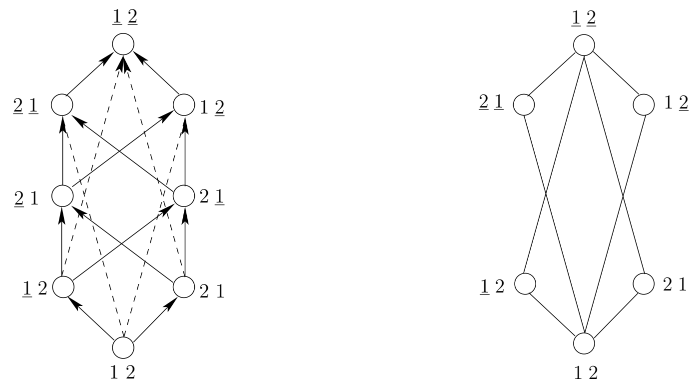


Fig. 1: $B(B_2)$ and $SP(B_2)$.

2.1 Finite Coxeter groups

For any finite Coxeter group W, there is a word w_0^W of maximal length. It is a well-known fact that $\ell(w_0^W) = |T|$. For any $w \in W$, one can write $t_1 t_2 \cdots t_n = w$ for some $t_1, t_2, \ldots, t_n \in T$. If n is minimal, then we say that w is T-*reduced*, and that the *absolute length* of w is n. We write $\ell_T(w) = n$.

Notice that for $w \in W$, if $\ell_T(w) = m$, then $t_1 t_2 \cdots t_m = w$ for some reflections in T, but this *does not* mean that $(t_1, t_2, \ldots, t_m)$ is a (directed) path in $B(e, w)$. Nevertheless, for finite W and $w = w_0^W$, $(t_1, t_2, \ldots, t_m)$ and any of its permutations $(t_{\tau(1)}, t_{\tau(2)}, \ldots, t_{\tau(m)})$, $\tau \in A_{m-1}$, are paths in $B(W)$ (see Theorem 2.3 below).

Let $SP(W)$ denote the poset $SP(e, w_0^W)$. The combinatorial structure of $SP(W)$ was described in (Bla09). For the sake of completeness, we include the main results therein.

Theorem 2.3 *Let W be a finite Coxeter group and $\ell_0 = \ell_T(w_0^W)$, the absolute length of the longest element of W. Then $SP(W)$ is isomorphic to the union of Boolean posets of rank ℓ_0. Each copy of $B(\ell_0)$ share at least e and w_0^W*

We summarize the number of Boolean posets that form $SP(W)$ and the rank of $SP(W)$ for each finite Coxeter group in Table 1.

where

$$b_n = 1 + \sum_{j=1}^{\lfloor \frac{n}{2} \rfloor} \frac{1}{j!} \prod_{i=0}^{j-1} \binom{n-2i}{2}.$$

and

$$d_m = \frac{1}{\lfloor \frac{m}{2} \rfloor !} \prod_{i=0}^{\lfloor \frac{m}{2} \rfloor - 1} \binom{m-2i}{2}$$

where $m = n$ if n is even, and $m = n-1$ if n is odd.

We point out that the union of the Boolean posets could share more elements than e and w_0^W. For instance, consider $SP(B_3)$ below.

Tab. 1: Finite coxeter groups W, rank$(SP(W))$, and the number of Boolean posets in $SP(W)$

W	rank$(SP(W))$	α_W = # of Boolean posets in $SP(W)$
A_{n-1}	$\lfloor \frac{n-1}{2} \rfloor$	1
B_n	n	b_n
D_n	n if n is even; $n-1$ if n is odd	d_n
$I_2(m)$	2 if m is even; 1 if m is odd	$\frac{m}{2}$ if m is even; 1 if m is odd
F_4	4	24
H_3	3	5
H_4	4	75
E_6	4	3
E_7	7	135
E_8	8	2025

While some elements other than e and $w_0^{B_3}$ are shared by more than one Boolean poset, each maximal chain belongs to a *unique* Boolean poset.

2.2 One rising chain

Since $[u,v]$ is *EL-shellable* (see (BW82) and (Dye93)), then $[u,v]$ has a unique maximal chain that is rising. So it is reasonable to study the structure of $SP(u,v)$ under the assumption that there is a unique rising chain. Even though this seems to be a strong assumption, there are several examples of Bruhat intervals where $SP(u,v)$ has a unique rising chain; for instance, $[21435, 53241]$.

An important tool in our study are the $\widetilde{R}$-*polynomials*, defined below.

Definition 2.4 ($\widetilde{R}$-polynomials) *Let $s \in S$ so that $\ell(vs) < \ell(v)$. Then define $\widetilde{R}_{u,v}(\alpha)$ by*

$$\widetilde{R}_{u,v}(\alpha) = \begin{cases} \widetilde{R}_{us,vs}(\alpha) & \text{if } \ell(us) < \ell(u), \\ \widetilde{R}_{us,vs}(\alpha) + \alpha \widetilde{R}_{u,vs}(\alpha) & \text{if } \ell(us) > \ell(vs). \end{cases}$$

Dyer (Dye01) provided an interpretation of $\widetilde{R}_{u,v}(\alpha)$ in terms of the number of rising paths of $B(u,v)$. Namely,

$$\widetilde{R}_{u,v}(\alpha) = \sum_{\substack{\Delta \in B(u,v) \\ D(\Delta)=\emptyset}} \alpha^{\ell(\Delta)}.$$

With this interpretation in mind, we have

Proposition 2.5 $\widetilde{R}_{u,y}(\alpha)\widetilde{R}_{y,v}(\alpha) \leq \widetilde{R}_{u,v}(\alpha)$.

We point out, in passing, that the above proposition generalizes Theorem 5.4, Corollary 5.5 and Theorem 5.6 in (Bre97) .

Proposition 2.5 yields the following theorem.

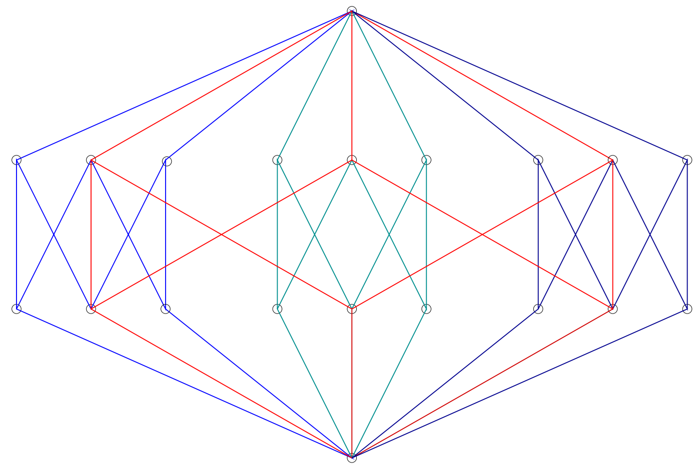


Fig. 2: $SP(B_3)$ has 4 copies of $B(3)$. Notice these copies intersect, but each maximal chain is in a unique Boolean poset.

Theorem 2.6 *If $SP(u,v)$ has a unique rising chain, then*
(a) $SP(u,v)$ is EL-shellable.
(b) $SP(u,v)$ is thin*, i.e., every subinterval of length two of $SP(u,v)$ has four elements.*

These topological properties will have consequences on the complete $\mathbf{cd}$-index, and it will be discussed in Section 3.

2.3 FLIP *algorithm*

Let $k+1 \stackrel{\text{def}}{=} \text{rank}(SP(u,v))$. An important distinction between $[u,v]$ and $SP(u,v)$ is that $[u,v]$ has a unique maximal, rising chain whereas $SP(u,v)$ could have more than one. So we propose an algorithm that splits the chains of $SP(u,v)$ into $[\mathbf{c}^k]_{u,v}$ posets P_i, $i = 1, \ldots, [\mathbf{c}^k]_{u,v}$. The structure of each P_i is easier to understand than $SP(u,v)$. So far we have been shown that the P_i have properties that resemble those of $[u,v]$.

We now follow (BB) to define the *flip* of $\Gamma \in B_2(u,v)$. Let (t_1,t_2) and (r_1,r_2) be in $B_2(u,v)$. We say that $(t_1,t_2) \leq_{lex} (r_1,r_2)$ if $t_1 <_T r_1$ or if $t_1 = r_1$ and $t_2 <_T r_2$, or $t_2 = r_2$. The existence of the complete $\mathbf{cd}$-index implies that there are as many paths with empty descent set in $B_2(u,v)$ as those with descent set $\{1\}$. Order all the paths in $B_2(u,v)$ lexicographically and let

$$r(\Gamma) = |\{\Delta \in B_2(u,v) : D(\Delta) = D(\Gamma), \Delta \leq_{lex} \Gamma\}|.$$

Definition 2.7 *With everything as above, we define the* flip *of Γ is the $r(\Gamma)$-th Bruhat path in $\{\Delta \in B_2(u,v) \mid D(\Delta) \neq D(\Gamma)\}$ ordered by $\leq_{lex}$. We denote this path by flip(Γ).*

Given $\Delta = (t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_k) \in B_k(u,v)$, we denote the path $(t_1, t_2, \ldots, t'_i, t'_{i+1}, \ldots, t_k)$, where $flip(t_i, t_{i+1}) = (t'_i, t'_{i+1})$, by $\text{FLIP}_i(\Delta)$. We are now ready to describe our algorithm.

The pseudocode of FLIP is given in Algorithm 1. In a few words, FLIP returns a (directed) graph G whose vertices are the maximal chains of $SP(u,v)$ and (C, C') is an edge if $\text{FLIP}_j(C) = C'$, where

Algorithm 1 FLIP$(SP(u, v))$

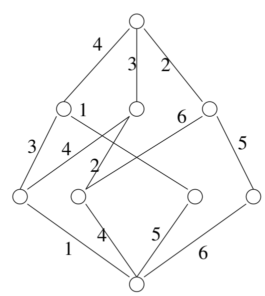
$G := (V, E)$, with V is the set of chains of $B(SP(u, v))$ and $E := \emptyset$.
$T := V$
for C a maximal chain of $SP(u, v)$ **do**
  if $D(C) \neq \emptyset$ **then**
    $i := \min D(C)$
    $C' :=$ FLIP$_i(C)$
    Add edge (C, C') to E.
  end if
end for
return G

$j = \min\{D(C)\}$. Notice that G has $[\mathbf{c}^k]_{u,v}$ connected components, say $G_1, G_2, \ldots, G_{[\mathbf{c}^k]_{u,v}}$. We define P_i to be the poset $SP(u, v)$ with all the chains (represented by vertices) *not* in G_i removed.

Let us illustrate FLIP with the following example. Notice that the chains in $SP(u, v)$ are represented by the labels assigned to the corresponding edges in the $B(u, v)$.

Example 1 *Consider the 10 elements of* $B_3(1234, 4312)$. *Then the output of* FLIP *is depicted below. In the first column we have the two components of* G*, and in the right column the posets* P_i *corresponding to each component.*

$$\begin{array}{l} 134 \leftarrow 143 \leftarrow 423 \leftarrow 462 \leftarrow 652 \\ \uparrow \\ 514 \end{array}$$



$$\begin{array}{l} 235 \leftarrow 251 \leftarrow 521 \\ \uparrow \\ 625 \end{array}$$

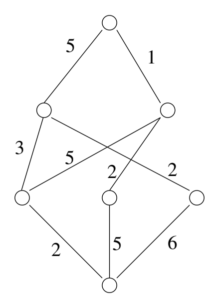


Fig. 3: On the left, we find the output of FLIP: two connected components. On the right the corresponding posets are depicted.

Each P_i satisfies properties resembling those of Bruhat intervals. Concretely, we have

Proposition 2.8 *(a) P_i is graded.*
(b) Every subinterval of P_i has at most one rising chain.
(c) Every subinterval of length two of P_i has at most two coatoms.

Bruhat intervals satisfy the properties above once we replace "at most" with "exactly".

2.4 FLIP *applied to* A_n, B_n *and* D_n

When applied to A_{n-1}, the output of FLIP is a unique graph G and the corresponding poset P is simply $SP(A_{n-1})$. Furthermore, one can choose a reflection order for the reflections of B_n (see (Bla11)) so that FLIP outputs b_n copies of $B(n)$ (see Table 1). For instance, FLIP($SP(B_3)$) separates $SP(B_3)$ into four copies of $B(3)$ (see Figure 2, where the four copies are drawn with different colors). The same holds, mutatis mutandis, for D_n.

So in these cases, FLIP produces the expected results: it divides $SP(W)$ into α_W subposets $P_1, \ldots, P_{\alpha_W}$ (where α_W is given in Table 1), and each P_i is a Boolean poset.

3 Connections to the complete $\mathbf{cd}$-index

In (Bla09), it is shown that the lowest-degree terms of $\widetilde{\psi}_{e,w_0^W}(\mathbf{c},\mathbf{d})$ are non-negative. Thus we have the theorem below.

Theorem 3.1 *If W is a finite Coxeter group, then he lowest degree terms of $\widetilde{\psi}_{e,w_0^W}(\mathbf{c},\mathbf{d})$ are nonnegative.*

In fact, these terms can be computed quite easily (see (Bla09) for details).

Now under the assumption of Theorem 2.6, $SP(u,v)$ is EL-shellable and thin. Thus Theorem 3.1.12 in (Wac07) yields the following proposition.

Proposition 3.2 *If $SP(u,v)$ has a unique rising chain, then it is a Gorenstein* poset.*

Now as a consequence of (Kar06, Theorem 4.10), we have the following theorem.

Theorem 3.3 *If $SP(u,v)$ has a unique rising chain, then the lowest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$ are non-negative.*

Moreover, in the case rank$(SP(u,v)) = 2$, the posets P_i described before Example 1 contribute a non-negative quantity to the lowest degree terms of $\widetilde{\psi}_{u,v}(\mathbf{c},\mathbf{d})$. We hope to extend this result to rank$(SP(u,v)) = 3$.

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I want to thank Louis Billera for many helpful discussions, as well as kind advice and encouragement throughout the years. I also want to thank Matthew Dyer for his hospitality during my visit to Notre Dame and his willingness to discuss his work.

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Relative Node Polynomials for Plane Curves

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Abstract. We generalize the recent work of Fomin and Mikhalkin on polynomial formulas for Severi degrees.

The degree of the Severi variety of plane curves of degree d and δ nodes is given by a polynomial in d, provided δ is fixed and d is large enough. We extend this result to generalized Severi varieties parametrizing plane curves which, in addition, satisfy tangency conditions of given orders with respect to a given line. We show that the degrees of these varieties, appropriately rescaled, are given by a combinatorially defined "relative node polynomial" in the tangency orders, provided the latter are large enough. We describe a method to compute these polynomials for arbitrary δ, and use it to present explicit formulas for $\delta \leq 6$. We also give a threshold for polynomiality, and compute the first few leading terms for any δ.

Résumé. Nous généralisons les travaux récents de Fomin et Mikhalkin sur des formules polynomiales pour les degrés de Severi.

Le degré de la variété de Severi des courbes planes de degré d et à δ nœuds est donné par un polynôme en d, pour δ fixé et d assez grand. Nous étendons ce résultat aux variétés de Severi généralisées paramétrant les courbes planes et qui, en outre, satisfont à des conditions de tangence d'ordres donnés avec une droite fixée. Nous montrons que les degrés de ces variétés, rééchelonnés de manière appropriée, sont donnés par un "polynôme de noeud relatif", défini combinatoirement, en les ordres de tangence, dès que ceux-ci sont assez grands. Nous décrivons une méthode pour calculer ces polynômes pour $delta$ arbitraire, et l'utilisons pour présenter des formules explicites pour $\delta \leq 6$. Nous donnons aussi un seuil pour la polynomialité, et calculons les premiers termes dominants pour tout δ.

Keywords: enumerative geometry, floor diagram, Gromov-Witten theory, node polynomial, tangency conditions

1 Introduction and Main Results

The *Severi degree* $N^{d,\delta}$ is the degree of the Severi variety of (possibly reducible) nodal plane curves of degree d with δ nodes. Equivalently, $N^{d,\delta}$ is the number of such curves passing through $\frac{(d+3)d}{2} - \delta$ generic points in the complex projective plane $\mathbb{CP}^2$. Severi varieties have received considerable attention since they were introduced by F. Enriques [Enr12] and F. Severi [Sev21] around 1915. Much later, in 1986, J. Harris [Har86] achieved a celebrated breakthrough by showing their irreducibility.

In 1994, P. Di Francesco and C. Itzykson [DFI95] conjectured that the numbers $N^{d,\delta}$ are given by a polynomial in d, for a fixed number of nodes δ, provided d is large enough. S. Fomin and G. Mikhalkin [FM10, Theorem 5.1] established this polynomiality in 2009. More precisely, they showed that there exists, for every $\delta \geq 1$, a *node polynomial* $N_\delta(d)$ which satisfies $N^{d,\delta} = N_\delta(d)$ for all $d \geq 2\delta$.

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The polynomiality of $N^{d,\delta}$ and the polynomials $N_\delta(d)$ were known in the 19th century for $\delta = 1, 2, 3$. For $\delta = 4, 5, 6$, this was only achieved by I. Vainsencher [Vai95] in 1995. In 2001, S. Kleiman and R. Piene [KP04] settled the cases $\delta = 7, 8$. In [Blo11], the author computed $N_\delta(d)$ for $\delta \leq 14$ and improved the threshold of S. Fomin and G. Mikhalkin by showing that $N^{d,\delta} = N_\delta(d)$ provided $d \geq \delta$.

Severi degrees can be generalized to incorporate tangency conditions to a fixed line $L \subset \mathbb{CP}^2$. More specifically, the *relative Severi degree* $N^\delta_{\alpha,\beta}$ is the number of (possibly reducible) nodal plane curves with δ nodes which have tangency of order i to L at α_i *fixed* points (chosen in advance) and tangency of order i to L at β_i *unconstrained* points, for all $i \geq 1$, and which pass through an appropriate number of generic points. Equivalently, $N^\delta_{\alpha,\beta}$ is the degree of the *generalized Severi variety* studied in [CH98, Vak00]. By Bézout's Theorem, the degree of a curve with tangencies of order (α, β) equals $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$. The number of point conditions (for a potentially finite count) is $\frac{(d+1)d}{2} - \delta + \beta_1 + \beta_2 + \cdots$. We recover non-relative Severi degrees by specializing to $\alpha = (0, 0, \dots)$ and $\beta = (d, 0, 0, \dots)$. The numbers $N^\delta_{\alpha,\beta}$ are determined by the rather complicated Caporaso-Harris recursion [CH98].

In this paper, we show that much of the story of (non-relative) node polynomials carries over to relative Severi degrees. Our main result is that, up to a simple combinatorial factor and for fixed $\delta \geq 0$, the relative Severi degrees $N^\delta_{\alpha,\beta}$ are given by a *multivariate polynomial* in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$, provided that $\beta_1 + \beta_2 + \dots$ is sufficiently large. For a sequence $\alpha = (\alpha_1, \alpha_2, \dots)$ of non-negative integers with finitely many α_i non-zero we write

$$|\alpha| \stackrel{\text{def}}{=} \alpha_1 + \alpha_2 + \cdots, \qquad \alpha! \stackrel{\text{def}}{=} \alpha_1! \cdot \alpha_2! \cdot \cdots.$$

Throughout the paper we use the grading $\deg(\alpha_i) = \deg(\beta_i) = 1$ (so that d and $|\beta|$ are homogeneous of degree 1). The following is our main result.

Theorem 1.1 *For every $\delta \geq 0$, there is a combinatorially defined polynomial $N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots)$ of (total) degree 3δ such that, for all $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ with $|\beta| \geq \delta$, the relative Severi degree $N^\delta_{\alpha,\beta}$ is given by*

$$N^\delta_{\alpha,\beta} = 1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot N_\delta(\alpha_1, \alpha_2, \dots; \beta_1, \beta_2, \dots). \tag{1.1}$$

We call $N_\delta(\alpha; \beta)$ the *relative node polynomial* and use the same notation as in the non-relative case if no confusion can occur. We do not need to specify the number of variables in light of the following stability condition.

Theorem 1.2 *For $\delta \geq 0$ and vectors $\alpha = (\alpha_1, \dots, \alpha_m)$, $\beta = (\beta_1, \dots, \beta_{m'})$ with $|\beta| \geq \delta$, it holds that*

$$N_\delta(\alpha, 0; \beta) = N_\delta(\alpha; \beta) \quad \textit{and} \quad N_\delta(\alpha; \beta, 0) = N_\delta(\alpha; \beta)$$

as polynomials. Therefore, there exists a formal power series $N^\infty_\delta(\alpha; \beta)$ in infinitely many variables $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ which specializes to all relative node polynomials under $\alpha_{m+1} = \alpha_{m+2} = \cdots = 0$ and $\beta_{m'+1} = \beta_{m'+2} = \cdots = 0$, for various $m, m' \geq 0$.

In fact, even more is true.

Proposition 1.3 *For $\delta \geq 0$ the relative node polynomial $N_\delta(\alpha, \beta)$ is a polynomial in d, $|\beta|$, $\alpha_1, \dots, \alpha_\delta$, and $\beta_1, \dots, \beta_\delta$, where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$.*

Using the combinatorial description we provide a method for computing the relative node polynomials for arbitrary δ (see Sections 3 and 4). We utilize it to compute $N_\delta(\alpha; \beta)$ for $\delta \leq 6$. The polynomials N_0 and N_1 already appeared (implicitly) in [FM10, Section 4.2].

Theorem 1.4 *The relative node polynomials $N_\delta(\alpha;\beta)$, for $\delta \le 3$ (resp., $\delta \le 6$) are as listed in [Blo10, Appendix A] (resp., as provided in the ancillary files accompanying [Blo10]).*

The polynomial $N_\delta(\alpha;\beta)$ is of degree 3δ by Theorem 1.1. We compute its terms of degree $\ge 3\delta - 2$.

Theorem 1.5 *The terms of $N_\delta(\alpha;\beta)$ of (total) degree $\ge 3\delta-2$ are given by*

$$\begin{aligned} N_\delta(\alpha;\beta) = \frac{3^\delta}{\delta!}\Big[& d^{2\delta}|\beta|^\delta + \frac{\delta}{3}\big[-\frac{3}{2}(\delta-1)d^2 - 8d|\beta| + |\beta|\alpha_1 + d\beta_1 + |\beta|\beta_1\big] d^{2\delta-2}|\beta|^{\delta-1} + \\ & + \frac{\delta}{9}\big[\frac{3}{8}(\delta-1)(\delta-2)(3\delta-1)d^4 + 12\delta(\delta-1)d^3|\beta| + (11\delta+1)d^2|\beta|^2 + \\ & - \frac{3}{2}\delta(\delta-1)(d^3\beta_1 + d^2|\beta|\alpha_1) - \frac{1}{2}(\delta+5)(3\delta-2)d^2|\beta|\beta_1 - 8(\delta-1)(d|\beta|^2\alpha_1 + d|\beta|^2\beta_1) + \\ & + \frac{1}{2}(\delta-1)(d^2\beta_1^2 + |\beta|^2\alpha_1^2 + |\beta|^2\beta_1^2) + (\delta-1)(d|\beta|\alpha_1\beta_1 + d|\beta|\beta_1^2 + |\beta|^2\alpha_1\beta_1)\big] d^{2\delta-4}|\beta|^{\delta-2} + \cdots \Big], \end{aligned}$$

where $d = \sum_{i\ge 1} i(\alpha_i + \beta_i)$.

Theorem 1.5 can be extended to terms of $N_\delta(\alpha,\beta)$ of degree $\ge 3\delta - 7$ (see Remark 5.2). We observe that all coefficients of $N_\delta(\alpha;\beta)$ in degree $\ge 3\delta-2$ are of the form $\frac{3^\delta}{\delta!}$ times a polynomial in δ. Without computating the coefficients, we can extended this further. It is conceivable to expect this property to hold for arbitrary degrees.

Proposition 1.6 *Every coefficient of $N_\delta(\alpha;\beta)$ in degree $\ge 3\delta - 7$ is given, up to a factor of $\frac{3^\delta}{\delta!}$, by a polynomial in δ with rational coefficients.*

Our approach to planar enumerative geometry is combinatorial and inspired by *tropical geometry*, a piecewise-linear analogue of algebraic geometry (see, for example, [Gat06]). By the celebrated Correspondence Theorem of G. Mikhalkin [Mik05, Theorem 1] one can replace the algebraic curve count in $\mathbb{CP}^2$ by an enumeration of certain *tropical curves*. E. Brugallé and G. Mikhalkin [BM07, BM09] introduced a class of decorated graphs, called *(marked) floor diagrams* (see Section 2), which, if counted correctly, are equinumerous to such tropical curves. We use a version of these results which incorporates tangency conditions due to S. Fomin and G. Mikhalkin [FM10] (see Theorem 2.4). S. Fomin and G. Mikhalkin also introduced a *template decomposition* of floor diagrams which we extend to be suitable for the relative case. This decomposition is crucial in the proofs of all results in this paper, as is the reformulation of algebraic curve counts in terms of floor diagrams.

For some related work see [AB10], where F. Ardila and the author generalized the polynomiality of Severi degrees to a class of toric surfaces which contains $\mathbb{CP}^1 \times \mathbb{CP}^1$ and Hirzebruch surfaces but which are non-smooth in general. A main feature is that we show polynomiality not only in the multi-degree of the curves but also in the parameters of the surface. In [BGM11], A. Gathmann, H. Markwig and the author defined *Psi-floor diagrams* which enumerate plane curves which satisfy point and tangency conditions, and conditions given by *Psi-classes*. We proved a Caporaso-Harris type recursion for Psi-floor diagrams, and show that *relative descendant Gromov-Witten invariants* equal their tropical counterparts.

This paper is organized as follows. In Section 2 we review the definition of floor diagrams and their markings. In Section 3 we introduce a new decomposition of floor diagrams which is compatible with tangency conditions. In Section 4 we discuss Theorems 1.1,1.2, 1.4 and Proposition 1.3. In Section 5 we discuss Theorem 1.5 and Proposition 1.6. For complete proofs of all statements see [Blo10].

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2 Floor diagrams and relative markings

Floor diagrams are a class of decorated graphs which, if counted with appropriate multiplicity, enumerate plane curves with prescribed properties. They were introduced by E. Brugallé and G. Mikhalkin [BM07, BM09] in the non-relative case and generalized to the relative setting by S. Fomin and G. Mikhalkin [FM10]. We begin with a review of the relative setup following notation of [FM10].

Definition 2.1 *A* floor diagram $\mathcal{D}$ *on a vertex set* $\{1,\dots,d\}$ *is a directed graph (possibly with multiple edges) with edge weights* $w(e)\in\mathbb{Z}_{>0}$ *satisfying:*

1. *The edge directions preserve the vertex order, i.e., for each edge* $i\to j$ *of* $\mathcal{D}$ *we have* $i<j$.
2. *(Divergence Condition) For each vertex* j *of* $\mathcal{D}$:

$$\mathit{div}(j) \overset{\mathit{def}}{=} \sum_{\substack{\text{edges } e\\ j\xrightarrow{e}k}} w(e) - \sum_{\substack{\text{edges } e\\ i\xrightarrow{e}j}} w(e) \le 1.$$

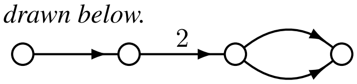
This means that at every vertex of $\mathcal{D}$ *the total weight of the outgoing edges is larger by at most 1 than the total weight of the incoming edges.*

The *degree* of a floor diagram $\mathcal{D}$ is the number of its vertices. It is *connected* if its underlying graph is. Note that in [FM10] floor diagrams are required to be connected. If $\mathcal{D}$ is connected its *genus* is the genus of the underlying graph (or the first Betti number of the underlying topological space). The *cogenus* of a connected floor diagram $\mathcal{D}$ of degree d and genus g is given by $\delta(\mathcal{D}) = \frac{(d-1)(d-2)}{2} - g$. If $\mathcal{D}$ is not connected let $d_1,d_2,\dots$ and $\delta_1,\delta_2,\dots$ be the degrees and cogenera, respectively, of its connected components. Then the *cogenus* of $\mathcal{D}$ is $\delta(\mathcal{D}) = \sum_j \delta_j + \sum_{j<j'} d_j d_{j'}$. Via the correspondence between algebraic curves and floor diagrams [BM09, Theorem 2.5] these notions correspond literally to the respective analogues for algebraic curves. Connectedness corresponds to irreducibility. Lastly, a marked floor diagram $\mathcal{D}$ has *multiplicity*[(i)]

$$\mu(\mathcal{D}) \overset{\text{def}}{=} \prod_{\text{edges } e} w(e)^2.$$

We draw floor diagrams using the convention that vertices in increasing order are arranged left to right. Edge weights of 1 are omitted.

Example 2.2 *An example of a floor diagram of degree* $d=4$, *genus* $g=1$, *cogenus* $\delta=2$, *divergences* $1,1,0,-2$, *and multiplicity* $\mu=4$ *is drawn below.*



To enumerate algebraic curves with floor diagrams we need the notion of markings of such diagrams. Our notation, which is more convenient for our purposes, differs slightly from [FM10] where S. Fomin and G. Mikhalkin define relative markings relative to the partitions $\lambda=\langle 1^{\alpha_1}2^{\alpha_2}\cdots\rangle$ and $\rho=\langle 1^{\beta_1}2^{\beta_2}\cdots\rangle$. In the sequel, all sequences are sequences of non-negative integers with finite support.

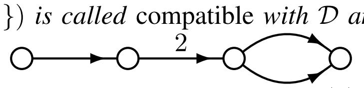
Definition 2.3 *For two sequences* α,β *we define an* (α,β)-marking *of a floor diagram* $\mathcal{D}$ *of degree* $d=\sum_{i\ge1} i(\alpha_i+\beta_i)$ *by the following four step process which we illustrate in the case of Example 2.2 for* $\alpha=(1,0,0,\dots)$ *and* $\beta=(1,1,0,0,\dots)$.

Step 1: *Fix a pair of collections of sequences* $(\{\alpha^i\},\{\beta^i\})$, *where* i *runs over the vertices of* $\mathcal{D}$, *with:*

(i) If floor diagrams are viewed as floor contractions of tropical plane curves this corresponds to the notion of multiplicity of tropical plane curves.

1. *The sums over each collection satisfy* $\sum_{i=1}^{d} \alpha^i = \alpha$ *and* $\sum_{i=1}^{d} \beta^i = \beta$.
2. *For all vertices* i *of* $\mathcal{D}$ *we have* $\sum_{j \geq 1} j(\alpha_j^i + \beta_j^i) = 1 - div(i)$.

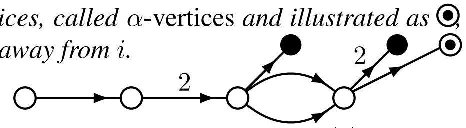
The second condition says that the "degree of the pair (α^i, β^i)*" is compatible with the divergence at vertex* i. *Each such pair* $(\{\alpha^i\}, \{\beta^i\})$ *is called* compatible *with* $\mathcal{D}$ *and* (α, β). *We omit writing down trailing zeros.*



$\alpha^i =$ (1)

$\beta^i =$ (1) $(0,1)$

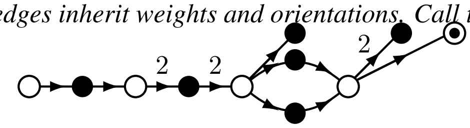
Step 2: *For each vertex* i *of* $\mathcal{D}$ *and every* $j \geq 1$ *create* β_j^i *new vertices, called* β-vertices *and illustrated as* ●*, and connected them to* i *with new edges of weight* j *directed away from* i. *For each vertex* i *of* $\mathcal{D}$ *and every* $j \geq 1$ *create* α_j^i *new vertices, called* α-vertices *and illustrated as* ⊙*, and connected them to* i *with new edges of weight* j *directed away from* i.



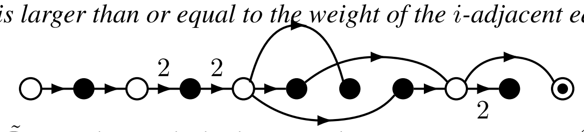
$\alpha^i =$ (1)

$\beta^i =$ (1) $(0,1)$

Step 3: *Subdivide each edge of the original floor diagram* $\mathcal{D}$ *into two directed edges by introducing a new vertex for each edge. The new edges inherit weights and orientations. Call the resulting graph* $\tilde{\mathcal{D}}$.



Step 4: *Linearly order the vertices of* $\tilde{\mathcal{D}}$ *extending the order of the vertices of the original floor diagram* $\mathcal{D}$ *such that, as in* $\mathcal{D}$, *each edge is directed from a smaller vertex to a larger vertex. Furthermore, we require that the* α*-vertices are largest among all vertices, and for every pair of* α*-vertices* $i' > i$, *the weight of the* i'*-adjacent edge is larger than or equal to the weight of the* i*-adjacent edge.*



We call the extended graph $\tilde{\mathcal{D}}$, *together with the linear order on its vertices, an* (α, β)-marked floor diagram*, or an* (α, β)-marking *of the floor diagram* $\mathcal{D}$.

We need to count (α, β)-marked floor diagrams up to equivalence. Two (α, β)-markings $\tilde{\mathcal{D}}_1$, $\tilde{\mathcal{D}}_2$ of a floor diagram $\mathcal{D}$ are *equivalent* if there exists a weight preserving automorphism of weighted graphs mapping $\tilde{\mathcal{D}}_1$ to $\tilde{\mathcal{D}}_2$ which fixes the vertices of $\mathcal{D}$. The *number of markings* $\nu_{\alpha,\beta}(\mathcal{D})$ is the number of (α, β)-marked floor diagrams $\tilde{\mathcal{D}}$ up to equivalence. Furthermore, we write $\mu_\beta(\mathcal{D})$ for the product $1^{\beta_1} 2^{\beta_2} \cdots \mu(\mathcal{D})$. The next theorem follows from [FM10, Theorem 3.18] by a straight-forward extension of the inclusion-exclusion procedure of [FM10, Section 1] which was used to conclude [FM10, Corollary 1.9] (the non-relative count of reducible curves via floor diagrams) from [FM10, Theorem 1.6] (the non-relative count of irreducible curves via floor diagrams).

Theorem 2.4 *For any* $\delta \geq 1$, *the relative Severi degree* $N^\delta_{\alpha,\beta}$ *is given by*

$$N^\delta_{\alpha,\beta} = \sum_{\mathcal{D}} \mu_\beta(\mathcal{D}) \nu_{\alpha,\beta}(\mathcal{D}),$$

where the sum is over (possibly disconnected) floor diagrams $\mathcal{D}$ *of degree* $d = \sum_{i \geq 1} i(\alpha_i + \beta_i)$ *and cogenus* δ.

3 Relative Decomposition of Floor Diagrams

In this section we introduce a new decomposition of floor diagrams compatible with tangency conditions. It is crucial in the proofs of all results stated in Section 1. The new decomposition is an extension of ideas of S. Fomin and G. Mikhalkin [FM10]. We start out by reviewing their key gadget.

Definition 3.1 *A* template Γ *is a directed graph (possibly with multiple edges) on vertices* $\{0,\dots,l\}$*, where* $l \geq 1$*, and edge weights* $w(e) \in \mathbb{Z}_{>0}$*, satisfying:*

1. *If* $i \to j$ *is an edge then* $i < j$*.*
2. *Every edge* $i \xrightarrow{e} i+1$ *has weight* $w(e) \geq 2$*. (No "short edges.")*
3. *For each vertex* j*,* $1 \leq j \leq l-1$*, there is an edge "covering" it, i.e., there exists an edge* $i \to k$ *with* $i < j < k$*.*

Every template Γ comes with some numerical data associated with it. Its *length* $l(\Gamma)$ is the number of vertices minus 1. The product of squares of the edge weights is its *multiplicity* $\mu(\Gamma)$. Its *cogenus* $\delta(\Gamma)$ is

$$\delta(\Gamma) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big].$$

For $1 \leq j \leq l(\Gamma)$ let $\varkappa_j = \varkappa_j(\Gamma)$ denote the sum of the weights of edges $i \xrightarrow{e} k$ with $i < j \leq k$ and define

$$k_{min}(\Gamma) \stackrel{\text{def}}{=} \max_{1 \leq j \leq l} (\varkappa_j - j + 1).$$

This makes $k_{min}(\Gamma)$ the smallest positive integer k such that Γ can appear in a floor diagram on $\{1,2,\dots\}$ with left-most vertex k. See [FM10, Figure 10] for a list of all templates Γ with $\delta(\Gamma) \leq 2$.

Our new decomposition of a floor diagram $\mathcal{D}$ depends on two (infinite) matrices A and B of non-negative integers. We require both to have only finitely many non-zero entries all of which lie above the respective dth row, where d is the degree of $\mathcal{D}$.

The triple $(\mathcal{D}, A, B)$ decomposes as follows. Let $l(A)$ and $l(B)$ be the largest row indices such that A and B have a non-zero entry in this row, respectively. After we remove all "short edges" from $\mathcal{D}$, i.e., all edges of weight 1 between consecutive vertices, the resulting graph is an ordered collection of templates $(\Gamma_1,\dots,\Gamma_r)$, listed left to right. Let k_s be the smallest vertex in $\mathcal{D}$ of each template Γ_s. Record all pairs (Γ_s, k_s) which satisfy $k_s + l(\Gamma_s) \leq d - \max(l(A), l(B))$. Record the remaining templates together with all vertices i, for $i \geq \max(l(A), l(B))$ in *one* graph Λ on vertices $0,\dots,l$ by shifting the vertex labels by $d-l$. See Example 3.4 for an example of this decomposition. Furthermore, by construction, if m is the number of recorded pairs (Γ_s, k_s), we have

$$\left\{ \begin{array}{rcll} k_i & \geq & k_{\min}(\Gamma_i) & \text{for } 1 \leq i \leq m, \\ k_{i+1} & \geq & k_i + l(\Gamma_i) & \text{for } 1 \leq i \leq m-1, \\ k_m + l(\Gamma_m) & \leq & d - l(\Lambda). & \end{array} \right. \tag{3.1}$$

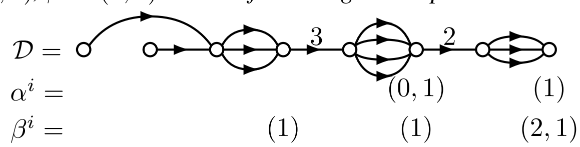
Fix a floor diagram $\mathcal{D}$. A partitioning of α and β into a compatible pair of collections $(\{\alpha^i, \beta^i\})$ (see Step 1 in Definition 2.3), where i runs over the vertices of $\mathcal{D}$, determines a pair of matrices A, B, if $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ are large enough as follows. Define the ith row a_i resp. b_i of the matrices A resp. B to be the sequence α^{d-i} resp. β^{d-i}, for $i \geq 1$, where d is the degree of $\mathcal{D}$. (If $d - i \leq 0$, set

$a_i = b_i = 0$.) In other words, the jth entry a_{ij} in row i of A is the number of α-edges of weight i adjacent to the $(j+1)$st vertex of Λ, counted from the right, and similarly for B (see Example 3.2). The sequences α^d and β^d (which are attached to the right-most vertex of $\mathcal{D}$) satisfy

$$\alpha^d = \alpha - \sum_{i\geq 1} a_i \quad \text{and} \quad \beta^d = \beta - \sum_{i\geq 1} b_i \tag{3.2}$$

if both expression are (component-wise) non-negative.

Example 3.2 *For $\alpha = (1,1)$, $\beta = (4,1)$ and the floor diagram $\mathcal{D}$ pictured below, the partitioning*



determines the matrices

$$A = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

In light of (3.2) we consider, for given tangency conditions α and β, only triples $(\mathcal{D}, A, B)$ with

$$\sum_{i\geq 1} a_i \leq \alpha \quad \text{and} \quad \sum_{i\geq 1} b_i \leq \beta \qquad \text{(always component-wise)}, \tag{3.3}$$

For fixed d, the decomposition

$$\big(\mathcal{D}, A, B\big) \longrightarrow \big(\{(\Gamma_s, k_s)\}, \Lambda, A, B\big). \tag{3.4}$$

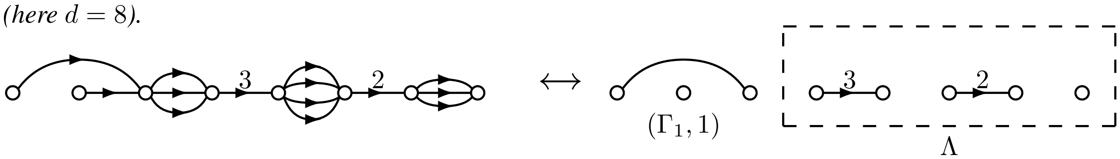
is reversible if the data on the right-hand side satisfies (3.1) and the tuple (Λ, A, B) is an "extended template."

Definition 3.3 *A tuple (Λ, A, B) is an* extended template *of* length $l = l(\Lambda) = l(\Lambda, A, B)$ *if Λ is a directed graph (possibly with multiple edges) on vertices $\{0, \ldots, l\}$, where $l \geq 0$, with edge weights $w(e) \in \mathbb{Z}_{>0}$, satisfying:*

1. *If $i \to j$ is an edge then $i < j$.*
2. *Every edge $i \xrightarrow{e} i+1$ has weight $w(e) \geq 2$. (No "short edges.")*

Moreover, we require A and B to be (infinite) matrices with non-negative integral entries and finite support, and we write $l(A)$ and $l(B)$ for the respective largest row indices of A and B of a non-zero entry. Additionally, we demand that $l(\Lambda) \geq \max(l(A), l(B))$ and that, for each $1 \leq j < l - \max(l(A), l(B))$, there is an edge $i \to k$ of Λ with $i < j < k$.

Example 3.4 *An example of a decomposition of a floor diagram $\mathcal{D}$ subject to the matrices A and B of Example 3.2 is pictured below. Once we fix the degree of the floor diagram the decomposition is reversible (here $d = 8$).*



The *cogenus* of an extended template (Λ, A, B) is the sum of the cogenera $\delta(\Lambda)$, $\delta(A)$ and $\delta(B)$, where

$$\delta(\Lambda) \stackrel{\text{def}}{=} \sum_{i \xrightarrow{e} j} \Big[(j-i)w(e) - 1\Big], \qquad \delta(A) \stackrel{\text{def}}{=} \sum_{i,j\geq 1} i \cdot j \cdot a_{i,j},$$

and similarly for B. It is not hard to see that the correspondence (3.4) is cogenus preserving in the sense that

$$\delta(\mathcal{D}) = \left(\sum_{i=1}^{m} \delta(\Gamma_i)\right) + \delta(\Lambda) + \delta(A) + \delta(B).$$

With an extended template (Λ, A, B) we associate the following numerical data. For $1 \leq j \leq l(\Lambda)$ let $\varkappa_j(\Lambda)$ denote the sum of the weights of edges $i \to k$ of Λ with $i < j \leq k$. Define $d_{min}(\Lambda, A, B)$ to be the smallest positive integer d such that (Λ, A, B) can appear (at the right end) in a floor diagram on $\{1, 2, \ldots, d\}$. We will see later that d_{min} is given by an explicit formula. For a matrix $A = (a_{ij})$ of non-negative integers with finite support define the "weighted lower sum sequence" $\mathrm{wls}(A)$ by

$$\mathrm{wls}(A)_i \stackrel{\text{def}}{=} \sum_{i' \geq i, j \geq 1} j \cdot a_{i'j}. \tag{3.5}$$

We now define the number of "markings" of templates and extended templates and relate them to the number of (α, β)-markings of the corresponding floor diagrams. To each template Γ we associate a polynomial as follows. For $k \geq k_{\min}(\Gamma)$ let $\Gamma_{(k)}$ denote the graph obtained from Γ by first adding $k + i - 1 - \varkappa_i$ short edges connecting $i - 1$ to i, for $1 \leq i \leq l(\Gamma)$, and then subdividing each edge of the resulting graph by introducing one new vertex for each edge. By [FM10, Lemma 5.6] the number of linear extensions (up to equivalence, see the paragraph after Definition 2.3) of the vertex poset of the graph $\Gamma_{(k)}$ extending the vertex order of Γ is given by a polynomial $P_\Gamma(k)$ in k, if $k \geq k_{min}(\Gamma)$(see [FM10, Figure 10]).

For each pair of sequences (α, β) and each extended template (Λ, A, B) satisfying (3.3) and $d \geq d_{min}$, where $d = \sum_{i\geq 1} i(\alpha_i + \beta_i)$, we define its "number of markings" as follows. Write $l = l(\Lambda)$ and let $\mathcal{P}(\Lambda, A, B)$ be the poset obtained from Λ by

1. first creating an additional vertex $l + 1$ $(> l)$,
2. then adding b_{ij} edges of weight j between $l - i$ and $l + 1$, for all $1 \leq i \leq l$ and $j \geq 1$,
3. then adding $\beta_j - \sum_{i\geq 1} b_{ij}$ edges of weight j between l and $l + 1$, for $j \geq 1$,
4. then adding
$$d - l(\Lambda) + i - 1 - \varkappa_i(\Lambda) - \mathrm{wls}(A)_{l+1-i} - \mathrm{wls}(B)_{l+1-i} \tag{3.6}$$
("short") edges of weight 1 connecting $i - 1$ and i, for $1 \leq i \leq l$, and finally
5. subdividing all edges of the resulting graph by introducing a midpoint vertex for each edge.

Denote by $Q_{(\Lambda,A,B)}(\alpha; \beta)$ the number of linear orderings on $\mathcal{P}(\Lambda, A, B)$ (up to equivalence) which extend the linear order on Λ. As $d \geq d_{\min}(\Lambda, A, B)$ if and only if (3.6) is non-negative, for $1 \leq i \leq l$, we have

$$d_{min}(\Lambda, A, B) = \max_{1\leq i\leq l(\Lambda)} (l(\Lambda) - i + 1 + \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}).$$

For sequences $s, t_1, t_2, \ldots$ with $s \geq \sum_i t_i$ (component-wise) we denote by $\binom{s}{t_1, t_2, \ldots} \stackrel{\text{def}}{=} \frac{s!}{t_1! t_2! \cdots (s - \sum_i t_i)!}$ the multinomial coefficient of sequences.

We obtain all (α, β)-markings of the floor diagram $\mathcal{D}$ that come from a compatible pair of sequences $(\{\alpha^i\}, \{\beta^i\})$ by independently ordering the α-vertices and the non-α-vertices. The number such markings is given (via the correspondence (3.4)) by

$$\Big(\prod_{s=1}^{m} P_{\Gamma_s}(k_s)\Big) \cdot \binom{\alpha}{a_1^T, a_2^T, \ldots} \cdot Q_{(\Lambda, A, B)}(\alpha; \beta), \tag{3.7}$$

where $a_1^T, a_2^T, \ldots$ are the *column* vectors of A. We conclude this section by recasting relative Severi degrees in terms of templates and extended templates.

Proposition 3.5 *For any $\delta \geq 1$, the relative Severi degree $N^{\delta}_{\alpha,\beta}$ is given by*

$$\sum_{\substack{(\Gamma_1, \ldots, \Gamma_m), \\ (\Lambda, A, B)}} \Big(\prod_{s=1}^{m} \mu(\Gamma_s) \sum_{k_1, \ldots k_m} \prod_{s=1}^{m} P_{\Gamma_s}(k_s)\Big) \cdot \Big(\mu(\Lambda) \prod_{i \geq 1} i^{\beta_i} \binom{\alpha}{a_1, a_2, \ldots} Q_{(\Lambda, A, B)}(\alpha; \beta)\Big), \tag{3.8}$$

where the first sum is over all collections $(\Gamma_1, \ldots, \Gamma_m)$ of templates and all extended templates (Λ, A, B) satisfying (3.3), $d \geq d_{min}(\Lambda, A, B)$ and

$$\sum_{i=1}^{m} \delta(\Gamma_i) + \delta(\Lambda) + \delta(A) + \delta(B) = \delta,$$

and the second sum is over all positive integers $k_1, \ldots, k_m$ which satisfy (3.1).

4 Relative Severi Degrees and Polynomiality

We now turn to the discussion of the proofs of our main results by first mentioning a number of technical lemmata whose proofs can be found in [Blo10]. For a graph G, we denote by $\#E(G)$ the number of edges of G. We write $||A||_1 = \sum_{i,j \geq 1} a_{ij}$ for the 1-norm of a (possibly infinite) matrix $A = (a_{ij})$.

Lemma 4.1 *For every extended template (Λ, A, B) there is a polynomial $q_{(\Lambda, A, B)}$ in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ of degree $\#E(\Lambda) + ||B||_1 + \delta(B)$ such that for all α and β satisfying (3.3) the number $Q_{(\Lambda, A, B)}(\alpha; \beta)$ of linear orderings (up to equivalence) of the poset $\mathcal{P}(\Lambda, A, B)$ is given by*

$$Q_{(\Lambda, A, B)}(\alpha; \beta) = \frac{(|\beta| - \delta(B))!}{\beta!} \cdot q_{(\Lambda, A, B)}(\alpha; \beta)$$

provided $\sum_{i \geq 1} i(\alpha_i + \beta_i) \geq d_{min}(\Lambda, A, B)$.

Recall that, for an extended template (Λ, A, B), we defined $d_{\min} = d_{\min}(\Lambda, A, B)$ to be the smallest $d \geq 1$ such that $d - l(\Lambda) + i - 1 \geq \varkappa_i(\Lambda) + \mathrm{wls}(A)_{l(\Lambda)+1-i} + \mathrm{wls}(B)_{l(\Lambda)+1-i}$ for all $1 \leq i \leq l(\Lambda)$. Let i_0 be the smallest i for which equality is attained (it is easy to see that equality is attained for some i). Define the quantity $s(\Lambda, A, B)$ to be the number of edges of Λ from $i_0 - 1$ to i_0 (of any weight). See [Blo10, Figure 2] for examples.

Lemma 4.2 *For any extended template* (Λ, A, B) *and any* $\alpha, \beta \geq 0$ *(component-wise) with*

$$d_{\min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \sum_{i \geq 1} i(\alpha_i + \beta_i) \leq d_{\min}(\Lambda, A, B) - 1$$

we have $q_{(\Lambda, A.B)}(\alpha; \beta) = 0$*, where* $q_{(\Lambda, A, B)}$ *is the polynomial of Lemma 4.1.*

The next lemma specifies which extended templates are compatible with a given degree.

Lemma 4.3 *For every extended template* (Λ, A, B) *we have*

$$d_{min}(\Lambda, A, B) - s(\Lambda, A, B) \leq \delta(\Lambda) + \delta(A) + \delta(B) + 1.$$

Proof of Theorems 1.1 and 1.2: The basic idea of the proof is to show that all summands of (3.8) are polynomial in $\alpha_1, \alpha_2, \dots, \beta_1, \beta_2, \dots$ (up to the combinatorial factor), thus contribute polynomially to the relative node polynomial $N_\delta(\alpha; \beta)$. The first factor of each summand is an iterated "discrete integral" of polynomials and thus polynomial in d. For the second factor we use Lemma 4.1. Then we use Lemmata 4.2 and 4.3 to reduce the polynomiality threshold. For a detailed proof see [Blo10]. □

Remark 4.4 *Expression (3.8) gives, in principle, an algorithm to compute the relative node polynomial* $N_\delta(\alpha; \beta)$*, for any* $\delta \geq 1$*. In [Blo11, Section 3] we explain how to generate all templates of a given cogenus, and how to compute the first factor in (3.8). The generation of all extended templates of a given cogenus from the templates is straightforward, as is the computation of the second factor in (3.8).*

Proof of Theorem 1.4: Proposition 3.5 gives a combinatorial description of relative Severi degrees. The proof of Lemma 4.1 (see [Blo10]) provides a method to calculate the polynomial $Q_{(\Lambda, A, B)}(\alpha; \beta)$. All terms of expression (3.8) are explicit or can be evaluated using the techniques of [Blo11, Section 3]. This reduces the calculation to a (non-trivial) computer calculation. □

5 Coefficients of Relative Node Polynomials

We now turn towards the computation of the coefficients of the relative node polynomial $N_\delta(\alpha; \beta)$ of large degree for any δ. We propose a method to compute all terms of $N_\delta(\alpha; \beta)$ of degree $\geq 3\delta - t$, for any given $t \geq 0$. This method was used (with $t = 2$) to compute the terms in Theorem 1.5.

The main idea of the algorithm is that, even for general δ, only a small number of summands of (3.8) contribute to the terms of $N_\delta(\alpha; \beta)$ of large degree. A summand of (3.8) is indexed by a collection of templates $\tilde{\Gamma} = \{\Gamma_s\}$ and an extended template (Λ, A, B). To determine whether this summand might contribute to $N_\delta(\alpha; \beta)$ we define the *(degree) defects*

- of the collection of templates $\tilde{\Gamma}$ by $\operatorname{def}(\tilde{\Gamma}) \overset{\text{def}}{=} \sum_{s=1}^{m} \left[\delta(\Gamma_i)\right] - m,$ and
- of the extended template (Λ, A, B) by $\operatorname{def}(\Lambda, A, B) \overset{\text{def}}{=} \delta(\Lambda) + 2\delta(A) + 2\delta(B) - ||A||_1 - ||B||_1.$

The following lemma restricts the indexing set of (3.8) to the relevant terms, if only the leading terms of $N_\delta(\alpha; \beta)$ are of interest. For a proof see [Blo10].

Lemma 5.1 *The summand of (3.8) indexed by $\tilde{\Gamma}$ and (Λ, A, B) is of the form*

$$1^{\beta_1} 2^{\beta_2} \cdots \frac{(|\beta| - \delta)!}{\beta!} \cdot P(\alpha; \beta),$$

where $P(\alpha;\beta)$ is a polynomial in $\alpha_1, \alpha_2, \ldots, \beta_1, \beta_2, \ldots$ of degree $\leq 3\delta - \operatorname{def}(\tilde{\Gamma}) - \operatorname{def}(\Lambda, A, B)$.

Therefore, to compute the coefficients of degree $\geq 3\delta - t$ of $N_\delta(\alpha;\beta)$ for some $t \geq 0$, it suffices to consider only summands of (3.8) with $\operatorname{def}(\tilde{\Gamma}) \leq t$ and $\operatorname{def}(\Lambda, A, B) \leq t$.

One can proceed as follows. First, we can compute, for some formal variable $\tilde{\delta}$, the terms of degree $\geq 2\tilde{\delta} - t$ of the first factor of (3.8) to $N_{\tilde{\delta}}(\alpha;\beta)$, that is the terms of degree $\geq 2\tilde{\delta} - t$ of

$$R_{\tilde{\delta}}(d) \stackrel{\text{def}}{=} \sum \prod_{i=1}^{m} \mu(\Gamma_i) \sum_{k_m = k_{min}(\Gamma_m)}^{d - l(\Gamma_m)} P_{\Gamma_m}(k_m) \cdots \sum_{k_1 = k_m in(\Gamma_1)}^{k_2 - l(\Gamma_1)} P_{\Gamma_1}(k_1), \tag{5.1}$$

where the first sum is over all collections of templates $\tilde{\Gamma} = (\Gamma_1, \ldots, \Gamma_m)$ with $\delta(\tilde{\Gamma}) = \tilde{\delta}$. The leading terms of $R_{\tilde{\delta}}(d)$ can be computed with a slight modification of [Blo11, Algorithm 2] (by replacing, in the notation of [Blo11], C^{end} by C and M^{end} by M). The algorithm relies on the polynomiality of solutions of certain polynomial difference equations, which has been verified for $t \leq 7$, see [Blo11, Section 5] for more details.

Finally, to compute the coefficients of degree $\geq 3\delta - t$, it remains to compute all extended templates (Λ, A, B) with $\operatorname{def}(\Lambda, A, B) \leq t$ and collect the terms of degree $\geq 3\delta - t$ of the polynomial

$$R_{\tilde{\delta}}(d - l(\Lambda)) \cdot \mu(\Lambda) \binom{\alpha}{a_1^T, a_2^T, \ldots} \prod_{i=\delta(B)}^{\delta-1} (|\beta| - i) \cdot q_{(\Lambda,A,B)}(\alpha;\beta), \tag{5.2}$$

where, as before, $a_1^T, a_2^T, \ldots$ denote the column vectors of the matrix A, $q_{(\Lambda,A,B)}(\alpha;\beta)$ is the polynomial of Lemma 4.1, and $\tilde{\delta} = \delta - \delta(\Lambda, A, B)$. Notice that, for an indeterminant x and integers $c \geq 0$ and $\delta \geq 1$, we have the expansion

$$\prod_{i=c}^{\delta-1} (x - i) = \sum_{t=0}^{\delta-c} s(\delta - c, \delta - c - t)(x - c)^{\delta - c - t},$$

where $s(n,m)$ is the *Stirling number of the first kind* [Sta97, Section 1.3] for integers $n, m \geq 0$. Furthermore, with $\delta' = \delta - c$ the coefficients $s(\delta', \delta' - t)$ of the sum equal $\delta'(\delta' - 1) \cdots (\delta' - t) \cdot S_t(\delta')$, where S_t is the t-th *Stirling polynomial* [GKP94, (6.45)], for $t \geq 0$, and thus are polynomial in δ'. Therefore, we can compute the leading terms of the product in (5.2) by collecting the leading terms in the sum expansion above. Theorem 1.5 is proved by an implementation of this method.

Proof of Proposition 1.6: Using [Blo11, Algorithm 2] we can compute the terms of the polynomial $R_{\tilde{\Gamma}}(d)$ of degree $\geq 2\tilde{\delta} - 7$ (see [Blo11, Section 5]) and observe that all coefficients are polynomial in $\tilde{\delta}$. By the previous paragraph the coefficients of the expansion of the sum of (5.2) are polynomial in δ. This completes the proof. □

Remark 5.2 *It is straight-forward to compute the coefficients of $N_\delta(\alpha;\beta)$ of degree $\geq 3\delta - 7$ (and thereby to extend Theorem 1.5). Algorithm 3 of [Blo11] computes the coefficients of the polynomials $R_{\tilde{\delta}}(d)$ of degree $\geq 2\tilde{\delta} - 7$, and thus the desired terms can be collected from (5.2). We expect this method to compute the leading terms of $N_\delta(\alpha, \beta)$ of degree $\geq 3\delta - t$ for arbitrary $t \geq 0$ (see [Blo11, Section 5], especially Conjecture 5.5).*

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Arc Spaces and Rogers-Ramanujan Identities

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Abstract. Arc spaces have been introduced in algebraic geometry as a tool to study singularities but they show strong connections with combinatorics as well. Exploiting these relations we obtain a new approach to the classical Rogers-Ramanujan Identities. The linking object is the Hilbert-Poincaré series of the arc space over a point of the base variety. In the case of the double point this is precisely the generating series for the integer partitions without equal or consecutive parts.

Résumé. Les espaces des arcs ont été introduit pour étudier les singularités, mais ils ont aussi un lien fort avec la combinatoire. Ce lien permet une nouvelle approche vers les identités de Rogers-Ramanujan. L'objet permettant cette approche est la série de Hilbert-Poincaré de l'algèbre des arcs centrés en un point de la variété de base. Dans le cas où cette variété est le point double, cette série est la série génératrice des partitions d'un nombre entier sans parties égales ou consécutives.

Resumen. Los espacios de arcos han sido introducidos en geometría algebraica como una herramienta para estudiar singularidades, sin embargo también han mostrado una robusta conexión con la combinatoria. Exprimiendo estas relaciones obtenemos un nuevo enfoque de las identidades de Rogers-Ramanujan. El objeto vinculante son las series de Hilbert-Poincaré de los espacios de arcos en un punto de la variedad base. En el caso del punto doble estas series son precisamente las series generadoras de las particiones enteras sin partes iguales o consecutivas.

Keywords: formal power series, Hilbert-Poincaré series, partitions, Rogers-Ramanujan Identities, arc spaces, infinite dimensional Gröbner basis

1 Introduction

Arc spaces have first appeared in the work of John Nash (see Nash (1995)) to study resolution of singularities of algebraic varieties. Besides their geometric usefulness (see Ein and Mustaţă (2004), and Ishii (2007) for an overview) arc spaces show strong relations with combinatorics. In this extended abstract we indicate how to exploit this connection both for algebraic as well as combinatorial benefit. Especially, we give a new approach towards the well-known Rogers-Ramanujan identities via these ideas. An extended version of this abstract including complete proofs can be found in Bruschek et al. (2011).

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Before starting with an overview of this note let us emphasize the main algebraic and combinatorial aspects presented here. First, we suggest to study algebro-geometric properties of algebraic (or analytic) varieties via natural Hilbert-Poincaré series attached to arc spaces. Second, we propose to derive identities between partitions by looking at suitable ideals in a polynomial ring in countably many variables endowed with a natural grading. Connecting both ideas will demand handling Gröbner basis in countably many variables, a problem which has been successfully dealt with in different contexts over the last years (see Hillar and Sullivant (2009), Draisma (2010)). In the present situation – that is for very specific ideals – salvation from the natural obstruction of being infinitely generated comes in the shape of a derivation making the respective ideals differential.

We start in Section 2 with the definition of the arc space of an algebraic variety as the set of formal power series solutions (in one variable) to the defining equations of the variety. This data can be encoded in conditions on the coefficients of the power series, thus yielding countably many equations in countably many variables. It turns out that these equations have very nice properties: they are homogeneous with respect to a grading which endows the ith coefficient of the power series with 'weight' i, and they are generated from the defining equations by applying a specific derivation (which will be introduced in the proof of Lemma 4.4). The first property involves that we can consider the coordinate algebra of the arc space as a graded algebra and especially we can try to compute its corresponding Hilbert-Poincaré series. This is elaborated in Section 3 where we also encounter a classical combinatorial object: partitions. These naturally arise when computing weights of monomials. Indeed, a monomial $y_1^{\alpha_1} \cdots y_j^{\alpha_j}$ has weight $\alpha_1 \cdot 1 + \cdots + \alpha_j \cdot j$. Asking for the number of monomials (up to coefficients) of some weight m is thus asking for the number of partitions of m. In Section 4 – using a well-known result from the theory of partitions – we are able to compute the Hilbert-Poincaré series of a simple, though already interesting, algebraic variety. On the other hand, we can use standard techniques from commutative algebra to compute the Hilbert-Poincaré series for the double point (i.e., the algebraic variety given by one polynomial equation $y^2 = 0$ in one variable y), thus retrieving the Rogers-Ramanujan identities. Using triple or even n-fold points we would obtain Gordon's generalizations of the Rogers-Ramanujan identities (see Andrews (1998) for a precise statement of those). We conclude with a short synopsis of the theory of Hilbert-Poincaré series.

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2 Arc spaces

Let us briefly recall the notion of *arc space*. Essentially arc spaces are sets of solutions to polynomial equations in a formal power series ring in one indeterminate. To be more precise, let $f \in k[x_1, \dots, x_n]$ be a polynomial in n variables $x_1, \dots, x_n$ with coefficients in a field k. The formal power series ring in one variable t over the field k is written as $k[[t]]$. The *arc space* of the algebraic variety X defined by f is the set of power series solutions $x(t) = (x_1(t), \dots, x_n(t)) \in k[[t]]^n$ to the equation $f(x(t)) = 0$. We denote it by X_∞. This set turns out to be eventually algebraic in the sense that it is given by polynomial equations (though there are countably many of them). Indeed, write $x_j(t) = \sum_{i=0}^{\infty} x_j^{(i)} t^i$ with new variables $x_j^{(i)}$, $1 \leq j \leq n$ and $i \in \mathbb{N} = \{0, 1, 2, \dots, \}$. Expanding $f(x(t))$ as a power series in t gives

$$f(x(t)) = F_0 + F_1 t + F_2 t^2 + \cdots$$

with $F_i \in k[x_j^{(i)}; 1 \leq j \leq n, i \in \mathbb{N}]$ polynomials in the coefficients of t in $x(t)$ (see Bruschek (2010) or Bruschek (2009) for more combinatorial properties of these polynomials). Therefore, a given vector of formal power series $a(t) \in k[[t]]^n$ is an element of the arc space X_∞ if and only if its coefficients $a_j^{(i)}$ fulfill the equations $F_0, F_1, \dots$. Algebraically the corresponding set of solutions is described by its *coordinate algebra*

$$J_\infty(X) = k[x_j^{(i)}; 1 \leq j \leq n, i \in \mathbb{N}]/(F_0, F_1, \dots).$$

Note, that the equation F_0 is nothing but our original polynomial f written in variables $x_j^{(0)}$, $1 \leq j \leq n$. We will mostly be interested in the case where we substitute in the F_i for $(x_1^{(0)}, \dots, x_n^{(0)})$ an n-tuple $p = (p_1, \dots, p_n)$ which fulfills $f(p) = 0$, i.e., which describes a point on our algebraic variety X. The resulting algebra is called the *focussed arc algebra* and denoted by $J_\infty^p(X)$. Without loss of generality we may assume that p is the origin. If we write f_i for the polynomial we obtain after substituting 0 for all $x_j^{(0)}$ in F_i then

$$J_\infty^0(X) = k[x_j^{(i)}; 1 \leq j \leq n, i \geq 1]/(f_1, f_2, \dots).$$

For sake of completeness we introduce also the *jet spaces of* X. These are usually defined as truncated power series solutions to the defining equations of the variety. In the present situation the m*th jet space* X_m *of* X is

$$X_m = \{x(t) \in \left(k[[t]]/(t)^{m+1}\right)^n ; f(x(t)) = 0 \text{ mod } (t)^{m+1}\}.$$

Its coordinate algebra is simply given by

$$J_m(X) = k[x_j^{(i)}; 1 \leq j \leq n, 0 \leq i \leq m]/(F_0, F_1, \dots, F_m).$$

with F_i as introduced above. Accordingly, we can also define the focussed mth jet space as

$$J_m^0(X) = k[x_j^{(i)}; 1 \leq j \leq n, 1 \leq i \leq m]/(f_1, f_2, \dots, f_m).$$

It is obvious how these notions extend to the case that our base variety X is not given by one polynomial f but by finitely many of them. For simplicity of notation we will restrict our considerations to the hypersurface case though.

3 The Hilbert Poincaré series of an arc algebra

Let X be the algebraic variety defined by a polynomial $f \in k[x_1, \dots, x_n]$ with $f(0) = 0$ and $J_\infty(X)$ respectively $J_\infty^0(X)$ the corresponding arc algebra respectively focussed arc algebra (at 0). Both are quotient algebras of the polynomial ring $k[x_j^{(i)}; 1 \leq j \leq n, i \in \mathbb{N}]$. Besides its natural grading via the classical 'degree' we will endow this polynomial ring with a grading induced by the following weight: $\text{wt}(x_j^{(i)}) = i$.

Proposition 3.1 *The ideal* $(f_1, f_2, \ldots) \subseteq k[x_j^{(i)}; 1 \leq j \leq n, i \in \mathbb{N} \setminus \{0\}]$ *is homogeneous with respect to the weight* wt*; hence the focussed arc algebra* $J_\infty^0(X)$ *is a graded algebra. Moreover, all homogeneous parts* $J_\infty^0(X)_i$ *of weight* i *are finite dimensional* k*-vector spaces.*

Remark 3.2 *The property that* f_i *is homogeneous with respect to* wt *is inherited from* F_i *which is homogeneous with respect to* wt *as well.*

Having a graded k-algebra with finite dimensional homogeneous components it is natural to ask for the corresponding Hilbert-Poincaré series. This is the generating series for the sequence of dimensions

$$\dim_k J_\infty^0(X)_i, i \in \mathbb{N}.$$

For a short synopsis of technical results in the theory of Hilbert-Poincaré series we refer to the Appendix, Section 5, or a text on commutative algebra like Greuel and Pfister (2002). In the remaining sections we will refer to Hilbert-Poincaré series in short as HP-series. The HP-series of the focussed arc algebra $J_\infty^0(X)$ will be denoted by $\mathrm{HP}_{J_\infty^0(X)}(t)$.

Example 3.3 *As a first example we compute the HP-series of the arc space of the affine line* $\mathbb{A}^1$ *focussed at* 0*. The affine line has coordinate ring* $k[y]$*, thus,* $J_\infty^0(\mathbb{A}^1) = k[y_1, y_2, \ldots]$*. The* m*th homogeneous piece of* $J_\infty^0(\mathbb{A}^1)$ *with respect to* wt *is the* k*-vector space spanned by all monomials of weight* m*. Let* y^α *with* $\alpha \in \mathbb{N}^{(\mathbb{N})}$ *denote a monomial in* $k[y_1, y_2, \ldots]$*. Clearly, it has weight* m *exactly if*

$$\alpha_1 \cdot 1 + \cdots + \alpha_m \cdot m = m. \tag{1}$$

Note, that no y_i *with* $i > m$ *can appear in* y^α *since then the weight of the monomial would exceed* m*. Equation (1) shows that every monomial of weight* m *corresponds to a partition of* m*, i.e.,*

$$\dim_k J_\infty^0(\mathbb{A}^1)_m = p(m)$$

where $p\colon \mathbb{Z} \to \mathbb{Z}$ *denotes the* partition function *mapping every integer* m *to its number of partitions. Therefore we obtain (see Andrews (1998)):*

$$\mathrm{HP}_{J_\infty^0(\mathbb{A}^1)}(t) = \prod_{i=1}^{\infty} \frac{1}{1-t^i}.$$

Henceforth we will denote the HP-series computed in the last example by $\mathbb{H}$. It is not hard to prove that for the focussed arc algebra of affine n-space $\mathbb{A}^n$ the following is true

Proposition 3.4 *With the above introduced notation we obtain*

$$\mathrm{HP}_{J_\infty^0(\mathbb{A}^n)}(t) = \mathbb{H}^n.$$

Remark 3.5 *For the more algebro-geometric inclined reader we note that if* X *and* Y *are analytically isomorphic at* $p \in X$ *and* $q \in Y$ *then the corresponding focussed arc algebras are isomorphic as graded algebras, thus having the same HP-series. Therefore, Proposition 3.4 also gives the HP-series for a smooth point of an arbitrary* n*-dimensional variety.*

Remark 3.6 *Recall that the* multiplicity *or* order of $f \in k[x_1, \ldots, x_n]$ *is defined as the minimal degree of a monomial appearing in f, i.e.,*

$$\text{ord } f = \min\{|\alpha|; \alpha \in \text{supp}(f)\},$$

where we define $|\alpha| = \alpha_1 + \cdots + \alpha_n$ as usual. Moreover, we consider the truncation operator

$$\tau_{\leq r} : k[[t]] \to k[t] : \sum_{i \geq 0} a_i t^i \mapsto \sum_{i=0}^{r} a_i t^i.$$

As one would expect

$$\tau_{\leq m} \mathrm{HP}_{J_m^0(X)}(t) = \tau_{\leq m} \mathrm{HP}_{J_\infty^0(X)}(t).$$

From this it is not hard to show the following:

Proposition 3.7 *Let $f \in k[x_1, \ldots, x_n]$ with $f(0) = 0$ define the algebraic variety X. Then f has multiplicity r if and only if r is the maximal number such that*

$$\tau_{\leq r-1} \mathrm{HP}_{J_\infty^0(X)} = \tau_{\leq r-1} \mathbb{H}^n.$$

Moreover

$$\tau_{\leq r} \mathrm{HP}_{J_\infty^0(X)} = \tau_{\leq r} \mathbb{H}^n - t^r.$$

Using results of Ein and Mustaţă (2004) one can show the following, which was obtained in Mourtada (2010) by explicit computation:

Proposition 3.8 *If X is a surface with a rational double point at the origin then*

$$\mathrm{HP}_{J_\infty^0(X)}(t) = \left(\frac{1}{1-t}\right)^3 \left(\prod_{i \geq 2} \frac{1}{1-t^i}\right)^2.$$

A similar result holds for normal crossings singularities. Indeed, using Theorem 2.2 from Goward and Smith (2006) one can show:

Proposition 3.9 *Let X be the hypersurface given by $x_1 \cdots x_e = 0$, $e \leq n$, in $\mathbb{A}_k^n$. Then*

$$\mathrm{HP}_{J_\infty^0(X)}(t) = \left(\prod_{i=1}^{e-1} \frac{1}{1-t^i}\right)^n \left(\prod_{i \geq e} \frac{1}{1-t^i}\right)^{n-1}.$$

4 Rogers-Ramanujan Identities

The Rogers-Ramanujan Identities are well-known relations between quantities of certain integer partitions. They will appear in what follows for two reasons: one can use them to compute HP-series for some varieties and in turn we are able to approach them by computing the very same HP-series by different techniques.

We will use the Rogers-Ramanujan Identities in the following form (sometimes called *(first) Rogers-Ramanujan identity* in the literature). For a classical proof and an account of its history, see Andrews (1998) Chpt. 7.

Theorem 4.1 (Rogers-Ramanujan Identity) *The number of partitions of n into parts congruent to 1 or 4 modulo 5 is equal to the number of partitions of n into parts that are neither repeated nor consecutive.*

Its analytic counterpart can be formulated as (see Corollary 7.9 in Andrews (1998)):

Corollary 4.2 (Rogers-Ramanujan Identity, analytic form) *Theorem 4.1 is equivalent to the identity*

$$1+\frac{t}{1-t}+\frac{t^4}{(1-t)(1-t^2)}+\frac{t^9}{(1-t)(1-t^2)(1-t^3)}+\cdots=\prod_{i=1,4 \bmod 5}\frac{1}{(1-t^i)}.$$

4.1 The Hilbert-Poincaré series of the double point

Let us first use Theorem 4.1 to compute the Hilbert-Poincaré series of the double point $X\colon y^2=0$ in $\mathbb{A}^1$. The corresponding focussed arc algebra looks as follows:

$$J^0_\infty(X)=k[y_1,y_2,\ldots]/(2y_1^2,6y_1y_2,6y_2^2+8y_1y_3,\ldots).$$

As before we denote by f_i, $i\geq 2$, the generators of the defining ideal I of $J^0_\infty(X)$. In order to compute the HP-series of this algebra it suffices to compute the HP-series of the algebra

$$k[y_1,y_2,\ldots]/L(I)$$

where $L(I)$ is the leading ideal of I with respect to some weight-compatible monomial ordering on $k[y_1,y_2,\ldots]$, see Theorem 5.3. We endow $k[y_0,y_1,\ldots]$ (and consequently $k[y_1,y_2,\ldots]$) with the following monomial ordering: for $\alpha,\beta\in\mathbb{N}^{(\mathbb{N})}$ we have $y^\alpha>y^\beta$ if and only if wt $\alpha>$ wt β or, in case of equality, the last non-zero entry of $\alpha-\beta$ is negative. The leading monomial of f_i with respect to this ordering is determined as

Proposition 4.3 *The leading monomial of f_i is ($j\geq 1$)*

$$\mathrm{lm}(f_i)=\begin{cases} y_jy_{j+1} & i=2j+1\\ y_j^2 & i=2j\end{cases}.$$

From this we can derive in the present situation $L(I)$:

Lemma 4.4 *The leading ideal of $I=(f_i;i\geq 2)$ is given by* $(\mathrm{lm}(f_i);i\geq 2)$.

Remark 4.5 *More precisely the following holds: the leading monomials of $(f_2,\ldots,f_q)$ of weight less equal q are generated by* $\mathrm{lm}(f_i)$*, $2\leq i\leq q$. In other words: If we extend $\{f_2,\ldots,f_q\}$ to a Gröbner basis of $(f_2,\ldots,f_q)$ all added elements will be of weight larger equal $q+1$.*

Before giving a short sketch of the proof we introduce the following terminology (cf. Cox et al. (1997)): let $g\in k[x_1,\ldots,x_n]$ and let $(h_1,\ldots,h_q)\subseteq k[x_1,\ldots,x_n]$ be an ideal. We say that *g reduces to 0 modulo* $(h_1,\ldots,h_q)$ if there exist $a_i\in k[x_1,\ldots,x_n]$, $1\leq i\leq q$, with $g=a_1h_1+\cdots+a_qh_q$ and any leading monomial of a_ih_i, $1\leq i\leq q$, is less or equal to the leading monomial of g.

Sketch of Proof: According to the theory of Gröbner basis it suffices to show that all S-polynomials of the generators f_i reduce to 0 modulo the ideal $(f_2,f_3,\ldots)$. In addition, we may restrict our considerations to $S(f_i,f_j)$ with coprime leading monomials $\mathrm{lm}(f_i)$ and $\mathrm{lm}(f_j)$ (see for example Cox et al. (1997), §9, Proposition 4). By Proposition 4.3 this reduces our investigation to S-polynomials of the following three types: $S(f_{2j-1}, f_{2j})$, $S(f_{2j}, f_{2j+1})$ and $S(f_{2j-1}, f_{2j+1})$. We prove that all these S-polynomials reduce to 0 by exploiting the differential structure of the arc ideal $F_0, F_1, \ldots$: every F_i can be obtained from F_{i-1} by applying the k-derivation

$$D\colon k[y_0, y_1, \ldots] \to k[y_0, y_1, \ldots]$$

given by $Dy_i = y_{i+1}$. Moreover, for $i \geq 2$ both F_i and f_i have the same leading term, thus, any S-polynomial of f_i and f_j lifts to an S-polynomial of F_i and F_j; it suffices to reduce $S(F_i, F_j)$ to 0. This can be achieved by applying an appropriate power of the derivation D to the simple relation

$$2y_1F_0 - y_0F_1 = 0.$$

To give an example, note that $S(F_3, F_4) = y_2F_3 - y_1F_4$, and

$$0 = D^4(2y_1F_0 - y_0F_1) = 2y_5F_0 + 7y_4F_1 + 8y_3F_2 + 2S(F_3, F_4) - y_0F_5.$$

This shows that $S(F_3, F_4)$ reduces to 0 modulo $(F_0, F_1, \ldots)$. □

Remark 4.6 *The task of proving Lemma 4.4 or more generally of determining the leading ideal for the defining ideal of an arc algebra is in essence the determination of a Gröbner basis for an ideal which has countably many generators. Fortunately, the infinitely many generators are not arbitrary but determined by a finite number of polynomials using the derivation D which was introduced in the above sketch of proof. Similar situations appeared in the work of Hillar and Sullivant (2009), and Draisma (2010).*

The computation of the HP-series of $k[y_1, y_2, \ldots]/L(I)$ allows an easy combinatorial interpretation: the weight of a monomial y^α can be interpreted, as we have seen already in Section 3, as an integer partition. By factoring out $L(I)$ we factor out all monomials y^α which contain as factors an y_i^2 or y_iy_{i+1}. Therefore, the weights of the remaining monomials correspond to integer partitions without repeated or consecutive parts. Thus, from Theorem 4.1 we deduce:

Theorem 4.7 *The Hilbert-Poincaré series of the focussed arc algebra $J_\infty^0(X)$ of the double line $X\colon y^2 = 0$ over the origin equals:*

$$\mathrm{HP}_{J_\infty^0(X)}(t) = \prod_{\substack{i=1,4 \\ \text{mod } 5}}^{\infty} \frac{1}{1-t^i}.$$

Remark 4.8 *We have recently learned from Edward Frenkel that this result can be obtained in a completely different way, namely by studying representations of the Virasoro algebra Feigin and Frenkel (1993).*

4.2 An alternative approach to Rogers-Ramanujan

In the previous section we used a combinatorial interpretation of the leading ideal of $I = (f_2, f_3, \ldots)$ to compute the HP-series of the corresponding graded algebra. There are commutative algebra methods to do this as well which yield a new approach to the Rogers-Ramanujan identity. By applying these we will obtain a recursion formula for the generating functions appearing therein which has already been considered by Andrews and Baxter (1989), though the present method gives a natural way to obtain it.

Consider the graded algebra $S = k[y_i; i \geq 1]/L(I)$. It is immediate (see the proof of Proposition 4.7) that its HP-series equals the generating series of the number of partitions of an integer n without repeated or consecutive parts. Differently, we compute the HP-series of S by recursively defining a sequence of formal power series (generating functions) in t which converges in the (t)-adic topology to the desired HP-series. We will simply write $k[\geq d]$ for the polynomial ring $k[y_i; i \geq d]$. It will be endowed with the grading wt $y_i = i$. The ideal generated by $y_i^2, y_i y_{i+1}$, $i \geq d$, in $k[\geq d]$ will be denoted by I_d. As usual, if E is an ideal in a ring R and $f \in R$ then we denote the ideal quotient, i.e.,

$$\{a \in R\,; a \cdot f \in E\}$$

by $(E : f)$. Corollary 5.2 implies in the present situation

$$\mathrm{HP}_{k[\geq d]/I_d}(t) = \mathrm{HP}_{k[\geq d+1]/I_{d+1}}(t) + t^d \cdot \mathrm{HP}_{k[\geq d+2]/I_{d+2}}(t).$$

For simplicity of notation let $h(d)$ stand for $\mathrm{HP}_{k[\geq d]/I_d}(t)$. Then the last equation reads as

$$h(d) = h(d+1) + t^d \cdot h(d+2) \tag{2}$$

and one deduces

Proposition 4.9 *For the HP-series* $\mathrm{HP}_{J^0_\infty(X)}(t) = h(1)$ *we obtain*

$$h(1) = A_d \cdot h(d) + B_{d+1} \cdot h(d+1),$$

for $A_i, B_i \in k[[t]]$ *fulfilling the following recursion*

$$\begin{aligned} A_d &= A_{d-1} + B_d \\ B_{d+1} &= A_{d-1} \cdot t^{d-1} \end{aligned}$$

with initial conditions $A_1 = A_2 = 1$ *and* $B_2 = 0, B_3 = t$.

If $(s_d)_{d \in \mathbb{N}}$ is a sequence of formal power series $s_d \in k[[t]]$ we will denote by $\lim s_d$ its limit – if it exists – in the (t)-adic topology. Since ord $B_d \geq d-2$ it is immediate that both $\lim A_d$ and $\lim B_d$ exist, in fact: $\lim B_d = 0$ and

$$h(1) = \lim A_d.$$

The recursion from Proposition 4.9 can easily be simplified. We obtain:

Corollary 4.10 *With the above introduced notation* $\mathrm{HP}_{J^0_\infty(X)}(t) = \lim A_d$ *where* A_d *fulfills*

$$A_d = A_{d-1} + t^{d-2} \cdot A_{d-2}$$

with initial conditions $A_1 = A_2 = 1$.

The recursion appearing in this corollary is well-known since Andrews and Baxter (1989). Its limit is precisely the infinite product

$$\prod_{\substack{i=1,4 \\ \bmod 5}}^{\infty} \frac{1}{1-t^i},$$

i.e., the generating series of the number of partitions with parts equal to 1 or 4 modulo 5. Note, that our construction gives the generating series G_i defined in Andrews and Baxter (1989) an interpretation as Hilbert-Poincaré series of the quotients $k[\geq i]/I_i$. This immediately implies that the series G_i are of the form $G_i = 1 + \sum_{j \geq i} G_{ij} t^j$.

5 Appendix

In this section we collect some of the basics about the theory of Hilbert-Poincaré series. For a detailed introduction, especially proofs, we refer to Greuel and Pfister (2002).

Let A be a ($\mathbb{Z}$-)graded k-algebra and let $M = \oplus_{i\in\mathbb{Z}} M_i$ be a graded A-module with ith graded pieces A_i and M_i of finite k-dimension. The *Hilbert function* $H_M\colon \mathbb{Z} \to \mathbb{Z}$ *of* M is defined by $H_M(i) = \dim_k M_i$, and its corresponding generating series

$$\mathrm{HP}_M(t) = \sum_{i\in\mathbb{Z}} H_M(i)t^i \in \mathbb{Z}[[t]]$$

is called the *Hilbert-Poincaré series of* M. It is well-known that if A is a Noetherian k-algebra generated by homogeneous elements $x_1, \dots, x_n$ of degrees $d_1, \dots, d_n$ and M is a finitely generated A-module then

$$\mathrm{HP}_M(t) = \frac{Q_M(t)}{\prod_{i=1}^n (1 - t^{d_i})}$$

for some $Q_M(t) \in \mathbb{Z}[t]$ which is called the *(weighted) first Hilbert series of* M. If A respectively M is non-Noetherian then the Hilbert-Poincaré series of M need not be rational anymore. For the rest of this section we assume that the polynomial ring $k[x_1, \dots, x_n]$ is graded (not necessarily standard graded). The notions of homogeneous ideal and degree are to be understood relative to this grading. If M is graded then for any integer d we write $M(d)$ for the *dth twist of* M, i.e., the graded A-module with $M(d)_i = M_{i+d}$.

The following lemma follows immediately from additivity of dimension:

Lemma 5.1 (Lemma 5.1.2 in Greuel and Pfister (2002)) *Let A and M be as above. Let d be a non-negative integer, $f \in A_d$ and $\varphi\colon M(-d) \to M$ be defined by $\varphi(m) = f \cdot m$; then* $\ker(\varphi)$ *and* $\operatorname{coker}(\varphi)$ *are graded $A/(f)$-modules with the induced gradings and*

$$\mathrm{HP}_M(t) = t^d \cdot \mathrm{HP}_M(t) + \mathrm{HP}_{\operatorname{coker}(\varphi)}(t) - t^d \cdot \mathrm{HP}_{\ker(\varphi)}(t).$$

As an immediate consequence we obtain the useful:

Corollary 5.2 (Lemma 5.2.2 in Greuel and Pfister (2002)) *Let $I \subseteq k[x_1, \dots, x_n]$ be a homogeneous ideal, and let $f \in k[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d then*

$$\mathrm{HP}_{k[x]/I}(t) = \mathrm{HP}_{k[x]/(I,f)}(t) + t^d \cdot \mathrm{HP}_{k[x]/(I:f)}(t).$$

For homogeneous ideals the leading ideal already determines the Hilbert-Poincaré series:

Theorem 5.3 (Theorem 5.2.6 in Greuel and Pfister (2002)) *Let $>$ be a graded monomial ordering on a polynomial ring $k[x_1, \dots, x_n]$, let $I \subseteq k[x]$ be a homogeneous ideal and denote by $L(I)$ its leading ideal with respect to $>$. Then*

$$\mathrm{HP}_{k[x]/I}(t) = \mathrm{HP}_{k[x]/L(I)}(t).$$

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Partition and composition matrices: two matrix analogues of set partitions

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Abstract. This paper introduces two matrix analogues for set partitions; partition and composition matrices. These two analogues are the natural result of lifting the mapping between ascent sequences and integer matrices given in Dukes & Parviainen (2010).

We prove that partition matrices are in one-to-one correspondence with inversion tables. Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

We show that composition matrices on the set X are in one-to-one correspondence with $(2+2)$-free posets on X. We show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $\{1, \ldots, n\}$.

Résumé. Ce papier introduit deux analogues matriciels des partitions d'ensembles: les matrices de composition et de partition. Ces deux analogues sont le produit naturel du relèvement de l'application entre suites de montées et matrices d'entiers introduite dans Dukes & Parviainen (2010).

Nous démontrons que les matrices de partition sont en bijection avec les tables d'inversion, les tables d'inversion croissantes correspondant aux matrices de partition avec une relation d'ordre sur les lignes. Les matrices de partition s-diagonales sont classées en fonction de leurs tables d'inversion. Les matrices de partition bidiagonales sont énumérées par la méthode de matrices de transfert et ont même cardinalité que les permutations triables par deux piles en parallèle.

Nous montrons que les matrices de composition sur l'ensemble X sont en bijection avec les ensembles ordonnés $(2+2)$-libres sur X. Nous prouvons que les paires de suites de montées et de permutations sont en bijection avec les ensembles ordonnés $(2+2)$-libres dont les éléments sont les cycles d'une permutation, et nous utilisons cette relation pour exprimer le nombre d'ensembles ordonnés $(2+2)$-libres sur $\{1, \ldots, n\}$.

Keywords: partition matrix, composition matrix, ascent sequence, inversion table

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1 Introduction

The recent papers [4, 2, 7] showed that four entirely different combinatorial structures – (2+2)-free posets, matchings with no neighbour nestings, a new class of pattern avoiding permutations, and a class of upper-triangular integer matrices – are in one-to-one correspondence with special sequences of non-negative integers called *ascent sequences*.

Building on this, Claesson & Linusson [6] presented a bijection from inversion tables (a superset of ascent sequences) to matchings with no left neighbour nestings (a superset of matchings with no neighbour nestings). Although the bijection of [2] is not recovered when the mapping in [6] is restricted to ascent sequences, we consider the second correspondence to be on a level above the first in a hierarchy.

This paper introduces two matrix analogues for set partitions, which does for the correspondence between ascent sequences and matrices what Claesson and Linusson's paper did for matchings. In fact it goes further by presenting a third layer in this yet-to-be-fully-determined hierarchy. Proofs of the results stated in this extended abstract can be found in [5].

Example 1 *Here is an instance of what we shall call a partition matrix:*

$$A = \begin{bmatrix} \{1,2,3\} & \emptyset & \{5,7,8\} & \{9\} \\ \emptyset & \{4\} & \{6\} & \{11\} \\ \emptyset & \emptyset & \emptyset & \{13\} \\ \emptyset & \emptyset & \emptyset & \{10,12\} \end{bmatrix}.$$

Definition 2 *Let X be a finite subset of $\{1,2,\dots\}$. A* partition matrix *on X is an upper triangular matrix over the powerset of X satisfying the following properties:*

(i) each column and row contain at least one non-empty set;

(ii) the non-empty sets partition X;

(iii) $\mathrm{col}(i) < \mathrm{col}(j) \implies i < j$,

where $\mathrm{col}(i)$ denotes the column in which i is a member. Let Par_n be the collection of all partition matrices on $[1,n] = \{1,\dots,n\}$.

For instance,

$$\mathrm{Par}_3 = \left\{ \begin{bmatrix} \{1,2,3\} \end{bmatrix}, \begin{bmatrix} \{1,2\} & \emptyset \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{2\} \\ \emptyset & \{3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \{3\} \\ \emptyset & \{2\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset \\ \emptyset & \{2,3\} \end{bmatrix}, \begin{bmatrix} \{1\} & \emptyset & \emptyset \\ \emptyset & \{2\} & \emptyset \\ \emptyset & \emptyset & \{3\} \end{bmatrix} \right\}.$$

In Section 2 we present a bijection between Par_n and the set of *inversion tables*

$$\mathcal{I}_n = [0,0] \times [0,1] \times \cdots \times [0,n-1], \text{ where } [a,b] = \{i \in \mathbb{Z} : a \le i \le b\}.$$

Non-decreasing inversion tables are shown to correspond to partition matrices with a row ordering relation. Partition matrices which are s-diagonal are classified in terms of inversion tables. Bidiagonal partition matrices are enumerated using the transfer-matrix method and are equinumerous with permutations which are sortable by two pop-stacks in parallel.

In Section 3 we define composition matrices to be the matrices that satisfy conditions (i) and (ii) of Definition 2. Composition matrices on X are shown to be in one-to-one correspondence with $(2+2)$-free posets on X.

Finally, in Section 4 we show that pairs of ascent sequences and permutations are in one-to-one correspondence with $(2+2)$-free posets whose elements are the cycles of a permutation, and use this relation to give an expression for the number of $(2+2)$-free posets on $[1, n]$.

Taking the entry-wise cardinality of the matrices in Par_n one gets the matrices of Dukes and Parviainen [7]. In that sense, we generalize the paper of Dukes and Parviainen in a similar way as Claesson and Linusson [6] generalized the paper of Bousquet-Mélou et al. [2]. We note, however, that if we restrict our attention to those inversion tables that enjoy the property of being an *ascent sequence*, then we do *not* recover the bijection of Dukes and Parviainen.

2 Partition matrices and inversion tables

For w a sequence let $\mathrm{Alph}(w)$ denote the set of distinct entries in w. In other words, if we think of w as a word, then $\mathrm{Alph}(w)$ is the (smallest) alphabet on which w is written. Also, let us write $\{a_1, \ldots, a_k\}_<$ for a set whose elements are listed in increasing order, $a_1 < \cdots < a_k$. Given an inversion table $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ with $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$ define the $k \times k$ matrix $A = \Lambda(w) \in \mathrm{Par}_n$ by

$$A_{ij} = \left\{ \ell : x_\ell = y_i \text{ and } y_j < \ell \leq y_{j+1} \right\},$$

where we let $y_{k+1} = n$. For example, for $w = (0,0,0,3,0,3,0,0,0,8,3,8) \in \mathcal{I}_{12}$ we have $\mathrm{Alph}(w) = \{0, 3, 8\}$ and

$$\Lambda(w) = \begin{bmatrix} \{1,2,3\} & \{5,7,8\} & \{9\} \\ \emptyset & \{4,6\} & \{11\} \\ \emptyset & \emptyset & \{10,12\} \end{bmatrix} \in \mathrm{Par}_{12}\,.$$

We now define a map $K : \mathrm{Par}_n \to \mathcal{I}_n$. Given $A \in \mathrm{Par}_n$, for $\ell \in [1, n]$ let $x_\ell = \min(A_{*i}) - 1$ where i is the row containing ℓ and $\min(A_{*i})$ is the smallest entry in column i of A. Define

$$K(A) = (x_1, \ldots, x_n).$$

Theorem 3 *The map $\Lambda : \mathcal{I}_n \to \mathrm{Par}_n$ is a bijection and K is its inverse.*

2.1 Statistics on partition matrices and inversion tables

Given $A \in \mathrm{Par}_n$, let $\mathrm{Min}(A) = \{\min(A_{*j}) : j \in [1, \dim(A)]\}$. For instance, the matrix A in Example 1 has $\mathrm{Min}(A) = \{1, 4, 5, 9\}$. From the definition of Λ the following proposition is apparent.

Proposition 4 *If $w \in \mathcal{I}_n$, $\mathrm{Alph}(w) = \{y_1, \ldots, y_k\}_<$ and $A = \Lambda(w)$, then*

$$\mathrm{Min}(A) = \{y_1 + 1, \ldots, y_k + 1\} \textit{ and } \dim(A) = |\mathrm{Alph}(w)|.$$

Corollary 5 *The statistic* $\dim$ *on* Par_n *is Eulerian.*

Let us say that i is a *special descent* of $w = (x_1, \ldots, x_n) \in \mathcal{I}_n$ if $x_i > x_{i+1}$ and i does not occur in w. Let $\mathrm{sdes}(w)$ denote the number of special descents of w, so

$$\mathrm{sdes}(w) = |\{\, i : x_i > x_{i+1} \text{ and } x_\ell \neq i \text{ for all } \ell \in [1, n] \,\}|.$$

Claesson and Linusson [6] conjectured that sdes has the same distribution on $\mathcal{I}_n$ as the, so-called, bivincular pattern $p = (231, \{1\}, \{1\})$ has on $\mathfrak{S}_n$. An occurrence of p in a permutation $\pi = a_1 \dots a_n$ is a subword $a_i a_{i+1} a_j$ such that $a_{i+1} > a_i = a_j + 1$. We shall define a statistic on partition matrices that is equidistributed with sdes. Given $A \in \mathrm{Par}_n$ let us say that i is a *column descent* if $i+1$ is in the same column as, and above, i in A. Let $\mathrm{cdes}(A)$ denote the number of column descents in A, so

$$\mathrm{cdes}(A) = |\{\, i : \mathrm{row}(i) > \mathrm{row}(i+1) \text{ and } \mathrm{col}(i) = \mathrm{col}(i+1) \,\}|.$$

Proposition 6 *The special descents of $w \in \mathcal{I}_n$ equal the column descents of $\Lambda(w)$. Consequently, the statistic* sdes *on $\mathcal{I}_n$ has the same distribution as* cdes *on* Par_n.

Let us write Mono_n for the collection of matrices in Par_n which satisfy

(iv) $\mathrm{row}(i) < \mathrm{row}(j) \implies i < j$,

where $\mathrm{row}(i)$ denotes the row in which i is a member. We say that an inversion table $(x_1, \dots, x_n)$ is *non-decreasing* if $x_i \le x_{i+1}$ for all $1 \le i < n$.

Theorem 7 *Under the map $\Lambda : \mathrm{Par}_n \to \mathcal{I}_n$, matrices in Mono_n correspond to non-decreasing inversion tables, and $|\mathrm{Mono}_n| = \binom{2n}{n}/(n+1)$, the nth Catalan number.*

2.2 s-diagonal partition matrices

Theorem 8 *Let $w = (x_1, \dots, x_n) \in \mathcal{I}_n$, $A = \Lambda(w)$ and $\mathrm{Alph}(w) = \{y_1, \dots, y_k\}_<$. Define $y_{k+1} = n$. The matrix A is s-diagonal if and only if for every $i \in [1, n]$ there exists an $a(i) \in [1, k]$ such that*

$$y_{a(i)} < i \le y_{a(i)+1} \text{ and } x_i \in \{y_{a(i)}, y_{a(i)-1}, \dots, y_{\max(1, a(i)-s+1)}\}.$$

Setting $s = 1$ in the above theorem gives us the class of diagonal matrices. These admit a more explicit description which we will now present.

In computer science, *run-length encoding* is a simple form of data compression in which consecutive data elements (runs) are stored as a single data element and its multiplicity. We shall apply this to inversion tables, but for convenience rather than compression purposes. Let $\mathrm{RLE}(w)$ denote the run-length encoding of the inversion table w. For example,

$$\mathrm{RLE}(0,0,0,0,1,1,0,2,3,3) = (0,4)(1,2)(0,1)(2,1)(3,2).$$

A sequence of positive integers $(u_1, \dots, u_k)$ which sum to n is called an *integer composition* of n and we write this as $(u_1, \dots, u_k) \models n$.

Corollary 9 *The set of diagonal matrices in Par_n is the image under Λ of*

$$\{\, w \in \mathcal{I}_n : (u_1, \dots, u_k) \models n \text{ and } \mathrm{RLE}(w) = (p_0, u_1) \dots (p_{k-1}, u_k) \,\},$$

where $p_0 = 0$, $p_1 = u_1$, $p_2 = u_1 + u_2$, $p_3 = u_1 + u_2 + u_3$, etc.

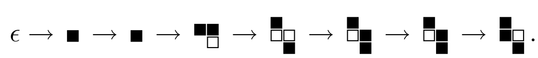
Since diagonal matrices are in bijection with integer compositions, the number of diagonal matrices in Par_n is 2^{n-1}. Although the bidiagonal matrices do not admit as compact a description in terms of the corresponding inversion tables, we can still count them using the so-called transfer-matrix method [11, §4.7]. Consider the matrix

$$B = \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}.$$

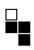
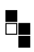
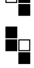
More specifically consider creating B by starting with the empty matrix, ϵ, and inserting the elements 1, $\ldots$, 7 one at a time:

$$\epsilon \to \begin{bmatrix} \{1\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} \\ \emptyset & \emptyset \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \emptyset \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4\} \end{bmatrix} \to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset \\ \emptyset & \emptyset & \{5\} \\ \emptyset & \emptyset & \{4,6\} \end{bmatrix}$$
$$\to \begin{bmatrix} \{1,2\} & \{3\} & \emptyset & \emptyset \\ \emptyset & \emptyset & \{5\} & \emptyset \\ \emptyset & \emptyset & \{4,6\} & \emptyset \\ \emptyset & \emptyset & \emptyset & \{7\} \end{bmatrix}$$

We shall encode (some aspects of) this process like this:



Here, ■ denotes any 1×1 matrix whose only entry is a non-empty set; denotes any 2×2 matrix whose black entries are non-empty;  denotes any matrix of dimension 3 or more, whose entries in the bottom right corner match the picture, that is, the black entries are non-empty; etc. The sequence of pictures does not, in general, uniquely determine a bidiagonal matrix, but each picture contains enough information to tell what pictures can possibly follow it. The matrix below gives all possible transitions (a q records when a new column, and row, is created):

	ϵ													
ϵ	0	q	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	q	q	0	0	0	0	0	0	0	0	0	0
	0	0	1	0	1	q	q	0	0	0	0	0	0	0
	0	0	0	1	1	0	0	0	q	q	0	0	0	0
	0	0	0	0	2	0	0	0	0	0	0	q	q	0
	0	0	0	0	0	$q+1$	q	1	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	q	q	0	0	0	0
	0	0	0	0	0	0	0	2	0	0	0	q	q	0
	0	0	0	0	0	0	0	0	1	0	1	0	0	0
	0	0	0	0	0	0	0	0	q	$q+1$	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	2	q	q	0
	0	0	0	0	0	q	q	0	0	0	0	1	0	1
	0	0	0	0	0	0	0	0	q	q	0	0	1	1
	0	0	0	0	0	0	0	0	0	0	0	q	q	2

We would like to enumerate paths that start with ϵ and end in a configuration with no empty rows or columns. Letting M denote the above transfer-matrix, this amounts to calculating the first coordinate in

$$(1 - xM)^{-1}[1\,1\,1\,0\,1\,1\,0\,1\,0\,0\,1\,1\,0\,1]^T.$$

Proposition 10 *We have*

$$\sum_{n\geq 0}\sum_{A\in \mathrm{BiPar}_n} q^{\dim(A)}x^n = \frac{2x^3-(q+5)x^2+(q+4)x-1}{2(q^2+q+1)x^3-(q^2+4q+5)x^2+2(q+2)x-1},$$

where BiPar_n *is the collection of bidiagonal matrices in* Par_n.

We find it interesting that the number of bidiagonal matrices in Par_n is given by the sequence [10, A164870], which corresponds to permutations of $[1,n]$ which are sortable by two pop-stacks in parallel. In terms of pattern avoidance those are the permutations in the class

$$\mathfrak{S}_n(3214, 2143, 24135, 41352, 14352, 13542, 13524).$$

See Atkinson and Sack [1]. A combinatorial proof of this correspondence would be interesting.

Moreover, there are exactly 2^{n-1} permutations of $[1,n]$ which are sortable by one pop-stack; hence equinumerous with the diagonal partition matrices. One might then wonder about permutations which are sortable by three pop-stacks in parallel. Are they equinumerous with tridiagonal partition matrices? Computations show that this is not the case: For $n=6$ there are 646 tridiagonal partition matrices, but only 644 permutations which are sortable by three pop-stacks in parallel.

3 Composition matrices and $(2+2)$-free posets

Consider Definition 2. Define a *composition matrix* to be a matrix that satisfies conditions (i) and (ii), but not necessarily (iii). Let $\mathrm{Comp}_n \supseteq \mathrm{Par}_n$ denote the set of all composition matrices on $[1, n]$. The smallest example of a composition matrix that is not a partition matrix is

$$\begin{bmatrix} \{2\} & \emptyset \\ \emptyset & \{1\} \end{bmatrix}.$$

In this section we shall give a bijection from Comp_n to the set of $(2+2)$-free posets on $[1, n]$. This bijection will factor through a certain union of Cartesian products that we now define. Given a set X, let us write $\binom{X}{x_1,\dots,x_\ell}$ for the collection of all sequences $(X_1, \dots, X_\ell)$ that are ordered set partitions of X and $|X_i| = x_i$ for all $i \in [1, \ell]$. For a sequence $(a_1, \dots, a_i)$ of numbers let

$$\mathrm{asc}(a_1, \dots, a_i) = |\{j \in [1, i-1] : a_j < a_{j+1}\}|.$$

Following Bousquet-Mélou et al. [2] we define a sequence of non-negative integers $\alpha = (a_1, \dots, a_n)$ to be an *ascent sequence* if $a_1 = 0$ and $a_{i+1} \in [0, 1 + \mathrm{asc}(a_1, \dots, a_i)]$ for $0 < i < n$. Let $\mathcal{A}_n$ be the collection of ascent sequences of length n. Define the *run-length record* of α to be the sequence that records the multiplicities of adjacent values in α. We denote it by $\mathrm{RLR}(\alpha)$. In other words, $\mathrm{RLR}(\alpha)$ is the sequence of second coordinates in $\mathrm{RLE}(\alpha)$, the run-length encoding of α. For instance,

$$\mathrm{RLR}(0,0,0,0,1,1,0,2,3,3) = (4,2,1,1,2).$$

Finally we are in a position to define the set which our bijection from Comp_n to $(2+2)$-free posets on $[1, n]$ will factor through. Let

$$\mathfrak{A}_n = \bigcup_{\alpha \in \mathcal{A}_n} \{\alpha\} \times \binom{[1, n]}{\mathrm{RLR}(\alpha)}.$$

Let $\mathcal{M}_n$ be the collection of upper triangular matrices that contain non-negative integers whose entries sum to n and such that there is no column or row of all zeros. Dukes and Parviainen [7] presented a bijection

$$\Gamma : \mathcal{M}_n \to \mathcal{A}_n.$$

Given $A \in \mathcal{M}_n$, let $\mathrm{nz}(A)$ be the number of non-zero entries in A. Since A may be uniquely constructed, in a step-wise fashion, from the ascent sequence $\Gamma(A)$, we may associate to each non-zero entry A_{ij} its time of creation $T_A(i, j) \in [1, \mathrm{nz}(A)]$. By defining $T_A(i, j) = 0$ if $A_{ij} = 0$ we may view T_A as a $\dim(A) \times \dim(A)$ matrix. Define $\mathrm{Seq}(A) = (y_1, \dots, y_{\mathrm{nz}(A)})$ where $y_t = A_{ij}$ and $T_A(i, j) = t$.

Example 11 *We have*

$$A = \begin{bmatrix} 3 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Longrightarrow T_A = \begin{bmatrix} 1 & 0 & 5 & 8 \\ 0 & 2 & 4 & 7 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

and $\mathrm{Seq}(A) = (3, 1, 2, 1, 3, 1, 1, 1)$.

The following is a straightforward consequence of the construction rules given by Dukes and Parviainen [7].

Lemma 12 *Given $A \in \mathcal{M}_n$, we have that* $\mathrm{Seq}(A) = \mathrm{RLR}(\Gamma(A))$.

For a matrix $A \in \mathrm{Comp}_n$ define $\mathrm{Card}(A)$ as the matrix obtained from A by taking the cardinality of each of its entries. Note that $A \mapsto \mathrm{Card}(A)$ is a surjection from Par_n onto $\mathcal{M}_n$. Define $E(A)$ as the ordered set partition $(X_1, \dots, X_{\mathrm{nz}(A)})$, where $X_t = A_{ij}$ for $t = T_{\mathrm{Card}(A)}(i,j)$. Finally, define $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ by

$$f(A) = \big(\Gamma(\mathrm{Card}(A)),\, E(A)\big).$$

Example 13 *Let us calculate $f(A)$ for*

$$A = \begin{bmatrix} \{3,8\} & \{6\} & \emptyset \\ \emptyset & \{2,5,7\} & \emptyset \\ \emptyset & \emptyset & \{1,4\} \end{bmatrix}.$$

We have

$$\mathrm{Card}(A) = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad T_{\mathrm{Card}(A)} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$f(A) = \big(\Gamma(\mathrm{Card}(A)),\, E(A)\big) = \big((0,0,1,1,1,0,2,2),\, \{3,8\}\{2,5,7\}\{6\}\{1,4\}\big).$$

We now define a map $g : \mathfrak{A}_n \to \mathrm{Comp}_n$. For $(w,\chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \dots, X_k)$ let $g(w,\chi) = A$, where $A_{ij} = X_t$, $t = T_B(i,j)$ and $B = \Gamma^{-1}(w)$. It is easy to verify that $f(\mathrm{Comp}_n) \subseteq \mathfrak{A}_n$, $g(\mathfrak{A}_n) \subseteq \mathrm{Comp}_n$, $g(f(w,\chi)) = (w,\chi)$ for $(w,\chi) \in \mathfrak{A}_n$, and $f(g(A)) = A$ for $A \in \mathrm{Comp}_n$. Thus the following theorem.

Theorem 14 *The map $f : \mathrm{Comp}_n \to \mathfrak{A}_n$ is a bijection and g is its inverse.*

Next we will give a bijection ϕ from $\mathfrak{A}_n$ to $\mathfrak{P}_n$, the set of $(2+2)$-free posets on $[1,n]$. Let $(\alpha,\chi) \in \mathfrak{A}_n$ with $\chi = (X_1, \dots, X_\ell)$. Assuming that $X_i = \{x_1, \dots, x_k\}_<$ define the word $\hat{X}_i = x_1 \dots x_k$ and let $\hat{\chi} = \hat{X}_1 \dots \hat{X}_\ell$. From this, $\hat{\chi}$ will be a permutation of the elements $[1,n]$. Let $\hat{\chi}(i)$ be the ith letter of this permutation.

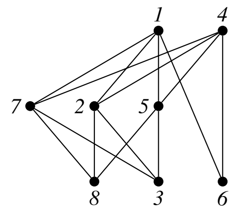
For $(\alpha,\chi) \in \mathfrak{A}_n$ define $\phi(\alpha,\chi)$ as follows: Construct the poset element by element according to the construction rules of [2] on the ascent sequence α. Label with $\hat{\chi}(i)$ the element inserted at step i.

The inverse of this map is also straightforward to state and relies on the following crucial observation [8, Prop. 3] concerning indistinguishable elements in an unlabeled $(2+2)$-free poset. Two elements in a poset are called *indistinguishable* if they obey the same relations relative to all other elements.

Let P be an unlabeled poset that is constructed from the ascent sequence $\alpha = (a_1, \dots, a_n)$. Let p_i and p_j be the elements that were created during the ith and jth steps of the construction given in [2, Sect. 3]. The elements p_i and p_j are indistinguishable in P if and only if $a_i = a_{i+1} = \cdots = a_j$.

Define $\psi : \mathfrak{P}_n \to \mathfrak{A}_n$ as follows: Given $P \in \mathfrak{P}_n$ let $\psi(P) = (\alpha,\chi)$ where α is the ascent sequence that corresponds to the poset P with its labels removed, and χ is the sequence of sets $(X_1, \dots, X_m)$ where X_i is the set of labels that corresponds to all the indistinguishable elements of P that were added during the ith run of identical elements in the ascent sequence.

Example 15 *Consider the $(2+2)$-free poset*

$P =$  $\in \mathfrak{P}_8.$

The unlabeled poset corresponding to P has ascent sequence $(0,0,1,1,1,0,2,2)$. There are four runs in this ascent sequence. The first run of two 0s inserts the elements 3 and 8, so we have $X_1 = \{3,8\}$. Next the run of three 1s inserts elements 2, 5 and 7, so $X_2 = \{2,5,7\}$. The next run is a run containing a single 0, and the element inserted is 6, so $X_3 = \{6\}$. The final run of two 2s inserts elements 1 and 4, so $X_4 = \{1,4\}$. Thus we have

$$\psi(P) = \big((0,0,1,1,1,0,2,2),\ \{3,8\}\{2,5,7\}\{6\}\{1,4\}\big).$$

It is straightforward to check that ϕ and ψ are each others inverses. Consequently, we have the following theorem.

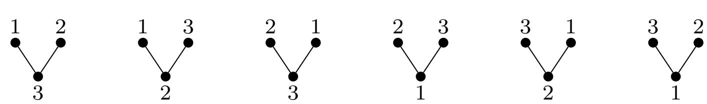
Theorem 16 *The map $\phi : \mathfrak{A}_n \to \mathfrak{P}_n$ is a bijection and ψ is its inverse.*

Let $\mathcal{C}_n$ be the collection of composition matrices M on $\{1,\ldots,n\}$ with the following property: in every row of M, the entries in the sets are increasing from left to right. For example $(\{1,2\},\emptyset,\{3,4\})$ is a valid row, but $(\{1,3\},\emptyset,\{2,4\})$ is not. Combining the properties of the map Λ from Section 2, Theorem 7, and the observation that relabeling the elements in a partition matrix which correspond to the same element of the inversion table when written from left to right in the rows in increasing order yield the above stated property of composition matrices, we have the following. The enumeration follows from Stanley [12, Exer. 5.49] or Haglund & Loehr [9, §2].

Corollary 17 *Let $\mathcal{G}_n$ be the collection of all pairs (α,χ) where α is a non-decreasing inversion table of length n, and χ is an ordered set partition of $\{1,\ldots,n\}$ where the sequences of sizes of the sets equals $RLR(\alpha)$. Then there is a bijection between $\mathcal{C}_n$ and $\mathcal{G}_n$, and $|\mathcal{C}_n| = (n+1)^{n-1}$.*

4 The number of $(2+2)$-free posets on $[1,n]$

Let us consider *plane (2+2)-free posets* on $[1,n]$. That is, $(2+2)$-free posets on $[1,n]$ with a canonical embedding in the plane. For instance, these are six *different* plane $(2+2)$-free posets on $[1,3] = \{1,2,3\}$:



By definition, if u_n is the number of unlabeled $(2+2)$-free posets on n nodes, then $u_n n!$ is the number of plane $(2+2)$-free posets on $[1,n]$. In other words, we may identify the set of plane $(2+2)$-free posets on $[1,n]$ with the Cartesian product $\mathcal{P}_n \times \mathfrak{S}_n$, where $\mathcal{P}_n$ denotes the set of unlabeled $(2+2)$-free posets on n nodes and $\mathfrak{S}_n$ denotes the set of permutations on $[1,n]$. We shall demonstrate the isomorphism

$$\bigcup_{\pi \in \mathfrak{S}_n} \mathfrak{P}(\mathrm{Cyc}(\pi)) \simeq \mathcal{P}_n \times \mathfrak{S}_n, \tag{1}$$

where $\mathrm{Cyc}(\pi)$ is the set of (disjoint) cycles of π and $\mathfrak{P}(\mathrm{Cyc}(\pi))$ is the set of $(2+2)$-free posets on those cycles. As an illustration we consider the case $n = 3$. On the right-hand side we have $|\mathcal{P}_3 \times \mathfrak{S}_3| = |\mathcal{P}_3||\mathfrak{S}_3| = 5 \cdot 6 = 30$ plane $(2+2)$-free posets. Taking the cardinality of the left-hand side we get

$$\begin{aligned} &|\mathfrak{P}\{(1),(2),(3)\}| + |\mathfrak{P}\{(1),(23)\}| + |\mathfrak{P}\{(12),(3)\}| + |\mathfrak{P}\{(2),(13)\}| + |\mathfrak{P}\{(123)\}| + |\mathfrak{P}\{(132)\}| \\ &= |\mathfrak{P}_3| + 3|\mathfrak{P}_2| + 2|\mathfrak{P}_1| = 19 + 3 \cdot 3 + 2 \cdot 1 = 30. \end{aligned}$$

Bousquet-Mélou et al. [2] gave a bijection Ψ from $\mathcal{P}_n$ to $\mathcal{A}_n$, the set of ascent sequences of length n. Recall also that in Theorem 16 we gave a bijection ϕ from $\mathfrak{P}_n$ to $\mathfrak{A}_n$. Of course, there is nothing special about the ground set being $[1,n]$ in Theorem 16; so, for any finite set X, the map ϕ can be seen as a bijection from $(2+2)$-free posets on X to the set

$$\mathfrak{A}(X) = \bigcup_{\alpha \in \mathcal{A}_{|X|}} \{\alpha\} \times \binom{X}{\mathrm{RLR}(\alpha)}.$$

In addition, the fundamental transformation [3] is a bijection between permutations with exactly k cycles and permutations with exactly k left-to-right minima. Putting these observations together it is clear that to show (1) it suffices to show

$$\bigcup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\mathrm{LMin}(\pi)) \simeq \mathcal{A}_n \times \mathfrak{S}_n, \tag{2}$$

where $\mathrm{LMin}(\pi)$ is the set of segments obtained by breaking π apart at each left-to-right minima. For instance, the left-to-right minima of $\pi = 5731462$ are 5, 3 and 1; so $\mathrm{LMin}(\pi) = \{57, 3, 1462\}$.

Let us now prove (2) by giving a bijection h from the left-hand side to the right-hand side. To this end, fix a permutation $\pi \in \mathfrak{S}_n$ and let $k = |\mathrm{LMin}(\pi)|$ be the number of left-to-right minima in π. Assume that $\alpha = (a_1, \dots, a_k)$ is an ascent sequence in $\mathcal{A}_k$ and that $\chi = (X_1, \dots, X_r)$ is an ordered set partition in $\binom{\mathrm{LMin}(\pi)}{\mathrm{RLR}(\alpha)}$. To specify the bijection h let

$$h(\alpha, \chi) = (\beta, \tau)$$

where $\beta \in \mathcal{A}_n$ and $\tau \in \mathfrak{S}_n$ are defined in the next paragraph.

For each $i \in [1,r]$, first order the blocks of X_i decreasingly with respect to first (and thus minimal) element, then concatenate the blocks to form a word $\hat{X}_i$. Define the permutation τ as the concatenation of the $\hat{X}_i$s:

$$\tau = \hat{X}_1 \dots \hat{X}_k.$$

Let $i_1 = 1$, $i_2 = i_1 + |X_1|$, $i_3 = i_2 + |X_2|$, etc. By definition, these are the indices where the ascent sequence α changes in value. Define β by

$$\mathrm{RLE}(\beta) = (a_{i_1}, x_1) \dots (a_{i_k}, x_k), \text{ where } x_i = |\hat{X}_i|.$$

Consider the permutation $\pi = \mathrm{A9B68D4F32C175E} \in \mathfrak{S}_{15}$ (in hexadecimal notation). Then $\mathrm{LMin}(\pi) = \{\mathrm{A}, \mathrm{9B}, \mathrm{68D}, \mathrm{4F}, 3, \mathrm{2C}, \mathrm{175E}\}$. Assume that

$$\begin{aligned}\alpha &= (0, 0, 1, 2, 2, 2, 0);\\ \chi &= \{\mathrm{2C}, \mathrm{68D}\}\{\mathrm{9B}\}\{3, \mathrm{175E}, \mathrm{4F}\}\{\mathrm{A}\}.\end{aligned}$$

Then we have $\hat{X}_1 = \mathrm{68D2C}$, $\hat{X}_2 = \mathrm{9B}$, $\hat{X}_3 = \mathrm{4F3175E}$ and $\hat{X}_4 = \mathrm{A}$. Also, $i_1 = 1$, $i_2 = 1 + 2 = 3$, $i_3 = 3 + 1 = 4$ and $i_4 = 4 + 3 = 7$. Consequently,

$$\begin{aligned}\beta &= (0, 0, 0, 0, 0, 1, 1, 2, 2, 2, 2, 2, 2, 2, 0);\\ \tau &= \;6\;\;8\;\;\mathrm{D}\;\;2\;\;\mathrm{C}\;\;9\;\;\mathrm{B}\;\;4\;\;\mathrm{F}\;\;3\;\;1\;\;7\;\;5\;\;\mathrm{E}\;\;\mathrm{A}.\end{aligned}$$

It is clear how to reverse this procedure: Split τ into segments according to where β changes in value when reading from left to right. With τ as above we get

$$(\mathrm{68D2C}, \mathrm{9B}, \mathrm{4F3175E}, \mathrm{A}) = (\hat{X}_1, \hat{X}_2, \hat{X}_3, \hat{X}_4)$$

We have thus recovered $\hat{X}_1$, $\hat{X}_2$, etc. Now $X_i = \mathrm{LMin}(\hat{X}_i)$, and we thus know χ. It only remains to recover α. Assume that $\mathrm{RLE}(\beta) = (b_1, x_1) \dots (b_k, x_k)$, then $\mathrm{RLE}(\alpha) = (b_1, |X_1|) \dots (b_k, |X_k|)$. This concludes the proof of (2). Let us record this result.

Theorem 18 *The map* $h : \cup_{\pi \in \mathfrak{S}_n} \mathfrak{A}(\mathrm{LMin}(\pi)) \to \mathcal{A}_n \times \mathfrak{S}_n$ *is a bijection.*

As previously explained, (1) also follows from this theorem. Let us now use (1) to derive an exponential generating function $L(t)$ for the number of $(2+2)$-free posets in $[1, n]$. Bousquet-Mélou et al. [2] gave the following *ordinary* generating function for *unlabeled* $(2+2)$-free posets on n nodes:

$$\begin{aligned}P(t) &= \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - (1-t)^i\right)\\ &= 1 + t + 2t^2 + 5t^3 + 15t^4 + 53t^5 + 217t^6 + 1014t^7 + 5335t^8 + O(t^9).\end{aligned}$$

This is, of course, also the exponential generating function for plane $(2+2)$-free posets on $[1, n]$. Moreover, the exponential generating function for cyclic permutations is $\log(1/(1-t))$. On taking the union over $n \geq 0$ of both sides of (1) it follows that $L(\log(1/(1-t))) = P(t)$; so $L(t) = P(1 - e^{-t})$.

Corollary 19 *The exponential generating function for* $(2+2)$*-free posets is*

$$\begin{aligned}L(t) &= \sum_{n \geq 0} \prod_{i=1}^{n} \left(1 - e^{-ti}\right)\\ &= 1 + t + 3\frac{t^2}{2!} + 19\frac{t^3}{3!} + 207\frac{t^4}{4!} + 3451\frac{t^5}{5!} + 81663\frac{t^6}{6!} + 2602699\frac{t^7}{7!} + O(t^8).\end{aligned}$$

This last result also follows from a result of Zagier [13, Eq. 24] and a bijection, due to Bousquet-Mélou et al. [2], between unlabeled $(2+2)$-free posets and certain matchings. See also Exercises 14 and 15 in Chapter 3 of the second edition of Enumerative Combinatorics Volume 1 (available on R. Stanley's homepage).

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Path tableaux and combinatorial interpretations of immanants for class functions on S_n

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Abstract. Let χ^λ be the irreducible S_n-character corresponding to the partition λ of n, equivalently, the preimage of the Schur function s_λ under the Frobenius characteristic map. Let ϕ^λ be the function $S_n \to \mathbb{C}$ which is the preimage of the monomial symmetric function m_λ under the Frobenius characteristic map. The irreducible character immanant $\mathrm{Imm}_\lambda(x) = \sum_{w \in S_n} \chi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n}$ evaluates nonnegatively on each totally nonnegative matrix A. We provide a combinatorial interpretation for the value $\mathrm{Imm}_\lambda(A)$ in the case that λ is a hook partition. The monomial immanant $\mathrm{Imm}_{\phi^\lambda}(x) = \sum_{w \in S_n} \phi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n}$ is conjectured to evaluate nonnegatively on each totally nonnegative matrix A. We confirm this conjecture in the case that λ is a two-column partition by providing a combinatorial interpretation for the value $\mathrm{Imm}_{\phi^\lambda}(A)$.

Résumé. Soit χ^λ le caractère irréductible de S_n qui correspond à la partition λ de l'entier n, ou de manière équivalente, la préimage de la fonction de Schur s_λ par l'application caractéristique de Frobenius. Soit ϕ^λ la fonction $S_n \to \mathbb{C}$ qui est la préimage de la fonction symétrique monomiale m_λ. La valeur du caractère irréductible immanent $\mathrm{Imm}_\lambda(x) = \sum_{w \in S_n} \chi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n}$ est non négative pour chaque matrice totalement non négative. Nous donnons une interprétation combinatoire de la valeur $\mathrm{Imm}_\lambda(A)$ lorsque λ est une partition en équerre. Stembridge a conjecturé que la valeur de l'immanent monomial $\mathrm{Imm}_{\phi^\lambda}(x) = \sum_{w \in S_n} \phi^\lambda(w) x_{1,w_1} \cdots x_{n,w_n}$ de ϕ^λ est elle aussi non négative pour chaque matrice totalement non négative. Nous confirmons cette conjecture quand λ satisfait $\lambda_1 \leq 2$, et nous donnons une interprétation combinatoire de $\mathrm{Imm}_{\phi^\lambda}(A)$ dans ce cas.

Keywords: total nonnegativity, Schur nonnegativity, planar network, symmetric group, class function, character

1 Introduction

A real matrix is called *totally nonnegative* (TNN) if the determinant of each of its square submatrices is nonnegative. Such matrices appear in many areas of mathematics and the concept of total nonnegativity has been generalized to apply not only to matrices, but also to other mathematical objects. (See e.g. [Lus08] and references there.) In particular, for an $n \times n$ matrix $x = (x_{i,j})$ of variables, a polynomial $p(x)$ in $\mathbb{C}[x] \underset{\text{def}}{=} \mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ is called *totally nonnegative* (TNN) if it satisfies

$$p(A) \underset{\text{def}}{=} p(a_{1,1}, \dots, a_{n,n}) \geq 0 \tag{1}$$

for every $n \times n$ TNN matrix $A = (a_{i,j})$. Obvious examples are the $n \times n$ determinant $\det(x)$, and each *minor* $\det(x_{I,J})$ of x, i.e., the determinant of a square submatrix

$$x_{I,J} \underset{\text{def}}{=} (x_{i,j})_{i \in I, j \in J}, \qquad I, J \subseteq [n] \underset{\text{def}}{=} \{1, \ldots, n\} \tag{2}$$

of x. Graph-theoretic interpretations of the nonnegativity of the numbers $\det(A_{I,J})$ for A TNN were given by Karlin and MacGregor [KM59] and Lindström [Lin73].

Close cousins of TNN matrices are the *Jacobi-Trudi* matrices indexed by pairs of partitions (λ, μ) and defined by $H_{\lambda/\mu} = (h_{\lambda_i - \mu_j + j - i})_{i,j=1}^{r}$ where h_k is the kth complete homogeneous symmetric function, and where we set $h_k = 0$ for $k < 0$. We declare a polynomial $p(x) \in \mathbb{C}[x]$ to be *monomial nonnegative* (MNN) or *Schur nonnegative* (SNN), if for each $n \times n$ Jacobi-Trudi matrix $H_{\lambda/\mu}$, the symmetric function $p(H_{\lambda/\mu})$ is MNN or SNN, respectively. Clearly, every SNN polynomial is MNN. Well-known examples of SNN polynomials are the $n \times n$ determinant $\det(x)$ and all minors $\det(x_{I,J})$. Graph-theoretic interpretations of the monomial nonnegativity of the symmetric functions $\det(H_{\lambda/\mu})$ were given by Gessel and Viennot [GV89]. No such interpretation of Schur nonnegativity is known.

Some recent interest in TNN, MNN, and SNN polynomials concerns the so-called *dual canonical basis* of $\mathbb{C}[x]$, which arose in the study of canonical bases of quantum groups. (See, e.g., [BZ93], [Du92], [Lus93, Sec. 29.5].) While the dual canonical basis has no elementary description, it includes the determinant $\det(x)$, all matrix minors $\det(x_{I,J})$, and by a result of Lusztig [Lus94], exclusively TNN polynomials. In [Du92], [Ska08], dual canonical basis elements were expressed in terms of functions $f : S_n \to \mathbb{C}$ and their generating functions

$$\mathrm{Imm}_f(x) \underset{\text{def}}{=} \sum_{v \in S_n} f(v) x_{1,v_1} \cdots x_{n,v_n} \tag{3}$$

in the complex span of $\{x_{1,v_1} \cdots x_{n,v_n} \,|\, v \in S_n\}$. Such generating functions had been named *immanants* in [Sta00], after Littlewood's term [Lit40] for the special cases in which f is one of the irreducible S_n-characters $\{\chi^\lambda \,|\, \lambda \vdash n\}$. This immanant formulation and results of Haiman [Hai93], showed that dual canonical basis elements are also SNN [RS06]. In general, given a TNN matrix A, Jacobi-Trudi matrix $H_{\lambda/\mu}$, and dual canonical basis element $\mathrm{Imm}_f(x)$, there is no known graph-theoretic interpretation for the nonnegative number $\mathrm{Imm}_f(A)$ or the MNN symmetric function $\mathrm{Imm}_f(H_{\lambda/\mu})$. On the other hand, Rhoades and the third author provided such interpretations for special dual canonical basis elements which they called *Temperley-Lieb immanants* in [RS05]. This led to the resolution in [LPP07] of several conjectures concerning Littlewood-Richardson coefficients.

Other interest in TNN, MNN, and SNN polynomials concerns immanants for class functions $f : S_n \to \mathbb{C}$ such as S_n-characters. Littlewood [Lit40], and Merris and Watkins [MW85] expressed immanants for induced sign characters $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$ as sums of products of matrix minors, making clear that these immanants are TNN and SNN. Goulden and Jackson [GJ92] conjectured irreducible character immanants to be MNN, leading Stembridge [Ste92] to conjecture the immanants to be TNN and SNN as well. The three conjectures were proved by Greene [Gre92], Stembridge [Ste91], and Haiman [Hai93]. (See also [Kos95].) Haiman, Stanley, and Stembridge formulated several stronger conjectures [Hai93], [SS93], [Ste92], including the total nonnegativity and Schur nonnegativity of immanants for class functions $\{\phi^\lambda \,|\, \lambda \vdash n\}$ called *monomial virtual characters*. Irreducible character immanants are nonnegative linear combinations [Ste92] of these. On the other hand, these conjectures have no graph-theoretic interpretation analogous to the results of Karlin-MacGregor, Lindström, and Gessel-Viennot.

In Section 2 we recall definitions and results relating total nonnegativity, planar networks, and immanants. We introduce *path tableaux*, a new generalization of Young tableaux, whose entries are paths in planar networks. In Section 3 we review facts about Temperley-Lieb immanants. In Section 4 we consider elementary immanants, known to be TNN and SNN. We use path tableaux to give new graph-theoretic interpretations of the nonnegative number $\mathrm{Imm}_{\epsilon^\lambda}(A)$ when A is TNN, and for the MNN symmetric function $\mathrm{Imm}_{\epsilon^\lambda}(H_{\mu/\nu})$ when $H_{\mu/\nu}$ is a Jacobi-Trudi matrix. In the special case $\lambda = (\lambda_1, \lambda_2)$, we state a new combinatorial interpretation of the coordinates of the elementary immanant $\mathrm{Imm}_{\epsilon^\lambda}(x)$ with respect to the Temperley-Lieb subset of the dual canonical basis. In Section 5, we consider monomial immanants, in general not known to be TNN, MNN, or SNN. We show that for $\lambda_1 \leq 2$, the monomial immanant $\mathrm{Imm}_{\phi^\lambda}(x)$ is TNN and SNN. This proves special cases of [Ste92, Conj. 2.1, Conj. 4.1]. For $\lambda_1 \leq 2$, we use path tableaux to provide new graph-theoretic interpretations of the nonnegative number $\mathrm{Imm}_{\phi^\lambda}(A)$ and of the MNN symmetric function $\mathrm{Imm}_{\phi^\lambda}(H_{\mu/\nu})$. Again we state a new combinatorial interpretation of the coordinates of $\mathrm{Imm}_{\phi^\lambda}(x)$ with respect to the Temperley-Lieb immanants. In Section 6, we consider irreducible character immanants, known to be TNN and SNN. For λ a hook shape, we use path tableaux to provide combinatorial interpretations of the nonnegative number $\mathrm{Imm}_{\chi^\lambda}(A)$ and of the MNN symmetric function $\mathrm{Imm}_{\chi^\lambda}(H_{\mu/\nu})$.

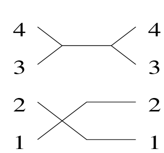
2 Planar networks, immanants, and path tableaux

Given a commutative $\mathbb{C}$-algebra R, we define an *R-weighted planar network of order n* to be an acyclic planar directed multigraph $G = (V, E, \omega)$, in which $2n$ distinguished boundary vertices are labeled clockwise as *source* $1, \cdots,$ *source* n, *sink* $n, \cdots,$ *sink* 1, and the function $\omega : E \to R$ associates a *weight* to each edge. For each multiset $F = e_1^{k_1} \cdots e_m^{k_m}$ we define the *weight* of F to be $\omega(F) = \omega(e_1)^{k_1} \cdots \omega(e_m)^{k_m}$. We draw planar networks with sources on the left, sinks on the right, and all edges understood to be oriented from left to right. Unlabeled edges are understood to have weight 1. We assume that all sources and sinks have indegree 0 and outdegree 0, respectively.

Given a planar network G of order n, we define the *path matrix* $A = A(G) = (a_{i,j})$ of G by letting $a_{i,j}$ be the sum

$$a_{i,j} = \sum_{(e_1,\ldots,e_m)} \omega(e_1) \cdots \omega(e_m)$$

of weights of all paths $(e_1, \ldots, e_m)$ from source i to sink j. It is easy to show that given planar networks G, H of order n, the concatenation $G \circ H$, constructed by identifying sinks of G with sources of H, satisfies $A(G \circ H) = A(G)A(H)$. It follows that every complex $n \times n$ matrix is the path matrix of some complex weighted planar network of order n. (See, e.g., [RS05, Obs. 2.2].) For example, one planar network of order 4 and its path matrix are



$$, \qquad \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \tag{4}$$

We call a sequence $\pi = (\pi_1, \ldots, \pi_k)$ of paths in G a *path family* if its component paths connect k distinct sources of G to k distinct sinks. We call the path family *bijective* if $k = n$, and for a permutation $w = w_1 \cdots w_n \in S_n$, we say that the path family has *type* w if path π_i originates at source i and terminates at sink w_i for $i = 1, \ldots, n$. If w is the identity permutation, we will say that π has *type* 1. We define the weight of a path family π to be the weight of the multiset of edges contained in its component paths,

$$\omega(\pi_1, \ldots, \pi_k) = \omega(\pi_1) \cdots \omega(\pi_k).$$

Following Stembridge, we will call this multiset of edges the *skeleton* of π. Conversely, we call a multiset F of edges of G a *bijective skeleton* if it is a multiset union of n source-to-sink paths. Sometimes it will be convenient to consider the endpoints of these edges to be part of the skeleton so that we may refer to the indegree or outdegree of a vertex in the skeleton as the cardinality of edges (including repetition) in the skeleton which originate or terminate at that vertex. We will refer to a planar network $G = (V, E)$ as a *bijective network* if E is the set of edges in some bijective skeleton. For example, the planar network in (4) is a bijective network.

Recall that Lindström's Lemma interprets the determinant of $A = A(G)$ as

$$\det(A) = \sum_{\pi} \omega(\pi), \tag{5}$$

where the sum is over all path families π of type 1 in which no two paths share a vertex [KM59], [Lin73]. Recall also that A is TNN if and only if we can choose the planar network G above to have nonnegative edge weights [ASW52], [Bre95], [Cry76], [Loe55]. Thus for each TNN matrix A, Lindström's Lemma provides a graph-theoretic interpretation of the nonnegative number $\det(A)$, indeed of the numbers $\det(A_{I,J})$ for $|I| = |J|$, since each submatrix of a TNN matrix is again TNN. Other families of immanants $\mathrm{Imm}_f(x)$ are known to be TNN, but only in few cases do the nonnegative numbers $\mathrm{Imm}_f(A)$ have graph-theoretic interpretations of the form

$$\mathrm{Imm}_f(A) = \sum_{\pi \in S} \omega(\pi) \tag{6}$$

where S is a set of path families in G having a certain property. By the Gessel-Viennot method [GV89], such an interpretation implies that $\mathrm{Imm}_f(x)$ is MNN, but not necessarily that it is SNN.

In order to prove that certain immanants $\mathrm{Imm}_f(x)$ are TNN and to state graph-theoretic interpretations of the nonnegative numbers $\mathrm{Imm}_f(A)$ for A TNN, it will be convenient to associate elements of the group algebra $\mathbb{C}[S_n]$ of S_n to planar networks and to bijective skeletons. We refer the reader to [Sag01] for standard information about $\mathbb{C}[S_n]$. Observing that any two path families π, π' satisfying $\mathrm{skel}(\pi) = \mathrm{skel}(\pi')$ must also satisfy $\omega(\pi) = \omega(\pi')$, we introduce the notation $\gamma(F, w)$ for the number of path families π of type w with skeleton F, and the $\mathbb{Z}[S_n]$-generating function

$$\beta(F) = \sum_{w \in S_n} \gamma(F, w) w \tag{7}$$

for these numbers. The importance of this generating function was stated by Goulden and Jackson [GJ92] and Greene [Gre92] as follows.

Observation 2.1 *Let R be a $\mathbb{C}$-algebra, and let G be an R-weighted planar network of order n having path matrix A. Then for any function $f : S_n \to \mathbb{C}$, we have*

$$\mathrm{Imm}_f(A) = \sum_{F \subseteq G} \omega(F) f(\beta(F)), \tag{8}$$

where the sum is over all bijective multisets F of edges in G. In particular, if we have $f(\beta(F)) \geq 0$ for all possible bijective networks F, then $\mathrm{Imm}_f(x)$ is TNN and MNN.

To see immediate consequences of Observation 2.1, suppose that p_0 is a property which applies to path families $\pi = (\pi_1, \ldots, \pi_n)$, let G be a planar network of order n, and define the sets

$$\begin{aligned} S(G, p_0) &= \{\pi \mid \pi \text{ bijective in } G, \pi \text{ has property } p_0\}, \\ T(G, p_0) &= \{\pi \in S(G, p_0) \mid \pi \text{ covers } G\}. \end{aligned} \tag{9}$$

Then we have the following.

Corollary 2.2 *Given a function $f : S_n \to \mathbb{C}$, and a property p_0, the following are equivalent:*

1. *For each planar network G of order n whose edges are weighted by distinct indeterminates, the path matrix $A = (a_{i,j})$ of G satisfies*

$$\mathrm{Imm}_f(A) = \sum_{\pi \in S(G, p_0)} \omega(\pi). \tag{10}$$

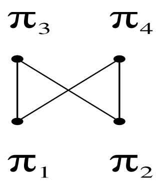
2. *For each bijective planar network G of order n, we have $f(\beta(G)) = |T(G, p_0)|$.*

Proof: Omitted. □

To facilitate the description of more TNN polynomials $p(x)$ and their evaluations at TNN matrices, we recall some standard terminology associated to integer partitions. For $\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n$, the (French) *Young diagram of* λ is the left-justified array of boxes with λ_i boxes in row i, and with rows numbered from bottom to top. We let $\lambda^\top$, the *transpose* or *conjugate* of λ, denote the partition of n whose ith part is equal to the number of boxes in column i of the Young diagram of λ. Filling the boxes of a Young diagram with any mathematical objects, we obtain a *tableau* T and we let $T(i, j)$ denote the object in row i and column j of T. We call a tableau containing nonnegative integers a *Young tableau*, and further classify it as *column-strict*, *row-semistrict*, or *semistandard* if its entries strictly increase upward in each column, weakly increase to the right in each row, or satisfy both of these conditions, respectively. If the multiset of entries of Young tableau T is $1^{\alpha_1} 2^{\alpha_2} \cdots$, we say that T has *content* α. We call a tableau containing elements of a poset P a *P-tableau*, and further classify it as *column-strict*, *row-semistrict*, or *semistandard* if its entries satisfy $T(1, j) <_P T(2, j) <_P \cdots$ in each column j, $T(i, 1) \not>_P T(i, 2) \not>_P \cdots$ in each row i, or both of these conditions, respectively.

Now observe that for any planar network G of order n, we may partially order the set $P(G)$ of paths F from source i to sink i in G, $i = 1, \ldots, n$, by declaring $F <_{P(G)} F'$ if F lies entirely below F' in G, i.e., if F and F' are disjoint paths connecting sources $i < j$ to sinks $i < j$, respectively. This definition leads to a set of $P(G)$-tableaux for each planar network G. Imposing one more condition on this set of tableaux, we declare a $P(G)$-tableau of shape $\lambda \vdash n$ to be a *G-tableau* if the paths in its n boxes form a bijective path family $\pi = (\pi_1, \ldots, \pi_n)$ such that π_i connects source and sink i, for $i = 1, \ldots, n$. Given a G-tableau T, we define its *weight* $\omega(T)$ to be the weight of the path family it contains. Thus we may restate Lindström's Lemma for a planar network G and its path matrix $A = A(G)$ as $\det(A) = \sum_T \omega(T)$, where the sum is over all column-strict G-tableaux of shape 1^n.

For example, let G be the planar network in (4), and let F be the multiset of its edges in which the upper central edge has multiplicity two and all other edges have multiplicity one. Then there is exactly one path family $(\pi_1, \pi_2, \pi_3, \pi_4)$ of type 1 having the skeleton F. The path poset $P(G)$ and six column-strict G-tableaux having skeleton F are



$$\begin{bmatrix} \pi_4 & & \\ \pi_1 & \pi_2 & \pi_3 \end{bmatrix}, \quad \begin{bmatrix} \pi_3 & & \\ \pi_1 & \pi_2 & \pi_4 \end{bmatrix}, \quad \begin{bmatrix} \pi_3 & & \\ \pi_2 & \pi_1 & \pi_4 \end{bmatrix},$$
$$\begin{bmatrix} \pi_3 & & \\ \pi_2 & \pi_1 & \pi_4 \end{bmatrix}, \quad \begin{bmatrix} \pi_3 & & \\ \pi_2 & \pi_4 & \pi_1 \end{bmatrix}, \quad \begin{bmatrix} \pi_4 & & \\ \pi_2 & \pi_3 & \pi_1 \end{bmatrix}. \tag{11}$$

The first four G-tableaux are also semistandard whereas the last two are not semistandard.

3 Temperley-Lieb immanants

The subspace of $\mathbb{C}[x]$ spanned by products of pairs of complementary principal minors of x,

$$\mathrm{span}\{\det(x_{I,I}) \det(x_{\overline{I},\overline{I}}) \mid I \subseteq [n]\}, \tag{12}$$

where $\overline{I} = [n] \setminus I$, is also equal to the span of certain immanants called *Temperley-Lieb immanants*, defined in [RS05]. These were shown to belong to the dual canonical basis of $\mathbb{C}[x]$ and therefore are TNN and SNN [RS06], [Ska08]. We call the space (12) the *Temperley-Lieb subspace* of $\mathbb{C}[x]$.

We define the *Temperley-Lieb algebra* $T_n(2)$ to be the quotient

$$T_n(2) = \mathbb{C}[S_n]/(1 + s_1 + s_2 + s_1 s_2 + s_2 s_1 + s_1 s_2 s_1), \tag{13}$$

and let $\theta_n : \mathbb{C}[S_n] \to T_n(2)$ denote the canonical projection. As a $\mathbb{C}$-algebra, $T_n(2)$ is generated by the $n-1$ elements $t_i = \theta(s_i + 1)$, $i = 1, \ldots, n-1$, subject to the relations

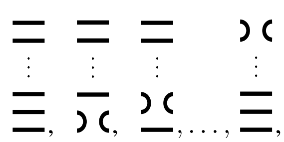
$$\begin{aligned} t_i^2 &= 2t_i, && \text{for } i = 1, \ldots, n-1, \\ t_i t_j t_i &= t_i, && \text{if } |i-j| = 1, \\ t_i t_j &= t_j t_i, && \text{if } |i-j| \geq 2. \end{aligned} \tag{14}$$

As a vector space, $T_n(2)$ has dimension $\frac{1}{n+1}\binom{2n}{n}$, and has a natural basis $\{\tau_v\}$ indexed by 321-avoiding permutations v in S_n. If $s_{i_1} \cdots s_{i_\ell}$ is a reduced expression for v, then τ_v is given by $\tau_v = \theta((1 + s_{i_1}) \cdots (1 + s_{i_\ell})) = t_{i_1} \cdots t_{i_\ell}$. This basis element is in fact independent of the chosen reduced expression for v. More generally, the relations (14) imply that for an arbitrary expression $s_{i_1} \cdots s_{i_\ell}$, we have

$$\theta((1 + s_{i_1}) \cdots (1 + s_{i_\ell})) = 2^k \tau, \tag{15}$$

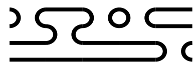
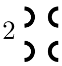
for some τ and some $k \geq 0$. Indeed, for F a bijective skeleton of order n with the property that all vertices in F have indegree and outdegree bounded by 2, $\beta(F)$ has the form $(1 + s_{i_1}) \cdots (1 + s_{i_\ell})$ by [RS05, Lem. 2.5], and apparently $\theta(\beta(F)) = 2^k \tau$ for some τ, k.

Computations in $T_n(2)$ can sometimes be simplified by the use of *Kauffman diagrams* for the natural basis elements. Kauffman [Kau87, Sec. 4] represents the identity and generators $1, t_1, \ldots, t_{n-1}$ of $T_n(2)$ by the diagrams



and represents multiplication by concatenating diagrams and replacing each resulting cycle by a factor of 2. For instance, the fourteen basis elements of $T_n(2)$ are

and the identity $t_3t_2t_3t_3t_1 = 2t_1t_3$ in $T_4(2)$ is represented by

 $= 2$  (16)

For each natural basis element τ of $T_n(2)$, we define the function $f_\tau : \mathbb{C}[S_n] \to \mathbb{C}$ by

$$f_\tau : w \mapsto \text{coefficient of } \tau \text{ in } \theta(w), \tag{17}$$

and the corresponding immanant $\mathrm{Imm}_\tau(x)$ by

$$\mathrm{Imm}_\tau(x) \underset{\text{def}}{=} \mathrm{Imm}_{f_\tau}(x) = \sum_{w \in S_n} f_\tau(w) x_{1,w_1} \cdots x_{n,w_n}. \tag{18}$$

Observe that for a bijective skeleton F satisfying $\theta(\beta(F)) = 2^k\tau$, the definition (17) implies the formula

$$f_\sigma(\beta(F)) = \delta_{\sigma,\tau} 2^k. \tag{19}$$

4 Elementary immanants

For $\lambda \vdash n$, let ϵ^λ be the S_n-character induced from the sign character of a Young subgroup of type λ. Equivalently, ϵ^λ is the S_n-class function which corresponds by the Frobenius characteristic map to the elementary symmetric function e_λ,

$$e_\lambda = \tfrac{1}{n!} \sum_{w \in S_n} \epsilon^\lambda(w) p_{\rho(w)},$$

where $\rho(w)$ is the cycle type of w. (See [Sag01].) We call the corresponding immanant $\mathrm{Imm}_{\epsilon^\lambda}(x)$ an *elementary immanant*. Since $\{\epsilon^\lambda \,|\, \lambda \vdash n\}$ is a basis of the space of S_n-class functions, the elementary immanants $\{\mathrm{Imm}_{\epsilon^\lambda}(x) \,|\, \lambda \vdash n\}$ are a basis of the space of class immanants. The Littlewood-Merris-Watkins identity [Lit40], [MW85] for elementary immanants provides a short proof that these immanants are TNN and SNN.

Proposition 4.1 *For $\mu = (\mu_1, \ldots, \mu_r) \vdash n$, we have*

$$\mathrm{Imm}_{\epsilon^\mu}(x) = \sum_Q \det(x_{Q_1,Q_1}) \cdots \det(x_{Q_r,Q_r}), \tag{20}$$

where the sum is over all sequences $Q = (Q_1, \ldots, Q_r)$ of disjoint subsets of $[n]$ satisfying $|Q_i| = \mu_i$ for $i = 1, \ldots, r$.

Combining this identity with Lindström's Lemma, we obtain the following combinatorial interpretation of elementary character immanants.

Corollary 4.2 *Let R be a $\mathbb{C}$-algebra, let G be an R-weighted planar network of order n with path matrix A, and let λ be a partition of n. Then we have*

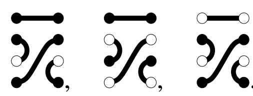
$$\mathrm{Imm}_{\epsilon^\lambda}(A) = \sum_T \omega(T), \tag{21}$$

where the sum is over all column-strict G-tableaux T of shape $\lambda^\top$.

Now let us closely examine the special cases of induced sign characters ϵ^λ indexed by partitions λ having at most two parts. The corresponding immanant $\mathrm{Imm}_{\epsilon^\lambda}(x)$ clearly belongs to the Temperley-Lieb subspace (12) of $\mathbb{C}[x]$. Expanding elementary immanants indexed by $\lambda = (\lambda_1, \lambda_2)$ in terms of the dual canonical basis, we have

$$\mathrm{Imm}_{\epsilon^\lambda}(x) = \sum_\tau b_{\lambda,\tau}\mathrm{Imm}_\tau(x), \tag{22}$$

with coefficients $b_{\lambda,\tau} \in \mathbb{N}$, by Proposition 4.1 and [RS05, Thm. 4.5]. We now reprove this result and interpret the coefficients in terms of colorings of natural basis elements of $T_n(2)$. We adjoin vertices to the $2n$ endpoints of the curves of the Kauffman diagram of a basis element τ, labeling these *source* 1, . . . , *source* n, *sink* n, . . . , *sink* 1, as in a planar network. We define a 2-coloring of τ to be an assignment of colors $\{1, 2\}$ to these vertices such that source i and sink i have equal colors for all i, and connected vertices have equal colors if and only if one is a source and one is a sink. More specifically, we call a 2-coloring a $(2, \lambda)$-*coloring* for $\lambda = (\lambda_1, \lambda_2) \vdash n$ if λ_1 sources have color 1 and λ_2 sources have color 2. For example, the basis element t_2t_1 of $T_4(2)$ has one $(2, 31)$-coloring and two $(2, 22)$-colorings:

 (23)

Now we may interpret the coordinates of elementary immanants with respect to the dual canonical basis as follows.

Proposition 4.3 *For $\lambda = (\lambda_1, \lambda_2) \vdash n$ and τ a standard basis element of $T_n(2)$, the coefficient $b_{\lambda,\tau}$ appearing in the expansion (22) is equal to the number of $(2, \lambda)$-colorings of τ.*

Proof: Omitted. □

5 Monomial immanants

For $\lambda \vdash n$, let ϕ^λ be the S_n-class function which corresponds by the Frobenius characteristic map to the monomial symmetric function m_λ,

$$m_\lambda = \tfrac{1}{n!} \sum_{w \in S_n} \phi^\lambda(w) p_{\rho(w)}. \tag{24}$$

Following [Ste92], we call ϕ^λ a *monomial virtual character* of S_n, and we call the corresponding immanant $\mathrm{Imm}_{\phi^\lambda}(x)$ a *monomial immanant*. (ϕ^λ is not an S_n-character.) Like induced sign characters and elementary immanants, monomial virtual characters and monomial immanants form bases of the space of S_n-class functions and the space of class immanants, respectively. While no simple formula analogous to (20) is known for monomial immanants, Stembridge has conjectured that they are TNN and SNN [Ste92].

Conjecture 5.1 *For $\lambda \vdash n$, the monomial immanant $\mathrm{Imm}_{\phi^\lambda}(x)$ is TNN and SNN.*

It is straightforward to show Stembridge's TNN conjecture to be the strongest possible for TNN class immanants, i.e., that a class immanant $\mathrm{Imm}_f(x)$ is TNN only if it is equal to a nonnegative linear combination of monomial immanants. For instance, one may deduce this from the following result.

Proposition 5.2 *For $\mu \vdash n$, let $A(\mu)$ be the $n \times n$ block-diagonal matrix whose ith diagonal block is a $\mu_i \times \mu_i$ matrix of ones. Then we have $\mathrm{Imm}_{\phi^\lambda}(A(\mu)) = \delta_{\lambda,\mu}$.*

Proof: Omitted. □

To prove the TNN and MNN cases of Conjecture 5.1, it would suffice to give an analog of Corollary 4.2 for monomial immanants.

Problem 5.3 *For $\lambda \vdash n$, find a graph-theoretic interpretation of $\mathrm{Imm}_{\phi^\lambda}(A)$ which holds for path matrices A of arbitrary R-weighted planar networks of order n.*

In Theorem 5.6, we prove the special case of Stembridge's conjectures for λ satisfying $\lambda_1 \leq 2$. In particular, we express each such monomial immanant $\mathrm{Imm}_{\phi^\lambda}(x)$ as a nonnegative linear combination of Temperley-Lieb immanants, with coefficients counting special 2-colorings of natural basis elements of $T_n(2)$. For each natural basis element τ and each index $j \leq \lfloor \frac{n}{2} \rfloor$, define $\mathcal{A}(\tau, j)$ to be the set of all $(2, 2^j 1^{n-2j})$-colorings of τ, define $\iota(\tau)$ to be the minimum i for which $\mathcal{A}(\tau, i) \neq \emptyset$, and define $\mathcal{B}(\tau) = \mathcal{A}(\tau, \iota(\tau))$. Let $M_{\lambda,\mu}$ denote the number of column-strict Young tableaux of shape λ and content μ. The cardinalities of these sets are related as follows.

Lemma 5.4 *For each natural basis element τ of $T_n(2)$, and each index $j \leq \lfloor \frac{n}{2} \rfloor$, we have $|\mathcal{A}(\tau, j)| = M_{2^j 1^{n-2j}, 2^{\iota(\tau)} 1^{n-2\iota(\tau)}} |\mathcal{B}(\tau)|$.*

Proof: Omitted. □

Proposition 5.5 *For each bijective skeleton F of order n, and each index $j \leq \lfloor \frac{n}{2} \rfloor$, we have $\phi^{2^i 1^{n-2i}}(\beta(F)) = \delta_{i,\iota(\tau)} |\mathcal{B}(\tau)|$, where k, τ are defined by $\theta(\beta(F)) = 2^k \tau$.*

Proof: Choose a reduced expression $s_{i_1} \cdots s_{i_\ell}$ for each $w \in S_n$ and define a corresponding element $D_w = (1 + s_{i_1}) \cdots (1 + s_{i_\ell})$ of $\mathbb{C}[S_n]$. (These elements depend upon the chosen reduced expressions, but we suppress this from the notation.) It is straightforward to show that there exist planar networks $\{F_w \,|\, w \in S_n\}$ satisfying $D_w = \beta(F_w)$, and that for each w, one may define the integer k_w and natural basis element σ_w of $T_n(2)$ by the equation $\theta(\beta(F_w)) = 2^{k_w} \sigma_w$. For $i = 0, \ldots, \lfloor \frac{n}{2} \rfloor$, define the functions $f_i : \mathbb{C}[S_n] \to \mathbb{C}$ by the conditions

$$f_i(D_w) = \delta_{i,\iota(\sigma_w)} 2^{k_w} |\mathcal{B}(\sigma_w)| \qquad \text{for all } w \in S_n, \tag{25}$$

and by linearly extending to all of $\mathbb{C}[S_n]$. Lemma 5.4 then implies that for each permutation w we have

$$\begin{aligned} \epsilon^{n-j,j}(D_w) = |\mathcal{A}(\sigma_w, j)| &= M_{2^j 1^{n-2j}, 2^{\iota(\sigma_w)} 1^{n-2\iota(\sigma_w)}} 2^{k_w} |\mathcal{B}(\sigma_w)| \\ &= \sum_{i=0}^{j} M_{2^j 1^{n-2j}, 2^i 1^{n-2i}} f_i(D_w). \end{aligned} \tag{26}$$

On the other hand, induced sign characters and monomial virtual characters are related by the (unitriangular) system of equations

$$\epsilon^{\mu^\top} = \sum_{\lambda \preceq \mu} M_{\mu,\lambda} \phi^\lambda. \tag{27}$$

Comparing this expression to (26) and using unitriangularity, we see that f_i and $\phi^{2^i 1^{n-2i}}$ agree on the basis $\{D_w \,|\, w \in S_n\}$ and therefore on all of $\mathbb{C}[S_n]$. □

Theorem 5.6 *For $j \leq \lfloor \frac{n}{2} \rfloor$ and $\lambda = 2^j 1^{n-2j}$, we have*

$$\mathrm{Imm}_{\phi^\lambda}(x) = \sum_{\substack{\tau \\ \iota(\tau)=j}} |\mathcal{B}(\tau)| \,\mathrm{Imm}_\tau(x). \tag{28}$$

Proof: Omitted. □

Now we solve the special case of Problem 5.3 for λ satisfying $\lambda_1 \leq 2$.

Theorem 5.7 *Let R be a $\mathbb{C}$-algebra, let G be an R-weighted planar network of order n with path matrix A, and fix $\lambda \vdash n$ with $\lambda_1 \leq 2$. Then we have*

$$\mathrm{Imm}_{\phi^\lambda}(A) = \sum_T \omega(T), \tag{29}$$

where the sum is over all column-strict G-tableaux T of shape λ, such that no column-strict G-tableau S of shape $\mu \prec \lambda$ satisfies $\mathrm{skel}(S) = \mathrm{skel}(T)$.

Proof: Omitted. □

6 Irreducible character immanants

For $\lambda \vdash n$, let χ^λ be the irreducible S_n-character traditionally indexed by λ. (See, e.g., [Sag01].) Equivalently, χ^λ is the S_n-class function which corresponds by the Frobenius characteristic map to the Schur function s_λ,

$$s_\lambda = \tfrac{1}{n!} \sum_{w \in S_n} \chi^\lambda(w) p_{\rho(w)}. \tag{30}$$

Let $\mathrm{Imm}_\lambda(x) = \mathrm{Imm}_{\chi^\lambda}(x)$ denote the corresponding *irreducible character immanant*. While no simple formula analogous to (20) is known for irreducible character immanants, Goulden and Jackson [GJ92] conjectured that they are MNN [Ste92], Stembridge conjectured that they are TNN and SNN, and these three conjectures were proved by Greene [Gre92], Stembridge [Ste91] and Haiman [Hai93]. In spite of these results, no analog of Corollary 4.2 is known for irreducible character immanants.

Problem 6.1 *For $\lambda \vdash n$, find a graph theoretic interpretation of* $\mathrm{Imm}_\lambda(A)$ *when A is the path matrix of an R-weighted planar network of order n.*

In Theorem 6.2 we solve the special case of Problem 6.1 for λ a *hook* partition, i.e., $\lambda = r1^{n-r}$. We will do so by expressing irreducible character immanants in terms of elementary immanants and by using combinatorial objects called *special ribbon diagrams* in [BRW96].

Theorem 6.2 *Let R be a $\mathbb{C}$-algebra, let G be an R-weighted planar network of order n with path matrix A, and let μ be the hook partition $r1^{n-r}$ of n. Then we have*

$$\mathrm{Imm}_\mu(A) = \sum_T \omega(T), \tag{31}$$

where the sum is over all semistandard G-tableaux T of shape μ.

Proof: Omitted. □

For example, when the planar network G and its path matrix A are as shown in (4), we have $\mathrm{Imm}_{31}(A) = 4$, counting the semistandard G-tableau in (11).

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Statistics on staircase tableaux, eulerian and mahonian statistics

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Abstract. We give a simple bijection between some staircase tableaux and tables of inversion. Some nice properties of the bijection allows us to define some q-Eulerian polynomials related to the staircase tableaux. We also give a combinatorial interpretation of these q-Eulerian polynomials in terms of permutations.

Résumé. Nous proposons une bijection simple entre certains tableaux escalier et les tables d'inversion. Cette bijection nous permet de montrer que les statistiques Euleriennes et Mahoniennes sont naturelles sur les tableaux escalier. Nous définissons des polynômes q-Eulériens et en donnons une interprétation combinatoire.

Keywords: staircase tableaux, bijection, permutations

1 Introduction

Staircase tableaux are new combinatorial objects defined by S. Corteel and L. Williams (10). They are related to the asymmetric exclusion process on a one-dimensional lattice with open boundaries (ASEP) and were also used to give a combinatorial formula for the moments of the Askey-Wilson polynomials defined in (1; 11). Those results are presented in (10; 7). The staircase tableaux are generalizations of the permutation tableaux (6; 16) coming from work and alternative tableaux (13; 17).

Definition 1 *(10) A staircase tableau of size n is a Young diagram of shape $(n, n-1, ..., 1)$ such that boxes are empty or filled with α,β,γ,δ and that*

- *the boxes along the diagonal are not empty*
- *a box in the same row and on the left of a β or a δ is empty*
- *a box in the same column and above a α or a γ is empty*

Definition 2 *(10) The* weight $\mathrm{wt}(\mathcal{T})$ *of a staircase tableau $\mathcal{T}$ is a monomial in $\alpha, \beta, \gamma, \delta, q$, and u, which we obtain as follows. Every blank box of $\mathcal{T}$ is assigned a q or u, based on the label of the closest labeled box to its right in the same row and the label of the closest labeled box below it in the same column, such that:*

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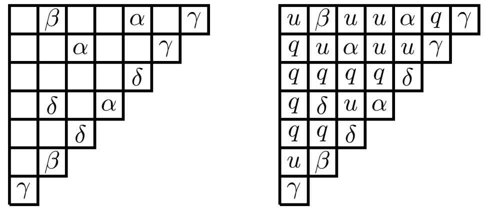

Fig. 1: A staircase tableau of size 7 and its weight

- *every blank box which sees a β to its right gets assigned a u;*
- *every blank box which sees a δ to its right gets assigned a q;*
- *every blank box which sees an α or γ to its right, and an α or δ below it, gets assigned a u;*
- *every blank box which sees an α or γ to its right, and a β or γ below it, gets assigned a q.*

After assigning a q or u to each blank box in this way, the weight *of $\mathcal{T}$ is then defined as the product of all labels in all boxes.*

The tableau on Figure 1 has weight $\alpha^3\beta^2\gamma^3\delta^3q^9u^8$.

Remark. The weight of a staircase tableau always has degree $n(n+1)/2$. For convenience, we will sometimes set $u=1$, since this results in no loss of information.

Definition 3 *The generating polynomial of staircase tableaux of size n is*

$$Z_n(\alpha,\beta,\gamma,\delta,q,u) = \sum_{T \text{ of size } n} \mathrm{wt}(T).$$

When $q=u=1$, this generating polynomial is simple (10) :

$$Z_n(\alpha,\beta,\gamma,\delta,1,1) = \prod_{i=0}^{n-1}(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta)). \quad (1)$$

In (7), the authors give an explicit formula for $Z_n(\alpha,\beta,\gamma,\delta,q,1)$. It is very complicated and is derived from a formula of the moments of the Askey-Wilson polynomials. In this paper, we show that there are other special cases of Z_n that have a very nice form. In particular, we show that

Theorem 1

$$Z_n(0,\beta,\gamma,0,q,u) = \prod_{i=0}^{n-1}(\beta u^i + \beta\gamma(u^{i-1}q+\ldots+uq^{i-1})+\gamma q^i).$$

Notice that as the definition of the tableaux implies that $Z_n(\alpha,\beta,\gamma,\delta,1,1) = Z_n(0,\beta+\delta,\alpha+\gamma,0,1,1)$, our result is a refinement of (1).

We will prove the results in two ways: a bijection and an inductive argument. We will see that both of these arguments are quite simple. This gives the simplest argument that there exist $4^n n!$ staircase tableaux of size n (10; 7). We will study (β/γ)-tableaux that are staircase tableaux which do not contain any α or δ. We show that our very simple bijection can be generalized to any family of staircase tableaux.

We continue the study of the (β/γ)-tableaux. When those tableaux have exactly n entries equal to γ, there exist exactly $n!$ such tableaux. In (7), it is shown that they are in bijection with permutation tableaux (16) or alternative tableaux (13; 17). We will show that the bijection allows us in this case to understand the statistic "number of βs on the diagonal" which is known to be related to the eulerian numbers (16; 18). Thanks to this we will introduce some new q-Eulerian polynomials and will give some combinatorial interpretation in terms of permutations.

We start in this paper by studying $Z_n(\alpha, \beta, \gamma, \delta, 1, 1)$ and some simple consequences and symmetries on staircase tableaux. We then study the (β/γ)-staircase tableaux and define the q-Eulerian polynomials. We show how the same type of arguments can be extended for type B staircase tableaux. We end this extended abstract with some concluding remarks and open problems.

2 Warm up on staircase tableaux

We first recall some simple recurrence to compute $Z_n(\alpha, \beta, \gamma, \delta, 1, 1)$ given in (7). Thanks to the definition of staircase tableaux, it is direct to see that

$$Z_n(\alpha, \beta, \gamma, \delta, 1, 1) = Z_n(\alpha + \gamma, \beta + \delta, 0, 0, 1, 1)$$

We then just need to count tableaux with αs and βs as done for permutation tableaux in (6). As in (10), we say that a line is indexed by α if the leftmost entry of the line is α. Let $Z_{n,k}(\alpha, \beta)$ be the number of tableaux counted by $Z_n(\alpha, \beta, 0, 0, 1, 1)$ with k rows indexed by α. Then if we add a new column to a staircase tableau, we see that :

$$Z_{n,k}(\alpha, \beta) = \sum_{\ell \geq k-1} \alpha\beta^{\ell-k+1} \binom{\ell}{k-1} Z_{n-1,\ell}(\alpha, \beta) + \sum_{\ell \geq k} \beta^{\ell-k+1} \binom{\ell}{k} Z_{n-1,\ell}(\alpha, \beta).$$

for $n > 0$ and $i \leq n$. The initial conditions are $Z_{0,0} = 1$ and $Z_{n,k} = 0$ if $k < 0$ or $n < 0$ or $k > n$. This implies that $Z_n(\alpha, \beta, x) = \sum_k Z_{n,k}(\alpha, \beta)x^k$ follows the recurrence for $n > 0$

$$Z_n(\alpha, \beta, x) = (\alpha x + \beta) Z_{n-1}(\alpha, \beta, x + \beta)$$

and with the initial condition $Z_0(\alpha, \beta, x) = 1$. The solution is $Z_n(\alpha, \beta, x) = \prod_{i=0}^{n-1}(\alpha x + \beta + i\alpha\beta)$ and therefore $Z_n(\alpha, \beta, 0, 0, 1, 1) = \prod_{i=0}^{n-1}(\alpha + \beta + i\alpha\beta)$.

This implies the following known results (7) :

1. The number of staircase tableaux of size n with αs and βs is $(n + 1)!$.
2. The number of staircase tableaux of size n is $4^n n!$
3. The number of staircase tableaux of size n with αs and βs and γs is $(2n + 1)!!$.

We get some other simple results.

Lemma 1 *1. The number of staircase tableaux of size n with a maximum number of α, β, γ or δ is $4^n(n-1)!$.*

2. *The number of staircase tableaux of size n with a minimum number of α, β, γ or δ is 4^n.*

From the definition of the weight of the tableaux, we also get:

Lemma 2 1. *The number of tableaux of size n with αs and βs and a minimum (resp. maximum) number of us is $3 \times 4^{\frac{n-1}{2}}$ (resp. n).*

2. *The number of tableaux of size n a minimum number of us is $\binom{n}{2}$.*

We can also define three involutions on tableaux. The proof that they are involutions follows directly from the definition of the tableaux.

Involution 1. Let ϕ be the involution on the staircase tableaux that takes a tableau T, exchanges αs and βs, and exchanges γs and δs, and conjugates the tableau. We can check that the tableau obtained is a staircase tableau, and that the number of α in T is the number of β in $\phi(T)$ and so on. This implies that:

$$Z_n(\alpha, \beta, \gamma, \delta, 1, 1) = Z_n(\beta, \alpha, \delta, \gamma, 1, 1).$$

An example is given on Figure 2.

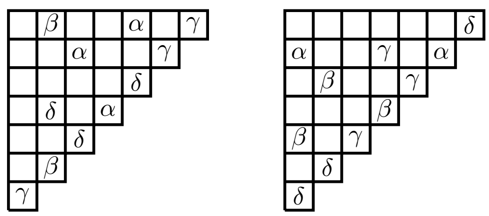

Fig. 2: Example of the involution 1

Involution 2. We can also use the involution ψ that exchanges αs with δs, and βs with γs and conjugates the obtained tableau. This gives :

$$Z_n(\alpha, \beta, \gamma, \delta, 1, 1) = Z_n(\delta, \gamma, \beta, \alpha, 1, 1).$$

Involution 3. Finally if we exchange αs and γs and βs and δs, we get

$$Z_n(\alpha, \beta, \gamma, \delta, q, u) = Z_n(\gamma, \delta, \alpha, \beta, \alpha, u, q).$$

Open problem Find a combinatorial proof of the fact that :

$$Z_n(\alpha, \beta, \gamma, \delta, q, u) = Z_n(\beta, \alpha, \delta, \gamma, q, u).$$

By a combinatorial proof, we mean a natural involution on the tableaux.

3 A bijection from tableaux to inversion table

3.1 Tableaux with n entries equal to γ and βs

We first recall that (β/γ)-tableaux are staircase tableaux with no α or δ. We start by enumerating the (β/γ)-tableaux of size n that contain exactly n entries equal to γ. Let $\tilde{Z}_n(\beta, \gamma, q)$ be their generating polynomial. We will show that :

Proposition 1 $\tilde{Z}_n(\beta, \gamma, q) = \gamma^n \prod_{i=0}^{n-1}(\beta(1 + q + \ldots + q^{i-1}) + q^i)$.

We define a bijection from those tableaux of size n to permutations of S_n via tables of inversion $T = [T[1], \ldots, T[n]]$ with $T[i] < i$ for $1 \leq i \leq n$.

A bijection from tableaux to permutations. There is one γ in each column, so we can number them: the leftmost γ will be designed by γ_1, the following by γ_2 and so on, until γ_n. Then, for each γ_i, we count the number t_i of cells that do not contain a Greek letter to the immediate left of γ_i. We can construct a table of inversion T with $T[i] = t_i$.

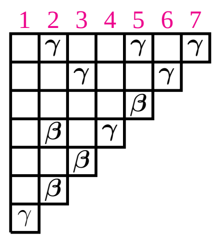

Fig. 3: A (β/γ)-staircase tableau

For example from the tableau on Figure 3, we obtain the table $T = (0, 1, 2, 1, 2, 2, 1)$. Then we can use any bijection from inversion table to permutations and obtain a permutation. For example if $T[i]$ corresponds to the number of elements $j < i$ such that $\sigma^{-1}(i) < \sigma^{-1}(j)$, we obtain the permutation (3,2,5,6,4,7,1).

Inverse of the bijection. We have an inversion table T of size n, we construct a staircase tableau of size n using the following algorithm :

- Put γ in the last column and first row, and mark $T[n]$ cells to its left (with q).
- For $i = n - 1$ to 1
  - Look at the topmost cell in the ith column which is not occupied and put a γ in it.
  - Mark the $T[i]$ cells to its left (with q)
  - Fill all the empty cells under it with β
  - Mark all the cells to the left of the βs (with u)

We have exactly one γ in each column. Each β has no Greek letter to its left and each γ has no Greek letter above itself. We have a staircase tableau, and it is obvious that the table of inversion obtained from this staircase tableau is exactly T. Therefore we defined a bijection. Moreover all the cells directly to the left of a γ get a weight q. Therefore

Proposition 2 *The number of (β/γ)-staircase tableaux of size n with n entries equal to γ, a entries equal to q, b rows indexed by γ is equal to the number of permutations of $\{1,\ldots,n\}$ with a inversions and b left-to-right minima.*

Using well known results on enumeration of permutations (see for example (15) Chapter 1), we get a proof of Proposition 1.

3.2 Generalization of the bijection

Now we can generalize the previous bijection to staircase tableaux. Start with a staircase tableau of size n and number the columns from 1 to n from left to right. Then for each column i, we look at the topmost Greek letter in column i and count the number of cells j directly to its left that does not contain any Greek letter. If this letter, say x, is topmost and leftmost, we record $T[i]=j_x$. Otherwise let y be the first Greek letter to the left of x and let z be the first Greek letter under y. Then $T[i]=j_{x,z}$.

For example, using the tableau of Figure 1, we obtain $T=(0_\gamma,1_\beta,2_\alpha,1_{\alpha,\beta},2_{\alpha,\delta},2_{\gamma,\delta},1_{\gamma,\delta})$.

For the general case, this is a bijection from staircase tableaux of size n and colored tables of inversion T such that $T[i]=(i-1)_x$ with $x\in\{\alpha,\beta,\gamma,\delta\}$ or $T[i]=j_{x,y}$ with $0\le j<i-1$ and $x\in\{\alpha,\gamma\}$ and $y\in\{\beta,\delta\}$.

This implies equation (1), that is:

$$Z_n(\alpha,\beta,\gamma,\delta,1,1)=\prod_{i=0}^{n-1}(\alpha+\beta+\gamma+\delta+i(\alpha+\gamma)(\beta+\delta)).$$

For the (β/γ)-tableaux, this is a bijection from (β/γ)-staircase tableaux of size n and colored tables of inversion T such that $T[i]=(i-1)_x$ with $x\in\{\beta,\gamma\}$ or $T[i]=j_{\gamma,\beta}$ with $0\le j<i-1$. The number of q of the tableau is equal to the sum of the $T[i]$ (except the ones that are equal to i_β). This implies that

$$Z_n(0,\beta,\gamma,0,q,1)=\prod_{i=0}^{n-1}(\beta+\beta\gamma(q+\ldots+q^{i-1})+\gamma q^i).$$

This is Theorem 1.

Remark. As in the previous section, we could have proven this by recurrence. Let $Z_{n,k}(\beta,\gamma,q)$ be the number of tableaux counted by $Z_n(0,\beta,\gamma,0,q,1)$ with k rows indexed by γ. We look at how many ways we can add a column to a tableau of size $n-1$. We get:

$$Z_{n,k}(\beta,\gamma,q)=\sum_{\ell\ge k-1}\gamma\beta^{\ell-k+1}q^{k-1}\binom{\ell}{k-1}Z_{n-1,\ell}(\beta,\gamma,q)+\sum_{\ell\ge k}\beta^{\ell-k+1}q^k\binom{\ell}{k}Z_{n-1,\ell}(\beta,\gamma,q).$$

for $n>0$ and $k\le n$. The initial condition are $Z_{0,0}=1$ and $Z_{n,k}=0$ if $k<0$ or $n<0$ or $k>n$. Let $Z_n(\beta,\gamma,q,x)=\sum_k Z_{n,k}(\beta,\gamma,q)x^k$. The recurrence implies that $Z_0(\beta,\gamma,q,x)=1$ and for $n>0$

$$Z_n(\beta,\gamma,q,x)=(\gamma x+\beta)Z_{n-1}(\beta,\gamma,q,xq+\beta)$$

The solution is

$$Z_n(\beta,\gamma,q,x) = \prod_{i=0}^{n-1}(\beta+\beta\gamma(q+\ldots+q^{i-1})+\gamma x q^i).$$

Therefore $Z_n(0,\beta,\gamma,0,q,1) = \prod_{i=0}^{n-1}(\beta+\beta\gamma(q+\ldots+q^{i-1})+\gamma q^i)$.

4 q-Eulerian polynomials

4.1 Entries equal to β on the diagonal

Again we number the columns of the tableau from left to right. In this section we use some properties of the bijection defined in Section 3.1. We need the following simple lemma.

Lemma 3 *Given a (β/γ)-tableau of size n with n entries equal to γ, there is a β on the diagonal in column i if and only if there is at least one γ in column $j > i$ that has $j-i-1$ entries equal to q to its immediate left.*

Proof: Direct from the definition of the tableau and the fact that those tableaux have exactly one γ in each column. □

We use the bijection of Section 3.1. We now transform the table of inversion $T = [T[1],\ldots,T[n]]$ obtained from the (β/γ)-tableau into the table $[0-T[1], 1-T[2],\ldots,n-1-T[n]]$. We still obtain a table of inversion. Moreover the distinct positive values of the table of inversion now correspond to the diagonal entries filled with β. We skip the proof of this claim. Therefore

Proposition 3 *There exists a bijection between*

- *(β/γ)-tableaux of size n with n entries equal to γ, entries equal to β in diagonals $\{i_1,\ldots,i_k\}$ and a entries equal to q*
- *table of inversion $T = [T[1],\ldots,T[n]]$ such that*
  - *for $1 \leq j \leq k$, there exists at least one ℓ such that $T[\ell] = i_j$*
  - *$\sum_{i=1}^{n} T[i] = \binom{n}{2} - a$.*

For fixed n and k, let $\mathcal{Z}_{n,k}(\beta,\gamma,q)$ be the generating polynomial of (β/γ)-tableaux of size n with n entries equal to γ and k entries equal to β on the diagonal.

Lemma 4 *The number $\mathcal{Z}_{n,k}(1,1,1)$ is equal to the Eulerian numbers $E_{n,k+1}$.*

Proof: This is direct as these staircase tableaux of size n are in bijection with permutation tableaux of length n. This bijection is such that the entries equal to β on the diagonal are in one-to-one correspondence with the columns of the permutation tableau. See (10) for the bijection from staircase tableaux to permutation tableaux. See (6; 16) for the bijection from permutation tableaux to permutations. □

We now interpret $\mathcal{Z}_{n,k}(\beta,\gamma,q)$ in terms of permutations.

4.2 Permutations with k descents

We have seen that staircase tableaux with k entries equal to β on the diagonal are in bijection with tables of inversion with k different positive values. We construct here a bijection between these tables of inversion and permutations with k descents.

4.2.1 From permutations with k descents to the tables of inversion with $(k+1)$ distinct values (including 0)

Let σ be a permutation with k descents. We construct a table of inversion T from σ. For i from 1 to n, let j be the first element to the right of i satisfying $j < i$. If such a j does not exist, set $T[i] = 0$ and $T[i] = j$ otherwise.

It is easy to check that for all i, $T[i] < i$. Moreover all the values of the table are either 0 or the values of the end of the descents of σ. Finally, for all descent in σ of index i, σ_{i+1} is in at least one index in T. Then the table has $k+1$ distinct values.

For example, let $\sigma = (5, 8, 2, 1, 6, 7, 3, 4, 9)$. We obtain $T = [0, 1, 0, 0, 2, 3, 3, 2, 0]$. The permutation σ has three descents that end in $1, 2$ and 3, and the table has four distinct values $0, 1, 2$ and 3.

4.2.2 From tables of inversion to permutations

We start a table T of inversions with $k+1$ distincts values. We create σ by inserting successively the letters $i = 1, 2, \ldots, n$. If $T[i] > 0$ then we insert i directly before $T[i]$ and add i at the end otherwise.

For example, if we have the table $T = [0, 0, 1, 0, 4, 1, 0]$, we get the permutation $\sigma = (3, 6, 1, 2, 5, 4, 7)$ which has two descents. This is clearly the reverse map of the previous subsection.

We now can interpret $\mathcal{Z}_{n,k}(\beta, \gamma, q)$ in terms of permutations. Given a permutation σ of S_n, we suppose that $\sigma(n+1) = 0$. Let $M(\sigma, i)$ be j if j is the first element to the right of i such that $j < i$. Let

$$\begin{aligned} M(i) &= \min\{j \mid j < i \text{ and } \sigma^{-1}(j) > \sigma^{-1}(i)\} \\ M(\sigma) &= \sum_i M(\sigma, i). \end{aligned}$$

Let $\mathrm{RLmin}(\sigma)$ be the number of right-to-left minima of σ. For example, if $\sigma = (3, 6, 1, 2, 5, 4, 7)$ then $M(\sigma, 3) = M(\sigma, 6) = 1$, $M(\sigma) = 6$ and $\mathrm{RLmin}(\sigma) = 4$. Let $S_{n,k}$ be the set of permutations in S_n with k descents. Thanks to the previous bijection, we get that

Proposition 4

$$\mathcal{Z}_{n,k}(\beta, \gamma, q) = \gamma^n \beta^n q^{\binom{n}{2}} \sum_{\sigma \in S_{n,k}} q^{-M(\sigma)} \beta^{-\mathrm{RLmin}(\sigma)}.$$

But also we get a refinement. Let $I = \{i_1, \ldots i_k\}$, let $Z_{n,I}(\beta, \gamma, q)$ be the generating polynomial of the (β/γ)-tableaux of size n with n entries equal to γ and where entries equal to β on the diagonals are indexed by I. Let $S_n(I)$ be the set of permutations of S_n such that $\sigma(j-1) > \sigma(j)$ if and only if $\sigma(j) \in I$. Then

Proposition 5

$$Z_{n,I}(\beta, \gamma, q) = q^{\binom{n}{2}} \beta^n \gamma^n \sum_{\sigma \in S_n(I)} q^{-M(\sigma)} \beta^{-\mathrm{RLmin}(\sigma)}.$$

Remark. The case $\beta = q = 1$ was already known for permutation tableaux (6; 18).

5 Type B tableaux

In this section, we study some type B staircase tableaux. They are the analogue of the type B permutation and alternative tableaux (12; 4; 5).

Definition 4 *A type B staircase tableau of size n is a staircase tableau of size $2n$ that is invariant under the involution 2 from Section 2.*

As the tableau is symmetric, we only keep half of it. A type B staircase tableau of size n is therefore of shape $(1, 2, \dots, n, n, n-1, \dots 1)$. We number the rows from top to bottom and the columns from left to right. We denote by *sign-diagonal* the cells (i, i), for $1 \leq i \leq n$. As in Section 1, we define the generating polynomial $Z_n^{(B)}(\alpha, \beta, \gamma, \delta, q, u)$. We only look at the case $u = q = 1$.

We first investigate tableaux with only βs and γs. We can construct a bijection from those tableaux the signed permutations, using the idea of the bijection between staircase tableaux and permutations:

- When column i does not contain a γ, we add a γ in cell (i, i).
- We number the γs from left to right.
- We create two tables, the table of inversion T and the table of sign θ
- For each i, $T[i]$ is the number of cells with no Greek letter immediately to the left of γ_i (in the column i). The sign of i is $\ominus$ if γ_i is in the sign-diagonal and $\oplus$ otherwise.

For example, starting from the tableau on Figure 4, we obtain the signed permutation given by the tables $T = [0, 0, 2, 1, 2]$ and $\theta = [\ominus, \oplus, \oplus, \ominus, \ominus]$.

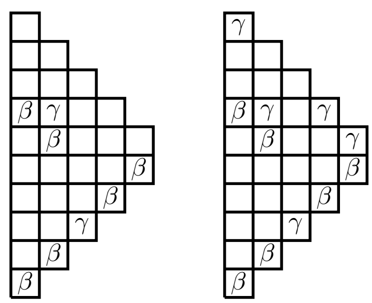

Fig. 4: A type B staircase tableau and the tableau when γs are inserted on the sign-diagonal

Therefore

Proposition 6 *The previous algorithm defines a bijection between type B staircase tableaux of size n with γs and βs and signed permutations of $\{1, \dots, n\}$. This bijection implies that*

$$Z_n^{(B)}(0, \beta, \gamma, 0, 1, 1) = (\gamma + \beta)^n \prod_{i=0}^{n-1} (1 + \beta i).$$

Proof: We obviously have a function that transforms a tableau into a signed permutation. To see that it is a bijection, we just have to notice that there is two choices for γ_n, and that knowing $\theta(n)$ allows us to know in which of these two cells is the γ of column n. Then for each i, if we know where are all the γ_j for $j > i$, we have two choices : γ_i may be on the left of a γ_j or on the diagonal or on the sign diagonal. The latter case corresponds to $\theta[i] = \ominus$. The others are identical to the construction between staircase tableaux and permutations. There is no other choice since for each γ on the column i that is not on the sign-diagonal the row i has to be empty (recall that the whole tableau is invariant under the involution 2 from Section 2). $\square$

Again, it is easy to see that:

$$Z_n^{(B)}(\alpha, \beta, \gamma, \delta) = Z_n^{(B)}(0, \beta + \delta, \gamma + \alpha, 0). \tag{2}$$

And we obtain the following corollary

Corollary 1 *There exist* $4^n(2n-1)!!$ *staircase tableaux of type* B *and size* n.

6 Conclusion

In this paper, we give a very simple bijection between (β/γ)-staircase tableaux and permutations. This bijection is such that the number of q in the tableaux is related to the number of inversions of the permutation. Thanks to this construction, we get some possibly new q-Eulerian polynomials. This work opens a set of natural open questions.

1. Is there a natural partially ordered set on (β/γ)-staircase tableaux that is isomorphic to the (weak) Bruhat order?
2. Can we compute these q-Eulerian polynomials as done in (18) for the permutation tableaux?
3. Can we compute the generating polynomial of (β/γ)-staircase tableaux when the diagonal is fixed as done in (18; 14) for the permutation tableaux?

Our goal in this study of the staircase tableaux is to understand the q-statistics in the general staircase tableaux. We know that this is related to crossings or 31-2 patterns in permutations for the case with only αs and βs (3; 6; 16), to inversions in permutation for the case with only βs and γs and to f-crossings in matchings (7) for the case with only αs, βs and γs. Similar results hold also for the type B analogue (12; 5; 4). The general case is still open for now.

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Polynomial functions on Young diagrams arising from bipartite graphs

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Abstract. We study the class of functions on the set of (generalized) Young diagrams arising as the number of embeddings of bipartite graphs. We give a criterion for checking when such a function is a polynomial function on Young diagrams (in the sense of Kerov and Olshanski) in terms of combinatorial properties of the corresponding bipartite graphs. Our method involves development of a differential calculus of functions on the set of generalized Young diagrams.

Résumé. Nous étudions la classe des fonctions sur l'ensemble des diagrammes de Young (généralisés) qui sont définies comme des nombres d'injections de graphes bipartites. Nous donnons un critère pour savoir si une telle fonction est une fonctions polynomiale sur les diagrammes de Young (au sens de Kerov et Olshanski) utilisant les propriétés combinatoires des graphes bipartites correspondants. Notre méthode repose sur le développement d'un calcul différentiel sur les fonctions sur les diagrammes de Young généralisés.

Keywords: Polynomial functions on Young diagrams, coloring of bipartite graphs, differential calculus on Young diagrams

The full version of this extended abstract will be published elsewhere.

1 Introduction

1.1 Prominent examples of polynomial functions on Young diagrams

The character $\chi^\lambda(\pi)$ of the symmetric group is usually considered as a function of the permutation π, with the Young diagram λ fixed. Nevertheless, it was observed by Kerov and Olshanski (1994) that for several problems of the asymptotic representation theory it is convenient to do the opposite: keep the permutation π fixed and let the Young diagram λ vary. It should be stressed the Young diagram λ is arbitrary, in particular there are no restrictions on the number of boxes of λ. In this way it is possible to study the structure of the series of the symmetric groups $\mathfrak{S}_1 \subset \mathfrak{S}_2 \subset \cdots$ and their representations in a uniform way. In order for this idea to be successful one has to replace the usual characters $\chi^\lambda(\pi)$ by, so called, *normalized characters* $\Sigma_\pi(\lambda)$, namely for partitions π, λ such that $|\pi| = k$, $|\lambda| = n$ we define

$$\Sigma_\pi(\lambda) = \begin{cases} \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}} \frac{\chi^\lambda(\pi, 1^{n-k})}{\chi^\lambda(1^n)} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

There are several other interesting examples of functions on the set of Young diagrams which in some sense — which will be specified in Section 1.2 — are similar to the normalized characters; we recall some of them in the following.

Free cumulants $R_k(\lambda)$, introduced by Kerov (2000a) and Biane (2003), are relatively simple functionals of the shape of the Young diagram λ. Their advantage comes from the fact that the normalized characters can be expressed in terms of free cumulants and that this expression takes a particularly simple form (Biane, 2003).

There are other interesting functionals of the shape of the Young diagram and *fundamental functionals of shape*, which are defined in Section 2.3, are very simple examples. These functionals are quite useful and powerful in the context of differential calculus on Young diagrams.

Jack symmetric functions (Jack, 1970/1971) are a generalization of Schur functions and are indexed by an additional parameter $\alpha > 0$. They can be used to define *Jack characters* $\Sigma_\pi^{(\alpha)}(\lambda)$ which are a natural generalization of the normalized characters of the symmetric groups. For some special values of α Jack symmetric functions become well-known objects. For example, for $\alpha = 2$ we obtain so-called *zonal polynomial* which is a zonal spherical function for the Gelfand pairs $(\mathfrak{S}_{2n}, H_n)$, where H_n denotes the hyperoctahedral group. This example has an important meaning in the representation theory for Gelfand pairs. Jack characters are related with α-anisotropic Young diagrams which are deformed Young diagrams with respect to the parameter α. More precisely, for a Young diagram $\lambda = (\lambda_1, \dots, \lambda_k)$ we can construct an α-anisotropic Young diagram $\alpha\lambda$ by rescaling the original diagram in one direction by parameter α, i.e. $\alpha\lambda = (\alpha\lambda_1, \dots, \alpha\lambda_k)$. The connection between functions on α-anisotropic Young diagrams and Jack polynomials was obtained by Kerov (2000b).

1.2 The algebra of polynomial functions on Young diagrams

Kerov and Olshanski (1994) defined the *algebra* $\mathcal{P}$ *of polynomial functions on the set* $\mathbb{Y}$ *of Young diagrams*. This algebra is generated by every example of a family of functions on Young diagrams which we presented in Section 1.1. The problem of how to express an element of one basis in terms of the elements of another basis of this algebra is very fascinating and it is related to Kerov polynomials, Goulden-Rattan polynomials and many other combinatorial objects which are sometimes well-known, but sometimes far away from being satisfactorily understood.

The algebra $\mathcal{P}$ of polynomial functions on $\mathbb{Y}$ turns out to be isomorphic to a subalgebra of the algebra of partial permutations of Ivanov and Kerov (1999). Therefore we can view the elements of $\mathcal{P}$ as (linear combinations of) partial permutations. Since the multiplication of functions on $\mathbb{Y}$ corresponds to convolution of central functions on partial permutations, we see that the algebra $\mathcal{P}$ turns out to be very closely related to the problems of computing connection coefficients and multiplication of conjugacy classes in the symmetric groups.

The algebra $\mathcal{P}$ is canonically isomorphic to the algebra of *shifted symmetric functions*. The algebra of shifted symmetric functions is an important object in the symmetric functions theory and the isomorphism with the algebra $\mathcal{P}$ gives some new results in this field due to Okounkov and Olshanski (1998).

1.3 Numbers of colorings of bipartite graphs

The set of vertices of a bipartite graph G will always be $V = V_1 \sqcup V_2$ with the elements of V_1 (respectively, V_2) referred to as white (respectively, black) vertices.

We consider a coloring h of the white vertices in V_1 by columns of the given Young diagram λ and of the black vertices in V_2 by rows of the given Young diagram λ. Formally, a coloring is a function

$h : V_1 \sqcup V_2 \to \mathbb{N}$ and we say that this coloring is *compatible* with a Young diagram λ if $(h(v_1), h(v_2)) \in \lambda$ (where $(h(v_1), h(v_2))$ denotes the box placed in $h(v_1)$th column and in $h(v_2)$th row) for each edge (v_1, v_2) of G with $v_1 \in V_1$, $v_2 \in V_2$. Alternatively, a coloring which is compatible with λ can be viewed as a function which maps the edges of the bipartite graph to boxes of λ with a property that if edges e_1, e_2 share a common white (respectively, black) vertex then $h(e_1)$ and $h(e_2)$ are in the same column (respectively, the same row). We can think that such a coloring defines an embedding of a graph G into the Young diagram λ. We denote by $N_G(\lambda)$ the number of colorings of G which are compatible with λ which is the same as the number of embeddings of G into λ by the above identification.

1.4 Polynomial functions on Young diagrams and bipartite graphs

Suppose that some interesting polynomial function $F \in \mathcal{P}$ is given. It turns out that it is very convenient to write F as a linear combination of the numbers of embeddings N_G for some suitably chosen bipartite graphs G:

$$F = \sum_G \alpha_G N_G \tag{1}$$

which is possible for any polynomial function F. This idea was initiated by Féray and Śniady (2011a) who found explicitly such linear combinations for the normalized characters $\Sigma_\pi(\lambda)$ and who used them to give new upper bounds on the characters of the symmetric groups. Another application of this idea was given by Dołęga, Féray, and Śniady (2010) who found explicitly the expansion of the normalized character $\Sigma_\pi(\lambda)$ in terms of the free cumulants $R_s(\lambda)$; such expansion is called Kerov polynomial (Kerov, 2000a; Biane, 2003).

The above-mentioned two papers concern only the case when $F = \Sigma_\pi$ is the normalized character, nevertheless it is not difficult to adapt them to other cases for which the expansion of F into N_G is known. For example, Féray and Śniady (2011b) found also such a representation for the zonal characters and in this way found the Kerov polynomial for the zonal polynomials.

It would be very tempting to follow this path and to generalize these results to other interesting polynomial functions on $\mathbb{Y}$. However, in order to do this we need to overcome the following essential difficulty.

Problem 1.1 *For a given interesting polynomial function F on the set of Young diagrams, how to find explicitly the expansion* (1) *of F as a linear combination of the numbers of colorings N_G?*

This problem is too ambitious and too general to be tractable. In this article we will tackle the following, more modest question.

Problem 1.2 *Which linear combinations of the numbers of colorings N_G are polynomial functions on the set of Young diagrams?*

Surprisingly, in some cases the answer to this more modest Problem 1.2 can be helpful in finding the answer to the more important Problem 1.1.

1.5 How characterization of polynomial functions can be useful?

Jack shifted symmetric functions $J_\mu^{(\alpha)}$ with parameter α are indexed by Young diagrams and are characterized (up to a multiplicative constant) by the following conditions:

(i) $J_\mu^{(\alpha)}(\mu) \neq 0$ and for each Young diagram λ such that $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$ we have $J_\mu^{(\alpha)}(\lambda) = 0$;

(ii) $J_\mu^{(\alpha)}$ is an α-anisotropic polynomial function on the set of Young diagrams, i.e. the function $\lambda \mapsto J_\mu^{(\alpha)}\left(\frac{1}{\alpha}\lambda\right)$ is a polynomial function;

(iii) $J_\mu^{(\alpha)}$ has degree equal to $|\mu|$ (regarded as a shifted symmetric function).

The structure on Jack polynomials remains mysterious and there are several open problems concerning them. The most interesting for us are introduced and investigated by Lassalle (2008, 2009).

One possible way to overcome these difficulties is to write Jack shifted symmetric functions in the form

$$J_\mu^{(\alpha)}(\lambda) = \sum_{\pi \vdash |\mu|} n_\pi^{(\alpha)} \, \Sigma_\pi^{(\alpha)}(\mu) \, \Sigma_\pi^{(\alpha)}(\lambda),$$

where $n_\pi^{(\alpha)}$ is some combinatorial factor which is out of scope of the current paper and where $\Sigma_\pi^{(\alpha)}$, called Jack character, is an α-anisotropic polynomial function on the set of Young diagrams. The problem is therefore reduced to finding the expansion (1) for Jack characters (which is a special case of Problem 1.1). It is tempting to solve this problem by guessing the right form of the expansion (1) and then by proving that so defined $J_\mu^{(\alpha)}$ have the required properties.

We expect that verifying a weaker version of condition (i), namely:

(i') For each Young diagram λ such that $|\lambda| < |\mu|$ we have $J_\mu^{(\alpha)}(\lambda) = 0$

should not be too difficult; sometimes it does not matter if in the definition of $N_G(\lambda)$ we count all embeddings of the graph into the Young diagram or we count only injective embeddings in which each edge of the graph is mapped into a different box of λ. If this is the case then condition (i') holds trivially if all graphs G over which we sum have exactly $|\mu|$ edges. Also condition (iii) would follow trivially. The true difficulty is to check that condition (ii) is fulfilled which is exactly the special case of Problem 1.2 (up to the small rescaling related to the fact that we are interested now with α-anisotropic polynomial functions).

1.6 The main result

The main result of this paper is Theorem 5.1 which gives a solution to Problem 1.2 by characterizing the linear combinations of N_G which are polynomial functions on $\mathbb{Y}$ in terms of a combinatorial property of the underlying formal linear combinations of bipartite graphs G.

1.7 Contents of this article

In this article we shall highlight just the main ideas of the proof of Theorem 5.1 because the whole proof is rather long and technical. In particular we will briefly show the main conceptual ingredients: differential calculus on $\mathbb{Y}$ and derivation of bipartite graphs.

Due to lack of space we were not able to show the full history of the presented results and to give to everybody the proper credits. For more history and bibliographical references we refer to the full version of this article Dołęga and Śniady (2010) which will be published elsewhere.

2 Preliminaries

2.1 Russian and French convention

We will use two conventions for drawing Young diagrams: the *French* one in the $0xy$ coordinate system and the *Russian* one in the $0zt$ coordinate system (presented on Figure 1). Notice that the graphs in the

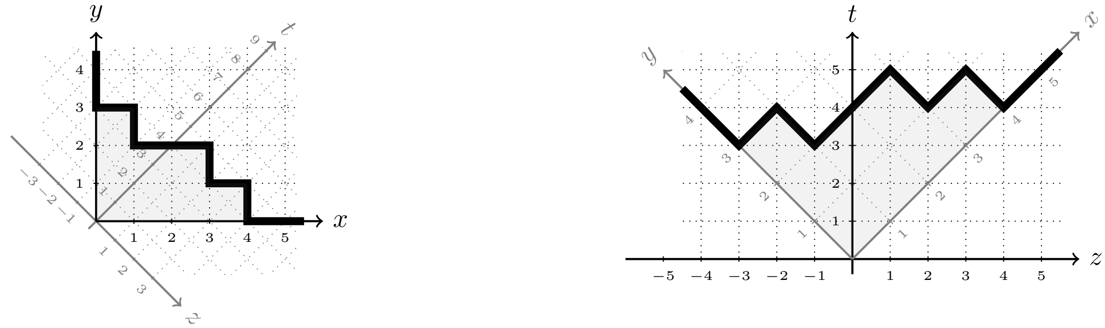

Fig. 1: Young diagram $(4, 3, 1)$ shown in the French and Russian conventions. The solid line represents the profile of the Young diagram. The coordinates system (z, t) corresponding to the Russian convention and the coordinate system (x, y) corresponding to the French convention are shown.

Russian convention are created from the graphs in the French convention by rotating counterclockwise by $\frac{\pi}{4}$ and by scaling by a factor $\sqrt{2}$. Alternatively, this can be viewed as choice of two coordinate systems on the plane: $0xy$, corresponding to the French convention, and $0zt$, corresponding to the Russian convention. For a point on the plane we will define its *content* as its z-coordinate.

In the French coordinates will use the plane $\mathbb{R}^2$ equipped with the standard Lebesgue measure, i.e. the area of a unit square with vertices (x, y) such that $x, y \in \{0, 1\}$ is equal to 1. This measure in the Russian coordinates corresponds to a the Lebesgue measure on $\mathbb{R}^2$ multiplied by the factor 2, i.e. i.e. the area of a unit square with vertices (z, t) such that $z, t \in \{0, 1\}$ is equal to 2.

2.2 Generalized Young diagrams

We can identify a Young diagram drawn in the Russian convention with its profile, see Figure 1. It is therefore natural to define the set of *generalized Young diagrams* $\mathbb{Y}$ *(in the Russian convention)* as the set of functions $\omega : \mathbb{R} \to \mathbb{R}_+$ which fulfill the following two conditions:

- ω is a Lipschitz function with constant 1, i.e. $|\omega(z_1) - \omega(z_2)| \leq |z_1 - z_2|$,
- $\omega(z) = |z|$ if $|z|$ is large enough.

We will define the support of ω in a natural way:

$$\operatorname{supp}(\omega) = \overline{\{z \in \mathbb{R} : \omega(z) \neq |z|\}}.$$

2.3 Functionals of shape

We define the fundamental functionals of shape λ for integers $k \geq 2$

$$S_k(\lambda) = (k-1) \iint_{(x,y)\in\lambda} (x-y)^{k-2}\, dx\, dy = \frac{1}{2}(k-1) \iint_{(z,t)\in\lambda} z^{k-2}\, dz\, dt,$$

where the first integral is written in the French and the second in the Russian coordinates. The family $(S_k)_{k\geq 2}$ generates the algebra $\mathcal{P}$ of polynomial functions on Young diagrams (Dołęga, Féray, and Śniady, 2010).

3 Differential calculus of functions on Young diagrams

3.1 Content-derivatives

Let F be a function on the set of generalized Young diagrams and let λ be a generalized Young diagram. We ask how quickly the value of $F(\lambda)$ would change if we change the shape of λ by adding infinitesimal boxes with content equal to z. In order to answer this informally formulated question we define a derivative of F with respect to content z; this definition is inspired by the Gâteaux derivative. We say that

$$\partial_{C_z} F(\lambda) = f(z)$$

if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that for any $\epsilon > 0$ and $C > 0$ there exists $\delta > 0$ such that for any generalized Young diagrams ω_1, ω_2 supported on $[-C, C]$ such that $\|\omega - \omega_i\|_{L^1} < \delta$ for $i \in \{1, 2\}$

$$\left| F(\omega_1) - F(\omega_2) - \frac{1}{2} \int_{\mathbb{R}} f(z) \big(\omega_1(z) - \omega_2(z)\big)\, dz \right| \leq \epsilon \, \|\omega_1 - \omega_2\|_{L^1}. \tag{2}$$

The strange constant $\frac{1}{2}$ in the above definition appears because of the fact that we are working with a Russian convention which rescales the length and the height of the Young diagram by a factor $\sqrt{2}$, hence

$$\operatorname{Area}(\lambda) = \frac{1}{2} \int_{\mathbb{R}} \big(\omega(z) - |z|\big)\, dz.$$

It can be shown using similar methods as in the case of a standard Gâteaux derivative that a content-derivative has the following properties:

(A) If the derivative $\partial_{C_z} F(\lambda)$ exists, then it is unique.

(B) The Leibniz rule holds, i.e. if F_1, F_2 are sufficiently smooth functions then

$$\partial_{C_z} F_1 F_2 = (\partial_{C_z} F_1)\, F_2 + F_1 \partial_{C_z} F_2.$$

(C) For any integer $k \geq 2$

$$\partial_{C_z} S_k = (k-1) z^{k-2}.$$

The next proposition shows important properties of derivation of a polynomial function on $\mathbb{Y}$.

Proposition 3.1 *Let F be a polynomial function on $\mathbb{Y}$.*

- *For any Young diagram λ the function $\mathbb{R} \ni z \mapsto \partial_{C_z} F(\lambda)$ is a polynomial.*
- *For any $z_0 \in \mathbb{R}$ the function $\mathbb{Y} \ni \lambda \mapsto \partial_{C_{z_0}} F(\lambda)$ is a polynomial function on $\mathbb{Y}$.*
- *For any integer $k \geq 0$ the function $\mathbb{Y} \ni \lambda \mapsto [z^k] \partial_{C_z} F(\lambda)$ is a polynomial function on $\mathbb{Y}$.*

Proof: By linearity it is enough to prove it for $F = \prod_{1 \leq i \leq n} S_{k_i}$. Then, thanks to the properties (B) and (C), we have that

$$\partial_{C_z} F = \sum_{1 \leq i \leq n} \prod_{\substack{1 \leq j \leq n \\ j \neq i}} S_{k_j} (k_i - 1) z^{k_i - 2},$$

which is a polynomial in z for fixed λ, and which is a polynomial function on $\mathbb{Y}$ for fixed $z = z_0$. Moreover $[z^k]\partial_{C_z} F(\lambda)$ is a linear combination of products of S_{k_i}, hence it is a polynomial function on $\mathbb{Y}$, which finishes the proof. □

The main result of this paper is that (in some sense) the opposite implication is true as well and thus it characterizes the polynomial functions on $\mathbb{Y}$.

In order to show it we would like to look at the content-derivative of $N_G(\lambda)$, hence it is necessary to extend the domain of the function N_G to the set of generalized Young diagrams. This extension is very natural. Indeed, we consider a coloring $h : V_1 \sqcup V_2 \to \mathbb{R}_+$ and we say that this coloring is *compatible* with generalized Young diagram λ if $(h(v_1), h(v_2)) \in \lambda$ for each edge $(v_1, v_2) \in V_1 \times V_2$ of G. If we fix the order of vertices in $V = V_1 \sqcup V_2$, we can think of a coloring h as an element of $\mathbb{R}_+^{|V|}$. Then we define

$$N_G(\lambda) = \mathrm{vol}\{h \in \mathbb{R}_+^{|V|} : h \text{ compatible with } \lambda\}.$$

Notice that this is really an extension, i.e. this function restricted to the set of ordinary Young diagrams is the same as N_G which was defined in Section 1.3.

Just before we finish this section, let us state one more lemma which will be helpful soon and which explains the connection between the usual derivation of a function on the set of Young diagrams when we change the shape of a Young diagram a bit, and the content-derivative of this function.

Lemma 3.2 *Let $\mathbb{R} \ni t \mapsto \lambda_t$ be a sufficiently smooth trajectory in the set of generalized Young diagrams and let F be a sufficiently smooth function on $\mathbb{Y}$. Then*

$$\frac{d}{dt} F(\lambda_t) = \int_{\mathbb{R}} \frac{1}{2} \frac{d\omega_t(z)}{dt}\, \partial_{C_z} F(\lambda_t)\, dz.$$

Proof: This is a simple consequence of equality (2). □

4 Derivatives on bipartite graphs

Let G be a bipartite graph. We denote

$$\partial_z G = \sum_e (G, e)$$

which is a formal sum (formal linear combination) which runs over all edges e of G. We will think about the pair (G, e) that it is graph G with one edge e decorated with the symbol z. More generally, if $\mathcal{G}$ is a linear combination of bipartite graphs, this definition extends by linearity.

If G is a bipartite graph with one edge decorated by the symbol z, we define

$$\partial_x G = \sum_f G_{f \equiv z}$$

which is a formal sum which runs over all edges $f \neq z$ which share a common black vertex with the edge z. The symbol $G_{f \equiv z}$ denotes the graph G in which the edges f and z are glued together (which means that also the white vertices of f and z are glued together and that from the resulting graph all multiple edges are replaced by single edges). The edge resulting from gluing f and z will be decorated by z. More generally, if $\mathcal{G}$ is a linear combination of bipartite graphs, this definition extends by linearity.

We also define

$$\partial_y G = \sum_f G_{f\equiv z}$$

which is a formal sum which runs over all edges $f \neq z$ which share a common white vertex with the edge z.

Conjecture 4.1 *Let $\mathcal{G}$ be a linear combination of bipartite graphs with a property that*

$$(\partial_x + \partial_y)\, \partial_z \mathcal{G} = 0.$$

Then for any integer $k \geq 1$

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = 0.$$

We are able to prove Conjecture 4.1 under some additional assumptions, however we believe it is true in general.

5 Characterization of functions arising from bipartite graphs which are polynomial

5.1 The main result

Theorem 5.1 *Let $\mathcal{G}$ be a linear combination of bipartite graphs such that*

$$\left(\partial_x^k - (-\partial_y)^k\right) \partial_z \mathcal{G} = 0 \tag{3}$$

for any integer $k > 0$. Then $\lambda \mapsto N_{\mathcal{G}}^{\lambda}$ is a polynomial function on the set of Young diagrams.

The main idea of the proof is to find a connection between content-derivative of a function $N_{\mathcal{G}}$ and a combinatorial derivation of the underlying linear combination of bipartite graphs $\mathcal{G}$; we present it in the following.

5.2 Colorings of bipartite graphs with decorated edges

Let a Young diagram λ and a bipartite graph G be given. If an edge of G is decorated by a real number z, we decorate its white end by the number $\frac{\omega(z)+z}{2}$ (which is the x-coordinate of the point at the profile of λ with contents equal to z) and we decorate its black end by the number $\frac{\omega(z)-z}{2}$ (which is the y-coordinate of the point at the profile of λ with contents equal to z). If some disjoint edges are decorated by n real numbers $z_1, \dots, z_n$, then we decorate white and black vertices in an analogous way.

For a bipartite graph G with some disjoint edges decorated we define $N_G(\lambda)$, the number of colorings of λ, as the volume of the set of functions from undecorated vertices to $\mathbb{R}_+$ such that these functions extended by values of decorated vertices are compatible with λ.

We will use the following lemma:

Lemma 5.2 *Let (G, z) be a bipartite graph with one edge decorated by a real number z. Then*

- $\frac{d}{dz} N_{(G,z)}(\lambda) = \frac{\omega'(z)+1}{2} N_{\partial_x (G,z)}(\lambda) + \frac{\omega'(z)-1}{2} N_{\partial_y (G,z)}(\lambda),$

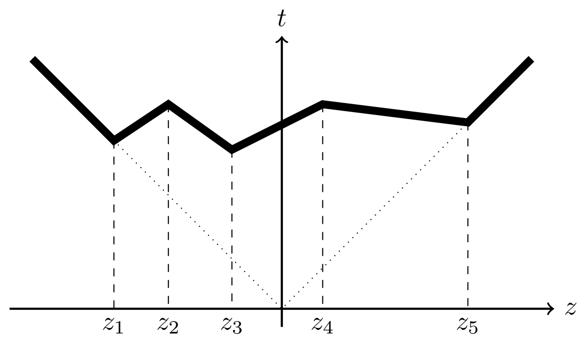

Fig. 2: Piecewise-affine generalized Young diagram.

- $\partial_{C_z} N_G = N_{\partial_z G}$.

The proof of this lemma is not difficult, but it is quite technical and we omit it.

Using Lemma 5.2, Theorem 4.1, and results from Section 3 one can prove the following lemma:

Lemma 5.3 *Let the assumptions of Theorem 5.1 be fulfilled. Then*

- $z \mapsto N_{\partial_z \mathcal{G}}(\lambda)$ *is a polynomial and*
- $\lambda \mapsto [z^i] N_{\partial_z \mathcal{G}}(\lambda)$ *is a polynomial function on* $\mathbb{Y}$ *for any* i.

Proof: The main ideas of the proof are the following. In order to show the first property we are looking at $\frac{d^i}{dz^i} N_{\partial_z \mathcal{G}}$ and using Lemma 5.2 we can show that $\frac{d^i}{dz^i} N_{\partial_z \mathcal{G}} = 0$ for any $i > |V| - 2$. The proof of the second property is going by induction on i and it uses an Lemma 3.2 in a similar way like the proof of Theorem 5.1 below. It is quite technical, so let us stop here. □

5.3 Proof of the main result

Proof of Theorem 5.1: We can assume without loss of generality that every graph which contributes to $\mathcal{G}$ has the same number of vertices, equal to m. Indeed, if this is not the case, we can write $\mathcal{G} = \mathcal{G}_2 + \mathcal{G}_3 + \cdots$ as a finite sum, where every graph contributing to $\mathcal{G}_i$ has i vertices; then clearly (3) is fulfilled for every $\mathcal{G}' := \mathcal{G}_i$.

Assume that λ is a piecewise affine generalized Young diagram such that $|\omega'(z)| < 1$ for any z in the support of ω (see Figure 2). For any $t \in \mathbb{R}_+$ we define a generalized Young diagram $t\lambda$ which is a dilation of λ by t. A profile $\tilde{\omega}$ of $t\lambda$ is given by $\tilde{\omega}(s) = t\omega(s/t)$. By Lemma 3.2 we can write:

$$N_{\mathcal{G}}(\lambda) = \frac{1}{m} \left. \frac{d}{dt} N_{\mathcal{G}}^{t\lambda} \right|_{t=1} = \frac{1}{2m} \int_{\mathbb{R}} \left. \frac{d}{dt} \big(t\omega(z/t)\big)\, \partial_{C_z} N_{\mathcal{G}}(t\lambda) \right|_{t=1} dz = \\ \frac{1}{2m} \int_{\mathbb{R}} \big(\omega(z) - z\omega'(z)\big)\, \partial_{C_z} N_{\mathcal{G}}(\lambda) dz.$$

Then, by Lemma 5.2 we have that

$$N_{\mathcal{G}}(\lambda) = \frac{1}{2m} \int_{\mathbb{R}} \big(\omega(z) - z\omega'(z)\big) \sum_{0 \leq i \leq m-2} (i+1) z^i \mathcal{F}_i(\lambda) dz = \\ \frac{1}{2m} \sum_{0 \leq i \leq m-2} \mathcal{F}_i(\lambda) \int_{\mathbb{R}} \big(\omega(z) - z\omega'(z)\big)(i+1) z^i dz = \frac{1}{2m} \sum_{0 \leq i \leq m-2} \mathcal{F}_i(\lambda) S_{i+2}(\lambda),$$

where $\mathcal{F}_i = \frac{1}{i+1}[z^i] N_{\partial_z \mathcal{G}}(\lambda)$ is a polynomial function on $\mathbb{Y}$ for each i by Lemma 5.3. It finishes the proof. □

6 Applications

6.1 Bipartite maps

A labeled (bipartite) graph drawn on a surface will be called a *(bipartite) map*. If this surface is orientable and its orientation is fixed, then the underlying map is called *oriented*; otherwise the map is *unoriented*. We will always assume that the surface is minimal in the sense that after removing the graph from the surface, the latter becomes a collection of disjoint open discs. If we draw an edge of such a graph with a fat pen and then take its boundary, this edge splits into two *edge-sides*. In the above definition of the map, by *'labeled'* we mean that each edge-side is labeled with a number from the set $[2n]$ and each number from this set is used exactly once.

Each bipartite labeled map can be constructed by the following procedure. For a partition $\lambda \vdash n$ we consider a family of $\ell(\lambda)$ bipartite polygons with the number of edges given by partition $2\lambda = (2\lambda_1, \dots, 2\lambda_{\ell(\lambda)})$. Then we label the edges of the polygons by elements of $[2n]$ in such a way that each number is used exactly once. A *pair-partition* of $[2n]$ is defined as a family $P = \{V_1, \dots, V_n\}$ of disjoint sets called *blocks* of P, each containing exactly two elements and such that $\bigcup P = [2n]$. For a given pair-partition P we glue together each pair of edges of the polygons which is matched by P in such a way that a white vertex is glued with the other white one, and a black vertex with the other black one.

6.2 Normalized and zonal characters

Theorem 6.1 (Féray and Śniady (2011b)) *Let $\Sigma_\mu^{(\alpha)}$ denote the Jack character with parameter α. Then:*

$$\Sigma_\mu^{(1)} = \sum_{\mathcal{M}} (-1)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}, \tag{4}$$

where the summation is over all labeled bipartite oriented maps with the face type μ and

$$\Sigma_\mu^{(2)} = \sum_{\mathcal{M}} (-2)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}, \tag{5}$$

where the summation is over all labeled bipartite maps (not necessarily oriented) with the face type μ.

Proof: Due to the characterization of a Jack symmetric function which was given in Section 1.5 it suffices to show that the right hand sides of (4) and (5) satisfy conditions (i), (ii), (iii). Due to lack of space, instead of (i) we will show a weaker condition (i').

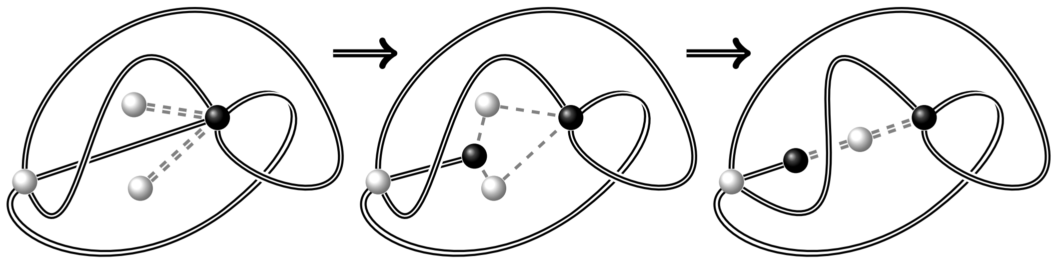

Fig. 3: Example of a construction of a map and its subtree $(\tilde{\mathcal{M}}, \tilde{\mathcal{T}})$ (on the right) from a given map with its subtree $(\mathcal{M}, \mathcal{T})$ (on the left), such that $\tilde{\mathcal{M}}/\tilde{\mathcal{T}} = \mathcal{M}/\mathcal{T}$. Face type of maps is given by $\mu = (12)$. As pair-partitions we have $\mathcal{M} = \{\{1,7\},\{2,3\},\{4,6\},\{5,11\},\{8,9\},\{10,12\}\}$, $\mathcal{T} = \{\{2,3\},\{8,9\}\}$, $\tilde{\mathcal{M}} = \{\{1,7\},\{2,8\},\{3,9\},\{4,6\},\{5,11\},\{10,12\}\}$, $\tilde{\mathcal{T}} = \{\{2,8\},\{3,9\}\}$.

Definition of N gives us property (iii) immediately. Property (i') can be shown, as it was mentioned in Section 1.5, by proving that if we change functions N on the right hand sides of (4) and (5) by some other functions $\tilde{N}$ which count 'injective embeddings', the equalities will still hold. The proof of that will be the same as in Féray and Śniady (2011b), hence we omit it.

The novelty in the current proof is showing the property (ii). First, we notice that

$$\sum_{\mathcal{M}} (-2)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}(\lambda) = \sum_{\mathcal{M}} (-1)^{|V_b(\mathcal{M})|} N_{\mathcal{M}}(2\lambda).$$

In the following we shall prove that condition (3) is fulfilled. Let us look at $\partial_x^k(\mathcal{M}, z)$ for some bipartite map $\mathcal{M}$ with one decorated edge by z. The procedure of derivation with respect to x can be viewed as taking all subtrees of $\mathcal{M}$ which consist of $k+1$ edges connected by a black vertex and where one edge is decorated by z and collapsing them to one decorated edge. Let us choose such a subtree $\mathcal{T}$. We can do the following procedure with $\mathcal{T}$: we unglue every edges corresponding to $\mathcal{T}$ locally in a way that we create k copies of a black vertex and local orientation of each vertex is preserved; in this way we obtained locally a bipartite $2k+2$-gon; then we glue it again but in such a way that we glue white vertices together in this $2k+2$-gon (see Figure 3). We obtained in this way a new bipartite map $\tilde{\mathcal{M}}$, such that $|V_b(\tilde{\mathcal{M}})| = |V_b(\mathcal{M})| + k$ and which contains a subtree $\tilde{\mathcal{T}}$ with $k+1$ edges and one white vertex. Moreover, collapsing of $\mathcal{T}$ in $\mathcal{M}$ to one decorated edge gives us the same bipartite graph as collapsing of $\tilde{T}$ in $\tilde{\mathcal{M}}$ to one decorated edge. We should check that this map has a face type μ, but this is clear from our construction. Of course we can do the same procedure if we start from $\partial_y^k(\mathcal{M}, z)$, because of the symmetry. These two procedures are inverses of each other hence (3) holds true. Applying the Main Theorem 5.1 to our case we obtain the property (ii), which finishes the proof. $\square$

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Critical Groups of Simplicial Complexes

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Abstract. We generalize the theory of critical groups from graphs to simplicial complexes. Specifically, given a simplicial complex, we define a family of abelian groups in terms of combinatorial Laplacian operators, generalizing the construction of the critical group of a graph. We show how to realize these critical groups explicitly as cokernels of reduced Laplacians, and prove that they are finite, with orders given by weighted enumerators of simplicial spanning trees. We describe how the critical groups of a complex represent flow along its faces, and sketch another potential interpretation as analogues of Chow groups.

Résumé. Nous généralisons la théorie des groupes critiques des graphes aux complexes simpliciaux. Plus précisément, pour un complexe simplicial, nous définissons une famille de groupes abéliens en termes d'opérateurs de Laplace combinatoires, qui généralise la construction du groupe critique d'un graphe. Nous montrons comment réaliser ces groupes critiques explicitement comme conoyaux des opérateurs de Laplace réduits combinatoires, et montrons qu'ils sont finis. Leurs ordres sont obtenus en comptant (avec des poids) des arbres simpliciaux couvrants. Nous décrivons comment les groupes critiques d'un complexe représentent le flux le long de ses faces, et esquissons une autre interprétation potentielle comme analogues des groupes de Chow.

Keywords: graph, simplicial complex, critical group, combinatorial Laplacian, chip-firing game, sandpile model, spanning trees

1 Introduction

Let G be a finite, simple, undirected, connected graph. The *critical group* of G is a finite abelian group $K(G)$ whose cardinality is the number of spanning trees of G. The critical group is an interesting graph invariant in its own right, and it also arises naturally in the theory of a discrete dynamical system with many essentially equivalent formulations — the *chip-firing game*, *dollar game*, *abelian sandpile model*, etc.— that has been discovered independently in contexts including statistical physics, arithmetic geometry, and combinatorics. There is an extensive literature on these models and their behavior: see, e.g., [3, 4, 7, 14, 20]. In all guises, the model describes a certain type of discrete flow along the edges of G. The elements of the critical group correspond to states in the flow model that are stable, but for which a small perturbation causes an instability.

The purpose of this paper is to extend the theory of the critical group from graphs to simplicial complexes. For a finite simplicial complex Δ of dimension d, we define its higher critical groups as

$$K_i(\Delta) := \ker \partial_i \,/\, \mathrm{im}(\partial_{i+1}\partial_{i+1}^*)$$

for $0 \leq i \leq d-1$; here ∂_j means the simplicial boundary map mapping j-chains to $(j-1)$-chains. The map $\partial_{i+1}\partial^*_{i+1}$ is called an *(updown) combinatorial Laplacian operator*. For $i = 0$, our definition coincides with the standard definition of the critical group of the 1-skeleton of Δ. Our main result (Theorem 3.4) states that, under certain mild assumptions on the complex Δ, the group $K_i(\Delta)$ is in fact isomorphic to the cokernel of a reduced version of the Laplacian. It follows from a simplicial analogue of the matrix-tree theorem [9, 10] that the orders $|K_i(\Delta)|$ of the higher critical groups are given by a torsion-weighted enumeration of higher-dimensional spanning trees (Corollary 4.2) and in terms of the eigenvalues of the Laplacian operators (Corollary 4.4). In the case of a simplicial sphere, we prove (Theorem 4.6) that the top-dimensional critical group is cyclic, with order equal to the number of facets, generalizing the corresponding statement [21, 22, 24] for cycle graphs. In the case that Δ is a skeleton of an n-vertex simplex, the critical groups are direct sums of copies of $\mathbb{Z}/n\mathbb{Z}$; as we discuss in Remark 4.7, this follows from an observation of Maxwell [23] together with our main result. We also give a model of discrete flow (Section 5) on the codimension-one faces along facets of the complex whose behavior is captured by the group structure. Finally, we outline (Section 6) an alternative interpretation of the higher critical groups as discrete analogues of the Chow groups of an algebraic variety.

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In this extended abstract, we omit the proofs; they can be found in the full version of the article [11].

2 Critical Groups of Graphs

2.1 The chip-firing game

We summarize the chip-firing game on a graph, omitting the proofs. For more details, see, e.g., Biggs [3].

Let $G = (V, E)$ be a finite, simple[(i)], connected, undirected graph, with $V = [n] \cup q = \{1, 2, \ldots, n, q\}$ and $E = \{e_1, \ldots, e_m\}$. The special vertex q is called the *bank* (or "root" or "government"). Let d_i be the degree of vertex i, i.e. the number of adjacent vertices. The chip-firing game is a discrete dynamical system whose state is described by a *configuration* vector $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{N}^n$. Each c_i is a nonnegative integer that we think of as the number of "chips" belonging to vertex i. (Note that the number c_q of chips belonging to the bank q is not part of the data of a configuration.)

Each non-root vertex is generous (it likes to donate chips to its neighbors), egalitarian (it likes all its neighbors equally), and prudent (it does not want to go into debt). Specifically, a vertex v_i is called *ready* in a configuration $\mathbf{c}$ if $c_i \geq d_i$. If a vertex is ready, it can *fire* by giving one chip to each of its neighbors. Unlike the other vertices, the bank is a miser. As long as other vertices are firing, the bank does not fire, but just collects chips.

As more and more chips accumulate at the bank, the game eventually reaches a configuration in which no non-bank vertex can fire. Such a configuration is called *stable*. At this point, the bank finally fires, giving one chip to each of its neighbors. Unlike the other vertices, the bank is allowed to go into debt: that is, we do not require that $c_q \geq d_q$ for the bank to be able to fire.

Denote by $\mathbf{c}(x_1, \ldots, x_r)$ the configuration obtained from $\mathbf{c}$ by firing the vertices $x_1, \ldots, x_r$ in order. This sequence (which may contain repetitions) is called a *firing sequence* for $\mathbf{c}$ if every firing is permissible: that is, for each i, either $x_i \neq q$ is ready to fire in the configuration $\mathbf{c}(x_1, \ldots, x_{i-1})$, or else $x_i = q$

(i) The chip-firing game and our ensuing results can easily be extended to allow parallel edges; we assume that G is simple for the sake of ease of exposition.

and $\mathbf{c}(x_1, \dots, x_{i-1})$ is stable. A configuration $\mathbf{c}$ is called *recurrent* if there is a nontrivial firing sequence X such that $\mathbf{c}(X) = \mathbf{c}$.

A configuration is called *critical* if it is both stable and recurrent. For every starting configuration $\mathbf{c}$, there is a uniquely determined critical configuration $[\mathbf{c}]$ that can be reached from $\mathbf{c}$ by some firing sequence [3, Thm. 3.8]. The *critical group* $K(G)$ is defined as the set of these critical configurations, with group law given by $[\mathbf{c}] + [\mathbf{c}'] = [\mathbf{c} + \mathbf{c}']$, where the right-hand addition is componentwise addition of vectors.

The *abelian sandpile model* was first introduced in [7] as an illustration of "self-organized criticality"; an excellent recent exposition is [20]. Here, grains of sand (analogous to chips) are piled at each vertex, and an additional grain of sand is added to a (typically randomly chosen) pile. If the pile reaches some predetermined size (for instance, the degree of that vertex), then it *topples* by giving one grain of sand to each of its neighbors, which can then topple in turn, and so on. This sequence of topplings is called an *avalanche* and the associated operator on states of the system is called an *avalanche operator*. (One can show that the avalanche operator does not depend on the order in which vertices topple; this is the reason for the use of the term "abelian".) The sandpile model itself is the random walk on the stable configurations, and the critical group is the group generated by the avalanche operators.

The critical group can also be viewed as a discrete analogue of the Picard group of an algebraic curve. This point of view goes back at least as far as the work of Lorenzini [21, 22] and was developed, using the language of divisors, by Bacher, de la Harpe, and Nagnibeda [1] (who noted that their "setting has a straightforward generalization to higher dimensional objects"). It appears in diverse combinatorial contexts including elliptic curves over finite fields (Musiker [25]), linear systems on tropical curves (Haase, Musker and Yu [15]), and Riemann-Roch theory for graphs (Baker and Norine [2]).

2.2 The algebraic viewpoint

The critical group can be defined algebraically in terms of the Laplacian matrix.

Definition 2.1 *Let G be a finite, simple, connected, undirected graph with vertices $\{1, \dots, n, q\}$. The Laplacian matrix of G is the symmetric matrix L (or, equivalently, linear self-adjoint operator) whose rows and columns are indexed by the vertices of G, with entries*

$$\ell_{ij} = \begin{cases} d_i & \textit{if } i = j, \\ -1 & \textit{if } ij \in E, \\ 0 & \textit{otherwise.} \end{cases}$$

Firing vertex i in the chip-firing game is equivalent to subtracting the i^{th} column of the Laplacian (ignoring the entry indexed by q) from the configuration vector $\mathbf{c}$. Equivalently, if $\mathbf{c}' = \mathbf{c}(x_1, \dots, x_r)$, then the configurations $\mathbf{c}$ and $\mathbf{c}'$ represent the same element of the cokernel of the Laplacian (that is, the quotient of $\mathbb{Z}^{n+1}$ by the column space of L).

It is immediate from the definition of L that $L(\mathbf{1}) = \mathbf{0}$, where $\mathbf{1}$ and $\mathbf{0}$ denote the all-ones and all-zeros vectors in $\mathbb{N}^{n+1}$. Moreover, it is not difficult to show that $\operatorname{rank} L = |V| - 1 = n$. In terms of homological algebra, we have a chain complex

$$\mathbb{Z}^{n+1} \xrightarrow{L} \mathbb{Z}^{n+1} \xrightarrow{S} \mathbb{Z} \to 0 \tag{1}$$

where $S(\mathbf{c}) = \mathbf{c} \cdot \mathbf{1} = c_q + c_1 + \cdots + c_n$. The equation $L(\mathbf{1}) = \mathbf{0}$ says that $\ker(S) \supseteq \operatorname{im}(L)$. Moreover, $\operatorname{rank} L = n = \operatorname{rank} \ker S$, so the abelian group $\ker(S)/\operatorname{im}(L)$ is finite.

Definition 2.2 *The* critical group *of a graph G is $K(G) = \ker(S)/\operatorname{im}(L)$.*

This definition of the critical group is equivalent to that in terms of the chip-firing game [3, Thm. 4.2]. The order of the critical group is the determinant of the *reduced Laplacian* formed by removing the row and column indexed by q [3, Thm. 6.2]. By the matrix-tree theorem, this is the number of spanning trees. As we will see, the algebraic description provides a natural framework for generalizing the critical group.

3 The Critical Groups of a Simplicial Complex

We assume familiarity with the basic algebraic topology of simplicial complexes; see, e.g., Hatcher [17]. Let Δ be a d-dimensional simplicial complex. For $-1 \leq i \leq d$, let $C_i(\Delta;\mathbb{Z})$ be the i^{th} simplicial chain group of Δ. We denote the simplicial boundary and coboundary maps respectively by

$$\begin{aligned} \partial_{\Delta,i} &: C_i(\Delta;\mathbb{Z}) \to C_{i-1}(\Delta;\mathbb{Z}), \\ \partial^*_{\Delta,i} &: C_{i-1}(\Delta;\mathbb{Z}) \to C_i(\Delta;\mathbb{Z}), \end{aligned}$$

where we have identified cochains with chains via the natural inner product. We will abbreviate the subscripts in the notation for boundaries and coboundaries whenever no ambiguity can arise.

Let $-1 \leq i \leq d$. The *i-dimensional combinatorial Laplacian*[(ii)] of Δ is the operator

$$L_{\Delta,i} = \partial_{i+1}\partial^*_{i+1} \colon C_i(\Delta;\mathbb{Z}) \to C_i(\Delta;\mathbb{Z}).$$

Combinatorial Laplacian operators seem to have first appeared in the work of Eckmann [12] on finite dimensional Hodge theory. As the name suggests, they are discrete versions of the Laplacian operators on differential forms on a Riemannian manifold. In fact, Dodziuk and Patodi [8] showed that for suitably nice triangulations of a manifold, the eigenvalues of the discrete Laplacian converge in an appropriate sense to those of the usual continuous Laplacian. For one-dimensional complexes, i.e., graphs, the combinatorial Laplacian is just the usual Laplacian matrix $L = D - A$, where D is the diagonal matrix of vertex degrees and A is the (symmetric) adjacency matrix.

In analogy to the chain complex of (1), we have the chain complex

$$C_i(\Delta;\mathbb{Z}) \xrightarrow{L} C_i(\Delta;\mathbb{Z}) \xrightarrow{\partial_i} C_{i-1}(\Delta;\mathbb{Z}),$$

where $L = L_{\Delta,i}$. (This is a chain complex because $\partial_i L = \partial_i\, \partial_{i+1}\partial^*_{i+1} = 0$.) We are now ready to make our main definition.

Definition 3.1 *The i-dimensional critical group of Δ is*

$$K_i(\Delta) := \ker \partial_i \,/\, \operatorname{im} L = \ker \partial_i \,/\, \operatorname{im}(\partial_{i+1}\partial^*_{i+1}).$$

Note that $K_0(\Delta)$ is precisely the critical group of the 1-skeleton of Δ.

(ii) In other settings, our Laplacian might be referred to as the "up-down" Laplacian, L^{ud}. The i^{th} *down-up Laplacian* is $L_i^{\mathrm{du}} = \partial_i^*\partial_i$, and the i^{th} *total Laplacian* is $L_i^{\mathrm{tot}} = L_i + L_i^{\mathrm{du}}$. We adopt the notation we do since, except for one application (Remark 4.7 below), we only need the up-down Laplacian.

3.1 Simplicial spanning trees

Our results about critical groups rely on the theory of simplicial and cellular spanning trees developed in [9], based on earlier work of Bolker [5] and Kalai [19]. Here we briefly review the definitions and basic properties, including the higher-dimensional analogues of Kirchhoff's matrix-tree theorem. For simplicity, we present the theory for simplicial complexes, the case of primary interest in combinatorics. Nevertheless, the definitions of spanning trees, their enumeration using a generalized matrix-tree theorem, and the definition and main result about critical groups are all valid in the more general setting of regular CW-complexes [10].

In order to define simplicial spanning trees, we first fix some notation concerning simplicial complexes and algebraic topology. The symbol Δ_i will denote the set of cells of dimension i. The *i-dimensional skeleton* $\Delta_{(i)}$ of a simplicial complex Δ is the subcomplex consisting of all cells of dimension $\leq i$. A complex is *pure* if all maximal cells have the same dimension. The i^{th} reduced homology group of Δ with coefficients in a ring R is denoted $\tilde{H}_i(\Delta; R)$. The *Betti numbers* of Δ are $\beta_i(\Delta) = \dim_{\mathbb{Q}} \tilde{H}_i(\Delta; \mathbb{Q})$. The *$f$-vector* is $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots)$, where $f_i(\Delta)$ is the number of faces of dimension i.

Definition 3.2 *Let Δ be a pure d-dimensional simplicial complex, and let $\Upsilon \subseteq \Delta$ be a subcomplex such that $\Upsilon_{(d-1)} = \Delta_{(d-1)}$. We say that Υ is a* (simplicial) spanning tree *of Δ if the following three conditions hold:*

1. $\tilde{H}_d(\Upsilon; \mathbb{Z}) = 0$;
2. $\tilde{H}_{d-1}(\Upsilon; \mathbb{Q}) = 0$ *(equivalently,* $|\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})| < \infty$*);*
3. $f_d(\Upsilon) = f_d(\Delta) - \beta_d(\Delta) + \beta_{d-1}(\Delta)$.

*More generally, an i-*dimensional spanning tree *of Δ is a spanning tree of the i-dimensional skeleton of Δ.*

In the case $d = 1$ (that is, Δ is a graph), we recover the usual definition of a spanning tree: the three conditions above say respectively that Υ is acyclic, connected, and has one more vertex than edge. Meanwhile, the 0-dimensional spanning trees of Δ are its vertices (more precisely, the subcomplexes of Δ with a single vertex), which are precisely the connected, acyclic subcomplexes of $\Delta_{(0)}$.

Just as in the graphical case, any two of the conditions of Definition 3.2 imply the third [9, Prop 3.5]. In order for Δ to have a d-dimensional spanning tree, it is necessary and sufficient that $\tilde{H}_i(\Delta; \mathbb{Q}) = 0$ for all $i < d$; such a complex is called *acyclic in positive codimension*, or APC. Note that a graph is APC if and only if it is connected.

Example 3.3 *Consider the* equatorial bipyramid*: the two-dimensional simplicial complex B with vertices $[5]$ and facets $123, 124, 125, 134, 135, 234, 235$. A geometric realization of B is shown in Figure 1. A 2-SST of B can be constructed by removing two facets F, F', provided that $F \cap F'$ contains neither of the vertices $4, 5$. A simple count shows that there are 15 such pairs F, F', so B has 15 two-dimensional spanning trees.*

A phenomenon arising only in dimension $d > 1$ is that spanning trees may have torsion: that is, $\tilde{H}_{d-1}(\Upsilon; \mathbb{Z})$ can be finite but nontrivial. For example, the 2-dimensional skeleton of a 6-vertex simplex has (several) spanning trees Υ that are homeomorphic to the real projective plane, and in particular have $\tilde{H}_1(\Upsilon; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. This cannot happen in dimension 1 (i.e., for graphs), in which every spanning tree is

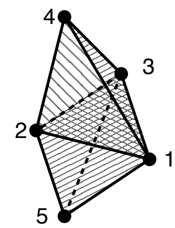

Fig. 1: The equatorial bipyramid B.

a contractible topological space. This torsion directly affects tree enumeration in higher dimension; see Section 4.

3.2 The main theorem

Our main result gives an explicit form for the critical group $K_i(\Delta)$ in terms of a reduced Laplacian matrix. This reduced form is both more convenient for computing examples, and gives a direct connection with the simplicial and cellular generalizations of the matrix-tree theorem [9, 10].

Let Δ be a pure, d-dimensional, APC simplicial complex, and fix $i < d$. Let Υ be an i-dimensional spanning tree of $\Delta_{(i)}$, and let $\Theta = \Delta_i \backslash \Upsilon$ (the set of i-dimensional faces of Δ *not* in Υ). Let $\tilde{L}$ denote the reduced Laplacian obtained from L by removing the rows and columns corresponding to Υ (equivalently, by restricting L to the rows and columns corresponding to Θ).

Theorem 3.4 *Suppose that $\tilde{H}_{i-1}(\Upsilon;\mathbb{Z}) = 0$. Then*

$$K_i(\Delta) \cong \mathbb{Z}^{\Theta} / \operatorname{im} \tilde{L}.$$

We omit the proof, which uses some basic homological algebra; for details, see [11].

Example 3.5 *We return to the bipyramid B from Example 3.3 to illustrate Theorem 3.4. We must first pick a 1-dimensional spanning tree Υ; we take Υ to be the spanning tree with edges $12, 13, 14, 15$. (In general, we must also make sure Υ is torsion-free, but this is always true for 1-dimensional trees.) Let $L = L_{B,1} \colon C_1(B;\mathbb{Z}) \to C_1(B;\mathbb{Z})$ be the full Laplacian; note that L is a 9×9 matrix whose rows and columns are indexed by the edges of B. The reduced Laplacian $\tilde{L}$ is formed by removing the rows and columns indexed by the edges of Υ:*

$$\tilde{L} \;=\; \begin{array}{c} \\ 23 \\ 24 \\ 25 \\ 34 \\ 35 \end{array} \!\!\begin{array}{c} \begin{array}{ccccc} 23 & 24 & 25 & 34 & 35 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 1 & 1 \\ -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & -1 \\ 1 & -1 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 2 \end{pmatrix} \end{array}.$$

The critical group $K_1(B)$ is the cokernel of this matrix, i.e., $K_1(B) \cong \mathbb{Z}^5 / \operatorname{im} \tilde{L}$. Since $\tilde{L}$ has full rank, it follows that $K_1(B)$ is finite; its order is $\det(\tilde{L}) = 15$.

4 The Order of the Critical Group

The matrix-tree theorem implies that the order of the critical group of a graph equals the number of spanning trees. In this section, we explain how this equality carries over to the higher-dimensional setting.

As before, let Δ be a pure d-dimensional simplicial complex. Let $\mathcal{T}_i(\Delta)$ denote the set of all i-dimensional spanning trees of Δ (that is, spanning trees of the i-dimensional skeleton $\Delta_{(i)}$). Define

$$\tau_i = \sum_{\Upsilon \in \mathcal{T}_i(\Delta)} |\tilde{H}_{i-1}(\Upsilon;\mathbb{Z})|^2,$$

$$\pi_i = \text{ product of all nonzero eigenvalues of } L_{\Delta,i-1}.$$

The following formulas relate the tree enumerators τ_i to the linear-algebraic invariants π_i.

Theorem 4.1 (The simplicial matrix-tree theorem) *[9, Thm. 1.3] For all $i \le d$, we have*

$$\pi_i = \frac{\tau_i \tau_{i-1}}{|\tilde{H}_{i-2}(\Delta;\mathbb{Z})|^2}.$$

Moreover, if Υ is any spanning tree of $\Delta_{(i-1)}$, then

$$\tau_i = \frac{|\tilde{H}_{i-2}(\Delta;\mathbb{Z})|^2}{|\tilde{H}_{i-2}(\Upsilon;\mathbb{Z})|^2} \det \tilde{L},$$

where $\tilde{L}$ is the reduced Laplacian formed by removing the rows and columns corresponding to Υ.

Recall that when $d = 1$, the number $\tau_1(\Delta)$ is simply the number of spanning trees of the graph Δ, and $\tau_0(\Delta)$ is the number of vertices (i.e., 0-dimensional spanning trees). Therefore, the formulas above specialize to the classical matrix-tree theorem.

Corollary 4.2 *Let $i < d$. Suppose that $\tilde{H}_{i-1}(\Delta;\mathbb{Z}) = 0$ and that Δ has an i-dimensional spanning tree Υ such that $\tilde{H}_{i-1}(\Upsilon;\mathbb{Z}) = 0$. Then the order of the i-dimensional critical group is the torsion-weighted number of $(i+1)$-dimensional spanning trees, i.e.,*

$$|K_i(\Delta)| = \tau_{i+1}.$$

Example 4.3 *Returning again to the bipyramid B, recall that 15 is both the number of its spanning trees (Example 3.3) and the order of its 1-dimensional critical group (Example 3.5), in each case because* $\det \tilde{L} = 15$.

Another formula for the orders of the critical groups of Δ is as follows. For $0 \le j \le d$, denote by π_j the product of the nonzero eigenvalues of the Laplacian $L^{ud}_{j-1} = \partial_j \partial_j^*$. Then Corollary 4.2, together with [10, Corollary 2.10], implies the following formula for $|K_i(\Delta)|$ as an alternating product:

Corollary 4.4 *Under the conditions of Corollary 4.2, for every $i \le d$, we have*

$$|K_i(\Delta)| = \prod_{j=0}^{i} \pi_j^{(-1)^{i-j}}.$$

The condition that Δ and Υ be torsion-free is not too restrictive, in the sense that many simplicial complexes of interest in combinatorics (for instance, all shellable complexes) are torsion-free and have torsion-free spanning trees.

Remark 4.5 *When every spanning tree of Δ is torsion-free, the order of the critical group is exactly the number of spanning trees. This is a strong condition on Δ, but it does hold for some complexes — notably for simplicial spheres, whose spanning trees are exactly the (contractible) subcomplexes obtained by deleting a single facet. Thus a given explicit bijection between spanning trees and elements of the critical group amounts to an abelian group structure on the set of facets of a simplicial sphere.*

Determining the structure of the critical group is not easy, even for very special classes of graphs; see, e.g., [6, 18]. One of the first such results is due to Lorenzini [21, 22] and Merris [24, Example 1(1.4)], who independently noted that the critical group of the cycle graph on n vertices is $\mathbb{Z}/n\mathbb{Z}$, the cyclic group on n elements. Simplicial spheres are the natural generalizations of cycle graphs from a tree-enumeration point of view. In fact, the theorem of Lorenzini and Merris carries over to simplicial spheres in the following way.

Theorem 4.6 *Let Σ be a d-dimensional simplicial sphere with n facets. Then $K_{d-1}(\Sigma) \cong \mathbb{Z}/n\mathbb{Z}$.*

For the proof, see [11].

The condition that Σ be a simplicial sphere can be relaxed: in fact, the proof goes through for any d-dimensional pseudomanifold Σ such that $\tilde{H}_{d-1}(\Sigma;\mathbb{Z}) = 0$. On the other hand, if Σ is APC in addition to being a pseudomanifold (for example, certain lens spaces—see [17, p. 144]), then it has the rational homology type of either a sphere or a ball (because $\tilde{H}_d(\Sigma;\mathbb{Q})$ is either $\mathbb{Q}$ or 0; see [26, p. 24]).

Remark 4.7 *Let Δ be the simplex on vertex set $[n]$, and let $k \leq n$. Kalai [19] proved that $\tau_k(\Delta) = n^{\binom{n-2}{k}}$ for every n and k, generalizing Cayley's formula n^{n-2} for the number of labeled trees on n vertices. Maxwell [23] studied the skew-symmetric matrix*

$$A = \left[\begin{array}{c} \tilde{\partial}_{\Delta,k} \\ \hline -\tilde{\partial}^*_{\Delta,k+1} \end{array}\right]$$

where $\tilde{\partial}_{\Delta,k}$ denotes the reduced boundary map obtained from the usual simplicial boundary $\partial_{\Delta,k}$ by deleting the rows corresponding to $(k-1)$-faces containing vertex 1, and $\tilde{\partial}^*_{\Delta,k+1}$ is obtained from $\partial^*_{\Delta,k+1}$ by deleting the rows corresponding to k-faces not *containing vertex 1. (Note that Maxwell and Kalai use the symbol $I^k_r(X)$ for what we call $\partial_{\Delta,k}$.)*

In particular, Maxwell [23, Prop. 5.4] proved that

$$\operatorname{coker} A \cong (\mathbb{Z}/n\mathbb{Z})^{\binom{n-2}{k}}.$$

The matrix A is not itself a Laplacian, but is closely related to the Laplacians of Δ. Indeed, Maxwell's result, together with ours, implies that all critical groups of Δ are direct sums of cyclic groups of order n, for the following reasons. We have

$$AA^T = -A^2 = \left[\begin{array}{c} \tilde{\partial}_{\Delta,k} \\ \hline -\tilde{\partial}^*_{\Delta,k+1} \end{array}\right] \left[\begin{array}{c|c} \tilde{\partial}^*_{\Delta,k} & -\tilde{\partial}_{\Delta,k+1} \end{array}\right] = \left[\begin{array}{c|c} \tilde{L}^{ud}_{k-1} & 0 \\ \hline 0 & \tilde{L}^{du}_{k+1} \end{array}\right]$$

where "ud" and "du" stand for "up-down" and "down-up" respectively (see footnote (ii)). Therefore

$$\begin{aligned}\operatorname{coker}(AA^T) &\cong \operatorname{coker}(\tilde{L}^{ud}_{k-1}) \oplus \operatorname{coker}(\tilde{L}^{du}_{k+1})\\ &\cong \operatorname{coker}(\tilde{L}^{ud}_{k-1}) \oplus \operatorname{coker}(\tilde{L}^{ud}_{k})\\ &\cong K_{k-1}(\Delta) \oplus K_k(\Delta),\end{aligned}$$

where the second step follows from the general fact that MM^T and M^TM have the same multisets of nonzero eigenvalues for any matrix M, and the third step follows from Theorem 3.4. On the other hand, we have $\operatorname{coker}(AA^T) = \operatorname{coker}(-A^2) = \operatorname{coker}(A^2) \cong (\operatorname{coker} A) \oplus (\operatorname{coker} A)$. *It follows from Maxwell's result that the kth critical group of the n-vertex simplex is a direct sum of $\binom{n-2}{k}$ copies of $\mathbb{Z}/n\mathbb{Z}$, as desired.*

5 The Critical Group as a Model of Discrete Flow

In this section, we describe an interpretation of the critical group in terms of flow, analogous to the chip-firing game. By definition of $K_i(\Delta)$, its elements may be represented as integer vectors $\mathbf{c} = (c_F)_{F\in\Delta_i}$, modulo an equivalence relation given by the Laplacian. These configurations are the analogues of the configurations of chips in the graph case ($i = 0$). When $i = 1$, it is natural to interpret c_F as a flow along the edge F, in the direction given by some predetermined orientation; a negative value on an edge corresponds to flow in the opposite direction. More generally, if F is an i-dimensional face, then we can interpret c_F as a generalized *i-flow*, again with the understanding that a negative value on a face means a i-flow in the opposite orientation. For instance, 2-flow on a triangle represents circulation around the triangle, and a negative 2-flow means to switch between clockwise and counterclockwise.

When $i = 1$, the condition $\mathbf{c} \in \ker \partial_i$ means that flow neither accumulates nor depletes at any vertex; intuitively, matter is conserved. In general, we call an i-flow *conservative* if it lies in $\ker \partial_i$. For instance, when $i = 2$, the ∂_i map converts 2-flow around a single triangle into 1-flow along the three edges of its boundary in the natural way; for a 2-flow on Δ to be conservative, the sum of the resulting 1-flows on each edge must cancel out, leaving no net flow along any edge. In general, the sum of (the boundaries of) all the i-dimensional flows surrounding an $(i-1)$-dimensional face must cancel out along that face.

That the group $K_i(\Delta)$ is a quotient by the image of the Laplacian means that two configurations are equivalent if they differ by an integer linear combination of Laplacians applied to i-dimensional faces. This is analogous to the chip-firing game, where configurations are equivalent when it is possible to get from one to the other by a series of chip-firings, each of which corresponds to adding a column vector of the Laplacian. When $i = 1$, it is easy to see that firing an edge e (adding the image of its Laplacian to a configuration) corresponds to diverting one unit of flow around each triangle containing e (see Example 5.1). More generally, to fire an i-face F means to divert one unit of i-flow from F around each $(i+1)$-face containing F.

By Theorem 3.4, we may compute the critical group as $\mathbb{Z}^{\Theta}$ modulo the image of the reduced Laplacian. In principle, passing to the reduced Laplacian means ignoring the i-flow along each facet of an i-dimensional spanning tree Υ. In the graph case ($i = 0$), this spanning tree is simply the bank vertex. The higher-dimensional generalization of this statement is that the equivalence class of a configuration $\mathbf{c}$ is determined by the subvector $(c_F)_{F\in\Delta\setminus\Upsilon}$.

A remaining open problem is to identify the higher-dimensional "critical configurations", i.e., a set of stable and recurrent configurations that form a set of coset representatives for the critical group. Recall that in the chip-firing game, when vertex i fires, every vertex other than i either gains a chip or stays unchanged. Therefore, we can define stability simply by the condition $\mathbf{c}_i < \deg(i)$ for every non-bank vertex i. On the other hand, when a higher-dimensional face fires, the flow along nearby faces can actually decrease. Therefore, it is not as easy to define stability. For instance, one could try to define stability by the condition that no face can fire without forcing some face (either itself or one of its neighbors) into debt. However, with this definition, there are some examples (such as the 2-skeleton of the tetrahedron) for which some of the cosets of the Laplacian admit more than one critical configuration. Therefore, it is not clear how to choose a canonical set of coset representatives analogous to the critical configurations of the graphic chip-firing game.

Example 5.1 *We return once again to the bipyramid B, and its 1-dimensional spanning tree Υ with edges* $12, 13, 14, 15$. *If we pick 1-flows on $\Theta = \Delta_1 \backslash \Upsilon$ as shown in Figure 2(a), it is easy to compute that we need 1-flows on Υ as shown in Figure 2(b) to make the overall flow 1-conservative. Since Theorem 3.4 implies we can always pick flows on Υ to make the overall flow 1-conservative, we only show flows on Θ in subsequent diagrams.*

If we fire edge 23, we get the configuration shown in Figure 2(c). One unit of flow on edge 23 has been diverted across face 234 to edges 24 and 34, and another unit of flow has been diverted across face 235 to edges 25 and 35. Note that the absolute value of flow on edge 25 has actually decreased, because of its orientation relative to edge 23. If we subsequently fire edge 24, we get the configuration shown in Figure 2(d). One unit of flow on edge 24 has been diverted across face 234 to edges 23 and 34, and another unit of flow has been diverted across face 124 to edges 12 and 14 (and out of the diagram of Θ).

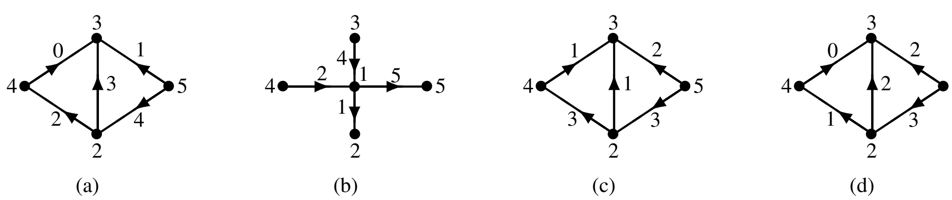

Fig. 2: Conservative 1-flows and firings

6 Critical Groups as Chow Groups

An area for further research is to interpret the higher-dimensional critical groups of a simplicial complex Δ as simplicial analogues of the Chow groups of an algebraic variety. (For the algebraic geometry background, see, e.g., [16, Appendix A] or [13].) We regard Δ as the discrete analogue of a d-dimensional variety, so that divisors correspond to formal sums of codimension-1 faces. Even more generally, algebraic cycles of dimension i correspond to simplicial i-chains. The critical group $K_i(\Delta)$ consists of closed i-chains modulo conservative flows (in the language of Section 5) is thus analogous to the Chow group of algebraic cycles modulo rational equivalence. This point of view has proved fruitful in the case of graphs [1, 2, 15, 21, 22]. In order to develop this analogy fully, the next step is to define a ring structure on $\bigoplus_{i \geq 0} K_i(\Delta)$ with a ring structure analogous to that of the Chow ring. The goal is to define a "critical ring" whose multiplication encodes a simplicial version of intersection theory on Δ.

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The topology of restricted partition posets

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Abstract. For each composition $\vec{c}$ we show that the order complex of the poset of pointed set partitions $\Pi^{\bullet}_{\vec{c}}$ is a wedge of $\beta(\vec{c})$ spheres of the same dimensions, where $\beta(\vec{c})$ is the number of permutations with descent composition $\vec{c}$. Furthermore, the action of the symmetric group on the top homology is isomorphic to the Specht module S^B where B is a border strip associated to the composition $\vec{c}$. We also study the filter of pointed set partitions generated by a knapsack integer partitions and show the analogous results on homotopy type and action on the top homology.

Résumé. Pour chaque composition $\vec{c}$ nous montrons que le complexe simplicial des chaînes de l'ensemble ordonné $\Pi^{\bullet}_{\vec{c}}$ des partitions pointées d'un ensemble est un bouquet de $\beta(\vec{c})$ sphères de même dimension, où $\beta(\vec{c})$ est le nombre de permutations ayant la composition de descentes $\vec{c}$. De plus, l'action du groupe symétrique sur le groupe d'homologie de degré maximum est isomorphe au module de Specht S^B où B est la bande frontalière associée à la composition $\vec{c}$. Nous étudions aussi le filtre des partitions pointées d'un ensemble, engendré par des partitions d'entiers de type "sac à dos" et nous démontrons des résultats analogues pour le type d'homotopie et pour l'action sur le groupe d'homologie de degré maximum.

Keywords: Pointed set partitions, descent set statistics, top homology group, Specht module, knapsack partitions.

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1 Introduction

The study of partitions with restrictions on their block sizes began in the dissertation by Sylvester (16), who studied the poset Π^2_n of partitions where every block has even size. He proved that the Möbius function of this poset is given by $\mu(\Pi^2_n \cup \{\hat{0}\}) = (-1)^{n/2} \cdot E_{n-1}$, where E_n denotes the nth Euler number. Recall that the nth Euler number enumerates the number of alternating permutations, that is, permutations $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ such that $\alpha_1 < \alpha_2 > \alpha_3 < \alpha_4 > \cdots$. Stanley (13) generalized this result to the d-divisible partition lattice Π^d_n, that is, the collection of partitions of $\{1, 2, \ldots, n\}$ where each block size is divisible by d. His results states that the Möbius function $\mu(\Pi^d_n \cup \{\hat{0}\})$ is, up to the sign $(-1)^{n/d}$, the number of permutations in $\mathfrak{S}_{n-1}$ with descent set $\{d, 2d, \ldots, n-d\}$, in other words, the number of permutations with descent composition $(d, \ldots, d, d-1)$.

Calderbank, Hanlon and Robinson (3) extended these results by considering the action of the symmetric group $\mathfrak{S}_{n-1}$ on the top homology group of the order complex of $\Pi^d_n - \{\hat{1}\}$. They showed this action is the Specht module on the border strip corresponding the composition $(d, \ldots, d, d-1)$.

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Wachs (17) showed that the d-divisible partition lattice has an EL-shelling, and hence as a corollary obtained that the homotopy type is a wedge of spheres of dimension $n/d - 2$. She then gave a more explicit proof of the representation of the top homology of $\Pi_n^d - \{\hat{1}\}$.

So far we note that the d-divisible partition lattice is closely connected with the permutations having the descent composition $(d, \ldots, d, d-1)$. We explain this phenomenon in this paper by introducing pointed partitions. They are partitions where one block is considered special, called the pointed block. We obtain such a partition by removing the element n from its block and making this block the pointed block. We now extend the family of posets under consideration. For each composition $\vec{c} = (c_1, \ldots, c_k)$ of n we define a poset $\Pi_{\vec{c}}^\bullet$ such that the Möbius function $\mu(\Pi_{\vec{c}}^\bullet \cup \{\hat{0}\})$ is the sign $(-1)^k$ times the number of permutations with descent composition $\vec{c}$. Furthermore, the order complex of $\Pi_{\vec{c}}^\bullet - \{\hat{1}\}$ is homotopy equivalent to a wedge of spheres of dimension $k-2$. Finally, we show the action of the symmetric group on the top homology group $\widetilde{H}_{k-2}(\Delta(\Pi_{\vec{c}}^\bullet - \{\hat{1}\}))$ is the Specht module corresponding to the composition $\vec{c}$.

Our techniques differ from Wachs' method for studying the d-divisible partition lattice. Using Quillen's fiber lemma, we are able to change the poset question into studying the subcomplexes of the complex of ordered partitions. These subcomplexes are in fact order complexes of rank-selected Boolean algebras and hence shellable. The homotopy equivalence given by Quillen's fiber lemma also carries the action of the symmetric group.

Ehrenborg and Readdy (4) studied the Möbius function of filters of the partition lattice. They defined the notion of a knapsack partition. For a filter of a knapsack partition they showed that its Möbius function was a sum of descent set statistics. We now extend these results topologically by showing that the associated order complex is a wedge of spheres. The proof follows the same outline as the previous study except that we use discrete Morse theory to determine the homotopy type of the associated complexes of ordered set partitions. Furthermore we obtain that the action of the symmetric group on the the top homology is a direct sum of Specht modules.

We end the paper with open questions for future research.

2 Preliminaries

For basic notions concerning partially ordered sets (posets), see the book by Stanley (14). For topological background, see the article by Björner (2) and the book by Kozlov (10). Finally, for representation theory for the symmetric group, see Sagan (12).

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and for $i \leq j$ let $[i, j]$ denote the interval $\{i, i+1, \ldots, j\}$. A *pointed set partition* π of the set $[n]$ is a pair (σ, Z), where Z is a subset of $[n]$ and $\sigma = \{B_1, B_2, \ldots, B_k\}$ is a partition of the set difference $[n] - Z$. We will write the pointed partition π as

$$\pi = \{B_1, B_2, \ldots, B_k, \underline{Z}\},$$

where we underline the set Z and we write $1358|4|\underline{267}$ as shorthand for $\{\{1,3,5,8\},\{4\},\underline{\{2,6,7\}}\}$. Moreover, we call the set Z the *pointed block*. Let $\Pi_n^\bullet$ denote the set of all pointed set partitions on the set $[n]$. The set $\Pi_n^\bullet$ has a natural poset structure. The order relation is given by

$$\begin{aligned} \{B_1, B_2, \ldots, B_k, \underline{Z}\} &< \{B_1 \cup B_2, \ldots, B_k, \underline{Z}\}, \\ \{B_1, B_2, \ldots, B_k, \underline{Z}\} &< \{B_2, \ldots, B_k, \underline{B_1 \cup Z}\}. \end{aligned}$$

The lattice $\Pi_n^\bullet$ is isomorphic to the partition lattice Π_{n+1} by the bijection $\{B_1, \ldots, B_k, \underline{Z}\} \longmapsto \{B_1, \ldots, B_k, Z \cup \{n+1\}\}$. However it is to our advantage to work with pointed set partitions.

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ define its *descent set* to be the set

$$\{i \in [n-1] \ : \ \alpha_i > \alpha_{i+1}\}.$$

Subsets of $[n-1]$ are in a natural bijective correspondence with compositions of n. Hence we define the *descent composition* of the permutation α to be the composition

$$\mathrm{Des}(\alpha) = (s_1, s_2 - s_1, s_3 - s_2, \ldots, s_{k-1} - s_{k-2}, n - s_{k-1}),$$

where the descent set of α is the set $\{s_1 < s_2 < \cdots < s_{k-1}\}$. We define an *integer composition* $\vec{c} = (c_1, \ldots, c_k)$ to be a list of positive integers $c_1, \ldots, c_{k-1}$ and a non-negative integer c_k with $c_1 + \cdots + c_k = n$. Note that the only part allowed to be 0 is the last part. Let $\beta(\vec{c})$ denote the number of permutations in $\alpha \in \mathfrak{S}_n$ with descent composition $\vec{c}$ for $c_k > 0$. If $c_k = 0$, let $\beta(\vec{c}) = 0$ for $k \geq 2$ and $\beta(\vec{c}) = 1$ for $k = 1$.

On the set of compositions on n we define an order relation by letting the cover relation be adding adjacent entries, that is,

$$(c_1, \ldots, c_i, c_{i+1}, \ldots, c_k) \prec (c_1, \ldots, c_i + c_{i+1}, \ldots, c_k).$$

Observe that this poset is isomorphic to the Boolean algebra B_n on n elements and the maximal and minimal elements are the two compositions (n) and $(1, \ldots, 1, 0)$.

An *integer partition* λ of a non-negative integer n is a multiset of positive integers whose sum is n. We will indicate multiplicities with a superscript. Thus $\{5, 3, 3, 2, 1, 1, 1\} = \{5, 3^2, 2, 1^3\}$ is a partition of 16. A *pointed integer partition* (λ, m) of n is pair where m is a non-negative integer and λ is a partition of $n - m$. We write this as $\{\lambda_1, \ldots, \lambda_p, \underline{m}\}$ where $\lambda = \{\lambda_1, \ldots, \lambda_p\}$ is the partition and m is the pointed part. This notion of pointed integer partition is related to pointed set partitions by defining the type of a pointed set partition $\pi = \{B_1, B_2, \ldots, B_k, \underline{Z}\}$ to be the pointed integer partition

$$\mathrm{type}(\pi) = \{|B_1|, |B_2|, \ldots, |B_k|, \underline{|Z|}\}.$$

Similarly, the type of a composition $\vec{c} = (c_1, \ldots, c_k)$ is the pointed integer partition

$$\mathrm{type}(\vec{c}) = \{c_1, \ldots, c_{k-1}, \underline{c_k}\}.$$

We now define the poset central to this paper.

Definition 2.1 *For $\vec{c}$ a composition of n, let $\Pi^\bullet_{\vec{c}}$ be the subposet of the pointed partition lattice $\Pi^\bullet_n$ described by*

$$\Pi^\bullet_{\vec{c}} = \left\{\pi \in \Pi^\bullet_n \ : \ \exists \vec{d} \geq \vec{c}, \ \mathrm{type}(\pi) = \mathrm{type}(\vec{d})\right\}.$$

In other words, the poset $\Pi^\bullet_{\vec{c}}$ consists of all pointed set partitions such that their type is the type of some composition $\vec{d}$ which is greater or equal to the composition $\vec{c}$ in the composition order.

Example 2.2 Consider the composition $\vec{c} = (d, \ldots, d, d-1)$ of the integer $n = d \cdot k - 1$. For a composition to be greater than or equal to $\vec{c}$, it is equivalent to all its parts must be divisible by d except the last part which is congruent to $d - 1$ modulo d. Hence $\Pi^\bullet_{\vec{c}}$ consists of all pointed set partitions where the block sizes are divisible by $\vec{d}$ except the pointed block whose size is congruent to $d - 1$ modulo d. Hence the poset $\Pi^\bullet_{\vec{c}}$ is isomorphic to the d-divisible partition lattice Π^d_{n+1}.

Example 2.3 We note that $\Pi^\bullet_{\vec{c}} \cup \{\hat{0}\}$ is in general not a lattice. Consider the composition $\vec{c} = (1, 1, 2, 1)$ and the four pointed set partitions

$$\pi_1 = 1|2|34|\underline{5}, \ \ \pi_2 = 2|5|34|\underline{1}, \ \ \pi_3 = 34|\underline{125} \ \text{ and } \ \pi_4 = 2|\underline{1345}$$

in $\Pi^\bullet_{(1,1,2,1)}$. In the pointed partition lattice $\Pi^\bullet_5$ we have that $\pi_1, \pi_2 < 2|34|\underline{15} < \pi_3, \pi_4$. Since the pointed set partition $2|34|\underline{15}$ does not belong to $\Pi^\bullet_{(1,1,2,1)}$, we conclude that $\Pi^\bullet_{(1,1,2,1)} \cup \{\hat{0}\}$ is not a lattice.

For a poset P define its *order complex* to be the simplicial complex $\Delta(P)$ where the vertices of the complex $\Delta(P)$ are the elements of the poset P and the faces are the chains in poset. In other words, the order complex of P is given by

$$\Delta(P) = \{\{x_1, x_2, \dots, x_k\} \ : \ x_1 < x_2 < \cdots < x_k, \ x_1, \dots, x_k \in P\}.$$

For the remainder of this section, we restrict ourselves to considering compositions of n where the last part is positive. Such a composition lies in the interval from $(1, \dots, 1)$ to (n). This interval is isomorphic to the Boolean algebra B_{n-1} which is a complemented lattice. Hence for such a composition $\vec{c}$ there exists a complementary composition $\vec{c}\,'$ such that $\vec{c} \wedge \vec{c}\,' = (1, \dots, 1)$ and $\vec{c} \vee \vec{c}\,' = (n)$. As an example, the complement of the composition $(1, 3, 1, 1, 4) = (1, 1+1+1, 1, 1, 1+1+1+1)$ is obtained by exchanging commas and plus signs, that is, $(1+1, 1, 1+1+1+1, 1, 1, 1) = (2, 1, 4, 1, 1, 1)$. Note that the complementary composition has $n - k + 1$ parts.

For a composition $\vec{c} = (c_1, \dots, c_k)$ define the intervals $R_1, \dots, R_k$ by $R_i = [c_1 + \cdots + c_{i-1} + 1, c_1 + \cdots + c_i]$. Define the subgroup $\mathfrak{S}_{\vec{c}}$ of the symmetric group $\mathfrak{S}_n$ by $\mathfrak{S}_{\vec{c}} = \mathfrak{S}_{R_1} \times \cdots \times \mathfrak{S}_{R_k}$. Let $K_1, \dots, K_{n-k+1}$ be the corresponding intervals for the complementary composition $\vec{c}\,'$. Define the subgroup $\mathfrak{S}'_{\vec{c}}$ by

$$\mathfrak{S}'_{\vec{c}} = \mathfrak{S}_{\vec{c}\,'} = \mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{n-k+1}}.$$

A *border strip* is a connected skew shape which does not contain a 2 by 2 square (15, Section 7.17). For each composition $\vec{c}$ there is a unique border strip B that has k rows and the ith row from below consists of c_i boxes. If we label the n boxes of the border strip from southwest to northeast, then the intervals $R_1, \dots, R_k$ correspond to the rows and the intervals $K_1, \dots, K_{n-k+1}$ correspond to the columns. Furthermore, the group $\mathfrak{S}_{\vec{c}}$ is the row stabilizer and the group $\mathfrak{S}'_{\vec{c}}$ is the column stabilizer of the border strip B.

3 The simplicial complex of ordered set partitions

An *ordered set partition* τ of set S is a list of blocks $(C_1, C_2, \dots, C_m)$ where the blocks are subsets of the set S satisfying:

(i) All blocks except possibly the last block are non-empty, that is, $C_i \neq \emptyset$ for $1 \leq i \leq m-1$.

(ii) The blocks are pairwise disjoint, that is, $C_i \cap C_j = \emptyset$ for $1 \leq i < j \leq m$.

(iii) The union of the blocks is S, that is, $C_1 \cup \cdots \cup C_m = S$.

To distinguish from pointed partitions we write 36-127-8-45 for $(\{3, 6\}, \{1, 2, 7\}, \{8\}, \{4, 5\})$. The *type* of an ordered set partitions, $\mathrm{type}(\tau)$, is the composition $(|C_1|, |C_2|, \dots, |C_m|)$.

Let Δ_n denote the collection of all ordered set partitions of the set $[n]$. We view Δ_n as a simplicial complex. The ordered set partition $\tau = (C_1, C_2, \ldots, C_m)$ is an $(m-2)$-dimensional face. It has $m-1$ facets, which are $\tau = (C_1, \ldots, C_i \cup C_{i+1}, \ldots, C_m)$ for $1 \leq i \leq m-1$. The empty face corresponds to the ordered partition $([n])$. The complex Δ_n has $2^n - 1$ vertices that are of the form (C_1, C_2) where $C_1 \neq \emptyset$. Moreover there are $n!$ facets corresponding to permutations in the symmetric group $\mathfrak{S}_n$, that is, for a permutation $\alpha = \alpha_1 \cdots \alpha_n$, the associated facet is $(\{\alpha_1\}, \{\alpha_2\}, \ldots, \{\alpha_n\}, \emptyset)$.

The permutahedron is the $(n-1)$-dimensional polytope obtained by taking the convex hull of the $n!$ points $(\alpha_1, \ldots, \alpha_n)$ where $\alpha = \alpha_1 \cdots \alpha_n$ ranges over all permutations in the symmetric group $\mathfrak{S}_n$. Let P_n denote the boundary complex of the dual of the $(n-1)$-dimensional permutahedron. Since the permutahedron is a simple polytope the complex P_n is a simplicial complex homeomorphic to an $(n-2)$-dimensional sphere. In fact, it is the boundary of the complex Δ_n.

For a permutation $\alpha = \alpha_1 \cdots \alpha_n$ in the symmetric group $\mathfrak{S}_n$ and a composition $\vec{c} = (c_1, \ldots, c_k)$ of n, define the ordered partition

$$\begin{aligned}\sigma(\alpha, \vec{c}) &= (\{\alpha_j \ : \ j \in R_i\})_{1 \leq i \leq k} \\ &= (\{\alpha_1, \ldots, \alpha_{c_1}\}, \{\alpha_{c_1+1}, \ldots, \alpha_{c_1+c_2}\}, \ldots, \{\alpha_{c_1+\cdots+c_{k-1}+1}, \ldots, \alpha_n\}).\end{aligned}$$

We write $\sigma(\alpha)$ when it is clear from the context what the composition $\vec{c}$ is.

For a composition $\vec{c}$ define the subcomplex $\Delta_{\vec{c}}$ to be

$$\Delta_{\vec{c}} = \{\tau \in \Delta_n \ : \ \vec{c} \leq \text{type}(\tau)\}.$$

This complex has all of its facets of type $\vec{c}$. Especially, each facet has the form $\sigma(\alpha, \vec{c})$ for some permutation α. As an example, note that $\Delta_{(1,1,\ldots,1)}$ is the complex P_n.

However for a facet F in $\Delta_{\vec{c}}$ there are $\vec{c}! = c_1! \cdots c_k!$ permutations that map to it by the function σ. Let $\sigma^{-1}(F)$ denote the unique permutation α that gets mapped to the facet F which satisfies the inequalities

$$\alpha_{c_1+\cdots+c_i+1} < \cdots < \alpha_{c_1+\cdots+c_{i+1}}$$

for $0 \leq i \leq k-1$. Furthermore, the descent composition of the permutation $\sigma^{-1}(F)$ is greater than or equal to the composition $\vec{c}$, that is, $\text{Des}(\sigma^{-1}(F)) \geq \vec{c}$.

Lemma 3.1 *If the composition $\vec{c} = (c_1, \ldots, c_k)$ ends with 0, then the simplicial complex $\Delta_{\vec{c}}$ is a cone over the complex $\Delta_{(c_1,\ldots,c_{k-1})}$ with apex $([n], \emptyset)$ and hence contractible.*

Theorem 3.2 *Let $\vec{c}$ be a composition not ending with a zero. Then the simplicial complex $\Delta_{\vec{c}}$ is shellable. The spanning facets (also known as the homology facets) are of the form $\sigma(\alpha)$ where α ranges over all permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{c}$, that is,* $\text{Des}(\alpha) = \vec{c}$. *Hence the complex $\Delta_{\vec{c}}$ is homotopy equivalent to wedge of $\beta(\vec{c})$ spheres of dimension $k-2$.*

By observing that $\Delta_{\vec{c}}$ is the order complex of a rank selection of the Boolean algebra B_n, this result is a direct consequence of that B_n is EL-shellable (1).

4 The homotopy type of the poset $\Pi^{\bullet}_{\vec{c}}$

We now will use Quillen's fiber lemma to show that the chain complex $\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\})$ is homotopy equivalent to the simplicial complex $\Delta_{\vec{c}}$. Recall that a *simplicial map* f from a simplicial complex Γ to a

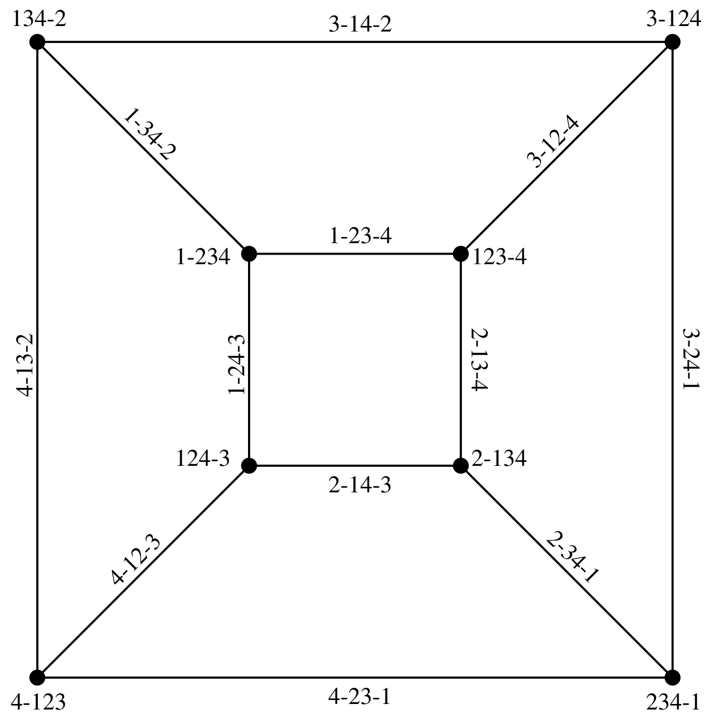

Fig. 1: The simplicial complex $\Delta_{(1,2,1)}$ of ordered partitions. Note that the ordered partition 1234 corresponds to the empty face.

poset P sends vertices of Γ to elements of P and faces of the simplicial complex to chains of P. We have the following result due to Quillen (11).

Theorem 4.1 (Quillen's Fiber Lemma) *Let f be a simplicial map from the simplicial complex Γ to the poset P such that for all x in P, the subcomplex $\Delta(f^{-1}(P_{\geq x}))$ is contractible. Then the order complex $\Delta(P)$ and the simplicial complex Γ are homotopy equivalent.*

Quillen's proof of this result uses homotopy colimits. The proof of this result due to Walker (19) shows that the continuous function $f : |\Gamma| \longrightarrow |\Delta(P)|$ has a homotopy inverse. We will use this later when we study the action of the symmetric group in Section 6.

Recall that the *barycentric subdivision* of a simplicial complex Γ is the simplicial complex $\mathrm{sd}(\Gamma)$ whose vertices are the non-empty faces of Γ and faces are subsets of chains of faces in Γ ordered by inclusion. It is well-known that Γ and $\mathrm{sd}(\Gamma)$ are homeomorphic (since they have the same geometric realization) and hence are homotopy equivalent.

Consider the map ϕ that sends an ordered set partition $(C_1, C_2, \ldots, C_k)$ to the pointed partition $\{C_1, C_2, \ldots, C_{k-1}, \underline{C_k}\}$. We call this map the *forgetful* map since it forgets about the order between the blocks except it keeps the last part as the pointed block. Observe that the inverse image of the pointed partition $\{C_1, C_2, \ldots, C_{k-1}, \underline{C_k}\}$ consists of $(k-1)!$ ordered set partitions.

Lemma 4.2 *Let π be the pointed partition $\{B_1, \ldots, B_{m-1}, \underline{B_m}\}$ where $m \geq 2$. Let Ω be the subcomplex* *of the complex $\Delta_{\vec{c}}$ whose facets are given by the inverse image*

$$\phi^{-1}\left(\left(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}\right)_{\geq \pi}\right).$$

Then the complex Ω is a cone over the apex $([n] - B_m, B_m)$ and hence is contractible.

Theorem 4.3 *The order complex $\Delta(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\})$ is homotopy equivalent to the barycentric subdivision $\mathrm{sd}(\Delta_{\vec{c}})$ and hence $\Delta_{\vec{c}}$.*

By considering the reduced Euler characteristic of the complex $\Delta\left(\Pi^{\bullet}_{\vec{c}} - \{\hat{1}\}\right)$, we have the following corollary.

Corollary 4.4 *The Möbius function of the poset $\Pi^{\bullet}_{\vec{c}} \cup \{\hat{0}\}$ is given by $(-1)^k \cdot \beta(\vec{c})$.*

We note that this corollary can be given a combinatorial proof which avoids Quillen's fiber lemma.

5 Cycles in the complex $\Delta_{\vec{c}}$

In this section and the next we assume that the last part of the composition $\vec{c}$ is non-zero, since in the case $c_k = 0$ the homology group is the trivial group; see Lemma 3.1.

For α a permutation in the symmetric group $\mathfrak{S}_n$, define the subcomplex Σ_α of the complex $\Delta_{\vec{c}}$ to be the simplicial complex whose facets are given by $\{\sigma(\alpha \circ \gamma) \ : \ \gamma \in \mathfrak{S}'_{\vec{c}}\}$.

Lemma 5.1 *The subcomplex Σ_α is isomorphic to the join of the duals of the permutahedra $P_{|K_1|} * \cdots * P_{|K_{n-k+1}|}$ and hence it is sphere of dimension $k-2$.*

Observe that the facets of $\Delta_{\vec{c}}$ are in bijection with permutations α such that $\mathrm{Des}(\alpha) \geq \vec{c}$ in the composition order.

Recall that the boundary map of the face $\sigma = (C_1, \ldots, C_r)$ in the chain complex of $\Delta_{\vec{c}}$ is defined by

$$\partial((C_1, \ldots, C_r)) = \sum_{i=1}^{r-1} (-1)^{i-1} \cdot (C_1, \ldots, C_i \cup C_{i+1}, \ldots, C_r).$$

Lemma 5.2 *For $\alpha \in \mathfrak{S}_n$, the element*

$$g_\alpha = \sum_{\gamma \in \mathfrak{S}'_{\vec{c}}} (-1)^\gamma \cdot \sigma(\alpha \circ \gamma)$$

in the chain group $C_{k-2}(\Delta_{\vec{c}})$ belongs to the kernel of the boundary map and hence to the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.

Theorem 5.3 *The cycles g_α, where α ranges over all permutations with descent composition $\vec{c}$, form a basis for the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.*

6 Representation of the symmetric group

The symmetric group $\mathfrak{S}_n$ acts naturally on the the poset $\Pi^\bullet_{\vec{c}}$ by relabeling the elements. Hence it also acts on the order complex $\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})$. Lastly, the symmetric group acts on the top homology group $\widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}))$. We show in this section that this action is a Specht module of the border strip B corresponding to the composition $\vec{c}$. For an overview on the representation theory of the symmetric group, we refer the reader to Sagan's book (12).

The forgetful map ϕ from $\mathrm{sd}(\Delta_{\vec{c}})$ to the order complex of the poset $\Pi^\bullet_{\vec{c}} - \{\hat{1}\}$ commutes with the action of the symmetric group $\mathfrak{S}_n$. In other words, we have the commutative diagram

$$\begin{array}{ccc} \mathrm{sd}(\Delta_{\vec{c}}) & \xrightarrow{\phi} & \Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}) \\ {\scriptstyle\gamma}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow \\ \mathrm{sd}(\Delta_{\vec{c}}) & \xrightarrow{\phi} & \Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}) \end{array}$$

where γ belongs to $\mathfrak{S}_n$. Observe that the map ϕ extends to a continuous function from the geometric realization $|\mathrm{sd}(\Delta_{\vec{c}})|$ to the geometric realization $|\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})|$, and hence we have the map between the homology groups $\phi_* : \widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}})) \longrightarrow \widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}))$. By applying homology, we obtain that the following diagram of homology groups commutes:

$$\begin{array}{ccc} \widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}})) & \xrightarrow{\phi_*} & \widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})) \\ {\scriptstyle\gamma_*}\big\downarrow & & {\scriptstyle\gamma_*}\big\downarrow \\ \widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}})) & \xrightarrow{\phi_*} & \widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})) \end{array}$$

From the Quillen's fiber lemma we know that the function ϕ has a homotopic inverse, say $\psi : |\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})| \longrightarrow |\mathrm{sd}(\Delta_{\vec{c}})|$. Hence we have $\psi_* : \widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})) \longrightarrow \widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}}))$. Note that ϕ_* and ψ_* are inverses of each other. The function ψ may not commute with the group action. However, since $\gamma_* \circ \psi_* = \psi_* \circ \phi_* \circ \gamma_* \circ \psi_* = \psi_* \circ \gamma_* \circ \phi_* \circ \psi_* = \psi_* \circ \gamma_*$ we have that ψ_* commutes with the group action. We formulate this statement as follows.

Proposition 6.1 *The two homology groups $\widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}}))$ and $\widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}))$ are isomorphic as $\mathfrak{S}_n$-modules.*

It is clear that $\widetilde{H}_{k-2}(\mathrm{sd}(\Delta_{\vec{c}}))$ and $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ are isomorphic as $\mathfrak{S}_n$ modules. Hence in the remainder of this section we will study the action the symmetric group $\mathfrak{S}_n$ on $\Delta_{\vec{c}}$ and its action on the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$.

Let B be the border strip that has k rows where the ith row consists of c_i boxes. Recall that a tableau is a filling of the boxes of the shape B with the integers 1 through n. A standard Young tableau is a tableau where the rows and columns are increasing. A tabloid is an equivalence class of tableaux under the relation of permuting the entries in each row. See (12, Section 2.1) for details.

Observe that there is a natural bijection between tabloids of shape B and facets of the complex $\Delta_{\vec{c}}$ by letting the elements in each row form a block and letting the order of the blocks go from lowest to highest row. Let M^B be the permutation module corresponding to shape B, that is, the linear span of all tabloids of shape B. Notice that the above bijection induces a $\mathfrak{S}_n$-module isomorphism between the permutation module M^B and the chain group $C_{k-2}(\Delta_{\vec{c}})$.

Furthermore, there is a bijection between tableaux of shape B and and permutations by reading the elements in the northeast direction from the border strip. Recall that the group $\mathfrak{S}'_{\vec{c}} = \mathfrak{S}_{K_1} \times \cdots \times \mathfrak{S}_{K_{n-k+1}}$ is the column stabilizer of the border strip B. Let t be a tableau and α its association permutation. Hence the polytabloid $\mathbf{e}_t$ corresponding to the tableau t is the element g_α presented in Lemma 5.2. Since the Specht module S^B is the submodule of M^B spanned by all polytabloids, Lemma 5.2 proves that the Specht module S^B is isomorphic to a submodule of the kernel of the boundary map ∂_{k-2}. Since the kernel is the top homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$, and the Specht module S^B and the homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ have the same dimension $\beta(\vec{c})$, we conclude that they are isomorphic. To summarize we have:

Proposition 6.2 *The top homology group $\widetilde{H}_{k-2}(\Delta_{\vec{c}})$ is isomorphic to the Specht module S^B as $\mathfrak{S}_n$-modules.*

By combining Propositions 6.1 and 6.2, the main result of this section follows.

Theorem 6.3 *The top homology group $\widetilde{H}_{k-2}(\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\}))$ is isomorphic to the Specht module S^B as $\mathfrak{S}_n$-modules.*

7 Knapsack partitions

We now turn our attention to filters in the pointed partition lattice $\Pi^\bullet_n$ that are generated by a pointed knapsack partition. These filters were introduced in (4).

Recall that we view an integer partition λ as a multiset of positive integers. Let $\lambda = \{\lambda_1^{e_1}, \ldots, \lambda_q^{e_q}\}$ be an integer partition, where we assume that the λ_i's are distinct. If all the $(e_1+1)\cdots(e_q+1)$ integer linear combinations

$$\left\{ \sum_{i=1}^{q} f_i \cdot \lambda_i \;:\; 0 \le f_i \le e_i \right\}$$

are distinct, we call λ a *knapsack partition*. A pointed integer partition $\{\lambda, \underline{m}\}$ is called a *pointed knapsack partition* if the partition λ is a knapsack partition.

Definition 7.1 *For a pointed knapsack partition $\{\lambda, \underline{m}\} = \{\lambda_1, \lambda_2, \ldots, \lambda_k, \underline{m}\}$ of n define the subposet $\Pi^\bullet_{\{\lambda,\underline{m}\}}$ to be the filter of $\Pi^\bullet_n$ generated by all pointed set partitions of type $\{\lambda, \underline{m}\}$ and define the subcomplex $\Lambda_{\{\lambda,\underline{m}\}}$ of the complex Δ_n by*

$$\Lambda_{\{\lambda,\underline{m}\}} = \{\tau = (C_1, \ldots, C_{r-1}, C_r) \in \Delta_n : \{C_1, \ldots, C_{r-1}, \underline{C_r}\} \in \Pi^\bullet_{\{\lambda,\underline{m}\}}\}.$$

For a pointed knapsack partition $\{\lambda, \underline{m}\}$ of n define F to be the filter in the poset of compositions of n generated by compositions $\vec{c}$ such that $\mathrm{type}(\vec{c}) = \{\lambda, \underline{m}\}$. Now define $V(\lambda, \underline{m})$ to be the collection of all pointed compositions $\vec{c} = (c_1, c_2, \ldots, c_r)$ in the filter F such that each $c_i, 1 \le i \le r-1$, is a sum of distinct parts of the partition λ and $c_r = m$. As an example, for $\lambda = \{1,1,3,7\}$ we have $(4,8,m) \in V(\lambda, \underline{m})$ but $(2,10,m) \notin V(\lambda, \underline{m})$.

For a composition $\vec{d}$ in $V(\lambda, \underline{m})$ define $\epsilon(\vec{d})$ to be the composition of type $\{\lambda, \underline{m}\}$, where each entry d_i of $\vec{d}$ has been replaced with a decreasing list of parts of λ. That is,

$$\epsilon(\vec{d}) = (\lambda_{1,1}, \ldots, \lambda_{1,t_1}, \ldots, \lambda_{s,1}, \ldots, \lambda_{s,t_s}, m),$$

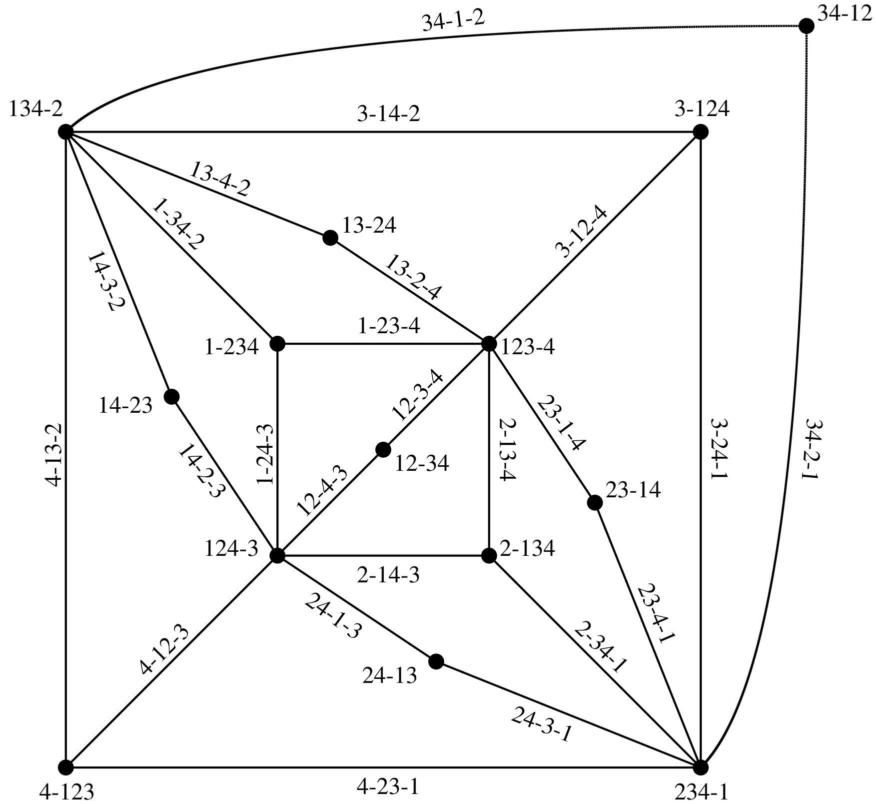

Fig. 2: The simplicial complex $\Lambda_{\{2,1,\underline{1}\}}$, corresponding to the knapsack partition $\{2,1,\underline{1}\}$. Notice that this complex is the union of two complexes $\Delta_{(1,2,1)}$ and $\Delta_{(2,1,1)}$.

where $\lambda_{i,1} > \lambda_{i,2} > \cdots > \lambda_{i,t_i}$, $\sum_{j=1}^{t_i} \lambda_{i,j} = d_i$ and

$$\{\lambda, \underline{m}\} = \{\lambda_{1,1}, \ldots, \lambda_{1,t_1}, \ldots, \lambda_{s,1}, \ldots, \lambda_{s,t_s}, \underline{m}\}.$$

As an example, for the pointed knapsack partition $\lambda = \{2,1,\underline{1}\}$ we have $\epsilon((3,1)) = (2,1,1)$, $\epsilon((2,1,1)) = (2,1,1)$ and $\epsilon((1,2,1)) = (1,2,1)$. Also note $\epsilon(\vec{d}) \leq \vec{d}$ in the partial order of compositions.

Similar to Theorem 3.2 we have the following topological conclusion. However, this time the tool is not shelling, but discrete Morse theory.

Theorem 7.2 *There is a Morse matching on the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$ such that the only critical cells are of the form $\sigma(\alpha, \epsilon(\vec{d}))$ where $\vec{d}$ ranges in the set $V(\lambda,\underline{m})$ and α ranges over all permutations in the symmetric group $\mathfrak{S}_n$ with descent composition $\vec{d}$. Hence, the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$ is homotopy equivalent to wedge of $\sum_{\vec{d} \in V(\lambda,\underline{m})} \beta(\vec{d})$ spheres of dimension $k-1$.*

Example 7.3 Consider the pointed knapsack partition $\{\lambda,\underline{m}\} = \{2,1,\underline{1}\}$, whose associated complex $\Lambda_{\{2,1,\underline{1}\}}$ is shown in Figure 2. Note that $V(\lambda,\underline{m}) = \{(1,2,1),(2,1,1),(3,1)\}$. The critical cells of the complex $\Lambda_{\{2,1,\underline{1}\}}$ are as follows:

$\vec{d}$	$\beta(\vec{d})$	$\epsilon(\vec{d})$	critical cells
$(1,2,1)$	5	$(1,2,1)$	2-14-3, 3-14-2, 3-24-1, 4-13-2, 4-23-1
$(2,1,1)$	3	$(2,1,1)$	14-3-2, 24-3-1, 34-2-1
$(3,1)$	3	$(2,1,1)$	12-4-3, 13-4-2, 23-4-1

Note that $\Lambda_{\{2,1,\underline{1}\}}$ is homotopy equivalent to a wedge of 11 circles.

Now by the same reasoning as in Section 4, that is, using the forgetful map ϕ and Quillen's fiber lemma, we obtain the homotopy equivalence between the order complex of pointed partitions $\Pi^\bullet_{\{\lambda,\underline{m}\}} - \{\hat{1}\}$ and the simplicial complex of ordered set partitions $\Lambda_{\{\lambda,\underline{m}\}}$.

Theorem 7.4 *The order complex $\Delta\left(\Pi^\bullet_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)$ is homotopy equivalent to the barycentric subdivision $\mathrm{sd}(\Lambda_{\{\lambda,\underline{m}\}})$ and hence the simplicial complex $\Lambda_{\{\lambda,\underline{m}\}}$.*

As a corollary we obtain the Möbius function of the poset $\Pi^\bullet_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}$; see (4).

Corollary 7.5 (Ehrenborg–Readdy) *The Möbius function of the poset $\Pi^\bullet_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}$ is given by*

$$\mu\left(\Pi^\bullet_{\{\lambda,\underline{m}\}} \cup \{\hat{0}\}\right) = (-1)^k \cdot \sum_{\vec{d}\in V(\lambda,\underline{m})} \beta(\vec{d}).$$

By the same reasoning as Sections 5 and 6, we have the following isomorphism.

Theorem 7.6 *The two homology groups $\widetilde{H}_{k-1}\left(\Delta\left(\Pi^\bullet_{\{\lambda,\underline{m}\}} - \{\hat{1}\}\right)\right)$ and $\widetilde{H}_{k-1}\left(\Lambda_{\{\lambda,\underline{m}\}}\right)$ are isomorphic as $\mathfrak{S}_n$-modules. Furthermore, they are isomorphic to the direct sum of Specht modules*

$$\bigoplus_{\vec{d}\in V(\lambda,\underline{m})} S^{B(\vec{d})}.$$

8 Concluding remarks

We have not dealt with the question whether the poset $\Pi^\bullet_{\vec{c}}$ is EL-shellable. Recall that Wachs proved that the d-divisible partition lattice $\Pi^d_n \cup \{\hat{0}\}$ has an EL-labeling. Ehrenborg and Readdy gave an extension of this labeling to prove that $\Pi^\bullet_{(d,\ldots,d,m)}$ is EL-shellable (5). Furthermore, Woodroofe (20) showed that the order complex $\Delta(\Pi^d_n - \{\hat{1}\})$ has a convex ear decomposition. This is not true in general for $\Delta(\Pi^\bullet_{\vec{c}} - \{\hat{1}\})$.

Can these techniques be used for studying other subposets of the partition lattice? One such subposet is the odd partition lattice, that is, the collection of all partition where each block size is odd. More generally, what can be said about the case when all the block sizes are congruent to r modulo d? These posets have been studied in (3) and (18). Moreover, what can be said about the poset $\Pi^\bullet_{\{\lambda,\underline{m}\}}$ when $\{\lambda,\underline{m}\}$ is not a pointed knapsack partition?

Another analogue of the partition lattice is the Dowling lattice. Subposets of the Dowling lattice have been studied in (5) and (8; 9). Here the first question to ask is what is the right analogue of the ordered set partitions.

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Allowed patterns of β-shifts

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Abstract. For a real number $\beta > 1$, we say that a permutation π of length n is allowed (or realized) by the β-shift if there is some $x \in [0, 1]$ such that the relative order of the sequence $x, f(x), \ldots, f^{n-1}(x)$, where $f(x)$ is the fractional part of βx, is the same as that of the entries of π. Widely studied from such diverse fields as number theory and automata theory, β-shifts are prototypical examples of one-dimensional chaotic dynamical systems. When β is an integer, permutations realized by shifts have been recently characterized. In this paper we generalize some of the results to arbitrary β-shifts. We describe a method to compute, for any given permutation π, the smallest β such that π is realized by the β-shift.

Résumé. Pour un nombre réel $\beta > 1$, on dit qu'une permutation π de longueur n est permise (ou réalisée) par β-*shift* s'il existe $x \in [0, 1]$ tel que l'ordre relatif de la séquence $x, f(x), \ldots, f^{n-1}(x)$, où $f(x)$ est la partie fractionnaire de βx, soit le même que celui des entrées de π. Largement étudiés dans des domaines aussi divers que la théorie des nombres et la théorie des automates, les β-*shifts* sont des prototypes de systèmes dynamiques chaotiques unidimensionnels. Quand β est un nombre entier, les permutations réalisées par décalages ont été récemment caractérisées. Dans cet article, nous généralisons certains des résultats au cas de β-*shifts* arbitraires. Nous décrivons une méthode pour calculer, pour toute permutation donnée π, le plus petit β tel que π soit réalisée par β-*shift*.

Keywords: beta-shift, forbidden pattern, consecutive pattern, shift map, real base expansion, dynamical system

1 Introduction

Forbidden order patterns in piecewise monotone maps on one-dimensional intervals are a powerful tool to distinguish random from deterministic time series. This contrasts with the fact that, from the viewpoint of symbolic dynamics, chaotic maps are able to produce any symbol pattern, and for this reason they are used in practice to generate pseudo-random sequences. However, this is no longer true when one considers order patterns instead, as shown in (1; 2). From now on, we will use the term patterns to refer to order patterns.

The allowed patterns of a map on a one-dimensional interval are the permutations given by the relative order of the entries in the finite sequences (usually called orbits) obtained by successively iterating the map, starting from any point in the interval. For any fixed piecewise monotone map, there are some permutations that do not appear in any orbit. These are called the forbidden patterns of the map. Understanding the forbidden patterns of chaotic maps is important because the absence of these patterns is what distinguishes sequences generated by chaotic maps from random sequences.

Determining the allowed and forbidden patterns of a given map is a difficult problem in general. The only non-trivial family of maps for which the sets of allowed patterns have been characterized are shift maps. The first results in this direction are found in (1), and a characterization and enumeration of the allowed patterns of shift maps appears in (6). For another family, the so-called logistic map, a few basic properties of their set of forbidden patterns have been studied in (7).

The focus of this paper are the allowed and forbidden patterns of β-shifts, which are a natural generalization of shifts. The combinatorial description of β-shifts is more elaborate than that of shifts, yet still simple enough for β-shifts to be amenable to the study of their allowed patterns. At the same time, β-shifts are good prototypes of chaotic maps because they exhibit important properties of low-dimensional chaotic dynamical systems, such as sensitivity to initial conditions, strong mixing, and a dense set of periodic points. The origin of β-shifts lies in the study of expansions of real numbers in an arbitrary real base $\beta > 1$, which were introduced by Rényi (10). Measure-theoretic properties of β-shifts and their connection to these expansions have been extensively studied in the literature (see for example (3; 8; 9; 11)). For instance, it is known that the base-β expansion of β itself determines the symbolic dynamics of the corresponding β-shift. Finally, β-shifts have also been considered in computability theory (12).

Related to the study of the allowed patterns of β-shifts, we are interested in the problem of determining, for a given permutation π, what is the largest β such that π is a forbidden pattern of the β-shift. We call this parameter the shift-complexity of the permutation. Putting technical details aside, this problem is equivalent to finding the smallest β such that π is realized by (i.e., is an allowed pattern of) the β-shift.

In Section 2 we formally define allowed and forbidden patterns of maps, and we describe shifts and β-shifts from a combinatorial perspective. In Section 3 we study some properties of the domain of β-shifts, we define shift-complexity, and we introduce two relevant real-valued statistics on words. Sections 4 and 5 describe how to determine the shift-complexity of a given permutation π, by expressing this parameter as a root of a certain polynomial whose coefficients depend on π in a non-trivial way. In Section 6 we give examples of the usage of our method for particular permutations.

2 Background and notation

Let $[n] = \{1, 2, \ldots, n\}$, and let $\mathcal{S}_n$ be the set of permutations of $[n]$. In the rest of the paper, the term permutation will always refer to an element of $\mathcal{S}_n$ for some n. For a real number x, we use $\lfloor x \rfloor$, $\lceil x \rceil$, and $\{x\}$ to denote the floor, ceiling, and fractional part of x, respectively. Most of the words considered in this paper will be infinite words over the alphabet $\{0, 1, 2, \ldots\}$ that use only finitely many different letters.

2.1 Allowed patterns of a map

Given a finite sequence $x_1, x_2, \ldots, x_n$ of different elements of a totally ordered set X, define its standardization $\mathrm{st}(x_1, x_2, \ldots, x_n)$ to be the permutation of $[n]$ that is obtained by replacing the smallest element in the sequence with 1, the second smallest with 2, and so on. For example, $\mathrm{st}(4, 7, 1, 6.2, \sqrt{2}) = 35142$.

Fix a map $f : X \to X$. For each $x \in X$ and $n \geq 1$, we define

$$\mathrm{Pat}(x, f, n) = \mathrm{st}(x, f(x), f^2(x), \ldots, f^{n-1}(x)) \in \mathcal{S}_n,$$

provided that there is no pair $1 \leq i < j \leq n$ such that $f^{i-1}(x) = f^{j-1}(x)$.

Given $\pi \in \mathcal{S}_n$, we say that π is *realized* by f, or that π is an *allowed pattern* of f, if there is some $x \in X$ such that $\mathrm{Pat}(x, f, n) = \pi$. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f) = \bigcup_{n\geq 1} \mathrm{Allow}_n(f)$, where $\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n$. The remaining permutations are called *forbidden patterns* of f.

2.2 Shift maps

Special cases of dynamical systems are shift systems. Shifts are interesting from a combinatorial perspective due to their simple definition, and at the same time they are important dynamical systems because they exhibit some key features of low-dimensional chaos.

For each $N \geq 2$, let $\mathcal{W}_N$ be the set of infinite words on the alphabet $\{0, 1, \ldots, N{-}1\}$, equipped with the lexicographic order. The *shift on N symbols* is defined to be the map

$$\begin{array}{rccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1w_2w_3\ldots & \mapsto & w_2w_3w_4\ldots. \end{array}$$

For a detailed description of the associated dynamical system, see (1). According to the above definitions, we have for example that $\mathrm{Pat}(2102212210\ldots, \Sigma_3, 7) = 4217536$.

Let $\Upsilon_N \subset \mathcal{W}_N$ be the set of all words of the form $u(N{-}1)^\infty$, where u is a finite word, and we use the notation $x^\infty = xxx\ldots$. Then $\mathcal{W}_N \setminus \Upsilon_N$ is closed under shifts, and the map

$$\begin{array}{rccc} \varphi : & \mathcal{W}_N \setminus \Upsilon_N & \longrightarrow & [0,1) \\ & w_1w_2w_3\ldots & \mapsto & \sum_{i\geq 1} w_i N^{-i} \end{array}$$

is an order-preserving bijection, also called an *order-isomorphism*. The map $M_N = \varphi \circ \Sigma_N \circ \varphi^{-1}$ from $[0,1)$ to itself is the so-called *sawtooth map*

$$M_N(x) = \{Nx\}.$$

We say in this case that Σ_N and M_N are order-isomorphic. As a consequence, Σ_N and M_N have the same allowed and forbidden patterns.

Allowed and forbidden patterns of shifts (equivalently, sawtooth maps) were first studied in (1), where the authors prove the following result.

Theorem 2.1 ((1)) *For $N \geq 2$, the shortest forbidden patterns of the shift Σ_N have length $N + 2$.*

For example, the shortest forbidden patterns of Σ_4 are $162534, 435261, 615243, 342516, 453621, 324156$. In fact, it was later shown in (6) that there are exactly six forbidden patterns of Σ_N of minimum length.

Proposition 2.2 ((6)) *For every $N \geq 2$, the shortest forbidden patterns of Σ_N, which have length $n = N + 2$, are $\{\rho, \rho^R, \rho^C, \rho^{RC}, \tau, \tau^C\}$, where*

$$\rho \;=\; 1\,n\,2\,(n{-}1)\,3\,(n{-}2)\;\ldots, \qquad \tau \;=\; \ldots\;4\,(n{-}1)\,3\,n\,2\,1,$$

and ^{R} and ^{C} denote the reversal (obtained by reading the entries from right to left) and complementation (obtained by replacing each entry i with $n{+}1{-}i$) operations, respectively.

A formula is given in (6) to compute, for any given permutation π, the minimum number of symbols needed in an alphabet in order for π to be realized by a shift, that is,

$$N(\pi) := \min\{N : \pi \in \mathrm{Allow}(\Sigma_N)\}. \tag{1}$$

The formula given to compute for $N(\pi)$ relies on a bijection between $\mathcal{S}_n$ and the set $\mathcal{T}_n$ of cyclic permutations of $[n]$ with a distinguished entry. For example, underlining the distinguished entry, we have

$$\mathcal{T}_3 = \{\underline{2}31, 2\underline{3}1, 23\underline{1}, \underline{3}12, 3\underline{1}2, 31\underline{2}\}.$$

Given $\pi = \pi(1)\pi(2)\dots\pi(n) \in \mathcal{S}_n$, let $\hat{\pi} \in \mathcal{T}_n$ be the permutation whose cycle decomposition is $(\pi(1), \pi(2), \dots, \pi(n))$, with the entry $\pi(1)$ distinguished. For example, if $\pi = 892364157$, then

$$\hat{\pi} = (\underline{8}, 9, 2, 3, 6, 4, 1, 5, 7) = 536174\underline{8}92.$$

For $\hat{\pi} \in \mathcal{T}_n$, let $\operatorname{des}(\hat{\pi})$ denote the number of descents of the sequence that we get by deleting the distinguished entry from the one-line notation of $\hat{\pi}$. For example, $\operatorname{des}(536174\underline{8}92) = 4$. With these definitions, we can now state the aforementioned formula for $N(\pi)$.

Theorem 2.3 ((6)) *Let $\pi \in \mathcal{S}_n$, and let $\hat{\pi}$ be defined as above. Then $N(\pi)$ is given by*

$$N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where $\epsilon(\hat{\pi}) = 1$ if the one-line notation of $\hat{\pi}$ starts with the distinguished entry followed by 1, or ends with n followed by the distinguished entry, and $\epsilon(\hat{\pi}) = 0$ otherwise.

The distribution of the descent sets of cyclic permutations is studied in (4). The goal of the present paper is to obtain a formula to compute the analogue of $N(\pi)$ for the more general case of β-shifts, which we define next.

2.3 β-shifts

These maps are a natural generalization of shift maps, and have been extensively studied in the literature (11; 8) from a measure-theoretic perspective. Let us begin by defining their order-isomorphic counterparts on the unit interval, which we call β-sawtooth maps. For any real number $\beta > 1$, define the β-sawtooth map

$$\begin{array}{rccc} M_\beta : & [0,1) & \longrightarrow & [0,1) \\ & x & \mapsto & \{\beta x\} \end{array}$$

(see Figure 1). In the rest of the paper we will assume, unless otherwise stated, that β is a real number with $\beta > 1$. Note that when β is an integer we recover the definition of standard sawtooth maps.

To describe the corresponding map on infinite words, called the β-shift, let us first define its domain, $W(\beta)$. As shown by Rényi (10), every nonnegative real number x has a β-expansion

$$x = w_0 + \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots ,$$

where $w_0 = \lfloor x \rfloor$, $w_1 = \lfloor \beta\{x\} \rfloor$, $w_2 = \lfloor \beta\{\beta\{x\}\} \rfloor$, $\dots$. This expansion has the property that $w_i \in \{1, 2, \dots, \lceil \beta - 1 \rceil\}$ for $i \geq 1$. If $0 \leq x < 1$, then $w_0 = 0$ and $w_i = \lfloor \beta M_\beta^{i-1}(x) \rfloor$ for $i \geq 1$.

Let $W_0(\beta)$ be the set of infinite words $w = w_1 w_2 w_3 \dots$ that are obtained in this way as β-expansions of numbers $x \in [0,1)$. The lexicographic order (which will be denoted by $<$ throughout the paper) makes $W_0(\beta)$ into a totally ordered set. The map $[0,1) \to W_0(\beta)$, $x \mapsto w$ is an order-isomorphism. For any $w = w_1 w_2 w_3 \cdots \in W_0(\beta)$, we can recover $x \in [0,1)$ as

$$x = \sum_{i \geq 1} w_i \beta^{-i}.$$

Of particular interest is the β-expansion of β itself, for which we use the notation

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots .$$

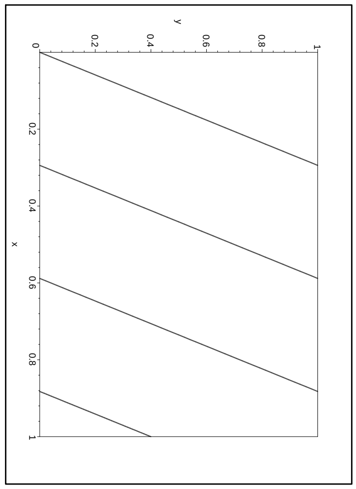

Fig. 1: The β-sawtooth map $M_\beta(x)$ for $\beta = 3.4$.

One can define $A_0 = \beta$, and for $i \geq 0$, $a_i = \lfloor A_i \rfloor$ and $A_{i+1} = \beta(A_i - a_i)$. Then it follows by induction that

$$A_i = \beta^{i+1} - a_0\beta^i - a_1\beta^{i-1} - \cdots - a_{i-1}\beta. \tag{2}$$

If β is such that its β-expansion is finite, i.e., it has only finitely many nonzero terms a_i, we let a_q be the last nonzero term of the expansion, so

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_q}{\beta^q},$$

and we let $y = (a_0a_1 \ldots a_{q-1}(a_q - 1))^\infty$. Define

$$W(\beta) = \{w_1w_2w_3 \ldots : w_kw_{k+1}w_{k+2} \ldots < a_0a_1a_2 \ldots \text{ for all } k \geq 1\}. \tag{3}$$

It follows from Parry (9) that $W_0(\beta)$ is precisely the set of words in $W(\beta)$ that do not end in y. For example, if $\beta = N \in \mathbb{Z}$, then $W(N) = \mathcal{W}_N$, the set of infinite words on the alphabet $\{0, 1, \ldots, N-1\}$, whereas $W_0(N)$ does not include words ending in $(N-1)^\infty$. If $\beta = 1 + \sqrt{2}$, then $a_0a_1a_2 \ldots = 210^\infty$, so $W(\beta)$ is the set of words over $\{0, 1, 2\}$ where every 2 is followed by a 0, but $W_0(\beta)$ does not include words ending in $y = (20)^\infty$. Clearly, if β has an infinite β-expansion, then $W(\beta) = W_0(\beta)$.

We define the β-shift Σ_β to be the map

$$\begin{array}{rccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1w_2w_3 \ldots & \mapsto & w_2w_3w_4 \ldots. \end{array}$$

For $x \in [0, 1)$, if $w \in W_0(\beta)$ is the word given by the β-expansion of x, then $\Sigma_\beta(w)$ is the word given by the β-expansion of $M_\beta(x)$. In particular, M_β and the restriction of Σ_β to $W_0(\beta)$ are order-isomorphic.

Besides, this restriction of the domain does not change the set of allowed patterns of Σ_β, and therefore $\text{Allow}(\Sigma_\beta) = \text{Allow}(M_\beta)$.

A well-studied problem is the connection between β-expansions and the ergodic properties of the corresponding β-shift (see (11) and references therein). In this paper, rather than the measure-theoretic properties of β-shifts, we are concerned with their allowed and forbidden patterns.

3 The shift-complexity of a permutation

In this section we establish some properties of the domain $W(\beta)$ of the β-shift, and we define a real-valued statistic on permutations, which we call the shift-complexity.

Proposition 3.1 *Let $1 < \beta \leq \beta'$. Then $W(\beta) \subseteq W(\beta')$ and $\text{Allow}(\Sigma_\beta) \subseteq \text{Allow}(\Sigma_{\beta'})$.*

Proof: Let $a = a_0 a_1 \dots$ be the β-expansion of β, and let $a' = a'_0 a'_1 \dots$ be the β'-expansion of β'. To prove that $W(\beta) \subseteq W(\beta')$, using the definition in equation (3) it is enough to show that $a \leq a'$. Let A_i be as defined in equation (2), and let A'_i be defined analogously for β'. Suppose that the first entry where a and a' differ is $a_i \neq a'_i$. We claim that $A_j \leq A'_j$ for $0 \leq j \leq i$. This follows by induction since $A_0 = \beta \leq \beta' = A'_0$, and if $A_j \leq A'_j$ for some $j < i$, then $\lfloor A_j \rfloor = a_j = a'_j = \lfloor A'_j \rfloor$ implies that $A_j - a_j \leq A'_j - a'_j$, so $A_{j+1} = \beta(A_j - a_j) \leq \beta'(A'_j - a'_j) = A'_{j+1}$. But then $a_i = \lfloor A_i \rfloor \leq \lfloor A'_i \rfloor = a'_i$, so $a_i < a'_i$ and we are done.

To prove that $\text{Allow}(\Sigma_\beta) \subseteq \text{Allow}(\Sigma_{\beta'})$, let $\pi \in \text{Allow}(\Sigma_\beta)$. By definition, there exists a word $w \in W(\beta)$ such that $\text{Pat}(w, \Sigma_\beta, n) = \pi$, where n is the length of π. But then $w \in W(\beta')$, and since $\text{Pat}(w, \Sigma_{\beta'}, n) = \pi$, we see that $\pi \in \text{Allow}(\Sigma_{\beta'})$. $\square$

Because of Proposition 3.1, it is clear from the definition of β-shifts that for $\beta < \beta'$, the restriction of $\Sigma_{\beta'}$ to $W(\beta)$ is equal to Σ_β. In the rest of the paper, we will write Σ instead of Σ_β when it creates no confusion. Now we can give the key definition of this section. We call $B(\pi)$ the *shift-complexity* of π.

Definition 3.2 *For any permutation π, let*

$$B(\pi) = \inf\{\beta : \pi \in \text{Allow}(\Sigma_\beta)\}.$$

Equivalently, $B(\pi)$ is the supremum of the set of values β such that π is a forbidden pattern of Σ_β. If we think of the β-shifts Σ_β as a family of functions parameterized by β, then the values of β for which there is a permutation π with $B(\pi) = \beta$ correspond to *phase transitions* where the set of allowed patterns of Σ_β changes.

To compute the value of $B(\pi)$ for given π, we will define two real-valued statistics on words. For an infinite word $w = w_1 w_2 \dots$, we use the notation $w_{i\rightarrow} = w_i w_{i+1} \dots$ for $i \geq 1$.

In the rest of this section, v and w denote words in $\mathcal{W}_N$ for some arbitrary positive integer N. We define the series

$$f_w(\beta) = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots .$$

This series is convergent for $\beta > 1$, and in this interval,

$$f'_w(\beta) = -\frac{w_1}{\beta^2} - \frac{2w_2}{\beta^3} - \cdots < 0,$$

assuming that $w \neq 0^\infty$. Since $\lim_{\beta\to\infty} f_w(\beta) = 0$, it follows that there is a unique solution to $f_w(\beta) = 1$ satisfying $\beta \geq 1$. Such value of β will be denoted by $\hat{b}(w)$. We define $\hat{b}(0^\infty) = 0$ by convention. Additionally, let

$$b(w) = \sup_{i\geq 1} \hat{b}(w_{i\to}).$$

Note that $\hat{b}(w) \leq b(w) \leq N$.

The following result describes the relationship between the permutation statistic B and the word statistic b. Its proof is omitted in this extended abstract, but it appears in (5).

Proposition 3.3 *For any $\pi \in \mathcal{S}_n$, $B(\pi) = \inf\{b(w) : \mathrm{Pat}(w, \Sigma, n) = \pi\}$.*

4 Computation of $B(\pi)$: from permutations to words

Suppose we are given $\pi \in \mathcal{S}_n$ with $n \geq 2$. The goal of this section and the next one is to describe a method to compute the shift-complexity of π. In the rest of the paper, we refer to the condition $\mathrm{Pat}(w, \Sigma, n) = \pi$ by saying that w *induces* π. In some cases we will be able to find a word w inducing π such that $b(w)$ is smallest for all such words; when this happens, $B(\pi) = b(w)$ and the infimum in Proposition 3.3 is a minimum. In other cases we will find a sequence of words $w^{(m)}$ inducing π where $b(w^{(m)})$ approaches $B(\pi)$ as m grows. This section is devoted to finding a word w or a sequence $w^{(m)}$ with the above properties. In Section 5 we show how to compute the values of the statistic b on these words in order to obtain $B(\pi)$.

Let $N = N(\pi)$ for the rest of this section. From the definitions, it is clear that $B(\pi) \leq N$, and that there is some word $z \in W(N) = \mathcal{W}_N$ that induces π. The explicit construction of such words z is given in (6). It is important to notice that to find words w and $w^{(m)}$ as described above, it is enough to consider only words in $\mathcal{W}_N$. Indeed, if $B(\pi) = N$ (we will later see in equation (5) that this case never happens, but cannot rule it out just yet), then any $z \in \mathcal{W}_N$ inducing π satisfies $b(z) = B(\pi)$, and we can just take $w = z$. On the other hand, to deal with the case $B(\pi) < N$, note that any word z with $b(z) < N$ must be in $\mathcal{W}_N$. This applies to $z = w$ for any word w satisfying $b(w) = B(\pi)$, and also to $z = w^{(m)}$ for words in the above sequence, provided that $b(w^{(m)})$ is close enough to $B(\pi)$. For convenience, a word w inducing π and satisfying

$$b(w) = B(\pi) \;\text{ or }\; B(\pi) \leq b(w) < N \tag{4}$$

will be called a *small* word. For words in $\mathcal{W}_N$ inducing π we can apply Corollary 2.13 from (6), which we restate here.

Proposition 4.1 ((6)) *Let $N = N(\pi)$ as above, and suppose that $z \in \mathcal{W}_N$ induces π. Then the entries $z_1 z_2 \ldots z_{n-1}$ are uniquely determined by π.*

In the rest of this section, we let $\zeta = \zeta(\pi) = z_1 z_2 \ldots z_{n-1}$ be the word defined in Proposition 4.1. It follows from (6) that the entries of ζ can be computed as follows:

- Write the sequence of (unassigned) variables $z_{\pi^{-1}(1)} z_{\pi^{-1}(2)} \cdots z_{\pi^{-1}(n)}$ in this order and remove z_n from it.
- For each pair $z_i z_j$ of adjacent entries in the sequence with z_i to the left of z_j, insert a vertical bar between them if and only if $\pi(i+1) > \pi(j+1)$.

- In the case that $\pi(n) = 1$ and $\pi(n-1) = 2$, insert a vertical bar before the first entry in the sequence (which is $z_{\pi^{-1}(2)}$ in this case).
- Set each z_i in the sequence to equal the number of vertical bars to its left.

For example, if $\pi = 892364157 \in \mathcal{S}_9$, the sequence with z_n removed is $z_7z_3z_4z_6z_8z_5z_1z_2$, which becomes $z_7|z_3z_4|z_6z_8|z_5z_1|z_2$ after inserting the bars, so $\zeta(\pi) = z_1z_2 \dots z_8 = 34113202$.

It is shown in (6, Lemma 2.8) that if $1 \leq i, j < n$ are such that $\pi(i) < \pi(j)$ and $\pi(i+1) > \pi(j+1)$, then the corresponding entries in $\zeta(\pi)$ satisfy $z_i < z_j$. This statement is logically equivalent to the following.

Lemma 4.2 ((6)) *If $1 \leq i, j < n$ are such that $z_j \leq z_i$ and $\pi(j) > \pi(i)$, then $\pi(j+1) > \pi(i+1)$ and $z_j = z_i$.*

The conclusion $z_j = z_i$ is clear from the fact that if $z_j < z_i$, then $z_{j\rightarrow} < z_{i\rightarrow}$, contradicting $\pi(j) < \pi(i)$. From Proposition 4.1 we see that $0 \leq z_1, \dots, z_{n-1} \leq N-1$. It is shown in (6) that if $\pi(n-1) = n-1$ and $\pi(n) = n$, then $0 \leq z_1, \dots, z_{n-1} \leq N-2$.

Proposition 4.1 and the paragraph preceding it imply that for any small word w (as defined by condition (4)) inducing π, the first $n-1$ entries of w are given by $\zeta = z_1z_2 \dots z_{n-1}$. The next theorem shows how to find the remaining entries $w_{n+1}w_{n+2} \dots$. Its proof, which is quite involved, is omitted in this extended abstract, but it can be found in (5).

Theorem 4.3 *Let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, and if $c \neq 1$, let $k = \pi^{-1}(c-1)$.*

(a) If $c = 1$, let $w = \zeta 0^\infty$. Then w induces π, and for any other word v that induces π, we have $b(v) > b(w)$. In particular,

$$B(\pi) = b(w) = \hat{b}(w_{\ell\rightarrow}).$$

(b) If $c \neq 1$ and $\ell > k$, let

$$w = z_1z_2 \cdots z_{n-1}\, z_kz_{k+1} \cdots z_{\ell-2}(z_{\ell-1}{+}1)0^\infty.$$

Then w induces π, and for any other word v that induces π, we have $b(v) \geq b(w)$. In particular,

$$B(\pi) = b(w) = \hat{b}(w_{\ell\rightarrow}).$$

(c) If $c \neq 1$ and $\ell < k$, let h be such that $\pi(h)$ is the maximum of $\pi(k+1), \pi(k+2), \dots, \pi(n)$. For each $m \geq 0$, let

$$w^{(m)} = z_1z_2 \cdots z_{n-1}\, (z_kz_{k+1} \cdots z_{n-1})^m z_kz_{k+1} \cdots z_{h-2}(z_{h-1}{+}1)0^\infty.$$

Then $w^{(m)}$ induces π for $m \geq \frac{n-2}{n-k}$, and for any word v that induces π, there exists an m_0 such that $b(v) > b(w^{(m)})$ for $m \geq m_0$. In particular,

$$B(\pi) = \lim_{m\to\infty} b(w^{(m)}).$$

Additionally, $b(w^{(m)}) = \hat{b}(w^{(m)}_{\ell\rightarrow})$.

Some examples of applications of the above results are given in Section 6. Section 5 deals with the problem of computing $b(w)$ and $\lim_{m\to\infty}\hat{b}(w^{(m)})$, where w and $w^{(m)}$ are the above words. We end this section looking in more detail at the phase transitions where new patterns become allowed for β-shifts, and discussing the relationship between $B(\pi)$ and $N(\pi)$.

Proposition 4.4 *For every $\pi \in \mathcal{S}_n$, $\pi \notin \mathrm{Allow}(\Sigma_{B(\pi)})$. In particular, the infimum in Definition 3.2 is never a minimum, and the shift-complexity of π is the maximum β such that π is a forbidden pattern of Σ_β.*

One can rephrase Proposition 4.4 by stating that $\pi \in \mathrm{Allow}(\Sigma_\beta)$ if and only if $\beta > B(\pi)$. It follows from this observation and the definition of $N(\pi)$ (see equation (1)) that

$$N(\pi) = \lfloor B(\pi) \rfloor + 1. \tag{5}$$

5 Computation of $B(\pi)$: the equations

In this section we find the shift-complexity of an arbitrary permutation π by expressing it as the unique real root greater than 1 of a certain polynomial $P_\pi(\beta)$. Given a finite word $u_1u_2\dots u_r$, define the polynomial

$$p_{u_1u_2\dots u_r}(\beta) = \beta^r - u_1\beta^{r-1} - u_2\beta^{r-2} - \cdots - u_r.$$

Theorem 5.1 *For any $\pi \in \mathcal{S}_n$ with $n \geq 2$, let $\zeta = \zeta(\pi) = z_1z_2\dots z_{n-1}$ as defined in Section 4. Let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, and if $c \neq 1$, let $k = \pi^{-1}(c-1)$. Define a polynomial $P_\pi(\beta)$ as follows. If $c = 1$, let*

$$P_\pi(\beta) = p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta);$$

if $c \neq 1$ and $\ell > k$, let

$$P_\pi(\beta) = p_{z_\ell z_{\ell+1}\dots z_{n-1} z_k z_{k+1}\dots z_{\ell-1}}(\beta) - 1;$$

if $c \neq 1$ and $\ell < k$, let

$$P_\pi(\beta) = \begin{cases} p_{z_\ell z_{\ell+1}\dots z_{n-c}}(\beta) & \text{if } \pi \text{ ends in } 12\dots c, \\ p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta) - p_{z_\ell z_{\ell+1}\dots z_{k-1}}(\beta) & \text{otherwise.} \end{cases}$$

Then $B(\pi)$ is the unique real root with $\beta \geq 1$ of $P_\pi(\beta)$.

Note that $P_\pi(\beta)$ is always a monic polynomial with integer coefficients. For $\pi \in \mathcal{S}_n$, its degree is never greater than the maximum of $n-\ell$ and $n-k$, and in particular never greater than $n-1$.

Proof: In the case $c = 1$, letting $w = \zeta 0^\infty$, we know by Theorem 4.3(a) that

$$B(\pi) = b(w) = \hat{b}(w_{\ell\to}) = \hat{b}(z_\ell z_{\ell+1}\dots z_{n-1}0^\infty).$$

Thus, $B(\pi)$ is the unique solution with $\beta \geq 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}} = 1,$$

which multiplying by $\beta^{n-\ell}$ is equivalent to $p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta) = 0$.

In the case $c \neq 1$ and $\ell > k$, Theorem 4.3(b) states that if we now let

$$w = z_1 z_2 \cdots z_{n-1}\, z_k z_{k+1} \cdots z_{\ell-2}(z_{\ell-1}{+}1)0^\infty,$$

then $B(\pi) = b(w) = \hat{b}(w_{\ell\to})$. Thus, $B(\pi)$ is the unique solution with $\beta \geq 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}} + \frac{z_k}{\beta^{n-\ell+1}} + \cdots + \frac{z_{\ell-2}}{\beta^{n-k-1}} + \frac{z_{\ell-1}+1}{\beta^{n-k}} = 1,$$

which multiplying by β^{n-k} is equivalent to $p_{z_\ell z_{\ell+1} \ldots z_{n-1} z_k z_{k+1} \ldots z_{\ell-1}}(\beta) - 1 = 0$.

Finally, if $c \neq 1$ and $\ell < k$, it follows from Theorem 4.3(b) that letting

$$w^{(m)} = z_1 z_2 \cdots z_{n-1}\,(z_k z_{k+1} \cdots z_{n-1})^m z_k z_{k+1} \cdots z_{h-2}(z_{h-1}{+}1)0^\infty,$$

where $\pi(h) = \max\{\pi(k+1), \pi(k+2), \ldots, \pi(n)\}$, we have $B(\pi) = \lim_{m\to\infty} b(w^{(m)})$ and $b(w^{(m)}) = \hat{b}(w^{(m)}_{\ell\to})$. Here $\hat{b}(w^{(m)}_{\ell\to})$ is the unique solution with $\beta \geq 1$ of

$$\begin{aligned} \frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{k-1}}{\beta^{k-\ell}} + \left(\frac{z_k}{\beta^{k-\ell+1}} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}}\right)\left(1 + \frac{1}{\beta^{n-k}} + \frac{1}{\beta^{2(n-k)}} + \cdots + \frac{1}{\beta^{m(n-k)}}\right) \\ + \frac{z_k}{\beta^{n-\ell+m(n-k)+1}} + \cdots + \frac{z_{h-2}}{\beta^{n-\ell+m(n-k)+h-k-1}} + \frac{z_{h-1}+1}{\beta^{n-\ell+m(n-k)+h-k}} = 1. \quad (6) \end{aligned}$$

For fixed m, it is clear that $\hat{b}(w^{(m)}_{\ell\to}) > 1$, because $w^{(m)}_{\ell\to}$ has at least two nonzero entries, since $z_\ell \geq 1$. Suppose first that not all of the entries $z_k, \ldots, z_{n-1}$ are zero. In this case, making m go to infinity in equation (6) and using that $B(\pi) = \lim_{m\to\infty} \hat{b}(w^{(m)}_{\ell\to})$, we see that $B(\pi)$ is the solution with $\beta > 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{k-1}}{\beta^{k-\ell}} + \left(\frac{z_k}{\beta^{k-\ell+1}} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}}\right)\frac{1}{1 - \frac{1}{\beta^{n-k}}} = 1.$$

Multiplying by $\beta^{k-\ell}(\beta^{n-k} - 1)$ we get

$$(\beta^{n-k} - 1)(z_\ell \beta^{k-\ell-1} + z_{\ell+1}\beta^{k-\ell-2} + \cdots + z_{k-1}) + z_k \beta^{n-k-1} + \cdots + z_{n-1} = \beta^{n-\ell} - \beta^{k-\ell},$$

which can be rearranged as $p_{z_\ell z_{\ell+1} \ldots z_{n-1}}(\beta) = p_{z_\ell z_{\ell+1} \ldots z_{k-1}}(\beta)$.

In the case where $z_k = \cdots = z_{n-1} = 0$, $B(\pi)$ is the solution with $\beta \geq 1$ of

$$z_\ell \beta^{k-\ell-1} + z_{\ell+1}\beta^{k-\ell-2} + \cdots + z_{k-1} = \beta^{k-\ell},$$

or equivalently $p_{z_\ell z_{\ell+1} \ldots z_{k-1}}(\beta) = 0$. This situation only happens when π ends in $123 \ldots c$. Indeed, one can use Lemma 4.2 to show that the condition $z_k = \cdots = z_{n-1}$ forces the sequence $\pi(k), \pi(k+1), \ldots, \pi(n)$ to be monotonic, which can only happen if $k = n-1$. Now, Lemma 4.2 again and the fact that $z_k = 0$ imply that if d_i is the entry following i in π, then $1 \neq d_1 < d_2 < \cdots < d_{c-1} = c$, which forces the ending of π to be $123 \ldots c$. We remark that since $z_{n-c+1} = \cdots = z_{n-1} = 0$ in this case, we have that $p_{z_\ell z_{\ell+1} \ldots z_{k-1}}(\beta) = \beta^{c-2} p_{z_\ell z_{\ell+1} \ldots z_{n-c}}(\beta)$. $\square$

6 Examples

In this section we give examples where Theorems 4.3 and 5.1 are used to construct words inducing a given permutation and to determine its shift-complexity.

(1) Let $\pi = 3421$. Using the construction from (6), described also right after Proposition 4.1 above, we get $\zeta(\pi) = 121$. Theorem 4.3(a) states that $w = 1210^\infty$ induces π and $B(\pi) = b(w) = \hat{b}(210^\infty)$. By Theorem 5.1, $B(\pi)$ is the root with $\beta \geq 1$ of $P_\pi(\beta) = p_{21}(\beta) = \beta^2 - 2\beta - 1$, so $B(3421) = 1 + \sqrt{2}$.

(2) Let $\pi = 735491826$. Using the construction from (6), $\zeta(\pi) = 42326051$. Applying Theorem 4.3(b) with $k = 3$ and $\ell = 5$, we get that $w = 42326051330^\infty$ induces π and $B(\pi) = b(w) = \hat{b}(6051330^\infty)$. By Theorem 5.1, $B(\pi)$ is the real root with $\beta \geq 1$ of

$$P_\pi(\beta) = p_{605132}(\beta) - 1 = \beta^6 - 6\beta^5 - 5\beta^3 - \beta^2 - 3\beta - 3,$$

so $B(735491826) \approx 6.139428921$.

(3) For $\pi = 892364157$, we have seen earlier that $\zeta(\pi) = 34113202$. Applying Theorem 4.3(c) with $k = 5$, $\ell = 2$, and $h = 9$, we have that $w^{(m)} = 34113202(3202)^m32030^\infty$ induces π for $m \geq 2$, and

$$B(\pi) = \lim_{m\to\infty} b(w^{(m)}) = \lim_{m\to\infty} \hat{b}(4113202(3202)^m32030^\infty).$$

By Theorem 5.1, $B(\pi)$ is the real root with $\beta \geq 1$ of

$$P_\pi(\beta) = p_{4113202}(\beta) - p_{411}(\beta) = \beta^7 - 4\beta^6 - \beta^5 - \beta^4 - 4\beta^3 + 2\beta^2 + \beta - 1,$$

so $B(892364157) \approx 4.327613926$.

(4) Let $\pi = (c+1)(c+2)\dots n12\dots c$ for any fixed $1 \leq c \leq n$. Here we get $\zeta(\pi) = 0^{n-c-1}10^{c-1}$. If $1 < c < n$, then $k = n-1$, $\ell = n-c$ and $h = n$, so by Theorem 4.3(c), $w^{(m)} = 0^{n-c-1}10^{c-1}0^m10^\infty$ induces π for $m \geq n-2$, and

$$B(\pi) = \lim_{m\to\infty} \hat{b}(10^{c-1}0^m10^\infty).$$

By Theorem 5.1, $B(\pi) = 1$ is the root of $P_\pi(\beta) = p_1(\beta) = \beta - 1$. If $c = n$, Theorem 4.3(b) gives $w = 0^{n-1}10^\infty$, and if $c = 1$, Theorem 4.3(a) yields $w = 0^{n-2}10^\infty$. In both cases, w induces π and $B(\pi) = \hat{b}(10^\infty) = 1$ as well. It is not hard to see that these are the only permutations with $B(\pi) = 1$.

The values of $B(\pi)$ for all permutations of length 2, 3, and 4 are given in Table 1. They have been computed using the implementation in *Maple* of Theorem 5.1 and the algorithm described in Section 4 to find $\zeta(\pi)$.

Using Theorem 5.1 one can show that for $n \geq 4$, the permutation $\rho = 1\,n\,2\,(n{-}1)\,3\,(n{-}2)\,\dots$ has the property that $B(\pi) < B(\rho)$ for all $\pi \in \mathcal{S}_n \setminus \{\rho\}$. For the proof of this result, as well as a method to determine the length of the shortest forbidden pattern of Σ_β for given $\beta > 1$, we refer the reader to the full version of this extended abstract (5).

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$\pi \in \mathcal{S}_2$	$\pi \in \mathcal{S}_3$	$\pi \in \mathcal{S}_4$	$B(\pi)$	$B(\pi)$ is a root of
$12, 21$	$123, 231, 312$	$1234, 2341, 3412, 4123$	1	$\beta - 1$
		$1342, 2413, 3124, 4231$	1.465571232	$\beta^3 - \beta^2 - 1$
	$132, 213, 321$	$1243, 1324, 2431, 3142, 4312$	$\frac{1+\sqrt{5}}{2} \approx 1.618033989$	$\beta^2 - \beta - 1$
		4213	1.801937736	$\beta^3 - \beta^2 - 2\beta + 1$
		$1432, 2143, 3214, 4321$	1.839286755	$\beta^3 - \beta^2 - \beta - 1$
		$2134, 3241$	2	$\beta - 2$
		4132	2.246979604	$\beta^3 - 2\beta^2 - \beta + 1$
		$2314, 3421$	$1 + \sqrt{2} \approx 2.414213562$	$\beta^2 - 2\beta - 1$
		1423	$\frac{3+\sqrt{5}}{2} \approx 2.618033989$	$\beta^2 - 3\beta + 1$

Tab. 1: The shift-complexity of all permutations of length up to 4.

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Polytopes from Subgraph Statistics

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Abstract. We study polytopes that are convex hulls of vectors of subgraph densities. Many graph theoretical questions can be expressed in terms of these polytopes, and statisticians use them to understand exponential random graph models.

Relations among their Ehrhart polynomials are described, their duals are applied to certify that polynomials are non-negative, and we find some of their faces.

For the general picture we inscribe cyclic polytopes in them and calculate volumes. From the volume calculations we conjecture that a variation of the Selberg integral indexed by Schur polynomials has a combinatorial formula. We inscribe polynomially parametrized sets, called curvy zonotopes, in the polytopes and show that they approximate the polytopes arbitrarily close.

Résumé. Nous étudions les polytopes qui sont les enveloppes convexes des vecteurs des densités de sous-graphe. Beaucop de questions théoriques de graphe peuvent être exprimées en termes de ces polytopes, et les statisticiens les utilisent pour comprendre les modèles de graphes aléatoires exponentielles

Des relations parmi leurs polynômes d'Ehrhart sont décrites leurs duals sont appliqués pour certifier que les polynômes sont non négatifs, et nous trouvons certaines de leurs faces.

Pour la description générale nous inscrivons les polytopes cycliques dans eux et calculons les volumes. D'après les calculs de volume, nous conjecturons qu'une variation de l'intégrale de Selberg indexés par des polynômes de Schur a une formule combinatoire. Nous inscrivons polynomialement les ensembles paramétrisés appelés "curvy zonotopes" dans les polytopes et montrons qu'ils sont arbitrairement proches de polytopes.

Keywords: polytopes, subgraph statistics, exponential random graph models, curvy zonotopes, graph limits

1 Introduction

In this paper we study polytopes from subgraph statistics. For any two graphs F and G, the F*-subgraph density of* G, denoted $t(F,G)$, is the proportion of injective maps from $V(F)$ to $V(G)$ sending edges of F onto edges of G. For any vector $\mathbf{F}$ of d graphs $F_1, F_2, \ldots, F_d$; and any graph G, we get a point

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$t(\mathbf{F}, G) = (t(F_1, G), t(F_2, G), \ldots, t(F_d, G))$ in $\mathbb{R}^d$. The convex hull of the collection of all such points from n vertex graphs is the *polytope from subgraph statistics*

$$P_{\mathbf{F};n} = \text{conv}\ \{t(\mathbf{F}, G) \mid G \text{ is a graph on } n \text{ vertices}\}.$$

The polytope $P_{(K_3,C_4,K_4\setminus e);6}$ is drawn in Figure 1, and in Figure 2 is a combinatorial representation of its vertices and edges. If larger examples of polytopes from subgraph statistics looks anything like in Figures 1 and 2, then it would be very difficult to give an explicit facet description. And indeed many hard theorems and conjectures in extremal graph theory can be rephrased as questions about these polytopes, making a complete facet description probably impossible in general. In Figure 2 we tabulated the vertices by the actual subgraph counts and not the proportions $t(F, G)$. This defines the lattice polytope $P^L_{\mathbf{F};n}$, a rescaling of $P_{\mathbf{F};n}$. It should be noted that several graphs could have the same subgraph statistics, and that

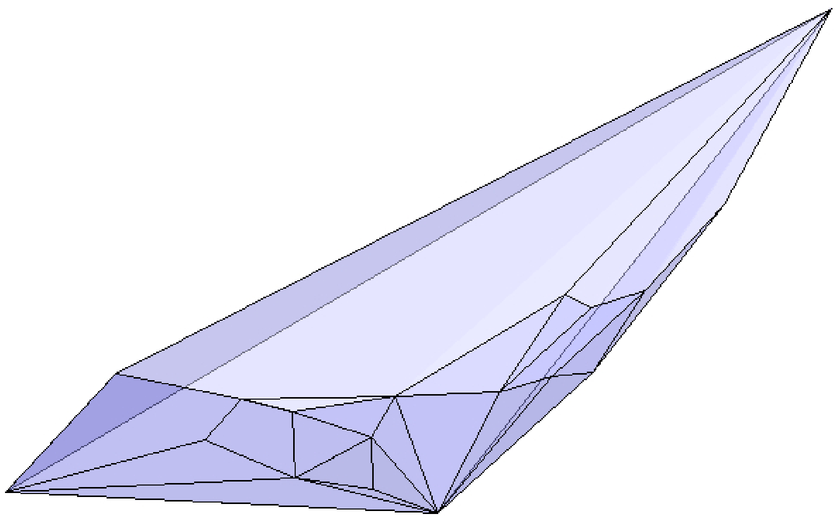

Fig. 1: The polytope $P_{(K_3,C_4,K_4\setminus e);6}$.

even if $t(\mathbf{F}, G_1)$ and $t(\mathbf{F}, G_2)$ are different vertices on the same facet, it is not necessary that G_1 and G_2 are related in any sense, for example as subgraphs. This is illustrated in Figure 3.

We got interested in studying the polytopes from subgraph statistics after several questions were raised about them by Rinaldo, Fienberg and Zhou [15]. They investigated maximum likelihood estimation for exponential random graph models and realized that its behavior is closely linked to the geometry of the polytopes. For some vector $\mathbf{F}$ of d graphs and model parameter $\gamma \in \mathbb{R}^d$, the probability of observing the n vertex graph G is

$$p_G = \frac{1}{Z(\gamma)} e^{\gamma \cdot t(\mathbf{F}, G)},$$

where $Z(\gamma)$ is the normalizing partition function. Given an empirical distribution of n vertex graphs, the object of a maximum likelihood estimation is to find the best parameter γ explaining the observations.

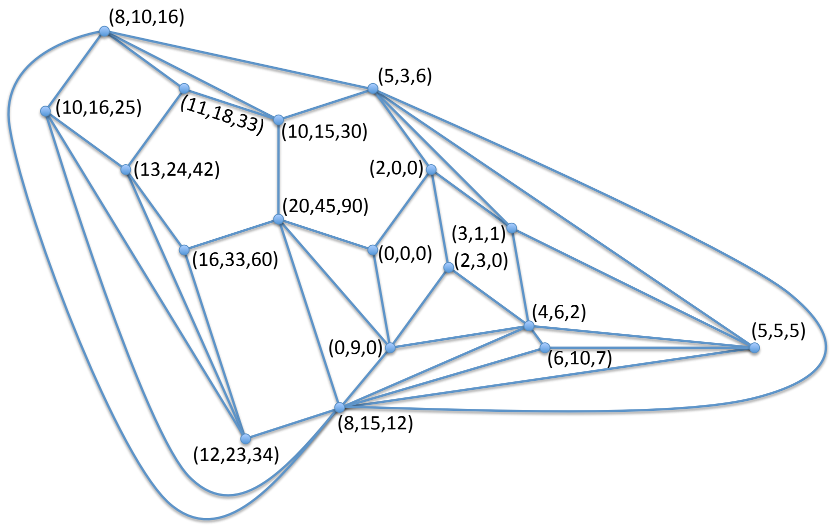

Fig. 2: A combinatorial representation of the vertices and edges of $P_{(K_3,C_4,K_4\setminus e);6}$, indexed by the actual subgraph counts.

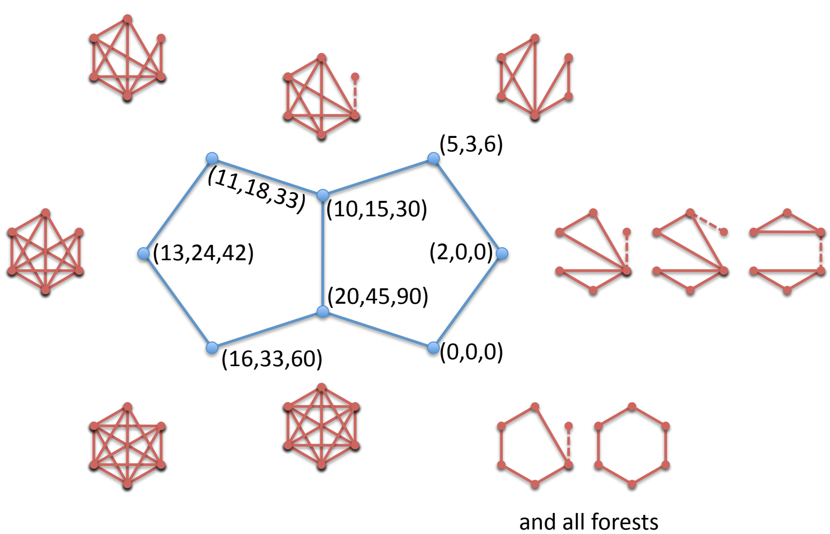

Fig. 3: The graphs underlying the statistics of a piece of the polytope in Figures 1 and 2. Dotted graph edges could be included or not. Recall that the subgraphs counted are $(K_3, C_4, K_4 \setminus e)$.

The graph vector $\mathbf{F}$ is usually determined by the applications of the model, in the social sciences small graphs as stars and triangles are common [16].

Before embarking on more general results about the polytopes from subgraph statistics, we point out some easy propositions about the facets of certain polytopes. Proving them is a good exercise to get acquainted with the polytopes.

Proposition 1.1 *Let $\mathbf{F}$ be a vector of d graphs of order at most n, with no pairs in a subgraph relation. Then $x_i = 0$ defines a facet of $P_{\mathbf{F};n}$ containing $\mathbf{0}$ for all $1 \leq i \leq d$.*

Proposition 1.2 *Let $\mathbf{F}$ be a vector of graphs including the edge K_2. Then the line from $\mathbf{1}$ to $t(\mathbf{F}, K_n \setminus e)$ is on an edge of $P_{\mathbf{F};n}$.*

Proposition 1.3 *Let $\mathbf{F}$ be a vector of d-regular graphs, and let G_i be the complete graph K_n with a star on i edges removed from it. Then the points $t(\mathbf{F}, G_i)$ are on a line in $P_{\mathbf{F};n}$. If this line is on the boundary of $P_{\mathbf{F};n}$, then so it is for any $n' > n$.*

The first non-trivial result follows from a graph reconstruction type argument.

Proposition 1.4 *Let $\mathbf{F}$ be a vector of graphs of order at most n. Then $P_{\mathbf{F};n''} \subseteq P_{\mathbf{F};n'}$ if $n'' \geq n' \geq n$.*

The vertices of the lattice polytope $P^L_{\mathbf{F};n}$ are the actual subgraph counts and not the relative densities. For any lattice polytope P the number of lattice points in kP is the *Ehrhart polynomial* $E_P(k)$ (see chapter 12 of [14]). This is the translation of Proposition 1.4 into the lattice polytope setting.

Proposition 1.5 *Let $\mathbf{F}$ be a vector of graphs of order l. Then $E_{P^L_{\mathbf{F};n''}}(\binom{n'}{l}k) \leq E_{P^L_{\mathbf{F};n'}}(\binom{n''}{l}k)$ for all positive integers k, if $n'' \geq n' \geq l$.*

In the proposition it is required that all graphs in $\mathbf{F}$ are of the same order, and this can partially be generalized by adding isolated vertices to get graphs of the same order. If $\mathbf{F}$ is the graph vector of the path on three vertices and the triangle, then

$$E_{P^L_{\mathbf{F};3}}\left(\binom{4}{3}k\right) = E_{P^L_{\mathbf{F};4}}\left(\binom{3}{3}k\right) = 8k^2 + 6k + 1$$

and

$$E_{P^L_{\mathbf{F};3}}\left(\binom{5}{3}k\right) = 50k^2 + 15k + 1 \geq 48k^2 + 13k + 1 = E_{P^L_{\mathbf{F};5}}\left(\binom{3}{3}k\right).$$

2 The spine of polytopes

Since it's hard to understand the polytopes exactly, we now try to inscribe more accessible polytopes and varieties within them. For a vector $\mathbf{F}$ of m graphs, the *spine* is the generalized moment curve

$$\{(p^{e_1}, p^{e_2}, \ldots, p^{e_m}) \mid 0 \leq p \leq 1\},$$

where e_i is the number of edges in F_i. In the Erdős-Rényi random graph model $\mathcal{G}(n,p)$ edges are included independently with probability p. The expected value of $t(\mathbf{F}, G)$ for $G \in \mathcal{G}(n,p)$ is $(p^{e_1}, p^{e_2}, \ldots, p^{e_m})$, proving the following proposition.

Proposition 2.1 *For any vector $\mathbf{F}$ of graphs of order at most n with $\mathbf{e}$ edges, the spine $\{(p^{e_1}, p^{e_2}, \ldots, p^{e_m}) \mid 0 \leq p \leq 1\}$ is in $P_{\mathbf{F};n}$.*

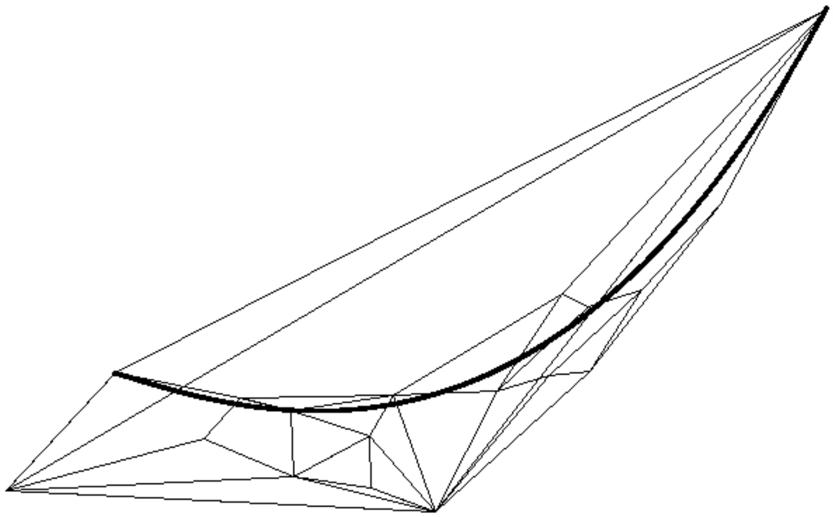

Fig. 4: The polytope in Figure 1 with its spine.

In Figure 4 is the polytope $P_{(K_3,C_4,K_4\setminus e);6)}$ from Figure 1 drawn with its spine. The point of Proposition 2.1 is that the spine is a generalized moment curve inside $P_{\mathbf{F};n}$. The convex hull of a finite number of points on the spine is a cyclic polytope (this can be seen directly by using generalized van der Monde matrices instead of the ordinary one in Ziegler's textbook derivation of the combinatorial structure of cyclic polytopes [21].) This shows that there is a cyclic polytope inscribed in $P_{\mathbf{F};n}$. The convex hull of all of the spine is not a polytope, but its boundary can be algebraically described. In Figure 5 is the spine from Figure 4 drawn with its convex hull. Since the boundary structure of the convex hull in Figure 5 is not very clear from this angle, we include in Figure 6 the same spine with its convex hull, but from another perspective.

2.1 Volumes

Inside our polytopes we have convex hulls of generalized moment curves and their volumes bound the volumes of polytopes from subgraph statistics. For the ordinary moment curve $\{(p, p^2, \cdots, p^d) \mid 0 \leq p \leq 1\}$ the volume of its convex hull was calculated by Karlin and Shapley [12]. An interesting curiosity is that Selberg and Shapley were in Princeton at the same time, and that this volume calculation was the first application of the now famous Selberg integral [8], by then only available in Norwegian and published by Selberg in a magazine for college math teachers [18].

Theorem 2.2 *The volume of the convex hull of the $2m$–dimensional spine*

$$\mathrm{Vol}(\mathrm{conv}\,\{(p^{e_1}, p^{e_2}, \ldots, p^{e_{2m}}) \mid 0 \leq p \leq 1\})$$

is

$$\frac{1}{(2m)!m!}\int_{[0,1]^m} S_\lambda(x_1, x_1, x_2, x_2, \ldots, x_m, x_m) \prod_{0\leq i<j\leq m} (x_i - x_j)^4 \; \mathrm{d}x_1 \cdots \mathrm{d}x_m,$$

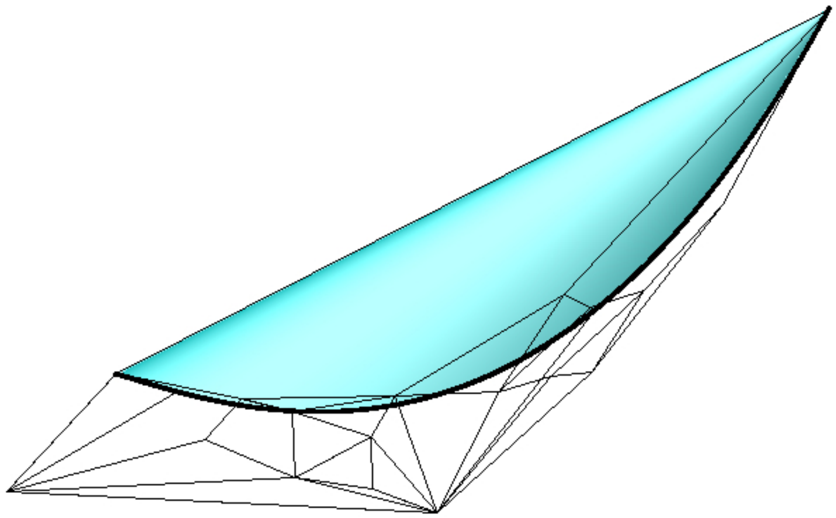

Fig. 5: The spine in Figure 4 drawn with its convex hull.

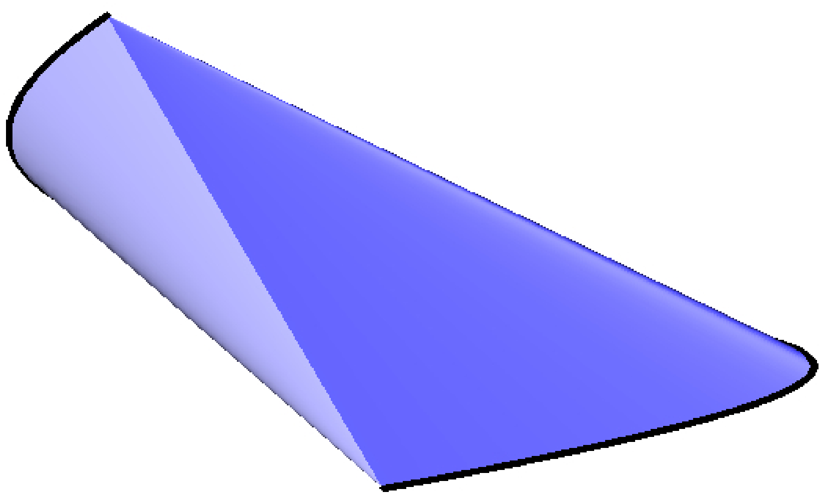

Fig. 6: The spine with its convex hull from Figure 5 drawn from another perspective.

where $\lambda_i = e_i - i + 1$ *in the Schur polynomial* S_λ*, and it is assumed that* $0 < e_1 < e_2 < \cdots < e_{2m}$.

For the definition and properties of the Schur polynomials we refer to Sagan [17]. For applications, it is perhaps most important that they are symmetric with non-negative coefficients, and are easy to compute. We proved Theorem 2.2 by approximating the convex hull by a cyclic polytope with n vertices and show that the volume converges to the integral above. The difference in volume between the cyclic polytope on the n vertices given by $p = 0, 1/(n-1), \ldots, 1$ and the convex hull of the spine is less than $4d(e_1 + e_2 + \cdots + e_d)n^{-1}$. Using the Selberg integral formula [8, 18] we get an explicit form for consecutive e_i's.

Corollary 2.3 *The volume of the convex hull of the* $2m$*–dimensional spine*

$$\mathrm{Vol}(\mathrm{conv}\,\{(p^e, p^{e+1}, \ldots, p^{e+2m-1}) \mid 0 \le p \le 1\})$$

is

$$\frac{1}{(2m)!m!} \prod_{j=0}^{m-1} \frac{(2e+2j)!(2j)!(2+2j)!}{(2e+2(m+j)-1)!2!}.$$

The integral in Theorem 2.2 evaluates, according to our extensive experiments, to combinatorial looking expressions like the one from Selberg's integral formula in Corollary 2.3. Even if special cases can be treated by variations of the Selberg's integral formula [8] there doesn't seem to be a general formula in the literature [9].

Conjecture 2.4 *For any Schur polynomial* S_λ *there is an explicit combinatorial formula for*

$$\int_{[0,1]^m} S_\lambda(x_1, x_1, x_2, x_2, \ldots, x_m, x_m) \prod_{0 \le i < j \le m} (x_i - x_j)^4 \,\mathrm{d}x_1 \cdots \mathrm{d}x_m.$$

For the odd dimensional spines we have similar results.

2.2 Duality

As for polytopes there is a duality theory for convex hulls of algebraic sets [3]. The dual of the convex hull of the moment curve $\{(p, p^2, \ldots, p^n) \mid 0 \le p \le 1\}$ parametrizes the degree n polynomials that are non-negative on the interval $[0, 1]$. The convex hulls of generalized moment curves are inside polytopes from subgraphs statistics, so the polytopes can be used to certify that polynomials are non-negative.

Proposition 2.5 *Let* P *be a polytope containing the generalized moment curve* $\{(p^{e_1}, p^{e_2}, \ldots, p^{e_d}) \mid 0 \le p \le 1\}$*. If* $(c_1, c_2, \ldots, c_d) \cdot v \ge -1$ *for all vertices* v *of* P *then the polynomial* $1 + c_1 x^{e_1} + c_2 x^{e_2} + \cdots + c_d x^{e_d}$ *is non-negative on the interval* $[0, 1]$.

Our running example $P_{(K_3, C_4, K_4 \backslash e);6}$ in Figure 1 is perhaps not the most interesting polytope to certify non-negativity with, but we will use it in an example anyways. The polynomial $p(x) = 1 - \frac{16}{3}x^3 + \frac{11}{2}x^4 - \frac{1}{2}x^5$ is non-negative on $[0, 1]$ since $(-\frac{16}{3}, \frac{11}{2}, -\frac{1}{2}) \cdot v \ge -1$ for all vertices v of $P_{(K_3, C_4, K_4 \backslash e);6}$. Note that the point $(-\frac{16}{3}, \frac{11}{2}, -\frac{1}{2})$ is dual to the facet with vertices $(8/20, 10/45, 16/90)$, $(10/20, 15/45, 30/90)$, $(5/20, 3/45, 6/90)$, which one can find using Figure 2.

3 Curvy zonotopes

In the previous section we used the spine to construct a cyclic polytope inscribed in the polytopes under investigation. We didn't prove any theorem about the ratio between the volume of the polytope and its inscribed cyclic polytope, and perhaps it could be arbitrarily bad for some large graph vectors $\mathbf{F}$. We got the spine as the expected value of subgraph densities for different p in the Erdős-Rényi random graph model $\mathcal{G}(n,p)$. Lovász and Szegedy [13] introduced an extremely general random graph model $\mathcal{G}(n,W)$ where W is any symmetric measurable function from $[0,1]^2$ to $[0,1]$, and hiding in the background of this section, and essential for all our proofs, is the theory of graph limits. But we state our results without this machinery to make them more accessible.

We remark that the route to proving these results, is not to use the graph limit results of Lovász and Szegedy [13] right off. It was realized by Diaconis and Janson [6] and Aldous [2] that the theory of graph limits is a reinterpretation of instances of old very abstract results on exchangeable sequences by Aldous [1] and Hoover [10]. Fortunately there is a new textbook by Kallenberg [11] that covers the relevant probability theory in chapter 7.

Before introducing the curvy zonotopes that generalizes spines, we define the *limit object*

$$P_{\mathbf{F};\infty} = \bigcap_{n' \geq n} P_{\mathbf{F};n'},$$

where n is some integer not smaller than the order of any graph in $\mathbf{F}$. Note that $P_{\mathbf{F};\infty}$ is closed and convex, and should be viewed as the limit of the infinite sequence

$$P_{\mathbf{F};n'} \supseteq P_{\mathbf{F};n'+1} \supseteq P_{\mathbf{F};n'+2} \supseteq \cdots.$$

The first non-trivial result on $P_{\mathbf{F};\infty}$ was proved by Bollobás [4, 5].

Proposition 3.1 *The limit object $P_{(K_2,K_m);\infty}$ is the convex hull of $(1,1)$ and*

$$\left\{ \left(1-\frac{1}{k}, \frac{m!}{k^m}\binom{k}{m}\right) \middle| \, k=1,2,3,\ldots \right\}.$$

Considering Turán's theorem it isn't strange that complete k-partite graphs are around: the number $\frac{m!}{k^m}\binom{k}{m}$ is the limit of $t((K_2,K_m);K_{t,t,\ldots,t})$ as the k-partite graph $K_{t,t,\ldots,t}$ grows larger as $t \to \infty$. The limit object $P_{\mathbf{F};\infty}$ is contained in $P_{\mathbf{F};n}$ in a fairly strong sense.

Theorem 3.2 *Let $\mathbf{x} \neq \mathbf{0}, \mathbf{1}$ be a point in $P_{\mathbf{F};\infty}$ for some vector of graphs $\mathbf{F}$. If n is not smaller than the order of some graph in $\mathbf{F}$, then there is a polytope $P_x \subset P_{\mathbf{F};n}$ with x in its interior.*

Note that the polytope P_x could be of lower dimension than $P_{\mathbf{F};n}$.

Definition 3.3 *Let $\mathbf{F}$ be a vector of d graphs and n a positive integer. The* curvy zonotope *is*

$$Z_{\mathbf{F};n} = \left\{ (p_{F_1;n}(\mathbf{x}), p_{F_2;n}(\mathbf{x}), \ldots, p_{F_d;n}(\mathbf{x}) \mid \mathbf{x} \in [0,1]^{n^2} \right\}$$

where

$$p_{F;n}(x_{11}, x_{12}, \ldots, x_{nn}) = \frac{1}{n^{|F|}} \sum_{\phi: V(F) \to [n]} \prod_{ij \in E(F)} x_{\phi(i)\phi(j)}.$$

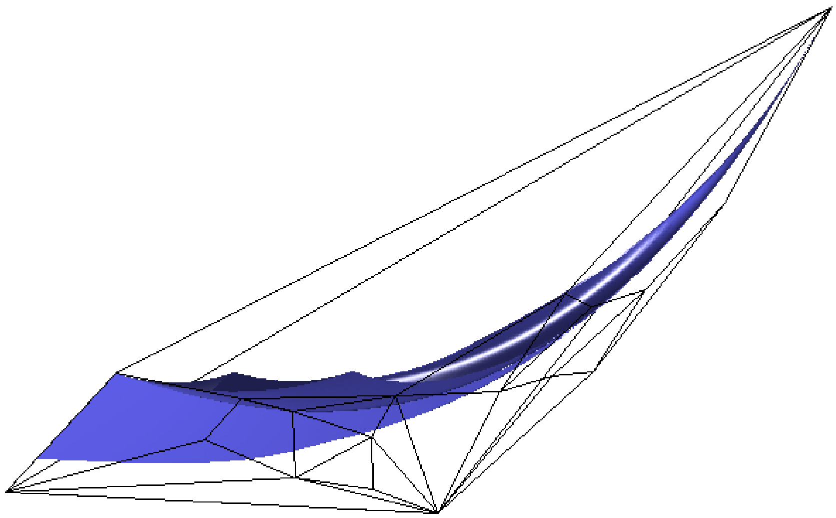

Fig. 7: The curvy zonotope $Z_{(K_3,C_4,K_4\backslash e);2}$ inscribed in $P_{(K_3,C_4,K_4\backslash e);6}$ from Figure 1.

When all the polynomials $p_{F;n}(\mathbf{x})$ are linear $Z_{\mathbf{F};n}$ is a zonotope. But usually they are polynomials and $Z_{\mathbf{F};n}$ is a curvy zonotope. The curvy zonotope $Z_{(K_3,C_4,K_4\backslash e);2}$, which looks like a melted toblerone, is drawn in Figure 7. As drawn in the Figure 7 the curvy zonotope is inscribed in the polytope $P_{(K_3,C_4,K_4\backslash e);6}$. But it can actually be inscribed in the limit object $P_{(K_3,C_4,K_4\backslash e);\infty}$ according to this proposition.

Proposition 3.4 *For any vector* $\mathbf{F}$ *of graphs and positive integer* n*,* $Z_{\mathbf{F};n} \subseteq P_{\mathbf{F};\infty}$.

Together with the following proposition, this shows that the polytopes from subgraph statistics and their limits are full dimensional.

Proposition 3.5 *For any vector* $\mathbf{F}$ *of* d *graphs and positive integer* n*,* $Z_{\mathbf{F};n}$ *is homeomorphic to a* d*-dimensional ball.*

The spine is inside the curvy zonotope, just set $x_{11} = \cdots = x_{nn}$ in all the polynomials to recover it.

The convex hull of the spine is possible to describe very explicit by taking the limit of cyclic polytopes and then Gale's evenness condition define the facets. For curvy zonotopes the convex hull is not as easily described, but the situation is fairly good. From an algebraic geometry perspective, calculating the algebraic boundary of the convex hull of a set parametrized by polynomials, is a nice situation [19, 20].

Since the curvy zonotopes are in the polytopes from subgraph statistics, so are their convex hulls. In Figure 8 is the convex hull of the curvy zonotope in Figure 7. Theorem 3.6 is our main result on curvy zonotopes, it states that their convex hulls converge towards the limit object $P_{\mathbf{F};\infty}$. In the core of the proof is a method for constructing convergent graph sequences from [13] by the weak Szemeredi regularity lemma as in [7].

Theorem 3.6 *Let* $\mathbf{F}$ *be a vector of* d *graphs on at most* e *edges. If* $\varepsilon > 0$ *and* $n > \lceil 2^{1600d^3e^2\varepsilon^{-2}} \rceil$ *then*

$$0 \leq \mathrm{Vol}(P_{\mathbf{F};\infty}) - \mathrm{Vol}(\mathrm{conv}\, Z_{\mathbf{F};n}) < \varepsilon.$$

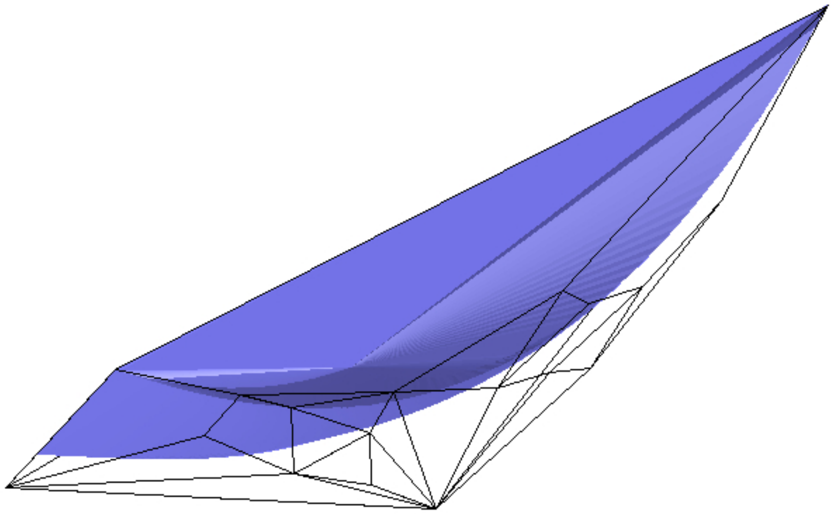

Fig. 8: The convex hull of the curvy zonotope $Z_{(K_3,C_4,K_4\setminus e);2}$ in Figure 7.

We end this section by stating a result for the readers who knows graph limits as presented in [13]: The convex hull of $t(\mathbf{F},W)$ for all symmetric measurable function $W:[0,1]^2\to[0,1]$ is $P_{\mathbf{F};\infty}$.

4 Conjectures about the limit object $P_{\mathbf{F};\infty}$.

We have previously inscribed cyclic polytopes in the limit objects. We will now define another cyclic polytope and conjecture that a particular class of limit objects actually are cyclic polytopes.

As described in Proposition 3.1 and the following discussion, the vertices of $P_{(K_2,K_n);\infty}$ are given by the limits of complete k-equipartite graphs. It is not hard to see that $P_{(K_2,K_n);\infty}$ is a cyclic polytope, and we believe that this is true in a more general setting.

For positive integers $e_1<e_2<\ldots<e_m$ define the *tail* $s^{\mathbf{e}}:[0,1]\to[0,1]^m$ by

$$s_i^{\mathbf{e}}(x)=\prod_{j=1}^{e_i}(1-jx).$$

Proposition 4.1 *The convex hull of any finite set of points on a tail is a cyclic polytope.*

Conjecture 4.2 *Let $e_1<e_2<\ldots<e_m$ be positive integers and $s^{\mathbf{e}}$ their tail. The convex hull of $\mathbf{1}$ and $\{s^{\mathbf{e}}(1/k)\mid k=1,2,3,\ldots\}$ is $P_{(K_{e_1},K_{e_2},\ldots,K_{e_m});\infty}$.*

The conjectured vertex description of $P_{(K_{e_1},K_{e_2},\ldots,K_{e_m});\infty}$ also gives a facet description since it's essentially a cyclic polytope. If we would chop off the vertex $\mathbf{1}$ from the convex hull described in Conjecture 4.2 with an hyperplane, then the remaining convex set would be an ordinary polytope. We believe this is true in the following general form.

Conjecture 4.3 *For any vector* $\mathbf{F}$ *of graphs there is a positive integer* m*, such that for any* $\varepsilon > 0$*, the limit object* $P_{\mathbf{F};\infty}$ *can be chopped down to a polytope with a finite number of vertices, by using* m *hyperplans to remove at most a volume* ε*.*

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Dual combinatorics of zonal polynomials

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Abstract. In this paper we establish a new combinatorial formula for zonal polynomials in terms of power-sums. The proof relies on the sign-reversing involution principle. We deduce from it formulas for zonal characters, which are defined as suitably normalized coefficients in the expansion of zonal polynomials in terms of power-sum symmetric functions. These formulas are analogs of recent developments on irreducible character values of symmetric groups. The existence of such formulas could have been predicted from the work of M. Lassalle who formulated two positivity conjectures for Jack characters, which we prove in the special case of zonal polynomials.

Résumé. Dans cet article, nous établissons une nouvelle formule combinatoire pour les polynômes zonaux en fonction des fonctions puissance. La preuve utilise le principe de l'involution changeant les signes. Nous en déduisons des formules pour les caractères zonaux, qui sont définis comme les coefficients des polynômes zonaux écrits sur la base des fonctions puissance, normalisés de manière appropriée. Ces formules sont des analogues de développements récents sur les caractères du groupe symétrique. L'existence de telles formules aurait pu être prédite à partir des travaux de M. Lassalle, qui a proposé deux conjectures de positivité sur les caractères de Jack, que nous prouvons dans le cas particulier des polynômes zonaux.

Keywords: zonal polynomials, zonal characters, Jack polynomials, Kerov polynomials, Stanley polynomials

The results of this extended abstract correspond to the ones of the full paper [FŚ11b], which will be published elsewhere. However, the method of proof used here is different and new.

1 Introduction

Zonal polynomials were introduced by James [Jam60, Jam61] (who credits also Hua [Hua63]) in order to solve some problems from statistics and multivariate analysis. They quickly became a fundamental tool in this theory as well as in the random matrix theory (an overview can be found in the book of Muirhead [Mui82]). They also play an important role in representation theory: they appear as zonal spherical functions of the pairs $(\mathfrak{S}_{2n}, H_n)$ (where H_n is the hyperoctahedral group) and $(\mathrm{GL}_d(\mathbb{R}), O_d)$, which means that they describe a canonical basis of the algebra of left and right H_n-invariant (resp. O_d-invariant) functions on $\mathfrak{S}_{2n}$ (resp. $\mathrm{GL}_d(\mathbb{R})$). This last property shows that zonal polynomials can be viewed as an analogue of Schur symmetric functions since the latter are zonal spherical functions for the Gelfand pairs $(\mathfrak{S}_n \times \mathfrak{S}_n, \mathfrak{S}_n)$ and $(\mathrm{GL}_d(\mathbb{C}), U_d)$. Besides, many of the properties of Schur functions can be extended to zonal polynomials and this article goes in this direction.

The main result of this article is a new combinatorial formula for zonal polynomials (Theorem 3). Note that, as the latter are a particular case of Jack symmetric functions, there already exists a combinatorial interpretation for them in terms of ribbon tableaux (due to Stanley [Sta89]). But our formula is of different kind: using it, one can describe combinatorially the coefficients of the zonal polynomial Z_λ expanded in the power-sum basis as a function of λ. In more concrete words, the combinatorial objects describing the coefficient of p_μ in Z_λ depend on μ, whereas the statistics on them depend on λ (in Stanley's result it is roughly the opposite). Note that the ring of shifted symmetric functions [OO97] is involved in this *dual* approach [Las08, Proposition 2]. The situation is analogous to recent developments concerning characters of the symmetric groups.

Zonal polynomials are known to be the special case of Jack symmetric functions. We conjecture that there exists some formula for Jack polynomials extending the one we present here for zonal polynomials, but unfortunately we have not been able to find it: even if our proof uses only general properties of Jack polynomials, we really need the special value of the parameter corresponding to zonal polynomials.

The paper is organized as follows. In Section 2 we recall the necessary definitions. In Section 3 we state our main theorem and in Section 4 we present its proof. Then, in Section 5 we sketch a few consequences of this theorem on zonal characters.

2 Notations

In this section, we give some definitions and notations on symmetric functions in general, on Jack polynomials and, in particular, zonal polynomials. We also give a few definitions of pair-partitions, which are the combinatorial objects involved in our formulas.

2.1 Symmetric functions

As much as possible, we use the notations of I.G. Macdonald's book [Mac95].

By definition, a *partition* $\lambda = (\lambda_1, \lambda_2, \dots)$ of the integer n is a non-increasing sequence of non-negative integers of sum n. Its *length* $\ell(\lambda)$ is the number of non-zero terms in the sequence. For $i \geq 1$ we denote by $m_i(\lambda)$ the number of occurrences of i in λ. Finally, let us define $\mathrm{aut}(\lambda) := \prod_{i\geq 1} m_i(\lambda)!$ and $z_\lambda = \mathrm{aut}(\lambda) \cdot \prod_j \lambda_j$.

The ring Λ of symmetric functions has several classical linear bases, all indexed by partitions:

- *monomial symmetric functions*: let us use for monomials the short notation $\mathbf{x}^{\mathbf{v}} = x_1^{v_1} x_2^{v_2} \dots$, then

$$m_\lambda := \sum_{\mathbf{v} \in S_\infty(\lambda)} \mathbf{x}^{\mathbf{v}},$$

  where the sum runs over all vectors $\mathbf{v}$ which are permutations of λ (without multiplicities);

- *power-sum symmetric functions*: they are defined by

$$p_0 = 1, \qquad p_k = \sum_i x_i^k, \qquad p_\mu = \prod_j p_{\mu_j};$$

- *Schur functions* (s_λ): they have several equivalent definitions, one of them will be given in the next paragraph.

In addition to the additive and multiplicative structures, the ring of symmetric functions can be endowed with a scalar product (called *Hall scalar product*), for which:

$$\langle p_\lambda, p_\mu \rangle = z_\lambda \delta_{\lambda,\mu}, \qquad \langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}.$$

2.2 Jack symmetric functions and zonal polynomials

The set of partitions of a given integer n can be endowed with a partial order, called dominance order: by definition,

$$\lambda \preceq \mu \iff \forall_{i \geq 1}\ \lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i.$$

With respect to this order, the matrix of Schur functions in the monomial basis is upper triangular: *i.e.*, if one writes a Schur functions s_λ on the monomial basis, the indices of the monomial symmetric functions with non-zero coefficients are dominated by λ. This triangularity property, together with the fact that they form an orthonormal basis with respect to Hall scalar product, determines entirely the Schur functions. Indeed, one can construct them by choosing any total order refining the dominance order and applying Gram-Schmidt orthogonalization process to the monomial symmetric function basis.

Hall scalar product can be deformed in the following way: $\langle \cdot, \cdot \rangle_\alpha$ is the bilinear form whose values on the power-sum basis is given by

$$\langle p_\lambda,\ p_\mu \rangle_\alpha = z_\lambda\ \alpha^{\ell(\lambda)}\ \delta_{\lambda,\mu}.$$

The following result corresponds to equations (VI,10.13) and (VI,10.14) of Macdonald's book [Mac95].

Proposition 1 *Fix $\alpha > 0$. There exists a (unique) family of functions $J_\lambda^{(\alpha)}$ such that the following three conditions are fulfilled:*

TR *The matrix of $J_\lambda^{(\alpha)}$ written in the monomial basis is upper triangular.*

OR *The $J_\lambda^{(\alpha)}$ form an orthogonal basis of the ring of symmetric functions endowed with the scalar product $\langle \cdot, \cdot \rangle_\alpha$.*

N *The coefficient of $p_1^{|\lambda|}$ in the power sum expansion of $J_\lambda^{(\alpha)}$ is equal to* 1.

These functions are called Jack symmetric functions. When $\alpha = 1$ they correspond, up to multiplication by a scalar, to Schur functions. When $\alpha = 2$ they are called zonal polynomials (because, as we already mentioned in Section 1, they appear in representation theory of some Gelfand pairs, see [Mac95, VII, 2]) and denoted Z_λ.

The uniqueness of such a family is easy to prove: it is essentially the same argument as the one given above for the characterization of Schur functions.

2.3 Pair partitions

Definition 2 *A pair-partition of $[2n] = \{1, \ldots, 2n\}$ is a set of pairwise disjoint two-element sets, such that their (disjoint) union is equal to $[2n]$.*

A pair-partition S can be also seen as an involution as the set $[2n]$. Therefore, we use the notation $S(i)$ for the partner of i, *i.e.* the other element in the same set of S.

For two pair-partitions S_1, S_2 of the same set $[2n]$ we consider the edge-bicolored graph G_{S_1,S_2} defined as follows: its vertex set is $[2n]$, its red (resp. blue) edges are the pairs in S_1 (resp. S_2). As every vertex has degree 2, this graph is a disjoint union of loops. We denote $\mathcal{L}(S_1, S_2)$ the corresponding set-partition of $[2n]$. Note that the length of each loop (or equivalently the size of each block of $\mathcal{L}(S_1, S_2)$) is an even number because of the edge bicoloration.

It is common to associate to a set-partition the integer partition formed by the non-increasing sequence of the part sizes. Here, as we deal only with set partitions Π with parts of even length, we will rather consider the non-increasing sequence of the *half-sizes* of the parts. This integer partition is called the type of the set partition Π and is denoted $\mu(\Pi)$.

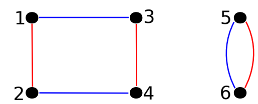
In other words, the type of $\mathcal{L}(S_1,S_2)$ is the partition $\ell_1,\ell_2,\dots$ where $(2\ell_i)_{i\geq 1}$ is the non-increasing sequence of the lengths of the loops of G_{S_1,S_2}. We also define the sign of a couple of pair-partitions as follows:

$$(-1)^{(S_1,S_2)} = (-1)^{(\ell_1-1)+(\ell_2-1)+\cdots} = (-1)^{n-|\mathcal{L}(S_1,S_2)|}.$$

Finally we use the short notations $p_\Pi = p_{\mu(\Pi)}$ and $m_\Pi = m_{\mu(\Pi)}$ for symmetric functions.

Example. We consider

$$S_1 = \big\{\{1,2\},\{3,4\},\{5,6\}\big\}; \qquad S_2 = \big\{\{1,3\},\{2,4\},\{5,6\}\big\}.$$

Then $G_{S_1,S_2} =$  .

In this case, $\mathcal{L}(S_1,S_2) = \big\{\{1,2,3,4\},\{5,6\}\big\}$ and $\mu(\mathcal{L}(S_1,S_2)) = (2,1)$.

3 The main result

3.1 Zonal polynomials in terms of pair-partition

Let $\lambda = (\lambda_1,\lambda_2,\dots)$ be a partition of n and T be a bijective filling of the Young diagram $2\lambda = (2\lambda_1,2\lambda_2,\dots)$ with the elements of $[2n]$. Any pair-partition of $[2n]$ can be viewed as a pairings of the boxes of T. We denote $S(T)$ the pair partition which matches the boxes $(i,2j-1)$ and $(i,2j)$ of T for all values of i and $1\leq j\leq \lambda_i$. Besides, a couple of pair-partitions (S_1,S_2) is called T*-admissible* if each pair of boxes matched by S_2 is in the same row and if S_1 matches elements of the $2j-1$-th column of T with elements of its $2j$-th column.

Theorem 3 *Let λ be a partition and T be a bijective filling of 2λ. One has:*

$$Z_\lambda = \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)}.$$

Example. Let $\lambda = (2,1)$ and $T = \begin{array}{|c|c|c|c|}\hline 1&2&3&4\\ \hline 5&6 \\ \cline{1-2}\end{array}$. Then (S_1,S_2) is T-admissible if and only if:

$$S_1 \in \Big\{ \big\{\{1,2\},\{3,4\},\{5,6\}\big\}, \big\{\{1,6\},\{3,4\},\{2,5\}\big\} \Big\} \quad \text{and} \quad S_2 \in \Big\{ \big\{\{1,2\},\{3,4\},\{5,6\}\big\}, \big\{\{1,3\},\{2,4\},\{5,6\}\big\}, \big\{\{1,4\},\{2,3\},\{5,6\}\big\} \Big\}.$$

The first possible value of S_1 gives $(-1)^{(S(T),S_1)} = 1$ and the types of $\mathcal{L}(S_1,S_2)$ for the three possible values of S_2 are, respectively, $(1,1,1)$, $(2,1)$ and $(2,1)$. For the second value of S_1 one has $(-1)^{(S(T),S_1)} = -1$ and the types of the corresponding set-partitions $\mathcal{L}(S_1,S_2)$ are, respectively, $(2,1)$, (3) and (3).

Finally, one obtains $Z_{(2,1)} = p_{(1,1,1)} + p_{(2,1)} - 2p_{(3)}$.

Remark. This theorem is an analogue of a known result on Schur symmetric functions:

$$\frac{n!\cdot s_\lambda}{\dim(\lambda)} = \sum (-1)^{|\sigma_1|}\, p_{\mathrm{type}(\sigma_1\circ\sigma_2)}, \tag{1}$$

where the sum runs over pairs of permutations (σ_1, σ_2) of the boxes of the diagram λ such that σ_1 (resp. σ_2) preserves the columns (resp. the rows) of λ and $\mathrm{type}(\sigma_1 \circ \sigma_2)$ denotes the partition describing the lengths of the cycles of $\sigma_1 \circ \sigma_2$. This formula is a consequence of the explicit construction of the representation associated to λ via the Young symmetrizer. For a detailed proof, see [FŚ11a, Theorem 4]. In [Han88], the authors tries unsuccessfully to generalize it to Jack polynomials by introducing some statistics on couples of permutations. Our result shows, that, at least for $\alpha = 2$, a natural way to generalize it consists in using other combinatorial objects than permutations.

3.2 Application

Let us look at the coefficients $\theta_\nu^{(\alpha)}(\lambda)$ of the power sum expansion of Jack polynomials:

$$J_\lambda^{(\alpha)} = \sum \theta_\nu^{(\alpha)}(\lambda)\, p_\nu.$$

In the case $\alpha = 1$ we obtain, up to a multiplicative constant, the irreducible characters of the symmetric groups. By analogy, in the general case, we call these quantities *Jack characters* (or *zonal characters* in the case $\alpha = 2$). With the right choice of the normalizing constant, namely

$$\Sigma_\nu^{(\alpha)}(\lambda) = \binom{|\lambda| - |\nu| + m_1(\nu)}{m_1(\nu)} z_\nu\, \theta^{(\alpha)}_{\nu, 1^{|\lambda|-|\nu|}}(\lambda), \tag{2}$$

they are α-shifted symmetric functions of the partition λ.

Recently, M. Lassalle stated two positivity conjectures [Las08, Las09] suggesting the existence of some combinatorial description of this coefficient. Theorem 3 can be used to obtain such a combinatorial description in the particular case of zonal polynomials. Note that this particular case is especially interesting because the quantities $\theta_\nu^{(2)}(\lambda)$ have representation-theoretical interpretations (see [Mac95, VII,2]). We give precise statements in Section 5.

3.3 Combinatorial interpretation

The triplet of pair-partitions $(S(T), S_1, S_2)$ can be seen as a map (bipartite graph drawn on a — possibly non-orientable — two-dimensional, borderless surface), see [GJ96]. The statistics considered in Theorem 3, Theorem 9, Theorem 10 and Theorem 11 are natural in this context: $\mathcal{L}(S(T), S_1)$ and $\mathcal{L}(S(T), S_2)$ correspond respectively to white and black vertices, while $\mathcal{L}(S_1, S_2)$ corresponds to the faces.

Therefore, our results can be rephrased in a natural way using graphs on surfaces. For more details and precise statements see the complete version of this paper [FŚ11b].

3.4 Links between pair-partitions and zonal characters

It should be stressed that a previous link between pair-partitions and zonal characters can be found in the work of Goulden and Jackson [GJ96]. But their result goes in the opposite direction than ours: they count pair-partitions using zonal characters, while we express zonal characters using pair-partitions. The same picture exists for permutations and usual characters. It would be nice to understand the link between these two dual approaches.

4 Proof of Theorem 3

From now on we fix $\alpha = 2$. Let us define Y_λ as the right-hand side of Theorem 3 (it obviously does not depend on the choice of T). In this section we prove that Y_λ satisfies the three properties of Proposition 1. In this way we prove Theorem 3 and we also give an alternative proof of Proposition 1 in the special case $\alpha = 2$.

4.1 Combinatorial lemmas

The proof relies on three simple combinatorial lemmas.

Lemma 4 *Let λ and μ be two partitions of the same integer n. If λ does not dominate μ then for any bijective fillings T_λ and T_μ of the diagrams λ and μ with integers in $[n]$ there exist two elements i and j which are in the same row of T_μ and in the same column of T_λ.*

Proof: Let us replace in the tableau T_λ each number i by the index of the row of T_μ containing i. We have now a filling T'_λ of λ in which any number k appears μ_k times and we have to prove the existence of a column with some number appearing (at least) twice. Of course, reordering each column of T'_λ in the increasing order does not change anything. As λ does not dominate μ, there exists an integer i such that:

$$\mu_1 + \cdots + \mu_i > \lambda_1 + \cdots + \lambda_i.$$

But $\mu_1 + \cdots + \mu_i$ is exactly the number of integers smaller than $i+1$ in T'_λ. There is not enough room to fit all these numbers in the first i rows of λ, therefore there exists a box $\Box$ in the other rows of λ which contains a number which is smaller or equal than i. The column containing $\Box$ has therefore at least $i+1$ boxes with entries smaller or equal to i. The pigeon-hole principle shows that this column must contain a repeated entry. $\Box$

Consider the following action of S_{2n} on the pair-partitions of $[2n]$: if σ is a permutation in S_{2n} and S a pair-partition of $[2n]$, we denote by $\sigma(S)$ the pair partition such that $\{\sigma(i), \sigma(j)\}$ is a part of $\sigma(S)$ if and only if $\{i, j\}$ is a part of S. Of course, the graphs G_{S_1,S_2} and $G_{\sigma(S_1),\sigma(S_2)}$ are isomorphic thus the corresponding set of loops have the same type. Conversely:

Lemma 5 *The couples (S_1, S_2) such that $\mu(\mathcal{L}(S_1, S_2)) = \nu$ form exactly one orbit under the diagonal action of the symmetric group S_{2n}. Moreover, there are exactly $\frac{(2n)!}{z_\nu 2^{\ell(\nu)}}$ of them.*

Proof: Let us consider two couples (S_1, S_2) and (S'_1, S'_2), such that both graphs $G := G_{S_1,S_2}$ and $G' := G_{S'_1,S'_2}$ are collections of loops of lengths $2\nu_1, 2\nu_2 \dots$. These two graphs are isomorphic as edge-bicolored graphs. Let φ be any isomorphism of them. As it sends vertices of G on vertices of G', it can be seen as a permutation in S_{2n}. As it sends red (resp. blue) edges of G on red (resp. blue) edges of G', one has: $\varphi(S_1) = S'_1$ (resp. $\varphi(S_2) = S'_2$). Thus all couples of pair-partitions of type ν are in the same orbit.

We will prove now that the size of the centralizer of a fixed couple (S_1, S_2) is equal to $z_\nu 2^{\ell(\nu)} = \prod_i m_i(\nu)!(2i)^{m_i(\nu)}$. Indeed, an element in the centralizer is entirely determined by the image of one given point in each loop of $\mathcal{L}(S_1, S_2)$. Moreover, the image of an element in a loop of length 2ℓ must be in a loop of the same length 2ℓ and the images of two elements in two different loops must be in two different loops. $\Box$

Lemma 6 *Let λ be a partition of n, let T be a bijective filling of 2λ and let S_1 be a pair-partition matching elements of the $2j-1$-th column of T with elements in its $2j$-th column. If $\tau = (i\ j)$ with i and j in the same column of T, and $S = S(T)$ then*

$$(-1)^{(S,\tau(S_1))} = (-1)^{(S,S_1)+1}.$$

Proof: Let us first remark that $G_{S,\tau(S_1)}$ can be obtained from G_{S,S_1} by replacing the blue edges $\{S_1(i), i\}$ and $\{S_1(j), j\}$ by $\{S_1(i), j\}$ and $\{S_1(j), i\}$. We distinguish two cases:

- if i and j are in different loops in G_{S,S_1} then these loops are unified into one new loop in $G_{S,\tau(S_1)}$. Thus $\big|\mathcal{L}(S,\tau(S_1))\big| = \big|\mathcal{L}(S,S_1)\big| - 1$ and the lemma holds in this case;
- suppose i and j are in the same loop in G_{S,S_1}. As i and j are in the same column of T and both S and S_1 match elements of the $2j-1$-th column of T with elements of its $2j$-th column, the distance between i and j is even, where the distance is regarded as the number of steps within one loop. It is easy to check in this case that the loop containing i and j in G_{S,S_1} is split into two in $G_{S,\tau(S_1)}$. Therefore, $\big|\mathcal{L}(S,\tau(S_1))\big| = \big|\mathcal{L}(S,S_1)\big| + 1$ and the lemma also holds in this case. □

4.2 Triangularity

In this paragraph we prove that the functions Y_λ satisfy property **TR**.

Let us fix a bijective filling T of 2λ and set $S = S(T)$. The first step of the proof consist of writing

$$Y_\lambda = \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S,S_1)}\, p_{\mathcal{L}(S_1,S_2)}$$

in the monomial basis. In order to do that we will write each $p_{\mathcal{L}(S_1,S_2)}$ on this basis.

Recall that, if Π and Π' are two set partitions, we say that Π is finer than Π' (and write $\Pi \leq \Pi'$) if each part of Π' is the union of some parts of Π.

Lemma 7 *Let S_1, S_2 be two pair partitions of $\{1, \dots, 2n\}$. Then*

$$p_{\mathcal{L}(S_1,S_2)} = \sum_{\Pi \geq \mathcal{L}(S_1,S_2)} \mathrm{aut}\big(\mu(\Pi)\big)\, m_\Pi.$$

Proof: It is a consequence of [Mac95, Chapter I, equation (6.9)]. □

Using this lemma, one can write Y_λ in the monomial basis:

$$\begin{aligned} Y_\lambda &= \sum_{(S_1,S_2)\ T\text{-admissible}} \ \sum_{\Pi \geq \mathcal{L}(S_1,S_2)} (-1)^{(S,S_1)}\ \mathrm{aut}\big(\mu(\Pi)\big)\, m_\Pi \\ &= \sum_{\substack{\Pi\text{set-}\\ \text{partition of } [2n]}} \left[\sum_{\substack{(S_1,S_2)\ T\text{-admissible} \\ \mathcal{L}(S_1,S_2) \leq \Pi}} (-1)^{(S,S_1)} \right] \mathrm{aut}\big(\mu(\Pi)\big)\, m_\Pi. \end{aligned} \tag{3}$$

Remark. The inequality $\mathcal{L}(S_1, S_2) \leq \Pi$ is equivalent to the following local condition: the partners of any integer i in the pair-partitions S_1 and S_2 are in the same part of Π as i.

Let us assume now that $\mu(\Pi)$ is not dominated by λ. By Lemma 4, there exist i and j in $[2n]$, which are in the same part of Π and in the same column of T. Let us denote $\tau = (i\ j) \in S_{2n}$. Then:

- $(\tau(S_1), S_2)$ is T-admissible $\iff$ (S_1, S_2) is T-admissible;
- $\mathcal{L}(\tau(S_1), S_2) \leq \Pi \iff \mathcal{L}(S_1, S_2) \leq \Pi$ (because of the remark above);
- $(-1)^{(S,\tau(S_1))} = (-1)^{(S,S_1)+1}$ (see Lemma 6).

Thus, $(S_1, S_2) \mapsto (\tau(S_1), S_2)$ is a sign reversing involution proving that the expression in the bracket in the right-hand side of (3) is equal to zero as soon as $\mu(\Pi)$ is not dominated by λ.

This ends the proof of property **TR**.

4.3 Orthogonality

In this section we prove that the functions Y_λ satisfy property **OR**.

As the definition of Y_λ does not depend on the choice of the bijective filling $T \in F(\lambda)$ ($F(\lambda)$ is by definition the set of bijective fillings of 2λ), one can write:

$$\begin{aligned}(2n)!\, Y_\lambda &= \sum_{T \in F(\lambda)} \sum_{(S_1,S_2)\ T\text{-admissible}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)} \\ &= \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}\, p_{\mathcal{L}(S_1,S_2)}. \end{aligned} \tag{4}$$

Let ν be a partition of n. A consequence of Lemma 5 is that each of the $\frac{(2n)!}{z_\nu 2^{\ell(\nu)}}$ couples of pair-partitions (S_1, S_2) such that $\mu(\mathcal{L}(S_1, S_2)) = \nu$ has the same contribution to the right-hand side of the previous equation. Therefore, if we fix such a couple (S_1, S_2), the coefficient of p_ν in Y_λ is given by:

$$(2n)!\, [p_\nu] Y_\lambda = \frac{(2n)!}{z_\nu 2^{\ell(\nu)}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}. \tag{5}$$

But, using equation (4) for Y_λ, one has the following expression for the scalar product:

$$(2n)!\, \langle Y_\lambda, Y_{\lambda'} \rangle_2 = \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} (-1)^{(S(T),S_1)}\, z_\nu\, 2^{\ell(\nu)}\, [p_\nu] Y_{\lambda'},$$

where $\nu = \mu\big(\mathcal{L}(S_1, S_2)\big)$ depends on the summation index. Now, $[p_\nu]Y_{\lambda'}$ can be evaluated *via* equation (5) and we obtain:

$$(2n)!\, \langle Y_\lambda, Y_{\lambda'} \rangle_2 = \sum_{\substack{S_1,S_2 \\ \text{pair-partitions}}} \sum_{\substack{T \in F(\lambda) \text{ s.t.} \\ (S_1,S_2)\ T\text{-admissible}}} \sum_{\substack{T' \in F(\lambda') \text{ s.t.} \\ (S_1,S_2)\ T'\text{-admissible}}} (-1)^{(S(T),S_1)}\, (-1)^{(S(T'),S_1)}. \tag{6}$$

Let us assume $\lambda \neq \lambda'$. One can assume without loss of generality that λ is not dominated by λ'; otherwise we swap λ and λ'. By Lemma 4 there exist two elements i and j which are in the same row of T' and in the same column of T. We choose the smallest ones in the lexicographic order. We denote by $\tau_{T,T'}$ the transposition $(i\ j)$ and we consider the function:

$$\varphi : (S_1, S_2, T, T') \mapsto (\tau_{T,T'}(S_1), S_2, T, T' \circ \tau_{T,T'}),$$

where $T' \circ \tau_{T,T'}$ is the tableau T' with the locations of i and j interchanged. It is easy to check that:

- φ is an involution (because $T' \circ \tau_{T,T'}$ and T' have the same rows and, therefore, the unordered pair $\{i,j\}$ is the same for the two pairs of fillings);
- (S_1, S_2) is T-admissible if and only if $(\tau_{T,T'}(S_1), S_2)$ is T-admissible;
- (S_1, S_2) is T'-admissible if and only if $(\tau_{T,T'}(S_1), S_2)$ is $T' \circ \tau_{T,T'}$-admissible;
- $(-1)^{(S(T),\tau_{T,T'}(S_1))} = (-1)^{(S(T),S_1)+1}$ (see Lemma 6);
- $(-1)^{\left(S(T'\circ\tau_{T,T'}),\tau_{T,T'}(S_1)\right)} = (-1)^{(S(T'),S_1)}$ (because $S(T' \circ \tau_{T,T'}) = \tau_{T,T'}(S(T'))$).

The contributions of the quadruplets (S_1, S_2, T, T') and $\varphi(S_1, S_2, T, T')$ in equation (6) cancel, which proves that the functions Y_λ are orthogonal with respect to the deformed Hall scalar product $\langle \cdot, \cdot \rangle_2$.

4.4 Normalization

In this paragraph we prove that the functions Y_λ satisfy property **N**.

Let λ be a partition of n. We fix a bijective filling T of 2λ. We have to find the T-admissible couples (S_1, S_2) with $\mu(\mathcal{L}(S_1, S_2)) = 1^n$. The last condition is equivalent to $S_1 = S_2$ and a couple (S_1, S_1) is T-admissible if and only if $S_1 = S(T)$. In this case, $(-1)^{(S(T),S_1)} = 1$. Therefore, $[p_{1^n}]Y_\lambda = 1$ and Theorem 3 is proved. □

5 Combinatorial formulas for zonal characters

In this section we give new formulas for zonal characters (the latter are defined in Section 3.2), which are analogues of some recent formulas for characters of the symmetric groups and establish some particular cases of some conjectures of Lassalle. These formulas can be derived relatively easily from our main result, as done in the complete version of this paper [FŚ11b].

5.1 *Via* numbers of embeddings

Let S_0, S_1, S_2 be three pair-partitions of the set $[2k]$. We consider the following function on the set of Young diagrams:

Definition 8 *$N^{(1)}_{S_0,S_1,S_2}(\lambda)$ is the number of functions f from $[2k]$ to the boxes of the Young diagram λ such that for every $l \in [2k]$:*

(Q0) $f(l) = f(S_0(l))$, in other words f can be viewed as a function on the set of pairs constituting S_0;

(Q1) $f(l)$ and $f(S_1(l))$ are in the same column;

(Q2) $f(l)$ and $f(S_2(l))$ are in the same row.

Theorem 9 *Let ν be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of the set $[2k]$ such that $\mu\big(\mathcal{L}(S_1, S_2)\big) = \nu$. Then one has the following equality between functions on the set of Young diagrams:*

$$\Sigma^{(2)}_\nu = \frac{1}{2^{\ell(\nu)}} \sum_{S_0} (-1)^{(S_0,S_1)}\, 2^{|\mathcal{L}(S_0,S_1)|}\, N^{(1)}_{S_0,S_1,S_2}, \tag{7}$$

where the sum runs over pair-partitions of $[2k]$.

5.2 In terms of Stanley's coordinates

The notion of Stanley's coordinates was introduced by Stanley [Sta04] who found a nice formula for normalized irreducible character values of the symmetric group corresponding to rectangular Young diagrams. In order to generalize this result, he defined, given two sequences $\mathbf{p}$ and $\mathbf{q}$ of positive integers of same size ($\mathbf{q}$ being non-increasing), the partition:

$$\mathbf{p} \times \mathbf{q} = (\underbrace{q_1, \dots, q_1}_{p_1 \text{ times}}, \dots, \underbrace{q_l, \dots, q_l}_{p_l \text{ times}}).$$

Then he suggested to consider the quantity $\Sigma_\nu^{(1)}(\mathbf{p} \times \mathbf{q})$ as a polynomial in $\mathbf{p}$ and $\mathbf{q}$. An explicit combinatorial interpretation of the coefficients was conjectured in [Sta06] and proved in [Fér10].

It is easy to deduce from the above expansion of $\Sigma_\nu^{(2)}$ in terms of the N functions a combinatorial description of the polynomial $\Sigma_\nu^{(2)}(\mathbf{p} \times \mathbf{q})$.

Theorem 10 *Let ν be a partition of the integer k and S_1, S_2 be two fixed pair-partitions of $[2k]$ such that $\mu\big(\mathcal{L}(S_1, S_2)\big) = \nu$. Then, one has:*

$$\Sigma_\nu^{(2)}(\mathbf{p} \times \mathbf{q}) = \frac{(-1)^k}{2^{\ell(\nu)}} \sum_{S_0} \left[\sum_{\phi : \mathcal{L}(S_1,S_0) \to \mathbb{N}^\star} \prod_{l \in \mathcal{L}(S_1,S_0)} (p_{\varphi(l)}) \cdot \prod_{l' \in \mathcal{L}(S_2,S_0)} (-2 q_{\psi(l')}) \right] \tag{8}$$

where $\psi(l') := \max_l \varphi(w)$ with l running over the loops of $\mathcal{L}(S_0, S_1)$ having at least one element in common with l'.

5.3 In terms of free cumulants

The following observation is due to Lassalle [Las09]. Let $k \geq 1$ be a fixed integer and let α be fixed. Since $\Sigma_k^{(\alpha)}$ is an α-shifted symmetric function and the anisotropic free cumulants $(R_l^{(\alpha)})_{l \geq 2}$ form an algebraic basis of the ring of α-shifted symmetric functions (see [Las09] for their definition), there exists a polynomial $K_k^{(\alpha)}$ such that, for any Young diagram λ,

$$\Sigma_k^{(\alpha)}(\lambda) = K_k^{(\alpha)}\big(R_2^{(\alpha)}(\lambda), R_3^{(\alpha)}(\lambda), \dots\big).$$

This polynomial is called *Jack Kerov polynomial*.

Thus Jack Kerov polynomials express Jack characters on cycles in terms of free cumulants. For more complicated conjugacy classes it turns out to be more convenient to express not directly the characters $\Sigma_{(k_1,\dots,k_\ell)}^{(\alpha)}$ but rather *cumulant*

$$(-1)^{\ell-1} \kappa^{\text{id}}(\Sigma_{k_1}^{(\alpha)}, \dots, \Sigma_{k_\ell}^{(\alpha)}). \tag{9}$$

This gives rise to *generalized Jack Kerov polynomials* $K_{(k_1,\dots,k_\ell)}^{(\alpha)}$. In the classical context $\alpha = 1$ these quantities have been introduced by one of us and Rattan [RŚ08]; in the Jack case they have been studied by Lassalle [Las09]. We skip the definitions and refer to the above papers for details since generalized Kerov polynomials are not of central interest for this paper.

Using the technology developed in [DFŚ10], one can deduce from Theorem 9 a combinatorial interpretation for the coefficients of generalized zonal Kerov polynomials (*i.e.* generalized Jack Kerov polynomials in the case $\alpha = 2$):

Theorem 11 *Let ν be a partition of an integer k and S_1, S_2 be two fixed pair-partitions of $[2k]$ such that $\mu\big(\mathcal{L}(S_1,S_2)\big)=\nu$. Consider also a sequence $s_2, s_3, \dots$ of non-negative integers with only finitely many non-zero elements. The rescaled coefficient*

$$(-2)^{\ell(\nu)}(-1)^{|\nu|+2s_2+3s_3+\cdots}\left[\left(R_2^{(2)}\right)^{s_2}\left(R_3^{(2)}\right)^{s_3}\cdots\right]K_\nu^{(2)}$$

of the (generalized) zonal Kerov polynomial is equal to the number of couples (S_0,q) with the following properties:

(a) the graph of vertex set $[2n]$ obtained by drawing an edge between i and j if they are in the same part of S_0, S_1 or S_2 is connected;

(b) the number of loops in G_{S_0,S_1} is equal to $s_2+s_3+\cdots$;

(c) the number of loops in G_{S_0,S_2} is equal to $s_2+2s_3+3s_4+\cdots$;

(d) q is a function from $\mathcal{L}(S_0,S_1)$ to the set $\{2,3,\dots\}$; we require that each number $i\in\{2,3,\dots\}$ is used exactly s_i times;

(e) for every subset $A\subset\mathcal{L}(S_0,S_1)$ of loops which is nontrivial (i.e., $A\neq\emptyset$ and $A\neq\mathcal{L}(S_0,S_1)$), there are more than $\sum_{v\in A}\big(q(v)-1\big)$ loops in $\mathcal{L}(S_0,S_2)$ which have a non-empty intersection with at least one loop from A.

6 Conclusion

Our result on zonal polynomials and zonal characters are analogs of recent developments on Schur polynomials and characters of symmetric groups. Lassalle's work suggests that this kind of results hold for generic values of the parameter α, but we have not been able (yet) to generalize our argument.

Note that, unlike the proof in the full version of the paper [FŚ11b], which relies on the representation theoretical interpretation of zonal polynomials, the one presented here only uses the fact that zonal polynomials are special cases of Jack polynomials. So, if we manage to guess the good combinatorial objects for other values of α, a similar proof technique could be used.

Another approach to the general case is proposed in [DŚ11]: M. Dołęga and the second author give a combinatorial condition for an expression of the kind of Therorem 9 to be an α-shifted symmetric function. As $\Sigma_\nu^{(\alpha)}(\lambda)$ belongs to this class, this could help guessing a generalization of Therorem 9.

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A Littlewood-Richardson type rule for row-strict quasisymmetric Schur functions

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Abstract. We establish several properties of an algorithm defined by Mason and Remmel (2010) which inserts a positive integer into a row-strict composition tableau. These properties lead to a Littlewood-Richardson type rule for expanding the product of a row-strict quasisymmetric Schur function and a symmetric Schur function in terms of row-strict quasisymmetric Schur functions.

Résumé. Nous établissons plusieurs propriétés d'un algorithme défini par Mason et Remmel (2010), qui insère un entier positif dans un tableau dont la forme est une composition, avec ordre strict sur les lignes (row-strict). Ces propriétés conduisent à une règle de type Littlewood-Richardson pour étendre le produit d'une fonction de Schur quasi-symétrique "row-strict" et d'une fonction de Schur symétrique en termes de fonctions de Schur quasi-symétriques "row-strict".

Keywords: Littlewood-Richardson rule, quasisymmetric function, Schur function

1 Introduction

Quasisymmetric functions were defined by Gessel in [5] where he developed many of their properties, although quasisymmetric functions had already appeared in earlier work of Stanley [14]. Since their introduction, quasisymmetric functions have become of increasing importance. They have appeared in such areas of mathematics as representation theory [9], symmetric function theory [3], and combinatorial Hopf algebras [1].

In [7], the authors define a new basis $\mathcal{CS}_\alpha$ for the algebra *QSym* of quasisymmetric functions, where α is a sequence of positive integers called a *strong composition*. In a fixed number of variables, the functions $\mathcal{CS}_\alpha$ are defined to be a certain positive integral sum of Demazure atoms. Demazure atoms first appeared in [10] and later were characterized as specializations of nonsymmetric Macdonald polynomials when $q = t = 0$ [12]. A subset of the functions $\mathcal{CS}_\alpha$, in a finite number of variables, were shown in [11] to give a basis of the coinvariant space of quasisymmetric polynomials, thus proving a conjecture of Bergeron and Reutenauer in [2].

In [8] the authors give a Littlewood-Richardson type rule for expanding the product $\mathcal{CS}_\alpha s_\lambda$, where s_λ is the symmetric Schur function, as a nonnegative integral sum of the functions $\mathcal{CS}_\beta$. This rule relied on their

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combinatorial definition for $\mathcal{CS}_\alpha$ as the generating function of column-strict composition tableaux, which are certain fillings of strong composition shape α with positive integers. These column-strict composition tableaux are defined by imposing three relations among certain sets of entries in the fillings of α. The proof of the Littlewood-Richardson type rule in [8] utilized an analogue of Schensted insertion on tableaux, which is an algorithm in classical symmetric function theory which inserts a positive integer b into a tableau T.

In [13], the authors provide a row-strict analogue of column-strict composition tableaux; specifically they interchange the roles of weak and strict in each of the three relations mentioned above. One of these relations requires the fillings to decrease strictly in each row, thus the name row-strict composition tableaux. Also contained in [13] is an insertion algorithm which inserts a positive integer b into a row-strict composition tableau, producing a new row-strict composition tableau.

This article establishes several new properties of the insertion algorithm given in [13]. If we define $\mathcal{RS}_\alpha$ to be the generating function of row-strict composition tableaux of shape α, then the properties of this algorithm lead directly to a Littlewood-Richardson type rule for expanding the product $\mathcal{RS}_\alpha s_\lambda$ as a nonnegative integral sum of the function $\mathcal{RS}_\beta$. The combinatorics of this rule contain many similarities to the classical Littlewood-Richardson rule for multiplying two Schur functions, see [4] for example.

1.1 Compositions and reverse lattice words

A *strong composition* $\alpha = (\alpha_1, \ldots, \alpha_k)$ with k parts is a sequence of positive integers. A *weak composition* $\gamma = (\gamma_1, \ldots, \gamma_k)$ with k parts is a sequence of nonnegative integers. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ with k parts is a weakly decreasing sequence of positive integers. Let $\lambda^* := (\lambda_k, \lambda_{k-1}, \ldots, \lambda_1)$ be the *reverse of* λ, and let λ^t denote the *transpose of* λ. Denote by $\widetilde{\alpha}$ the unique partition obtained by placing the parts of α in weakly decreasing order. Denote by γ^+ the unique strong composition obtained by removing the zero parts of γ. For any sequence $\beta = (\beta_1, \ldots, \beta_k)$ with k parts let $\ell(\beta) := k$ be the *length of* β. For γ and β arbitrary (possibly weak) compositions of the same length k we say γ is *contained in* β, denoted $\gamma \subseteq \beta$, if $\gamma_i \leqslant \beta_i$ for all $1 \leqslant i \leqslant k$. For α and β strong compositions, we say β is a *refinement* of α, denote $\beta \preceq \alpha$, if α can be obtained by summing consecutive parts of β. That is, $\beta \preceq \alpha$ if $\alpha_1 = \beta_1 + \cdots + \beta_i$, $\alpha_2 = \beta_{i+1} + \cdots + \beta_j$, $\alpha_3 = \beta_{j+1} + \cdots + \beta_m$, and so on.

A finite sequence $w = w_1 w_2 \cdots w_n$ of positive integers with largest part size m is called a *reverse lattice word* if in every prefix of w there are at least as many i's as $(i-1)$'s for each $1 < i \leqslant m$. The *content* of a word w is the sequence $\mathrm{cont}(w) = (\mathrm{cont}(w)_1, \ldots, \mathrm{cont}(w)_m)$ with the property that $\mathrm{cont}(w)_i$ equals the number of times i appears in w. A reverse lattice word is called *regular* if $\mathrm{cont}(w)_1 \neq 0$. Note that if w is a regular reverse lattice word, then $\mathrm{cont}(w) = \lambda^*$ for some partition λ.

1.2 Diagrams and fillings

To any sequence β of nonnegative integers we may associate a diagram, also denoted β, of left justified boxes with β_i boxes in the ith row from the top. In the case $\beta = \lambda$ is a partition, the diagram of λ is the usual Ferrers diagram in English notation. Given a diagram β, let (i, j) denote the box in the ith row and jth column.

Given two sequences γ and β of the same length k such that $\gamma \subseteq \beta$, define the *skew diagram* β/γ to be the array of boxes that are in β and not in γ. The boxes in γ are called the *skewed boxes.* For each skew diagram contained in this article, an extra column with k boxes will be added strictly to the left of each existing column so that the ith row of β/γ has $(\beta_i + 1) - (\gamma_i + 1)$ boxes. This new column will be called the 0*th column.*

A *filling* U of a diagram β is an assignment of positive integers to the boxes of β. Given a filling U of β, let $U(i,j)$ be the entry in the box (i,j). A *reverse row-strict tableau*, or just *tableau*, T is a filling of partition shape λ such that each row strictly decreases when read left to right and each column weakly decreases when read top to bottom. If λ is a partition with $\lambda_1 = m$, then let T_λ be the tableau of shape λ which has the entire ith column filled with the entry $(m+1-i)$ for all $1 \leqslant i \leqslant m$.

A filling of a skew diagram β/γ is an assignment of positive integers to the boxes that are in β and not in γ. We follow the convention that each box in the 0th column and each skewed box is assigned a virtual ∞ symbol. Once filled, two such boxes in the same row are defined to strictly decrease, while two such boxes in the same column are defined to be equal.

The *column reading order* of a (possibly skew) diagram is the total order $<_{col}$ on its boxes where $(i,j) <_{col} (i',j')$ if $j < j'$ or ($j = j'$ and $i > i'$). This is the total order obtained by reading the boxes from bottom to top in each column, starting with the left-most column and working rightwards. The *column reading word* of a (possibly skew) filling U is the sequence of integers $w_{col}(U)$ obtained by reading the entries of U in column reading order, where we ignore entries from skewed boxes and entries in the 0th column. The content of any filling U of partition or composition shape, denoted $\mathrm{cont}(U)$, is the content of its column reading word $w_{col}(U)$. To any filling U we may associate a monomial $\mathbf{x}^U = \prod_{i \geqslant 1} x_i^{\mathrm{cont}(U)_i}$.

The following definition first appeared in [13].

Definition 1 *Let α be a strong composition with k parts with largest part size m. A row-strict composition tableau (RCT) U is a filling of the diagram α such that*

1. *The first column is weakly increasing when read top to bottom.*

2. *Each row strictly decreases when read left to right.*

3. *Triple Rule: Supplement U with zeros added to the end of each row so that the resulting filling $\hat{U}$ is of rectangular shape $k \times m$. Then for $1 \leqslant i_1 < i_2 \leqslant k$ and $2 \leqslant j \leqslant m$,*

$$\left(\hat{U}(i_2,j) \neq 0 \textit{ and } \hat{U}(i_2,j) > \hat{U}(i_1,j)\right) \Rightarrow \hat{U}(i_2,j) \geqslant \hat{U}(i_1,j-1).$$

If we let $\hat{U}(i_2,j) = b$, $\hat{U}(i_1,j) = a$, and $\hat{U}(i_1,j-1) = c$, then the Triple Rule ($b \neq 0$ and $b > a$ implies $b \geqslant c$) can be pictured as

$$\begin{array}{cc} \boxed{c} & \boxed{a} \\ & \vdots \\ & \boxed{b} \end{array} \;.$$

A row-strict composition tableau is called *standard* if each of the entries $\{1,2,\ldots,n\}$ appears exactly once. Given a standard row-strict composition tableau U, define its *descent set* $D(U)$ to be the set of all entries b such that the entry $b+1$ appears in a column strictly to the right of the column containing b.

Inversion triples were originally introduced by Haglund, Haiman, and Loehr in [6] to describe a combinatorial formula for symmetric Macdonald polynomials. In this article inversion triples are defined as follows. Let γ be a (possibly weak) composition and let β be a strong composition with $\gamma \subseteq \beta$. Let U be some arbitrary filling of β/γ. A *Type A triple* is a triple of entries

$$U(i_1, j-1) = c, U(i_1, j) = a, U(i_2, j) = b$$

in U with $\beta_{i_1} \geqslant \beta_{i_2}$ for some rows $i_1 < i_2$ and some column $j > 0$. A *Type B triple* is a triple of entries

$$U(i_1, j) = b, U(i_2, j) = c, U(i_2, j+1) = a$$

in U with $\beta_{i_1} < \beta_{i_2}$ for some rows $i_1 < i_2$ and some column $j \geqslant 0$. A triple of either type is said to be an *inversion triple* if either $b \leqslant a < c$ or $a < c \leqslant b$. Note that triples of either type may involve boxes in the 0th column. Type A and Type B triples can be visualized as

$$\begin{array}{cc} \text{Type A} & \text{Type B} \\ \begin{array}{|c|c|}\hline c & a \\ \hline \end{array} & \begin{array}{|c|}\hline b \\ \hline \end{array} \\ \quad\vdots & \vdots\quad \\ \quad\begin{array}{|c|}\hline b \\ \hline \end{array} & \begin{array}{|c|c|}\hline c & a \\ \hline \end{array} \end{array} .$$

Central to Theorem 13 in Section 3 is the following definition.

Definition 2 *Let β and α be strong compositions. Let γ be some (possibly weak) composition satisfying $\gamma^+ = \alpha$. A Littlewood-Richardson skew row-strict composition tableau S, or LR skew RCT, of shape β/α is a filling of a diagram of skew shape β/γ such that*

1. *Each row strictly decreases when read left to right.*
2. *Every Type A and Type B triple is an inversion triple.*
3. *The column reading word of S, $w_{col}(S)$, is a regular reverse lattice word.*

Note that in Definition 2, the filling is defined to be of a diagram of skew shape β/γ where $\gamma^+ = \alpha$ for some fixed α. Thus, we define the *shape of a LR skew RCT* to be β/α.

Example 3 *Below is a RCT, U, which has shape $(1,3,2,2)$, and a LR skew RCT, S, which has shape $(1,2,3,1,5,3)/(1,3,2,2)$ with $w_{col}(S) = 4433421$.*

$$U = \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-1} 4 & 3 & 2 \\ \hline 5 & 4 \\ \cline{1-2} 5 & 3 \\ \cline{1-2} \end{array} \qquad S = \begin{array}{c|c|c|c|c|c|} \infty & \infty \\ \infty & 4 & 3 \\ \infty & \infty & \infty & \infty \\ \infty & 4 \\ \infty & \infty & \infty & 4 & 2 & 1 \\ \infty & \infty & \infty & 3 \end{array}$$

1.3 Generating functions

The algebra of symmetric functions Λ has many bases, of which the Schur functions s_λ are arguably the most important. The Schur function s_λ can be defined in a number of ways. In this article it is advantageous to define s_λ as the generating function of reverse row-strict tableaux of shape λ^t. That is

$$s_\lambda = \sum \mathbf{x}^T$$

where the sum is over all reverse row-strict tableaux T of shape λ^t.

The algebra *QSym* of quasisymmetric functions also has several interesting bases, two of which we recall here. The *monomial quasisymmetric function basis* M_α is given by

$$M_\alpha = \sum_{i_1 < i_2 < \cdots i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

where α is a strong composition. In [5], Gessel defines the *fundamental quasisymmetric function basis*, which can be expressed as

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta$$

where α and β are strong compositions such that β is a refinement of α.

The generating function of row-strict composition tableaux of shape α are denoted $\mathcal{RS}_\alpha$. That is,

$$\mathcal{RS}_\alpha = \sum \mathbf{x}^U$$

where the sum is over all row-strict composition tableaux U of shape α. The generating functions $\mathcal{RS}_\alpha$ are called *row-strict quasisymmetric Schur functions* and were originally defined in [13].

In [13] the authors show $\mathcal{RS}_\alpha$ are indeed quasisymmetric, and furthermore the collection of all $\mathcal{RS}_\alpha$, as α ranges over all strong compositions, forms a basis of the algebra *QSym* of quasisymmetric functions. This result is obtained by expressing the functions $\mathcal{RS}_\alpha$ in terms of the fundamental basis F_β of quasisymmetric functions.

Proposition 4 *[13] Let α and β be strong compositions of n. Then*

$$\mathcal{RS}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ is equal to the number of standard row-strict composition tableaux U of shape α such that $comp(D(U)) := (b_1, b_2 - b_1, \ldots, b_k - b_{k-1}, n - b_k) = \beta$. Here, $D(U) = \{b_1, b_2, \ldots, b_k\}$ is the descent set of U.

In [13] the authors show the transition matrix given by the coefficients $d_{\alpha\beta}$ is upper uni-triangular. Hence the collection of all $\mathcal{RS}_\alpha$ form a basis of *QSym*.

The relation between row-strict quasisymmetric Schur functions and symmetric Schur functions is given in [13] by

$$s_\lambda = \sum_{\alpha:\ \tilde{\alpha} = \lambda^t} \mathcal{RS}_\alpha.$$

The well-known involution ω on symmetric functions acts on the Schur basis by the formula $\omega(s_\lambda) = s_{\lambda^t}$. An extension of ω to quasisymmetric functions appears in [5], and can be defined on the fundamental quasisymmetric functions by the formula $\omega(F_\alpha) = F_{\alpha^*}$. Using Proposition 4 the authors in [13] describe the action of ω on row-strict quasisymmetric Schur functions.

Theorem 5 *[13] Let α be a strong composition of n. Let $\mathcal{CS}_\alpha$ be the column-strict quasisymmetric Schur function indexed by α (see [7]). Then*

$$\omega(\mathcal{RS}_\alpha(x_1, x_2, \ldots, x_n)) = \mathcal{CS}_\alpha(x_n, x_{n-1}, \ldots, x_1).$$

In a separate but related work, we show that a certain subset of $\mathcal{RS}_\alpha$ are a basis for the coinvariant space of quasisymmetric functions. This proof is analogous to the proof appearing in [11]. Unlike the functions $\mathcal{CS}_\alpha$, there is to date no representation theoretic interpretation of the functions $\mathcal{RS}_\alpha$. It would be interesting to know whether such an interpretation exists.

2 Insertion algorithms

Let A be a matrix with finitely many nonzero entries, each entry in $\mathbb{N}$. Associate to A a *two-line array* w_A by letting

$$w_A = \begin{pmatrix} i_1 & i_2 & \cdots & i_m \\ j_1 & j_2 & \cdots & j_m \end{pmatrix}$$

where i_r, j_r are positive integers for $1 \leqslant r \leqslant m$, and (a) $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_m$, (b) if $i_r = i_s$ and $r \leqslant s$ then $j_r \leqslant j_s$, and (c) there are exactly a_{ij} numbers r such that $(i_r, j_r) = (i, j)$ for each pair (i, j). Denote by $\widehat{w}_A$ the sequence $i_1, i_2, \ldots, i_m$ and denote by $\check{w}_A$ the sequence $j_1, j_2, \ldots, j_m$.

The classical Robinson-Schensted-Knuth (RSK) correspondence gives a bijection between two-line arrays w_A and pairs of (reverse row-strict) tableaux (P, Q) of the same shape [4]. The basic operation of RSK is Schensted insertion on tableaux, which is an algorithm that inserts a positive integer b into a tableau T to produce a new tableau T'. In our setting, Schensted insertion can be stated as

Definition 6 *Given a tableau T and b a positive integer one can obtain $T' := b \to T$ by inserting b as follows:*

1. *Let $\tilde{b}$ be the largest entry less than or equal to b in the first row of T. If no such $\tilde{b}$ exists, simply place b at the end of the first row.*
2. *If $\tilde{b}$ does exists, replace (bump) $\tilde{b}$ with b and proceed to insert $\tilde{b}$ into the second row using the method just described.*

The authors in [13] provide an analogous algorithm on row-strict composition tableaux.

Definition 7 *(RCT Insertion) Let U be a RCT with longest row of length m, and let b be a positive integer. One can obtain $U' := U \leftarrow b$ by inserting b as follows. Scan the entries of U in reverse column reading order, that is top to bottom in each column starting with the right-most column and working leftwards, starting with column $m+1$ subject to the conditions:*

1. *In column $m+1$, if the current position is at the end of a row of length m, and b is strictly less than the last entry in that row, then place b in this empty position and stop. If no such position is found, begin scanning at the top of column m.*
2. *(a) Inductively, suppose some entry b_j begins scanning at the top of column j. In column j, if the current position is empty and at the end of a row of length $j-1$, and b_j is strictly less than the last entry in that row, then place b_j in this empty position and stop.*

(b) *If a position in column j is nonempty and contains $\tilde{b}_j \leqslant b_j$ such that b_j is strictly less than the entry immediately to the left of $\tilde{b}_j$, let b_j bump $\tilde{b}_j$ and continue scanning column j with the entry $\tilde{b}_j$, bumping whenever possible. After scanning the last entry in column j, begin scanning column $j-1$.*

3. *If an entry b_1 is bumped into the first column, then place b_1 in a new row that appears after the last entry in the first column that is weakly less than b_1.*

In [13] the authors show $U' = U \leftarrow b$ is a row-strict composition tableau. The algorithm of inserting b into U determines a set of boxes in U' called the *insertion path of* b and denoted $I(b)$, which is precisely the set of boxes in U' which contain an entry bumped during the algorithm. We call the row in U' in which the new box is added the *row augmented by insertion.* We establish several new lemmas concerning RCT insertion that are instrumental in proving the main theorem in Section 3.

Lemma 8 *Let U be a RCT and b be a positive integer. Then each row of $U' = U \leftarrow b$ contains at most one box from $I(b)$.*

Lemma 9 *Let U be a RCT and b be a positive integer. Let $U' = U \leftarrow b$ with row i of U' being the row augmented by insertion. Then for all rows $r > i$ of U', the length of row r is not equal to the length of row i.*

Lemma 10 *(Main Bumping Lemma) Let U be a RCT, and let a, b, and c be positive integers with $a < b \leqslant c$. Consider successive insertions $U_1 := (U \leftarrow b) \leftarrow c$ and $U_2 := (U \leftarrow b) \leftarrow a$. Let $B_a = (i_a, j_a)$, $B_b = (i_b, j_b)$, and $B_c = (i_c, j_c)$ be the new boxes created after inserting $a, b,$ and c, respectively, into the appropriate RCT. Let i_1 be a row in U_1 which contains a box (i_1, j_1) from $I(b)$ and a box (i_1, j_1') from $I(c)$. Similarly, let i_2 be a row in U_2 which contains a box (i_2, j_2) from $I(b)$ and a box (i_2, j_2') from $I(a)$. Then*

1. *In U_1, $j_c \leqslant j_b$. In U_2, $j_a > j_b$.*

2. *In U_1, $j_1' \leqslant j_1$. In U_2, $j_2' > j_2$.*

Part (1.) of Lemma 10 says the new box B_c is weakly left of the new box B_b in U_1, and the new box B_a is strictly right of the new box B_b in U_2. Part (2.) of Lemma 10 says that in any row which contains a box from both $I(b)$ and $I(c)$, or from both $I(b)$ and $I(a)$, then in this row $I(c)$ is weakly left of $I(b)$ in U_1, and $I(a)$ is strictly right of $I(b)$ in U_2.

Lemma 11 *Consider the RCT obtained after n successive insertions*

$$U_n := (\cdots((U \leftarrow b_1) \leftarrow b_2)\cdots) \leftarrow b_n$$

with $b_1 \leqslant b_2 \leqslant \cdots \leqslant b_n$. Let $B_1, B_2, \ldots, B_n$ be the corresponding new boxes. Then in U_n,

$$B_n <_{col} B_{n-1} <_{col} \cdots <_{col} B_1.$$

Knuth's contribution to the RSK algorithm included describing Schensted insertion in terms of two *elementary transformations $\mathcal{K}_1$ and $\mathcal{K}_2$* which act on words w. Let $a, b,$ and c be positive integers. Then

$$\begin{array}{lll} \mathcal{K}_1 : & bca \to bac & \text{if } a < b \leqslant c \\ \mathcal{K}_2 : & acb \to cab & \text{if } a \leqslant b < c \end{array} .$$

The relations $\mathcal{K}_1, \mathcal{K}_2$, and their inverses $\mathcal{K}_1^{-1}, \mathcal{K}_2^{-1}$, act on words w by transforming triples of consecutive letters. Denote by $\overset{1}{\cong}$ the equivalence relation defined by using $\mathcal{K}_1$ and $\mathcal{K}_1^{-1}$. That is, $w \overset{1}{\cong} w'$ if and only if w can be transformed into w' using a finite sequence of transformations $\mathcal{K}_1$ or $\mathcal{K}_1^{-1}$. The following lemma is extremely useful in proving Theorem 13.

Lemma 12 *Let U be a RCT and let w and w' be two words such that $w \overset{1}{\cong} w'$. Then*

$$U \leftarrow w = U \leftarrow w'.$$

3 Littlewood-Richardson type rule

The main theorem of this article is the following.

Theorem 13 *Let s_λ be the Schur function indexed by the partition λ, and let $\mathcal{RS}_\alpha$ be the row-strict quasisymmetric Schur function indexed by the strong composition α. We have*

$$\mathcal{RS}_\alpha \cdot s_\lambda = \sum_\beta C^\beta_{\alpha,\lambda} \mathcal{RS}_\beta \tag{1}$$

where $C^\beta_{\alpha,\lambda}$ is the number of Littlewood-Richardson skew RCT of shape β/α and content λ^*.

Theorem 13 is established by constructing a bijection ρ between pairs (U, T) and (V, S), where U is a RCT of shape α, T a tableau of shape λ^t, V is a RCT of shape β, and S is a LR skew RCT of shape β/α and content λ^*.

Specifically, the bijection ρ is obtained in the following way. First, use RSK to map the pair (T, T_λ) to a two-line array w_A. Then perform the insertion $U \leftarrow \check{w}_A$ while simultaneously recording in a skew shape each new box, using the letters of $\hat{w}_A$ in order. The result is a pair (V, S). To invert this procedure, perform the inverse of insertion on V using S in the following way. Find each occurrence of the entry 1 in S. Un-insert the entries in the corresponding boxes in V according to the order they appear with respect to $<_{col}$; that is, the smallest box in column reading order is un-inserted first. After each entry is un-inserted we get a pair $(1, j)$. Inductively, find each occurrence of i in S and un-insert the entries in the corresponding boxes of V in the order they appear with respect to $<_{col}$, each time producing a pair (i, j) for some j. When all entries have been removed from S, the result is a pair (U, T).

Below is an example of the bijection ρ using the RCT of Example 3 and following the notation established above.

$$\text{U=} \begin{array}{|c|c|c|} \hline 1 \\ \cline{1-1} 4 & 3 & 2 \\ \hline 5 & 4 \\ \cline{1-2} 5 & 3 \\ \cline{1-2} \end{array} \qquad \text{T=} \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline 4 & 3 \\ \cline{1-2} 2 \\ \cline{1-1} \end{array} \qquad T_\lambda = \begin{array}{|c|c|c|c|} \hline 4 & 3 & 2 & 1 \\ \hline 4 & 3 \\ \cline{1-2} 4 \\ \cline{1-1} \end{array} \qquad (T, T_\lambda) \overset{RSK}{\Leftrightarrow} \begin{pmatrix} 4 & 4 & 4 & 3 & 3 & 2 & 1 \\ 2 & 4 & 4 & 3 & 3 & 2 & 1 \end{pmatrix}$$

$$
V = U \leftarrow 2443321 = \begin{array}{|c|c|c|c|c|}
\hline
1 \\
\hline
3 & 2 \\
\hline
4 & 3 & 2 \\
\hline
4 \\
\hline
5 & 4 & 3 & 2 & 1 \\
\hline
5 & 4 & 3 \\
\hline
\end{array}
\qquad
S = \begin{array}{c|c|c|c|c|c|}
\infty & \infty \\
\infty & 4 & 3 \\
\infty & \infty & \infty & \infty \\
\infty & 4 \\
\infty & \infty & \infty & 4 & 2 & 1 \\
\infty & \infty & \infty & 3 \\
\end{array}
$$

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K-classes for matroids and equivariant localization

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Abstract. To every matroid, we associate a class in the K-theory of the Grassmannian. We study this class using the method of equivariant localization. In particular, we provide a geometric interpretation of the Tutte polynomial. We also extend results of the second author concerning the behavior of such classes under direct sum, series and parallel connection and two-sum; these results were previously only established for realizable matroids, and their earlier proofs were more difficult.

Résumé. À chaque matroïde, nous associons une classe dans la K-théorie de la grassmannienne. Nous étudions cette classe en utilisant la méthode de localisation équivariante. En particulier, nous fournissons une interprétation géométrique du polynôme de Tutte. Nous étendons également les résultats du second auteur concernant le comportement de ces classes pour la somme directe, les connexions série et parallèle et la 2-somme; ces résultats n'ont été déjà établis que pour les matroïdes réalisables, et leurs preuves précédentes étaient plus difficiles.

Keywords: matroid, Tutte polynomial, K-theory, equivariant localization, Grassmannian

1 Introduction

Let $H_1, H_2, \ldots, H_n$ be a collection of hyperplanes through the origin in $\mathbb{C}^d$. The study of such hyperplane arrangements is a major field of research, resting on the border between algebraic geometry and combinatorics. There are two natural objects associated to a hyperplane arrangement.

The first is the ***matroid*** of the hyperplane arrangement, which can be thought of as encoding the combinatorial structure of the arrangement.

The second, which captures the geometric structure of the arrangement, is a point in the Grassmannian $G(d, n)$. There is ambiguity in the choice of this point; it is only determined up to the action of an n-dimensional torus on $G(d, n)$. So more precisely, to any hyperplane arrangement, we associate an orbit in $G(d, n)$ for this torus action. It is technically more convenient to work with the closure of this orbit. In [19], the second author suggested that the K-class of this orbit could give rise to useful invariants of matroids, thus exploiting the geometric structure to study the combinatorial one. In this paper, we continue that project.

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One of our main results, Theorem 1.1, is a formula for the Tutte polynomial, the most famous of matroid invariants, in terms of the K-class of Y. The other, Theorem 1.2, relates it to the invariants of [19]. In addition, we rewrite all of the K-theoretic definitions in terms of moment graphs, as [19] also began to do. This makes our theory purely combinatorial and in principle completely computable. Many results which were shown for realizable matroids in [19] are now extended to all matroids.

We review the definitions of these invariants now; the necessary K-theoretic definitions will be given in the following section. Let M be a rank d matroid on the ground set $[n]$, and let ρ_M be the rank function of M. The rank generating function of M is

$$r_M(u,v) := \sum_{S \subset [n]} u^{d-\rho_M(S)} v^{|S|-\rho_M(S)}.$$

The Tutte polynomial is defined by $t_M(z,w) = r_M(z-1, w-1)$. See [3] for background on the Tutte polynomial, including several alternate definitions. Let $h_M(s)$ be the invariant defined in [19], which we introduce after Lemma 3.3.

Given integers $0 < d_1 < \cdots < d_s < n$, let $\mathcal{F}\ell(d_1,\ldots,d_s;n)$ be the partial flag manifold of flags of dimensions $(d_1,\ldots,d_s)$. For instance, $\mathcal{F}\ell(d;n) = G(d,n)$.

We will be concerned with the maps

$$\pi_d : \mathcal{F}\ell(1,d,n-1;n) \to G(d,n), \qquad \pi_{1(n-1)} : \mathcal{F}\ell(1,d,n-1;n) \to G(1,n) \times G(n-1,n) \tag{1}$$

given respectively by forgetting the 1 and $(n-1)$-planes, and as the product of maps forgetting the 1 and d-planes and the d and $(n-1)$-planes. Note that $\pi_{1(n-1)}$ factors through the hypersurface $\mathcal{F}\ell(1,n-1;n)$ in $G(1,n) \times G(n-1,n) \cong \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$.

Let T be the torus $(\mathbb{C}^*)^n$, which acts on the spaces in (1) in an obvious way. Let x be a point of $G(d,n)$, M the corresponding matroid, and $\overline{Tx}$ the closure of the T orbit through x. Let Y be the class of the structure sheaf of $\overline{Tx}$ in $K^0(G(d,n))$. Write $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}) = \mathbb{Q}[\alpha,\beta]/(\alpha^n,\beta^n)$, where α and β are the structure sheaves of hyperplanes.

Theorem 1.1 *With the above notations,*

$$(\pi_{1(n-1)})_* \pi_d^* (Y \cdot [\mathcal{O}(1)]) = t_M(\alpha,\beta)$$

where t_M is the Tutte polynomial.

The constant term of t_M is zero; this corresponds to the fact that $\pi_{1(n-1)}$ is not surjective onto $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$ but, rather, has image lying in $\mathcal{F}\ell(1,n-1;n)$. The linear term of Tutte is $(\alpha+\beta)$ times the beta invariant of M, corresponding to the fact that the map $\pi_d^{-1}(\overline{Tx}) \to \mathcal{F}\ell(1,n-1;n)$ is finite of degree the beta invariant.

Theorem 1.2 *Also with the above notations,*

$$(\pi_{1(n-1)})_* \pi_d^* (Y) = h_M(\alpha+\beta-\alpha\beta)$$

where h_M is the polynomial from [19].

1.1 Notation

We write $[n]$ for $\{1,2,\ldots,n\}$. For any set S, we write $\binom{S}{k}$ for the set of k-element subsets of S.

1.2 Acknowledgments

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2 Background on K-theory

In this section, we will introduce the requisite background on K-theory, emphasizing equivariant methods and localization. Our treatment here adopts a geometric focus.

We reassure combinatorial readers[(i)] that the computations can be carried out in a purely combinatorial setting, via ***moment graphs***. We recommend [9] as a reference for the use of moment graphs in equivariant cohomology, which is extremely similar to the K-theory setup. In short, it is possible to read Theorem 2.6 as constructing the ring $K^0_T(X)$ as certain functions on graphs Γ whose vertices (the T-fixed points) are elements of a free abelian group $\mathrm{Char}(T)$, and whose edges (the one dimensional orbits) are labelled with the minimal lattice vectors in the respective directions (associated characters); no further geometry is necessary. The operations in K-theory rings that conclude Section 2.3 can also be carried out using these graphs.

2.1 Groundwork

If V is a finite dimensional representation of T, the Hilbert series of V is the sum

$$\mathtt{hilb}(V) := \sum_{\chi \in \mathrm{Char}(T)} \dim \mathrm{Hom}(\chi, V) \cdot \chi$$

in $\mathbb{Z}[\mathrm{Char}(T)]$. If V isn't finite dimensional, but $\mathrm{Hom}(\chi, V)$ is for every character χ, then we can still consider this as a formal sum.

Here is one example of particular interest: let W be a finite dimensional representation of T with character $\sum \chi_i$. Suppose that all of the χ_i lie in an open half space in $\mathrm{Char}(T) \otimes \mathbb{R}$; if this condition holds, we say that W is ***contracting***. Then the Hilbert series of $\mathrm{Sym}(W)$, defined as a formal power series, represents the rational function $1/(1-\chi_1)\cdots(1-\chi_r)$. If M is a finitely generated $\mathrm{Sym}(W)$ module, then the Hilbert series of M will likewise represent an element of $\mathrm{Frac}(\mathbb{Z}[\mathrm{Char}(T)])$ [16, Theorem 8.20].

Sign conventions are potentially confusing. Here are ours: if a group G acts on a ring A, we let G act on $\mathrm{Spec}\, A$ by $g(a) = (g^{-1})^* a$. This definition is necessary in order to make sure that both actions are *left* actions. In examples where T acts on various partial flag varieties, our convention is that T acts on $\mathbb{A}^n$ by the characters $t_1^{-1}, \ldots, t_n^{-1}$. Grassmannians, and other partial flag varieties, are flags of *subspaces*, not quotient spaces, and T acts on them by acting on the subobjects of $\mathbb{A}^n$. The advantage of this convention is that, for any ample line bundle L on $\mathcal{F}\ell(n)$, the action on $\int^T L$ will be by nonnegative powers of the t_i, i.e. the equivariant K-class of $\int^T L$ will be a polynomial.

2.2 Definition of K^0 and K^0_T

If X is any algebraic variety, then $K_0(X)$ denotes the free abelian group generated by isomorphism classes of coherent sheaves on X, subject to the relation $[A] + [C] = [B]$ whenever there is a short exact

(i) Section 2.4 of the full paper is dedicated to this audience, in case our small reassurance here is insufficient.

sequence $0 \to A \to B \to C \to 0$. The subspace generated by the classes of vector bundles is denoted $K^0(X)$. If X is smooth, as all the spaces we deal with will be, the inclusion $K^0(X) \hookrightarrow K_0(X)$ is an equality. (See [18, Proposition 2.1] for this fact, and its equivariant generalization.)

We put a ring structure on $K^0(X)$, generated by the relations $[E][F] = [E \otimes F]$ for any vector bundles E and F on X. The group $K_0(X)$ is a module for $K^0(X)$, with multiplication given by $[E][F] = [E \otimes F]$ where E is a vector bundle and F a coherent sheaf.

For any map $f : X \to Y$, there is a pull back map $f^* : K^0(Y) \to K^0(X)$ given by $f^*[E] = [f^*E]$. This is a ring homomorphism. If $f : X \to Y$ is a proper map, there is also a pushforward map $f_* : K_0(X) \to K_0(Y)$ given by

$$f_*[E] = \sum (-1)^i [R^i f_* E].$$

These two maps are related by the projection formula, which asserts that

$$f_*\big((f^*[E])[F]\big) = [E] f_*[F]. \tag{2}$$

That is, f_* is a $K^0(Y)$-module homomorphism, if $K^0(X)$ has the module structure induced by f^*.

We always have a map from X to a point. We denote the pushforward along this map by $\int$, or by $\int_X$, by analogy with de Rham cohomology. Notice that $K_0(\mathtt{pt}) = K^0(\mathtt{pt}) = \mathbb{Z}$, and $\int [E]$ is the holomorphic Euler characteristic of the sheaf E.

If T is a torus acting on X, then we can form the analogous constructions using T-equivariant vector bundles and sheaves. These are denoted $K_T^0(X)$ and $K_0^T(X)$. Writing $\mathrm{Char}(T)$ for the lattice of characters, $\mathrm{Hom}(T, \mathbb{C}^*)$, we have $K_0^T(\mathtt{pt}) = K_T^0(\mathtt{pt}) = \mathbb{Z}[\mathrm{Char}(T)]$. Explicitly, a T-equivariant sheaf on $\mathtt{pt}$ is simply a vector space with a T-action, and the corresponding element of $\mathbb{Z}[\mathrm{Char}(T)]$ is the character. We write $[E]^T$ for the class of the sheaf E in $K_T^0(X)$.

2.3 Localization

The results in this section are well known to experts, but it seems difficult to find a reference that records them all in one place. The reader may want to compare the description of equivariant cohomology in [13].

Here we will be only concerned with $K_0^T(X)$ for extremely nice spaces X — in fact, only partial flag manifolds and products thereof. All of these spaces are ***equivariantly formal*** spaces, meaning that their K-theory can be described using the method of ***equivariant localization***, which we now explain. We will gradually add niceness hypotheses on X as we need them.

Condition 2.1 Let X be a smooth projective variety with an action of a torus T.

Writing X^T for the subvariety of T-fixed points, we have a restriction map

$$K_T^0(X) \to K_T^0(X^T) \cong K^0(X^T) \otimes K_T^0(\mathtt{pt}).$$

Theorem 2.2 ([18, Theorem 3.2], see also [12, Theorem A.4] and [21, Corollary 5.11]) *In the presence of Condition 2.1, the restriction map $K_T^0(X) \to K_T^0(X^T)$ is an injection.*

If we have Condition 2.1 and

Condition 2.3 X has finitely many T-fixed points, then $K^0_T(X^T)$ is simply the ring of functions from X^T to $K^0_T(\mathtt{pt})$.

Let x be a fixed point of the torus action on X, so we have a restriction map $K^0_T(X) \to K^0_T(x) \cong K^0_T(\mathtt{pt})$. It is important to understand how this map is explicitly computed. For $\xi \in K^0_T(X)$, we write $\xi(x)$ for the image of ξ in $K^0_T(x)$.

In all of our examples, there will exist a T-equivariant neighborhood U of x isomorphic to a contracting T-representation on $\mathbb{A}^n$. Let $\chi_1, \ldots, \chi_N$ be the characters by which T acts on U. Let E be a T-equivariant coherent sheaf on U, corresponding to a graded, finitely generated $\mathcal{O}(U)$-module M. Then the Hilbert series of M lies in $\operatorname{Frac}(\mathbb{Z}[\operatorname{Char}(T)])$; it is a rational function of the form $k(E)/\prod(1-\chi_i^{-1})$ for some polynomial $k(E)$ in $\mathbb{Z}[\operatorname{Char}(T)]$.

Theorem 2.4 *If U is an open neighborhood of x as above then $K^0_T(U) \cong K^0_T(\mathtt{pt})$. With the above notations, $[E]^T(x) = k(E)$.*

In particular, if E is a vector bundle on U, and T acts on the fiber over x with character $\sum \eta_i$, then $[E]^T(x) = \sum \eta_i$.

We have now described, given a T-equivariant sheaf E in $K_0^T(X)$, how to express it as a function from X^T to $K^0_T(\mathtt{pt})$. It will also be worthwhile to know, given a function from X^T to $K^0_T(\mathtt{pt})$, when it is in $K^0_T(X)$. For this, we need

Condition 2.5 There are finitely many 1-dimensional T-orbits in X, each of which has closure isomorphic to $\mathbb{P}^1$ (and thus contains two T-fixed points).

Theorem 2.6 ([21, Corollary 5.12], see also [12, Corollary A.5]) *Assume conditions 2.1, 2.3 and 2.5. Let f be a function from X^T to $K^0_T(\mathtt{pt})$. Then f is of the form $\xi(\cdot)$ for some $\xi \in K^0_T(X)$ if and only if the following condition holds: For every one dimensional orbit, on which T acts by character χ and for which x and y are the T-fixed points in the closure of the orbit, we have*

$$f(x) \equiv f(y) \mod 1-\chi.$$

Example 2.7 Let's see what this theorem means for the Grassmannian $G(d,n)$. Here $K^0_T(\mathtt{pt})$ is the ring of Laurent polynomials $\mathbb{Z}[t_1^{\pm}, t_2^{\pm}, \ldots, t_n^{\pm}]$. The fixed points $G(d,n)^T$ are the linear spaces of the form $\operatorname{Span}(e_i)_{i\in I}$ for $I \in \binom{[n]}{d}$. We will write this point as x_I for $I \in \binom{[n]}{d}$. So an element of $K^0_T(G(d,n))$ is a function $f : \binom{[n]}{d} \to K^0_T(\mathtt{pt})$ obeying certain conditions. What are those conditions? Each one-dimensional torus orbit joins x_I to x_J where $I = S \sqcup \{i\}$ and $J = S \sqcup \{j\}$ for some S in $\binom{[n]}{d-1}$. Thus an element of $K^0_T(G(d,n))$ is a function $f : \binom{[n]}{d} \to K^0_T(\mathtt{pt})$ such that

$$f(S \sqcup \{i\}) \equiv f(S \sqcup \{j\}) \mod 1 - t_i/t_j$$

for all $S \in \binom{[n]}{d-1}$ and $i, j \in [n] \setminus S$.

We now describe how to compute tensor products, pushforwards and pullbacks in the localization description. The first two are simple: tensor product corresponds to multiplication in the ring of functions, and pullback to pullback of functions. The formula for pushforward is somewhat more complex, and is more conveniently stated in terms of multi-graded Hilbert series. If $\mathtt{hilb}(E_x)$ is the multi-graded Hilbert series of the stalk E_x, then

$$\mathtt{hilb}(\pi_*(E)_y) = \sum_{x \in X^T,\ \pi(x)=y} \mathtt{hilb}(E_x) \tag{3}$$

This yields a formula for $(\pi_*[E]^T)(y)$ by Theorem 2.4. The special case of this formula for pushforward to a point,

$$\int_X^T [E]^T = \sum_{x \in X^T} \mathtt{hilb}(E_x), \tag{4}$$

is more prominent in the literature than the general result; see for example [18, Section 4].

Finally, we describe the relation between ordinary and T-equivariant K-theories. There is a map from equivariant K-theory to ordinary K-theory by forgetting the T-action. In particular, the map $K_T^0(\mathtt{pt}) \to K^0(\mathtt{pt}) = \mathbb{Z}$ just sends every character of T to 1. In this way, $\mathbb{Z}$ becomes a $K_T^0(\mathtt{pt})$-module. Thus, for any space X with a T-action, we get a map $K_T^0(X) \otimes_{K_T^0(\mathtt{pt})} \mathbb{Z} \to K^0(X)$. All we will need is that this map exists, but in fact given Condition 2.1 it is an isomorphism [15, Theorem 4.3].

3 Matroids and Grassmannians

Let $\mathbb{E}$ be a finite set (the ***ground set***), which we will usually take to be $[n]$. For $I \subseteq \mathbb{E}$, we write e_I for the vector $\sum_{i \in I} e_i$ in $\mathbb{Z}^{\mathbb{E}}$. Let M be a collection of d-element subsets of $\mathbb{E}$. Let $\mathtt{Poly}(M)$ be the convex hull of the vectors e_I, as I runs through M. The collection M is called a matroid if it obeys any of a number of equivalent conditions (see [17] for these, and [5] for motivation). Our favorite is due to Edmonds:

Theorem 3.1 ([7]; see also [8, Theorem 4.1]) *M is a matroid if and only if M is nonempty and every edge of $\mathtt{Poly}(M)$ is in the direction $e_i - e_j$ for some i and $j \in \mathbb{E}$.*

We now explain the connection between matroids and Grassmannians. We assume basic familiarity with Grassmannians (see [16, Chapter 14] for background). Given a point x in $G(d,n)$, the set of I for which the Plücker coordinate $p_I(x)$ is nonzero forms a matroid, which we denote $\mathrm{Mat}(x)$. (A matroid of this form is called ***realizable***.) Let T be the torus $(\mathbb{C}^*)^n$, which acts on $G(d,n)$ in the obvious way, so that $p_I(tx) = t^{e_I} p_I(x)$ for $t \in T$. Clearly, $\mathrm{Mat}(tx) = \mathrm{Mat}(x)$ for any $t \in T$.

We now discuss how we will bring K-theory into the picture. Consider the torus orbit closure $\overline{Tx}$. The orbit Tx is a translate (by x) of the image of the monomial map given by the set of characters $\{t^{-e_I} : p_I(x) \neq 0\}$. Essentially by definition, $\overline{Tx}$ is the toric variety associated to the polytope $\mathtt{Poly}(\mathrm{Mat}(x))$ (see [4, Section 5], and [22] regarding normality). In the appendix to [19], the second author checked that the class of the structure sheaf of $\overline{Tx}$ in $K_T^0(G(d,n))$ depends only on $\mathrm{Mat}(x)$, and gave the following natural way to define a class $y(M)$ in $K_T^0(G(d,n))$ for any matroid M of rank d on $[n]$, nonrealizable matroids included.

For a polyhedron P and a point $v \in P$, define $\mathtt{Cone}_v(P)$ to be the positive real span of all vectors of the form $u - v$, with $u \in P$; if v is not in P, define $\mathtt{Cone}_v(P) = \emptyset$. Let $M \subseteq \binom{[n]}{d}$ be a matroid. We will abbreviate $\mathtt{Cone}_{e_I}(\mathtt{Poly}(M))$ by $\mathtt{Cone}_I(M)$. For a pointed rational polyhedron C in $\mathbb{R}^n$, define $\mathtt{hilb}(C)$ to be the Hilbert series

$$\mathtt{hilb}(C) := \sum_{a \in C \cap \mathbb{Z}^n} t^a.$$

This is a rational function with denominator dividing $\prod_{i \in I} \prod_{j \notin I} (1 - t_i^{-1} t_j)$ [20, Theorem 4.6.11]. We define the class $y(M)$ in $K_T^0(G(d,n))$ by

$$y(M)(x_I) := \mathtt{hilb}(\mathtt{Cone}_I(M)) \prod_{i \in I} \prod_{j \notin I} (1 - t_i^{-1} t_j),$$

Note that $\texttt{hilb}(\texttt{Cone}_I(M)) = 0$ for $I \notin M$.

To motivate this definition, suppose M is of the form $\mathrm{Mat}(x)$ for some $x \in G(d,n)$. For I in M, the toric variety $\overline{Tx}$ is isomorphic near x_I to $\mathrm{Spec}\,\mathbb{C}[\texttt{Cone}_I(M) \cap \mathbb{Z}^n]$. In particular, the Hilbert series of the structure sheaf of $\overline{Tx}$ near x_I is $\texttt{hilb}(\texttt{Cone}_I(M))$. So in this situation $y(M)$ is exactly the T-equivariant class of the structure sheaf of $\overline{Tx}$.

Proposition 3.2 *Whether or not M is realizable, the function $y(M)$ from $G(d,n)^T$ to $K_T^0(\texttt{pt})$ defines a class in $K_T^0(G(d,n))$.*

This follows from the following, more general, polyhedral result.

Lemma 3.3 *Let P be a lattice polytope in $\mathbb{R}^n$ and let u and v be vertices of P connected by an edge of P. Let e be the minimal lattice vector along the edge pointing from u to v, with $v = u + ke$. Then $\texttt{hilb}(\texttt{Cone}_u(P)) + \texttt{hilb}(\texttt{Cone}_v(P))$ is a rational function whose denominator is **not** divisible by $1 - t^e$.*

Having defined $y(M)$, we can give the definition of $h_M(s)$ from [19]. Let i be an index between 1 and d. Choose a line ℓ in n-space and an $n-i$ plane M containing ℓ. Let $\Omega_i \subset G(d,n)$ be the Schubert cell of those d-planes L such that $\ell \subset L$ and $L + M$ is contained in a hyperplane. If $i > d$, we define Ω_i to be Ω_d. $h_M(s)$ is defined by

$$\frac{h_M(s)}{1-s} = \sum_{i=1}^{\infty} \int_{G(d,n)} y(M)[\mathcal{O}_{\Omega_i}] s^i.$$

Example 3.4 We work through these definitions for the case of a matroid in $G(2,4)$, namely $M = \{13, 14, 23, 24, 34\}$. This M is realizable, arising as $\mathrm{Mat}(x)$ when for instance x is the rowspan of $\begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$, with Plücker coordinates $(p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}) = (0, 1, 1, 1, 1, -1)$.

Computing $y(M)$ entails finding the Hilbert functions $\texttt{hilb}(\texttt{Cone}_I(M))$ for each $I \in M$. We leave the reader to find these and confirm that $y(M)$ is sent under the localization map of Theorem 2.2 to

$$(0, 1 - t_2 t_3^{-1}, 1 - t_2 t_4^{-1}, 1 - t_1 t_3^{-1}, 1 - t_1 t_4^{-1}, 1 - t_1 t_2 t_3^{-1} t_4^{-1}),$$

ordering the coordinates lexicographically. We see that this satisfies the congruences in Theorem 2.6.

3.1 Valuations

A subdivision of a polyhedron P is a polyhedral complex $\mathcal{D}$ with $|\mathcal{D}| = P$. We use the names $P_1, \ldots, P_k$ for the facets of a typical subdivision $\mathcal{D}$ of P, and for $J \subseteq [k]$ nonempty we write $P_J = \bigcap_{j \in J} P_j$, which is a face of $\mathcal{D}$. We also put $P_\emptyset = P$. Let $\mathcal{P}$ be a set of polyhedra (for us, the set of matroid polytopes), and A an abelian group. We say that a function $f : \mathcal{P} \to A$ is a ***valuation*** (or is ***valuative***) if, for any subdivision such that $P_J \in \mathcal{P}$ for all $J \subseteq [k]$, we have

$$\sum_{J \subseteq [k]} (-1)^{|J|} f(P_J) = 0.$$

For example, one valuation of fundamental importance to the theory is the function $\mathbf{1}(\cdot)$ mapping each polytope P to its characteristic function. Namely, $\mathbf{1}(P)$ is the function $V \to \mathbb{Z}$ which takes the value 1 on P and 0 on $V \setminus P$. Also, many important functions of matroids, including the Tutte polynomial, are valuations.

We discuss how valuations arise from K-theory. Let $\mathcal{D}$ be a subdivision of matroid polytopes, with facets $P_1, \ldots, P_k$, and let $P_J = \mathtt{Poly}(M_J)$. From the definition of $y(M)$ it is not hard to show

Proposition 3.5 *The function y is a valuation of matroids.*

That is, we have a linear relation of K-theory classes

$$\sum_{J \subseteq [k]} (-1)^{|J|} y(M_J) = 0. \tag{5}$$

As a corollary, any function built using K-theory, for example the functions in our main theorems, is a valuation.

Theorem 1.5a of [6] asserts that the group of valuative matroid invariants is free of rank $\binom{n}{d}$. The group $K^0(G(d,n))$ is also free of rank $\binom{n}{d}$. This gives rise to the hope that every valuative matroid invariant might factor through $M \mapsto y(M)$, i.e. that every matroid valuation might come from K-theory. This hope is quite false, however. The reason is that no torus orbit closure can have dimension greater than that of T, namely $n-1$. Therefore, $\int y(M)[E]$ vanishes whenever E is supported in codimension n or greater. This imposes nontrivial linear constraints on $y(M)$, so the classes $y(M)$ span a proper subspace of $K^0(G(d,n))$. The reader may check that for $(d,n) = (2,4)$, an explicit valuative invariant not extending to a linear function on $K^0(G(d,n))$ is z given by $z(M)$ is 1 if $\mathtt{Poly}(M)$ contains $(1/2, 1/2, 1/2, 1/2)$ and 0 otherwise.

4 A fundamental lemma

Recall from section 1 the maps $\pi_d : \mathcal{F}\ell(1, d, n-1; n) \to G(d,n)$ and $\pi_{1(n-1)} : \mathcal{F}\ell(1, d, n-1; n) \to \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$, and the notations α and β for the hyperplane classes in $K^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. Over $G(d,n)$, we have the tautological exact sequence

$$0 \to S \to \mathbb{C}^n \to Q \to 0. \tag{6}$$

Over each point of $G(d,n)$, the fiber of S is the corresponding d-dimensional vector space.

The following lemma is central to the proofs of Theorems 1.1 and 1.2.

Lemma 4.1 *Given $[E] \in K^0(G(d,n))$, define a formal polynomial in u and v by*

$$R(u,v) := \int_{G(d,n)} [E] \sum [\textstyle\bigwedge^p S][\textstyle\bigwedge^q (Q^\vee)] u^p v^q. \tag{7}$$

Then

$$(\pi_{1(n-1)})_* \pi_d^* [E] = R(\alpha - 1, \beta - 1).$$

We do not have an equivariant generalization of Lemma 4.1, relating classes in $K_T^0(G(d,n))$ to those in $K_T^0(\mathbb{P}^{n-1} \times \mathbb{P}^{n-1})$. However, Lemma 4.1 provides an alternate way to obtain equivariant versions of our main theorems.

By Lemma 4.1, the content of Theorem 1.1 is that

$$\int y(M) \cdot [\mathcal{O}(1)] \cdot \sum_{p=0}^{d} \sum_{q=0}^{n-d} [\textstyle\bigwedge^p S]\ [\textstyle\bigwedge^q (Q^\vee)] u^p v^q = r_M(u,v).$$

In fact, something stronger is true.

Theorem 4.2 *In equivariant K-theory, we have*

$$\int \sum_{p=0}^{d} \sum_{q=0}^{n-d} y(M)\, [\mathcal{O}(1)]^T\, [\textstyle\bigwedge^p S]^T\, [\textstyle\bigwedge^q (Q^\vee)]^T\, u^p v^q = \sum_{S \subset [n]} t^{e_S} u^{d-\rho_M(S)} v^{|S|-\rho_M(S)}. \tag{8}$$

That is, the integral (8) is a generating function in $K_T^0(\mathtt{pt})[u,v]$ recording the subsets of $[n]$ which $r_M(u,v)$ enumerates.

Question 4.3 *Is there an equivariant version of Lemma 4.1 which provides a generating function in $K_T^0(\mathtt{pt})[u,v]$ for the bases of given activity, parallel to Theorem 4.2 for the rank generating function?*

We do not obtain any enriched equivariant version of h via Lemma 4.1.

Lemma 4.4 *In the setup of Theorem 1.2, $\int y(M)[\bigwedge^p S]^T[\bigwedge^q (Q^\vee)]^T \in \mathbb{Z}$ for any p and q, and equals 0 when $p \neq q$.*

Example 4.5 We outline the computation of t_M and h_M for the matroid M of Example 3.4. At a fixed point x_I, equivariant localisation of the summation in (7) yields

$$s'_I := \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{-e_P + e_Q} u^{|P|} v^{|Q|} = \prod_{i \in I} (1 + u t_i^{-1}) \prod_{j \in J} (1 + v t_j)$$

and the same sum multiplied by $\mathcal{O}(1)$ gives

$$s_I := \sum_{P \subseteq I} \sum_{Q \subseteq J} t^{e_P + e_Q} u^{d-|P|} v^{|Q|} = \prod_{i \in I} (u + t_i) \prod_{j \in J} (1 + v t_j),$$

where $J = [n] \setminus I$. Define also

$$h_I := \prod_{i \in I} \prod_{j \in J} (1 - t_i^{-1} t_j)^{-1}.$$

The reader or their computer algebra system may check that the $R(u,v)$ of Lemma 4.1 in the two cases is

$$\sum_I s_I h_I y(M)(x_I) = 5 + 4u + 4v + u^2 + uv + v^2$$

$$\sum_I s'_I h_I y(M)(x_I) = 1 - uv$$

yielding $t_M(z,w) = w + z + w^2 + wz + z^2$ and $h_M(t) = t$.

5 Flipping cones

Let f be a rational function in $\mathbb{Q}(z_1, z_2, \ldots, z_n)$. It is possible that many different Laurent power series represent f on different domains of convergence. This section discusses some implications of this phenomenon. By Lemma 4.1 and equivariant localization, the computations in our main results are reduced to manipulating sums of Hilbert series of certain infinite-dimensional T-representations. This section's results are of importance for gaining control over their coefficients.

The results can be thought of as generalizations of the relationships between the lattice point enumeration formulas of Brianchon-Gram, Brion and Lawrence-Varchenko [2]. We recommend [1] as a general introduction to generating functions for lattice points in cones.

Let $\mathcal{P}_n$ be the sub-vector space of real-valued functions on $\mathbb{Z}^n$ spanned by characteristic functions of lattice polytopes. If P is a pointed polytope, then the sum $\sum_{e\in P} z^e$ converges somewhere, and the value it converges to is a rational function in $\mathbb{Q}(z_1,\ldots,z_n)$ which we denote $\mathtt{hilb}(P)$. It is a theorem of Lawrence [14], and later Khovanski-Pukhlikov [11], that $\mathbf{1}(P)\mapsto \mathtt{hilb}(P)$ extends to a linear map $\mathtt{hilb}:\mathcal{P}_n\to\mathbb{Q}(z_1,\ldots,z_n)$.

Lemma 5.1 *The vector space $\mathcal{P}_n$ is spanned by the classes of simplicial cones.*

Let $\boldsymbol{\zeta} := (\zeta_1,\zeta_2,\ldots,\zeta_n)$ be a basis for $\mathbb{R}^n$, which is given the standard inner product. Define an order $<_{\boldsymbol{\zeta}}$ on $\mathbb{Q}^n$ by $x <_{\boldsymbol{\zeta}} y$ if, for some index i, we have $\langle\zeta_1,x\rangle=\langle\zeta_1,y\rangle$, $\langle\zeta_2,x\rangle=\langle\zeta_2,y\rangle$, $\ldots$, $\langle\zeta_{i-1},x\rangle=\langle\zeta_{i-1},y\rangle$ and $\langle\zeta_i,x\rangle<\langle\zeta_i,y\rangle$. (In fact we lose no strength in our applications if we reduce to the case of a single vector ζ, but the freedom to use a tuple of vectors with integer entries is convenient.)

We'll say that a polytope P is ***$\boldsymbol{\zeta}$-pointed*** if, for every $a\in\mathbb{R}^n$, the intersection $P\cap\{e : e <_{\boldsymbol{\zeta}} a\}$ is bounded. We'll say that an element in $\mathcal{P}_n$ is $\boldsymbol{\zeta}$-pointed if it is supported on a finite union of $\boldsymbol{\zeta}$-pointed polytopes. Let $\mathcal{P}_n^{\boldsymbol{\zeta}}$ be the vector space of $\boldsymbol{\zeta}$-pointed elements in $\mathcal{P}_n$.

Lemma 5.2 *The restriction of* $\mathtt{hilb}$ *to $\mathcal{P}_n^{\boldsymbol{\zeta}}$ is injective.*

Corollary 5.3 *Suppose that we have functions f_1, f_2, $\ldots$, f_r, g_1, g_2, $\ldots$, g_s in $\mathcal{P}_n^{\boldsymbol{\zeta}}$ and scalars a_1, $\ldots$, a_r, b_1, $\ldots$, b_s such that $\sum a_i\,\mathtt{hilb}(f_i)=\sum b_j\,\mathtt{hilb}(g_j)$. Let e be any lattice point in $\mathbb{Z}^n$. Then $\sum a_i f_i(e)=\sum b_j g_j(e)$.*

The next lemma, in the case of a single ζ, is the main result of [10].

Lemma 5.4 *Let $\boldsymbol{\zeta}=(\zeta_1,\ldots,\zeta_n)$ be as above. For every $f\in\mathcal{P}_n$, there is a unique $f^{\boldsymbol{\zeta}}\in\mathcal{P}_n^{\boldsymbol{\zeta}}$ such that $\mathtt{hilb}(f)=\mathtt{hilb}(f^{\boldsymbol{\zeta}})$. The map $f\mapsto f^{\boldsymbol{\zeta}}$ is linear.*

For example $\sum_{i\geq 0} z^i$ and $-\sum_{i<0} z^i$ both converge to $1/(1-z)$, on different domains, and correspond to $\{x : x\geq 0\}$ and $\{x : x<0\}$. In general, if C is a simplicial cone, $\mathbf{1}(C)^{\boldsymbol{\zeta}}$ can be computed by "flipping" defining inequalities in this sense, and possibly negating.

Lemma 5.5 *Let C be a pointed cone with vertex at w. Then $\mathbf{1}(C)^{\boldsymbol{\zeta}}$ is contained in the half space $\{x : \langle\zeta_1,x\rangle\geq\langle\zeta_1,w\rangle\}$. Furthermore, if C is not contained in $\{x : \langle\zeta_1,x\rangle\geq\langle\zeta_1,w\rangle\}$, then $\mathbf{1}(C)^{\boldsymbol{\zeta}}$ is in the open half space $\{x : \langle\zeta_1,x\rangle>\langle\zeta_1,w\rangle\}$.*

Corollary 5.6 *Let C_i be a finite sequence of pointed cones in $\mathbb{R}^n$, with the vertex of C_i at w_i. Let a_i be a finite sequence of scalars. Suppose that we know $\sum a_i\,\mathtt{hilb}(C_i)$ is a Laurent polynomial. Then its Newton polytope is contained in the convex hull of the w_i.*

6 Geometric interpretations of matroid operations

Our techniques give combinatorial proofs a number of facts about the behavior of h_M under standard matroid operations, originally proved geometrically in [19]. Before stating them, we introduce slightly more general polynomials for which they hold. Following section 4, we define

$$F_M^{m,T}(u,v) := \int y(M)[\mathcal{O}(m)]^T \sum_{p,q}[\textstyle\bigwedge^p S]^T[\textstyle\bigwedge^q(Q^\vee)]^T u^p v^q.$$

Its nonequivariant counterpart $F_M^m(u,v)$ is the unique polynomial, of degree $\leq n$ in u and v, such that

$$F_M^m(\mathcal{O}(1,0),\mathcal{O}(0,1)) = (\pi_{1(n-1)})_*\pi_d^*\left([\mathcal{O}(m)]y(M)\right). \tag{9}$$

We have seen that $F_M^{0,T}(u,v) = h_M(1-uv)$, that $F_M^{1,T}(u,v)$ and $F_M^1(u,v)$ are the weighted and unweighted rank generating functions, and that $F_M^1(u-1,v-1)$ is the Tutte polynomial. The entire collection of $F_M^{m,T}$ can be seen as a generalization of the Ehrhart polynomial of $\texttt{Poly}(M)$. Specifically, $F_M^m(0,0) = \#(m\cdot\texttt{Poly}(M)\cap\mathbb{Z}^n)$ for $m\geq 0$.

We denote the matroid dual to M by M^*. Given matroids M_1 and M_2 on respective ground sets $\mathbb{E}_1$ and $\mathbb{E}_2$, we denote their direct sum by $M_1\oplus M_2$. For $i_k\in\mathbb{E}_k$, there are three further standard matroid operations that join M_1 and M_2 together after identifying $i_1=i_2$: they are the series connection M_{ser}, the parallel connection M_{par}, and the two-sum $M_{2\mathrm{sum}}$.

Theorem 6.1 *We have*

(a) $F_M^m(u,v) = F_{M^*}^m(v,u)\in\mathbb{Z}[u,v]$

(b) $F_{M_1}^m F_{M_2}^m = F_{M_1\oplus M_2}^m$

(c) $F_{M_1\oplus M_2}^m = (1+v)F_{M_{\mathrm{ser}}}^m + (1+u)F_{M_{\mathrm{par}}}^m - (1+v)(1+u)F_{M_{2\mathrm{sum}}}^m$.

In particular, $F_{M_{2\mathrm{sum}}}^0 = F_{M_{\mathrm{ser}}}^0 = F_{M_{\mathrm{par}}}^0 = F_{M_1\oplus M_2}^0/(1-uv)$.

The series, respectively parallel, extension of a matroid M_1 along i_1 is its series, respectively parallel, connection to the uniform matroid $U_{1,2}$. Two-sum with $U_{1,2}$ leaves M_1 unchanged. Since $h_{U_{1,2}} = 1-uv$, Proposition 6.1(b,c) implies one of the most characteristic combinatorial properties of h from [19].

Corollary 6.2 *The values of h_M and F_M^0 are unchanged by series and parallel extensions.*

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Counting Shi regions with a fixed separating wall

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Abstract. Athanasiadis introduced separating walls for a region in the extended Shi arrangement and used them to generalize the Narayana numbers. In this paper, we fix a hyperplane in the extended Shi arrangement for type A and calculate the number of dominant regions which have the fixed hyperplane as a separating wall; that is, regions where the hyperplane supports a facet of the region and separates the region from the origin.

Résumé. Athanasiadis a introduit la notion d'hyperplan de séparation pour une région dans l'arrangement de Shi et l'a utilisée pour généraliser les numéros de Narayana. Dans cet article, nous fixons un hyperplan dans l'arrangement de Shi pour le type A et calculons le nombre de régions dominantes qui ont l'hyperplan fixe pour mur de séparation, c'est-à-dire les régions où l'hyperplan soutient une facette de la région et sépare la région de l'origine.

Keywords: Shi arrangement, partitions

1 Introduction

A hyperplane arrangement dissects its ambient vector space into regions. The regions have walls – hyperplanes which support facets of the region – and the walls may or may not separate the region from the origin. The regions in the extended Shi arrangement are enumerated by well-known sequences: all regions by the extended parking functions numbers, the dominant regions by the extended Catalan numbers, dominant regions with a given number of separating walls by the Narayana numbers. In this paper we study the extended Shi arrangement by fixing a hyperplane in it and calculating the number of regions for which that hyperplane is a separating wall. For example, suppose we are considering the mth extended Shi arrangement in dimension $n-1$, with highest root θ. Let $H_{\theta,m}$ be the mth translate of the hyperplane through the orgin with θ as normal. Then we show there are m^{n-2} regions which abut $H_{\theta,m}$ and are separated from the origin by it.

At the heart of this paper is a well-known bijection from certain integer partitions to dominant alcoves (and regions). One particularly nice aspect of our work is that we are able to use the bijection to enumerate regions. We characterize the partitions associated to the regions in question by certain interesting features and easily count those partitions, whereas it would be difficult to count the regions directly.

We rely on work from several sources. Shi (1986) introduced what is now called the Shi arrangement while studying the affine Weyl group of type A, and Stanley (1998) extended it. We also use the work on alcoves in Shi (1987). The work in Richards (1996) on decomposition numbers for Hecke algebras has been very useful. The Catalan numbers have been extended and generalized; see Athanasiadis (2005) for the history. Fuss-Catalan numbers is another name for the extended Catalan numbers. The Catalan numbers can be written as a sum of Narayana numbers. Athanasiadis (2005) generalized the Narayana numbers. He showed they enumerated several types of objects; one of them was the number of dominant Shi regions with a fixed number of separating walls. This led us to investigate separating walls. All of our work is for type A.

In Section 2, we introduce notation, define the Shi arrangement, certain partitions, and the bijection between them which we use to count regions. In Section 3, we characterize the partitions assigned to the regions which have $H_{\theta,m}$ as separating wall. Finally, we give a recursion for counting the regions which have other separating walls $H_{\alpha,m}$ in Section 4, by using generating functions.

2 Preliminaries

Here we introduce notation and review some constructions.

2.1 Root system notation and extended Shi arrangements

Let $\{\varepsilon_1, \ldots, \varepsilon_n\}$ be the standard basis of $\mathbb{R}^n$ and $\langle \mid \rangle$ be the bilinear form for which this is an orthonormal basis. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. Then $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$ is a basis of

$$V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i = 0\}.$$

We let $\alpha_{ij} = \alpha_i + \ldots + \alpha_j$, the highest root $\alpha_{1,n-1} = \theta$, and note that $\alpha_{ii} = \alpha_i$ and $\alpha_{ij} = \varepsilon_i - \varepsilon_{j+1}$.

The elements of $\Delta = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$ are called roots and we say a root α is positive, written $\alpha > 0$, if $\alpha \in \Delta^+ = \{\varepsilon_i - \varepsilon_j \mid i < j\}$.

A *hyperplane arrangement* is a set of hyperplanes, possibly affine hyperplanes, in V. We are interested in certain sets of hyperplanes of the following form. For each $\alpha \in \Delta^+$, we define its reflecting hyperplane

$$H_{\alpha,0} = \{v \in V \mid \langle v \mid \alpha \rangle = 0\} \text{ and for } k \in \mathbb{Z},\ H_{\alpha,0}\text{'s } k\text{th translate, } H_{\alpha,k} = \{v \in V \mid \langle v \mid \alpha \rangle = k\}.$$

Note $H_{-\alpha,-k} = H_{\alpha,k}$ so we usually take $k \in \mathbb{Z}_{\geq 0}$. Then the extended Shi arrangement, here called the *m-Shi arrangement*, is the collection of hyperplanes

$$\mathcal{H}_m = \{H_{\alpha,k} \mid \alpha \in \Delta^+, -m < k \leq m\}.$$

This arrangement is defined for crystallograhic root systems of all finite types.

Regions of the m-Shi arrangement are the connected components of the hyperplane arrangement complement $V \setminus \bigcup_{H \in \mathcal{H}_m} H$.

We denote the closed half-spaces $\{v \in V \mid \langle v \mid \alpha \rangle \geq k\}$ and $\{v \in V \mid \langle v \mid \alpha \rangle \leq k\}$ by ${H_{\alpha,k}}^+$ and ${H_{\alpha,k}}^-$ respectively. The *dominant* chamber of V is $V \cap \bigcap_{i=1}^{n-1} {H_{\alpha_i,0}}^+$ and is also referred to as the fundamental chamber in the literature. This paper primarily concerns regions and alcoves in the dominant chamber.

A *dominant region* of the m-Shi arrangement is a region that is contained in the dominant chamber. We call the collection of dominant regions in the m-Shi arrangement $\mathcal{S}_{n,m}$.

Each connected component of

$$V \setminus \bigcup_{\substack{\alpha\in\Delta^+ \\ k\in\mathbb{Z}}} H_{\alpha,k}$$

is called an *alcove* and the *fundamental alcove* is $\mathcal{A}_0$, the interior of $H_{\theta,1}{}^- \cap \bigcap_{i=1}^{n-1} H_{\alpha_i,0}{}^+$, where $\theta = \alpha_1 + \cdots + \alpha_{n-1} = \varepsilon_1 - \varepsilon_n$. A *dominant alcove* is one contained in the dominant chamber. Denote the set of dominant alcoves by $\mathfrak{A}_n$.

A *wall* of a region is a hyperplane in $\mathcal{H}_m$ which supports a facet of that region or alcove. Two open regions are *separated* by a hyperplane H if they lie in different closed half-spaces relative to H. Please see Athanasiadis (2005) for details. We study dominant regions with a fixed separating wall. A *separating wall* for a region R is a wall of R which separates R from $\mathcal{A}_0$.

2.2 The affine symmetric group

Definition 2.1 *The affine symmetric group, denoted $\widehat{\mathfrak{S}}_n$, is defined as*

$$\widehat{\mathfrak{S}}_n = \langle s_1, \ldots, s_{n-1}, s_0 \mid s_i^2 = 1, \quad s_i s_j = s_j s_i \text{ if } i \not\equiv j \pm 1 \mod n, \\ s_i s_j s_i = s_j s_i s_j \text{ if } i \equiv j \pm 1 \mod n\rangle$$

for $n > 2$, but $\widehat{\mathfrak{S}}_2 = \langle s_1, s_0 \mid s_i^2 = 1\rangle$.

The affine symmetric group $\widehat{\mathfrak{S}}_n$ acts freely and transitively on the set of alcoves. We thus identify each alcove $\mathcal{A}$ with the unique $w \in \widehat{\mathfrak{S}}_n$ such that $\mathcal{A} = w^{-1}\mathcal{A}_0$. Each simple generator s_i, $i > 0$, acts by reflection with respect to the simple root α_i. In other words, it acts by reflection over the hyperplane $H_{\alpha_i,0}$. The element s_0 acts as reflection with respect to the affine hyperplane $H_{\theta,1}$.

2.3 Shi coordinates and Shi tableaux.

Every alcove $\mathcal{A}$ can be written as $w^{-1}\mathcal{A}_0$ for a unique $w \in \widehat{\mathfrak{S}}_n$ and additionally, for each $\alpha \in \Delta^+$, there is a unique integer k_α such that $k_\alpha < \langle \alpha \mid x\rangle < k_\alpha + 1$ for all $x \in \mathcal{A}$. Shi characterized the integers k_α which can arise in this way and the next lemma gives the conditions for type A.

Lemma 2.2 (Shi (1987)) *Let $\{k_{\alpha_{ij}}\}_{1\le i\le j\le n-1}$ be a set of $\binom{n}{2}$ integers. There exists a $w \in \widehat{\mathfrak{S}}_n$ such that*

$$k_{\alpha_{ij}} < \langle \alpha_{ij} \mid x\rangle < k_{\alpha_{ij}} + 1$$

for all $x \in w^{-1}\mathcal{A}_0$ if and only if

$$k_{\alpha_{it}} + k_{\alpha_{t+1,j}} \le k_{\alpha_{ij}} \le k_{\alpha_{it}} + k_{\alpha_{t+1,j}} + 1,$$

for all t such that $i \le t < j$.

From now on, we write k_{ij} for $k_{\alpha_{ij}}$. These $\{k_{ij}\}_{1\le i\le n-1}$ are the *Shi coordinates* of the alcove. We arrange the coordinates for an alcove $\mathcal{A}$ in the Young's diagram (see Section 2.4) of a staircase partition $(n-1, n-2, \ldots, 1)$ by putting k_{ij} in the box in row i, column $n-j$. See Krattenthaler et al. (2002) for a similar arrangement of sets indexed by positive roots. For alcoves in $\mathfrak{A}_n$, the entries are nonincreasing along rows and columns and are nonnegative.

We can also assign coordinates to regions in the Shi arrangement. In each region of the m-Shi hyperplane arrangement, there is exactly one "representative," or m-minimal, alcove closest to the fundamental alcove $\mathcal{A}_0$. See Shi (1986) for $m = 1$ and Athanasiadis (2005) for $m \geq 1$. Let $\mathcal{A}$ be an alcove with Shi coordinates $\{k_{ij}\}_{1 \leq i \leq n-1}$ and suppose it is the m-minimal alcove for the region R. We define coordinates $\{e_{ij}\}_{1 \leq i \leq j \leq n-1}$ for R by $e_{ij} = \min(k_{ij}, m)$.

Again, we arrange the coordinates for a region R in the Young's diagram (see Section 2.4) of a staircase partition $(n-1, n-2, \ldots, 1)$ by putting e_{ij} in the box in row i, column $n-j$. For dominant regions, the entries are nonincreasing along rows and columns and are nonnegative.

Example 2.3 *For $n = 5$, the coordinates are arranged*

k_{14}	k_{13}	k_{12}	k_{11}
k_{24}	k_{23}	k_{22}	
k_{34}	k_{33}		
k_{44}			

e_{14}	e_{13}	e_{12}	e_{11}
e_{24}	e_{23}	e_{22}	
e_{34}	e_{33}		
e_{44}			

Example 2.4 *The dominant chamber for the* 2*-Shi arrangement for $n = 3$ is illustrated in Figure 1 The yellow region has coordinates $e_{12} = 2$, $e_{11} = 1$, and $e_{22} = 2$. Its* 2*-minimal alcove has coordinates $k_{12} = 3$, $k_{11} = 1$, and $k_{22} = 2$.*

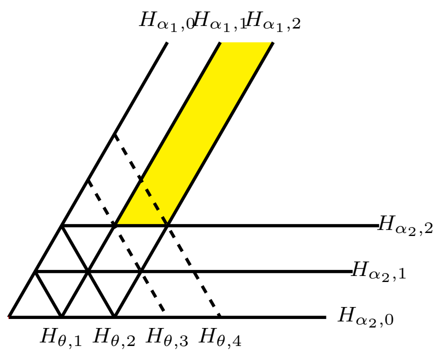

Fig. 1: $\mathcal{S}_{3,2}$ consists of 12 regions

Denote the *Shi tableau* for the alcove $\mathcal{A}$ by $T_{\mathcal{A}}$ and for the region R by T_R.

Both Richards (1996) and Athanasiadis (2005) characterized the Shi tableaux for dominant m-Shi regions.

Lemma 2.5 *Let $T = \{e_{ij}\}_{1 \leq i \leq j \leq n-1}$ be a collection of integers such that $0 \leq e_{ij} \leq m$. Then T is the Shi tableau for a region $R \in \mathcal{S}_{n,m}$ if and only if*

$$e_{ij} = \begin{cases} e_{it} + e_{t+1,j} \text{ or } e_{it} + e_{t+1,j} + 1 & \text{if } m - 1 \geq e_{it} + e_{t+1,j} \text{ for } t = i, \ldots, j-1 \\ m & \text{otherwise} \end{cases} \tag{2.1}$$

Proof: Proof omitted in abstract. □

Lemma 3.9 from Athanasiadis (2005) is crucial to our work here. He characterizes the co-filtered chains of ideals for which $H_{\alpha,m}$ is a separating wall. We translate that into our set-up in Lemma 2.6, using entries from the Shi Tableau.

Lemma 2.6 (Athanasiadis (2005)) *A region $R \in \mathcal{S}_{n,m}$ has $H_{\alpha_{uv},m}$ as a separating wall if and only if $e_{uv} = m$ and for all t such that $u \leq t < v$, $e_{ut} + e_{t+1,v} = m - 1$.*

2.4 Partitions

A *partition* is a non-increasing sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ of nonnegative integers. $\lambda_1, \lambda_2, \ldots$ are called the *parts* of λ. We identify a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with its Young diagram, that is the array of boxes with coordinates $\{(i, j) : 1 \leq j \leq \lambda_i \text{ for all } \lambda_i\}$. The *conjugate* of λ is the partition λ' whose diagram is obtained by reflecting λ's diagram about the diagonal. The *length* of a partition λ, $\ell(\lambda)$, is the number of positive parts of λ.

2.4.1 Core partitions

The (k, l)-*hook* of any partition λ consists of the (k, l)-box of λ, all the boxes to the right of it in row k together with all the nodes below it and in column l. The *hook length* h^{λ}_{kl} of the box (k, l) is the number of boxes in the (k, l)-hook. Let n be a positive integer. An *n-core* is a partition λ such that $n \nmid h^{\lambda}_{(k,l)}$ for all $(k, l) \in \lambda$. We let $\mathcal{C}_n$ denote the set of partitions which are n-cores.

2.5 Abacus diagrams

In Section 3, we use a bijection, called Φ, to describe certain regions. We will need abacus diagrams to define Φ. We associate to each partition λ its abacus diagram. When λ is an n-core, its abacus has a particularly nice form.

The *β-numbers* for a partition $\lambda = (\lambda_1, \ldots, \lambda_r)$ are the hook lengths from the boxes in its first column:

$$\beta_k = h^{\lambda}_{(k,1)}.$$

Each partition is determined by its β-numbers and $\beta_1 > \beta_2 > \cdots > \beta_{\ell(\lambda)} > 0$.

An *abacus diagram* is a diagram, with integer entries arranged in n columns labeled $0, 1, \ldots, n-1$, called *runners*. The horizontal cross-sections or rows will be called *levels* and runner k contains the integer entry $qn + r$ on level q where $-\infty < q < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, and we always put runner 0 in the leftmost position, increasing to runner $n - 1$ in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. The *level* of a bead labelled by $qn + r$ is q and its runner is r. Entries which are not circled will be called *gaps*. Two abacus diagrams are equivalent if one can be obtained by adding a constant to each entry of the other.

See Example 2.8 below.

Given a partition λ its abacus is any abacus diagram equivalent to the one with beads at entries $\beta_k = h^{\lambda}_{(k,1)}$ and all entries $j \in \mathbb{Z}_{<0}$.

Given the abacus for the partition λ with beads at $\{\beta_k\}_{1 \leq k \leq \ell(\lambda)}$, let b_i be one more than the largest level number of a bead on runner i; that is, the level of the first gap. Then $(b_0, \ldots, b_{n-1})$ is the *vector of level numbers* for λ.

Remark 2.7 *It is well-known that λ is an n-core if and only if its abacus is flush, that is to say whenever there is a bead at entry j there is also a bead at $j-n$. Additionally, if $(b_0,\ldots,b_{n-1})$ is the vector of level numbers for λ, then $b_0=0$, $\sum_{i=0}^{n-1} b_i = \ell(\lambda)$, and since there are no gaps, $(b_0\ldots,b_{n-1})$ describes λ completely.*

Example 2.8 *The abacus below in Figure 2 represents the 4-core $\lambda=(5,2,1,1,1)$. The levels are indicated to the left of the abacus and below each runner is the largest level number of a bead in that runner. The boxes of the Young diagram of λ have been filled with their hooklengths. The vector of level numbers for λ is $(0,3,1,1)$.*

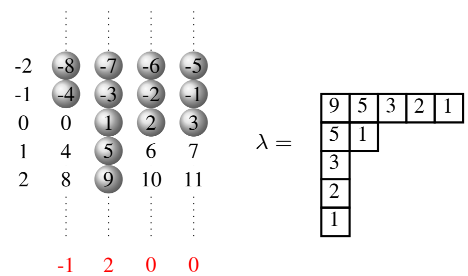

Fig. 2: The abacus represents the 4-core λ.

2.6 Bijection

We describe here a bijection Φ from the set of n-cores to dominant alcoves. It is a slightly modified version of the bijection given in Richards (1996). Given an n-core λ, let $(b_0=0,b_1,\ldots,b_{n-1})$ be the level numbers for its abacus. Now let $\tilde{p}_i = b_{i-1}n + i - 1$, which is the entry of the first gap on runner i, for i from 1 to n, and then let $p_1 = 0 < p_2 < \cdots < p_n$ be the $\{\tilde{p}_i\}$ written in ascending order. Finally we define $\Phi(\lambda)$ to be the alcove whose Shi coordinates are given by

$$k_{ij} = \lfloor \frac{p_{j+1}-p_i}{n} \rfloor$$

for $1 \le i \le j \le n-1$.

Example 2.9 *We continue Example 2.8. We have $n=4$, $\lambda=(5,2,1,1,1)$, and $(b_0,b_1,b_2,b_3) = (0,3,1,1)$. Then $\tilde{p}_1=0$, $\tilde{p}_2=13$, $\tilde{p}_3=6$, and $\tilde{p}_4=7$ and $p_1=0$, $p_2=6$, $p_3=7$, and $p_4=13$. Thus $\Phi(\lambda)$ is the alcove with coordinates $k_{13}=3$, $k_{12}=1$, $k_{11}=1$, $k_{23}=1$,$k_{22}=0$, and $k_{33}=1$.*

Proposition 2.10 *The map Φ from n-cores to dominant alcoves is a bijection.*

Proof: We first show that we indeed produce an alcove by the process above. By Lemma 2.2, it is enough to show that $k_{it}+k_{t+1,j} \le k_{ij} \le k_{it}+k_{t+1,j}+1$ for all t such that $1 \le t < j$. Write $p_i = nq_i + r_i$ for $1 \le i \le n$. Then

$$k_{it} = \begin{cases} q_{t+1}-q_i & \text{if } r_{t+1} > r_i \\ q_{t+1}-q_i-1 & \text{if } r_{t+1} < r_i, \end{cases} \quad k_{t+1,j} = \begin{cases} q_{j+1}-q_{t+1} & \text{if } r_{j+1} > r_{t+1} \\ q_{j+1}-q_{t+1}-1 & \text{if } r_{j+1} < r_{t+1}. \end{cases}$$

and

$$k_{ij} = \begin{cases} q_{j+1} - q_i & \text{if } r_{j+1} > r_i \\ q_{j+1} - q_i - 1 & \text{if } r_{j+1} < r_i. \end{cases} . \tag{2.2}$$

Therefore

$$k_{ij} = \begin{cases} k_{it} + k_{t+1,j} & \text{if } r_i < r_{t+1} < r_{j+1} \text{ or } r_{j+1} < r_i < r_{t+1} \text{ or } r_{t+1} < r_{j+1} < r_i \\ k_{it} + k_{t+1,j} + 1 & \text{if } r_i < r_{j+1} < r_{t+1} \text{ or } r_{t+1} < r_i < r_{j+1} \text{ or } r_{j+1} < r_{t+1} < r_i \end{cases},$$

so that the conditions in Lemma 2.2 are satisfied and we have the Shi coordinates of an alcove. Since each $k_{ij} \geq 0$, it is an alcove in the dominant chamber.

Now we reverse the process described above to show that Φ is a bijection. Let $\{k_{ij}\}_{1\leq i \leq j \leq n-1}$ be the Shi coordinates of a dominant alcove. Again, write $p_i = nq_i + r_i$ for the intermediate values $\{p_i\}$, which we first calculate. Then $p_1 = q_1 = r_1 = 0$ and $q_i = k_{1,i-1}$. We must now determine $r_2, \ldots, r_n$, a permutation of $1, \ldots, n-1$. However, by (2.2) we can determine the inversion table for this permutation, using k_{ij} for $2 \leq i \leq j \leq n-1$ and $q_1, \ldots, q_n$, so we can compute $r_2, \ldots, r_n$ and therefore $p_1, p_2, \ldots, p_n$. We can now sort the $\{p_i\}$ according to their residue mod n, giving us $\tilde{p}_1, \ldots, \tilde{p}_n$; from this, $(b_0, \ldots, b_{n-1})$. Note that $(b_0, \ldots, b_{n-1})$ is a permutation of $q_1, \ldots, q_n$. □

Example 2.11 *We continue Examples 2.8 and 2.9 here. Suppose we are given that $n = 4$ and the alcove coordinates $k_{13} = 3$, $k_{12} = 1$, $k_{11} = 1$, $k_{23} = 1$, $k_{22} = 0$, and $k_{33} = 1$. We demonstrate Φ^{-1} and calculate (b_0, b_1, b_2, b_3) and thereby the 4-core λ. We have $q_1 = 0$, $q_2 = 1$, $q_3 = 1$, and $q_4 = 3$, and $r_1 = 0$, from k_{13}, k_{12}, and k_{11}. We must determine r_2, r_3, r_4, a permutation of $1, 2, 3$.*

$$\begin{array}{lllll} k_{23} & = & 1 & = & q_4 - q_2 - 1 \quad \textit{so } r_4 < r_2 \\ k_{22} & = & 0 & = & q_3 - q_2 \quad \textit{so } r_3 > r_2 \\ k_{33} & = & 1 & = & q_4 - q_3 - 1 \quad \textit{so } r_4 < r_3 \end{array}$$

Therefore we have $r_3 = 3$, $r_2 = 2$, and $r_4 = 1$, which means $b_1 = q_4 = 3$, $b_2 = q_2 = 1$, and $b_3 = q_3 = 1$.

Remark 2.12 *There is a well-known action of $\widehat{\mathfrak{S}}_n$ on n-cores; please see Misra and Miwa (1990), Lascoux (2001), Lapointe and Morse (2005), for more details and history. This leads to a bijection Ψ from n-cores to dominant alcoves, where $w\emptyset \mapsto w^{-1}\mathcal{A}_0$. We mention that $\Phi = \Psi$, a fact which we will neither use nor prove here. We also remark that the column (or row) sums of the Shi tableau of an alcove give us a partition whose conjugate is $(n-1)$-bounded, as in the bijections of Lapointe and Morse (2005) or Björner and Brenti (1996)*

3 Separating wall $H_{\theta,m}$

Separating walls were defined in Section 2.1 as a wall of a region which separates the region from $\mathcal{A}_0$. Equivalently for alcoves, $H_{\alpha,k}$ is a separating wall for the alcove $w^{-1}\mathcal{A}_0$ if there is a simple reflection s_i, where $0 \leq i < n$, such that $w^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^+$ and $(s_i w)^{-1}\mathcal{A}_0 \subseteq {H_{\alpha,k}}^-$. We want to count the regions which have $H_{\alpha,m}$ as a separating wall, for any $\alpha \in \Delta^+$. We do this by induction and the base case will be $\alpha = \theta$. Our main result in this section characterizes the regions which have $H_{\theta,m}$ as a separating wall by describing the n-core partitions associated to them under the bijection Φ described in Section 2.6.

Theorem 3.1 *Let $\Phi : \mathcal{C}_n \to \mathfrak{A}_n$ be the bijection described in Section 2.6, let $R \in \mathcal{S}_{n,m}$ have m-minimal alcove $\mathcal{A}$, and let λ be the n-core such that $\Phi(\lambda) = \mathcal{A}$. Then $H_{\theta,m}$ is a separating wall for the region R if and only if the Shi coordinates of the region R are the same as the Shi coordinates of its m-minimal alcove $\mathcal{A}$ and $h_{11}^{\lambda} = n(m-1)+1$.*

Proof: Proof omitted in abstract. □

We have the following corollary to Theorem 3.1.

Corollary 3.2 *There are m^{n-2} regions in $\mathcal{S}_{n,m}$ which have $H_{\alpha_{1n-1},m}$ as a separating wall.*

Proof: Let $\vec{b}(\lambda) = (b_0, b_1, \ldots, b_{n-1})$ be the vector of level numbers for the n-core λ. Note that $h_{11} = n(m-1)+1$ if and only if $b_1 = m$ and $b_i < m$ for $1 < i \leq n-1$. There are m^{n-2} vectors of level numbers $(b_0, b_1, \ldots, b_{n-1})$ such that $b_0 = 0$, $b_1 = m$, and $0 \leq b_i \leq m-1$ for $2 \leq i \leq n-1$. □

4 Arbitrary separating wall

We use $\mathfrak{h}^n_{\alpha k}$ to denote the set of regions in $\mathcal{S}_{n,m}$ which have $H_{\alpha,k}$ as a separating wall. See Figure 3. In the language of Athanasiadis (2005), these are the regions whose corresponding co-filtered chain of ideals have α as an indecomposable element of rank k.

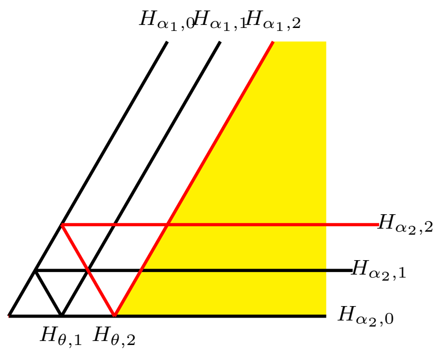

Fig. 3: There are three regions in $\mathfrak{h}^3_{\alpha_1 2}$

In this section, we present a generating function for regions in $\mathfrak{h}^n_{\alpha k}$. We use two statistics r() and c() on regions in the extended Shi arrangement. Let $R \in \mathcal{S}_{n,m}$ and define

$$\mathrm{r}(R) = |\{(j,k) : R \in \mathfrak{h}^n_{\alpha_{1j}k} \text{ and } 1 \leq k \leq m\}| \text{ and } \mathrm{c}(R) = |\{(i,k) : R \in \mathfrak{h}^n_{\alpha_{i,n-1}k} \text{ and } 1 \leq k \leq m\}|.$$

$\mathrm{r}(R)$ counts the number of translates of $H_{\alpha_{1j},0}$ which separate R from $\mathcal{A}_0$, for $1 \leq j \leq n-1$. Similarly for $\mathrm{c}(R)$ and translates of $H_{\alpha_{i,n-1},0}$.

The generating function is

$$f^n_{\alpha_{ij}m}(p,q) = \sum_{R \in \mathfrak{h}^n_{\alpha_{ij}m}} p^{\mathrm{c}(R)} q^{\mathrm{r}(R)}.$$

Example 4.1 $f^3_{\alpha_1 2}(p,q) = p^4q^2 + p^4q^3 + p^4q^4$.

We let $[k]_{p,q} = \sum_{j=0}^{k-1} p^j q^{k-1-j}$ and $[k]_q = [k]_{1,q}$. We will also need to truncate polynomials and the notation we use for that is

$$\left(\sum_{j=0}^{j=n} a_j q^j\right)_{\leq q^N} = \sum_{j=0}^{j=N} a_j q^j.$$

The statistics are related to the n-core partition assigned to the m-minimal alcove for the region.

Claim 4.2 *Let λ be an n-core with vector of level numbers $(b_0, \ldots, b_{n-1})$ and suppose $\Phi(\lambda) = R$ and $R \in \mathfrak{h}^n_{\theta m}$. Then $r(R) = m + \sum_{i=2}^{n-1} b_i$ and $c(R) = m + \sum_{i=2}^{n-1}(m-1-b_i)$.*

Proof: Proof omitted in abstract. $\square$

We thus obtain another corollary to Theorem 3.1.

Corollary 4.3

$$f^n_{\theta,m}(p,q) = p^m q^m (p^{m-1} + p^{m-2}q + \cdots + pq^{m-1} + q^{m-1})^{n-2} = p^m q^m [m]^{n-2}_{p,q}.$$

Corollary 4.3 follows from Claim 4.2 and the abacus representation of n-cores which have the prescribed hook length.

Corollary 3.2 can be derived from Corollary 4.3 by evaluating at $p = q = 1$.

Given a Shi tableau $T_R = \{e_{ij}\}_{1 \leq i \leq j \leq n-1}$, where $R \in \mathcal{S}_{n,m}$, let $\tilde{T}_R = \{\tilde{e}_{ij}\}_{1 \leq i \leq j \leq n-2}$ denote the tableau where $\tilde{e}_{ij}$ is given by e_{ij}. That is, $\tilde{T}_R$ is T_R with the first column removed.

Example 4.4 *Suppose $R \in \mathcal{S}_{5,m}$ and*

$$T_R = \begin{array}{|c|c|c|c|} \hline e_{14} & e_{13} & e_{12} & e_{11} \\ \hline e_{24} & e_{23} & e_{22} \\ \cline{1-3} e_{34} & e_{33} \\ \cline{1-2} e_{44} \\ \cline{1-1} \end{array}. \text{ Then } \tilde{T}_R = \begin{array}{|c|c|c|} \hline e_{13} & e_{12} & e_{11} \\ \hline e_{23} & e_{22} \\ \cline{1-2} e_{33} \\ \cline{1-1} \end{array}$$

The next lemma tells us that $\tilde{T}_R$ is always the Shi tableau for a region in one less dimension.

Lemma 4.5 *If T_R is the tableau of a region $R \in \mathcal{S}_{n,m}$ and $1 \leq u \leq v \leq n-1$, then $\tilde{T}_R = T_{\tilde{R}}$ for some $\tilde{R} \in \mathcal{S}_{n-1,m}$.*

Proof: This follows from Lemma 2.5. $\square$

Lemma 4.6 *Let T_R be the Shi tableau for the region $R \in \mathcal{S}_{n,m}$ and let $\tilde{R}$ be defined by $T_{\tilde{R}} = \tilde{T}_R$, where $\tilde{R} \in \mathcal{S}_{n-1,m}$ by Lemma 4.5. Then $R \in \mathfrak{h}^n_{\alpha_{i,n-2}m}$ if and only if $\tilde{R} \in \mathfrak{h}^{n-1}_{\alpha_{i,n-2}m}$.*

Proof: This follows from Lemma 2.6. $\square$

In terms of generating functions, Lemma 4.6 states:

$$f^n_{\alpha_{i,n-2}m}(p,q) = \sum_{R\in\mathfrak{h}^n_{\alpha_{i,n-2}m}} p^{\mathrm{c}(R)}q^{\mathrm{r}(R)} = \sum_{\hat{R}\in\mathfrak{h}^{n-1}_{\alpha_{i,n-2}m}} \sum_{R\in\mathcal{S}_{n,m}:\tilde{R}=\hat{R}} p^{\mathrm{c}(R)}q^{\mathrm{r}(R)} \tag{4.1}$$

If $\hat{R} \in \mathfrak{h}^{n-1}_{\alpha_{i,n-n}}$, so that $e_{i,n-2} = m$, and $\tilde{R} = \hat{R}$, then $\mathrm{r}(R) = \mathrm{r}(\hat{R}) + m$ and $\mathrm{c}(R) = \mathrm{c}(\hat{R}) + k$, for some k. We need to establish the possible values for k.

We will use Proposition 3.5 from Richards (1996) to do this. His "pyramids" correspond to our Shi tableaux for regions, with his e and w being our n and $m+1$. He does not mention hyperplanes, but with the conversion ${}_ua_v = m - e_{u+1,v}$ his conditions in Proposition 3.4 become our conditions in Lemma 2.5.

In our language, his Proposition 3.5 [(i)] becomes

Lemma 4.7 (Richards (1996)) *Let $s_1, s_2, \ldots, s_n$ be non-negative integers with*

$$s_1 \geq s_2 \geq \ldots s_n = 0 \textit{ and } s_i \leq (n-i)m.$$

Then there is a unique region $R \in \mathcal{S}_{n,m}$ with Shi tableau $T_R = \{e_{ij}\}_{1\leq i\leq jn-1}$ such that

$$s_j = s_j(R) = \sum_{i=1}^{n-j} e_{i,n-j} \textit{ for } 1 \leq j \leq n-1$$

We include a proof for compeleteness. **Proof:** Proof omitted in abstract. □

Lemma 4.7 means for all pairs $(T_{\hat{R}}, k)$, where $T_{\hat{R}} = \{\hat{e}_{ij}\}_{1\leq i\leq j\leq n-2}$ and $\hat{R} \in \mathfrak{h}^{n-1}_{\alpha m}$, and k is an integer such that $\sum_{i=1}^{n-2} \hat{e}_{i,n-2} \leq k \leq (n-1)m$, there is a region $R \in \mathcal{S}_{n,m}$ whose Shi tableau has first column sum is k and gives $T_{\hat{R}}$ when its first column is removed; that is, $\tilde{T}_R = T_{\hat{R}}$.

Continuing (4.1), keeping in mind that $s_1(R) = \mathrm{c}(R)$,

$$\begin{aligned} f^n_{\alpha_{i,n-2}m}(p,q) &= \sum_{\hat{R}\in\mathfrak{h}^{n-1}_{\alpha_{i,n-2}m}} \sum_{s_1(\hat{R})\leq k\leq n(m-1)} p^{\mathrm{r}(\hat{R})+m}q^{\mathrm{c}(\hat{R})+k} && (4.2)\\ &= \left(p^m[(n-2)m+1]_q f^{n-1}_{\alpha_{i,n-2}m}(p,q)\right)_{\leq q^{(n-1)m}}. && (4.3)\end{aligned}$$

Example 4.8 *Consider R_1, R_2, and R_3 in $\mathcal{S}_{3,2}$ with tableaux*

2	2	1
2	2	
2		

2	2	1
2	2	
1		

2	2	1
2	2	
0		

respectively. Then $\tilde{R_1} = \tilde{R_2} = \tilde{R_2} = R$, where R is the region in $\mathcal{S}_{2,2}$ with tableau

2	1
2	

[(i)] In the statement of Proposition 3.5, Richards makes the claim for a unique Shi tableau for $0 \leq j \leq n-2$. However, in the proof, he shows the result for $0 \leq j \leq n-1$.

The next proposition will provide a method for determining whether or not $H_{\alpha_{1n-j},m}$ is a separating wall for R. Given a Shi tableau $T = \{b_{ij}\}_{1\le i\le j\le n-1}$ for a region in $\mathcal{S}_{n,m}$, let T' be its *conjugate* given by $T' = \{b'_{ij}\}_{1\le i\le j\le n-1}$, where $b'_{ij} = b_{n-j,n-i}$.

Example 4.9

By Lemma 2.5, T' will also be Shi tableau of a region in $\mathcal{S}_{n,m}$. Additionally, by Lemma 2.6, we have the following proposition.

Proposition 4.10 *Suppose the regions R and R' are related by*

$$(T_R)' = T_{R'}.$$

Then $R \in \mathfrak{h}^n_{\alpha_{ij}m}$ if and only if $R \in \mathfrak{h}^n_{\alpha_{n-j,n-i}m}$.

In terms of generating functions, this becomes the following:

$$f^n_{\alpha_{ij}m}(p,q) = f^n_{\alpha_{n-j,n-i}m}(q,p). \tag{4.4}$$

We will now combine Theorem 3.1, Proposition 4.6, and Proposition 4.10 to produce an expression for the generating function for regions with a given separating wall.

Given a polynomial $f(p,q)$ in two variables, let $\phi_{km}(f)$ be the polynomial

$$\left(p^m[m(k-2)+1]_q f(p,q)\right)_{\le q^{(k-1)m}}$$

and let $\rho(f)$ be the original polynomial with p and q reversed: $f(q,p)$. Then (4.2) is

$$f^n_{\alpha_{ij}m}(p,q) = \phi_{nm}(f^{n-1}_{\alpha_{ij}m}(p,q)) \text{ and (4.4) is } f^n_{im}(j,p)q = \rho(f^n_{\alpha_{n-j,n-i}m}(p,q)).$$

Finally, the full recursion is

Theorem 4.11

$$f^n_{\alpha_{uv}m}(p,q) = \phi_n(\phi_{n-1}(\ldots\phi_{v+2}(\rho(\phi_{v+1}(\ldots(\phi_{v-u+3}(p^m q^m [m]^{v-u}_{p,q})\ldots).$$

The idea behind the theorem is that, given a root α_{uv} in dimension $n-1$, we remove columns using Lemma 4.7 until we are in dimension $(v+1)-1$, then we conjugate, then remove columns again until our root is $\alpha_{1,v-u+1}$ and we are in dimension $(v-u+2)-1$.

Example 4.12 *We would like to know how many elements there are in $\mathfrak{h}^7_{\alpha_{24}2}$; that is, how many dominant regions in the 2-Shi arrangement for $n=7$ have $H_{\alpha_{24},2}$ as a separating wall.*

$$\begin{aligned}
f^7_{\alpha_{24}2}(p,q) &= \left(p^2[13]_q f^6_{\alpha_{24}2}(p,q)\right)_{\le q^{12}} \\
&= \left(p^2[13]_q \left(p^2[11]_q f^5_{\alpha_{24}2}(p,q)\right)_{\le q^{10}}\right)_{\le q^{12}} \\
&= \left(p^2[13]_q \left(p^2[11]_q f^5_{\alpha_{13}2}(q,p)\right)_{\le q^{10}}\right)_{\le q^{12}} \\
&= \left(p^2[13]_q \left(p^2[11]_q \left(q^2[9]_p f^4_{\alpha_{13}2}(q,p)\right)_{\le p^8}\right)_{\le q^{10}}\right)_{\le q^{12}} \\
&= \left(p^2[13]_q \left(p^2[11]_q \left(q^2[9]_p \left(p^2q^2[2]^2_{p,q}\right)\right)_{\le p^8}\right)_{\le q^{10}}\right)_{\le q^{12}}
\end{aligned}$$

After expanding this polynomial and evaluating at $p = q = 1$, we see there are 781 such regions.

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Cofree compositions of coalgebras

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Abstract. We develop the notion of the composition of two coalgebras, which arises naturally in higher category theory and the theory of species. We prove that the composition of two cofree coalgebras is cofree and give conditions which imply that the composition is a one-sided Hopf algebra. These conditions hold when one coalgebra is a graded Hopf operad $\mathcal{D}$ and the other is a connected graded coalgebra with coalgebra map to $\mathcal{D}$. We conclude with examples of these structures, where the factor coalgebras have bases indexed by the vertices of multiplihedra, composihedra, and hypercubes.

Résumé. Nous développons la notion de composition de coalgèbres, qui apparaît naturellement dans la théorie des catégories d'ordre supérieur et dans la théorie des espèces. Nous montrons que la composée de deux coalgèbres colibres est colibre et nous donnons des conditions qui impliquent que la composée est une algèbre de Hopf unilatérale. Ces conditions sont valables quand l'une des coalgèbres est une opérade de Hopf graduée $\mathcal{D}$ et l'autre est une coalgèbre graduée connexe avec un morphisme vers $\mathcal{D}$. Nous concluons avec des exemples de ces structures, où les coalgèbres composées ont des bases indexées par les sommets de multiplièdres, de composièdres, et d'hypercubes.

Keywords: multiplihedron, cofree coalgebra, Hopf algebra, operad, species

1 Introduction

The Hopf algebras of ordered trees (Malvenuto and Reutenauer (1995)) and of planar binary trees (Loday and Ronco (1998)) are cofree coalgebras that are connected by cellular maps from permutahedra to associahedra. Related polytopes include the multiplihedra (Stasheff (1970)) and the composihedra (Forcey (2008b)), and it is natural to study what Hopf structures may be placed on these objects. The map from permutahedra to associahedra factors through the multiplihedra, and in (Forcey et al. (2010)) we used this factorization to place Hopf structures on bi-leveled trees, which correspond to vertices of multiplihedra.

Multiplihedra form an operad module over associahedra. This leads to painted trees, which also correspond to the vertices of the multiplihedra. In terms of painted trees, the Hopf structures of (Forcey et al. (2010)) are related to the operad module structure. We generalize this in Section 3, defining the functorial construction of a graded coalgebra $\mathcal{D} \circ \mathcal{C}$ from graded coalgebras $\mathcal{C}$ and $\mathcal{D}$. We show that this composition of coalgebras preserves cofreeness. In Section 4 we give sufficient conditions, when $\mathcal{D}$ is a Hopf algebra,

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for the composition of coalgebras $\mathcal{D} \circ \mathcal{C}$ (and $\mathcal{C} \circ \mathcal{D}$) to be a one-sided Hopf algebra. These conditions also guarantee that a composition is a Hopf module and a comodule algebra over $\mathcal{D}$.

This composition (also known as substitution) is familiar from the theories of operads and species. If a species is a monoid with respect to $\circ$ then it is also an operad (Aguiar and Mahajan, 2010, App. B). In Section 4 we show that an operad $\mathcal{D}$ of connected graded coalgebras is automatically a Hopf algebra.

We discuss three examples related to well-known objects from category theory and algebraic topology and show that the Hopf algebra of simplices of (Forcey and Springfield (2010)) is cofree as a coalgebra.

2 Preliminaries

We work over a fixed field $\mathbb{K}$ of characteristic zero. For a graded vector space $V = \bigoplus_n V_n$, we write $|v| = n$ and say v has *degree* n if $v \in V_n$.

2.1 Hopf algebras and cofree coalgebras

A coalgebra $\mathcal{C}$ is a vector space $\mathcal{C}$ equipped with a coassociative coproduct $\Delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ and counit $\varepsilon\colon \mathcal{C} \to \mathbb{K}$. For $c \in \mathcal{C}$, write $\Delta(c)$ as $\sum_{(c)} c' \otimes c''$. Coassociativity means that

$$\sum_{(c),(c')} (c')' \otimes (c')'' \otimes c'' \;=\; \sum_{(c),(c'')} c' \otimes (c'')' \otimes (c'')'' \;=\; \sum_{(c)} c' \otimes c'' \otimes c''' \,,$$

and the counital condition reads $\sum_{(c)} \varepsilon(c')c'' = \sum_{(c)} c'\varepsilon(c'') = c$. A Hopf algebra is a unital associative algebra H that is also a coalgebra whose structure maps (coproduct Δ and counit ε) are algebra homomorphisms, with the additional condition of having an antipode. See (Montgomery (1993)) for more details. Takeuchi (1971) showed that a graded bialgebra $H = (\bigoplus_{n\geq 0} H_n, \cdot, \Delta, \varepsilon)$ that is connected ($H_0 = \mathbb{K}$) is a Hopf algebra. A *one-sided* Hopf algebra $H = (H, u, m, \Delta, \varepsilon, S)$ is allowed to have only a one-sided unit u and to satisfy only one of $m(S\otimes 1)\Delta = u\varepsilon$ and $m(1\otimes S)\Delta = u\varepsilon$. This relaxes the standard notion appearing in the literature (Green et al. (1980)), where only the antipode is allowed to be one-sided.

The *graded cofree coalgebra* on a vector space V is $\mathsf{C}(V) := \bigoplus_{n\geq 0} V^{\otimes n}$ with counit the projection $\varepsilon\colon \mathsf{C}(V) \to \mathbb{K} = V^{\otimes 0}$ and the *deconcatenation coproduct*: writing "\" for the tensor product in $V^{\otimes n}$, we have

$$\Delta(c_1\backslash \cdots \backslash c_n) \;=\; \sum_{i=0}^{n} (c_1\backslash \cdots \backslash c_i) \otimes (c_{i+1}\backslash \cdots \backslash c_n)\,.$$

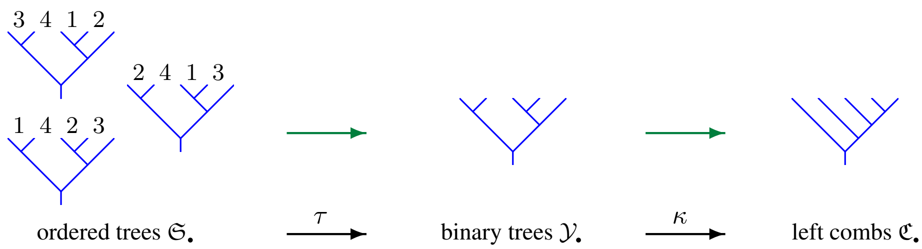
Observe that V is the set of primitive elements of $\mathsf{C}(V)$. A graded coalgebra $\mathcal{C}$ is *cofree* if $\mathcal{C} \simeq \mathsf{C}(P_{\mathcal{C}})$, where $P_{\mathcal{C}}$ is the space of primitive elements of $\mathcal{C}$. Many coalgebras arising in combinatorics are cofree.

2.2 Cofree Hopf algebras on trees

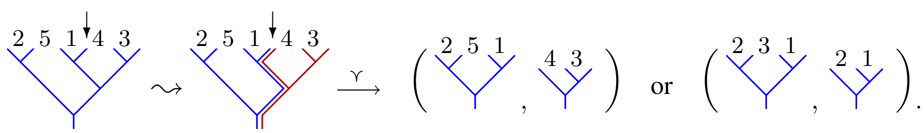
We describe three cofree Hopf algebras built on rooted planar binary trees: *ordered trees* $\mathfrak{S}_n$, *binary trees* $\mathcal{Y}_n$, and *(left) combs* $\mathfrak{C}_n$ on n internal nodes. Set $\mathfrak{S}_{\bullet} := \bigcup_{n\geq 0} \mathfrak{S}_n$ and define $\mathcal{Y}_{\bullet}$ and $\mathfrak{C}_{\bullet}$ similarly.

2.2.1 Constructions on trees

The nodes of a tree $t \in \mathcal{Y}_n$ form a poset. An *ordered tree* $w = w(t)$ is a linear extension of this node poset of t. This linear extension is indicated by placing a permutation in the gaps between its leaves, which gives a bijection between ordered trees and permutations. The map $\tau\colon \mathfrak{S}_n \to \mathcal{Y}_n$ sends an ordered tree $w(t)$ to its underlying tree t. The map $\kappa\colon \mathcal{Y}_n \to \mathfrak{C}_n$ shifts all nodes of a tree to the right branch from the root. Set $\mathfrak{S}_0 = \mathcal{Y}_0 = \mathfrak{C}_0 = |$. Note that $|\mathfrak{C}_n| = 1$ for all $n \geq 0$.

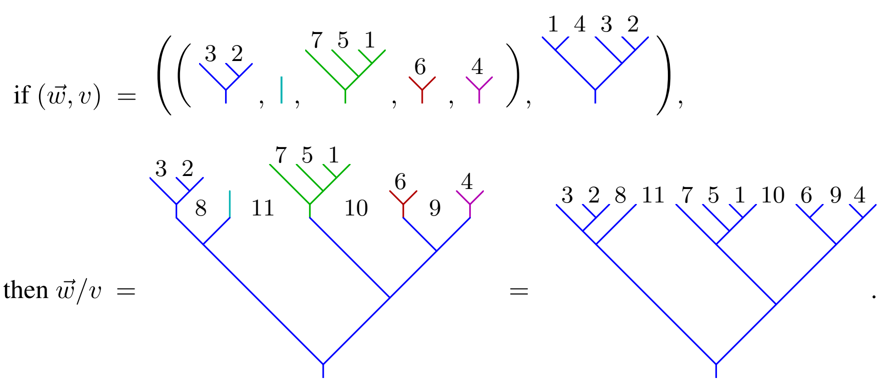


Splitting an ordered tree w along the path from a leaf to the root yields an ordered forest (where the nodes in the forest are totally ordered) or a pair of ordered trees,



Write $w \xrightarrow{\curlyvee} (w_0, w_1)$ when the ordered forest (w_0, w_1) (or pair of ordered trees) is obtained by splitting w. (Context will determine how to interpret the result.)

We may *graft* an ordered forest $\vec{w} = (w_0, \ldots, w_n)$ onto an ordered tree $v \in \mathfrak{S}_n$, obtaining the tree $\vec{w}/v$ as follows. First increase each label of v so that its nodes are greater than the nodes of $\vec{w}$, and then graft tree w_i onto the i^{th} leaf of v. For example,



Splitting and grafting make sense for trees in $\mathcal{Y}_\bullet$. They also work for $\mathfrak{C}_\bullet$ if, after grafting a forest of combs onto the leaves of a comb, one applies κ to the resulting planar binary tree to get a new comb.

2.2.2 Three cofree Hopf algebras

Let $\mathfrak{S}Sym := \bigoplus_{n\geq 0} \mathfrak{S}Sym_n$ be the graded vector space whose n^{th} graded piece has basis $\{F_w \mid w \in \mathfrak{S}_n\}$. Define $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$ similarly. The set maps τ and κ induce vector space maps $\boldsymbol{\tau}$ and $\boldsymbol{\kappa}$, $\boldsymbol{\tau}(F_w) = F_{\tau(w)}$ and $\boldsymbol{\kappa}(F_t) = F_{\kappa(t)}$. Fix $\mathfrak{X} \in \{\mathfrak{S}, \mathcal{Y}, \mathfrak{C}\}$. For $w \in \mathfrak{X}_\bullet$ and $v \in \mathfrak{X}_n$, set

$$F_w \cdot F_v \ := \sum_{w \xrightarrow{\curlyvee} (w_0,\dots,w_n)} F_{(w_0,\dots,w_n)/v}\ ,$$

the sum over all (ordered) forests obtained by splitting w at a multiset of n leaves. For $w \in \mathfrak{X}_{\bullet}$, set

$$\Delta(F_w) \ := \sum_{w \xrightarrow{\curlyvee} (w_0,w_1)} F_{w_0} \otimes F_{w_1}\ ,$$

the sum over all splittings of w at one leaf. The counit ε is the projection onto the 0^{th} graded piece, spanned by the unit element $1 = F_{|}$ for the multiplication.

Proposition 2.1 *For $(\Delta, \cdot, \varepsilon)$ above, $\mathfrak{S}Sym$ is the Malvenuto–Reutenauer Hopf algebra of permutations, $\mathcal{Y}Sym$ is the Loday–Ronco Hopf algebra of planar binary trees, and $\mathfrak{C}Sym$ is the divided power Hopf algebra. Moreover, $\mathfrak{S}Sym \xrightarrow{\boldsymbol{\tau}} \mathcal{Y}Sym$ and $\mathcal{Y}Sym \xrightarrow{\boldsymbol{\kappa}} \mathfrak{C}Sym$ are surjective Hopf algebra maps.* □

The part of the proposition involving $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ is found in (Aguiar and Sottile (2005, 2006)); the part involving $\mathfrak{C}Sym$ is straightforward and we leave it to the reader.

Typically (Montgomery, 1993, Ex 5.6.8), the divided power Hopf algebra is defined to be $\mathbb{K}[x] := \text{span}\{x^{(n)} \mid n \ge 0\}$, with basis vectors $x^{(n)}$ satisfying $x^{(m)} \cdot x^{(n)} = \binom{m+n}{n} x^{(m+n)}$, $1 = x^{(0)}$, $\Delta(x^{(n)}) = \sum_{i+j=n} x^{(i)} \otimes x^{(j)}$, and $\varepsilon(x^{(n)}) = 0$ for $n > 0$. An isomorphism between $\mathbb{K}[x]$ and $\mathfrak{C}Sym$ is given by $x^{(n)} \mapsto F_{c_n}$, where c_n is the unique comb in $\mathfrak{C}_n$.

Proposition 2.2 *The Hopf algebras $\mathfrak{S}Sym$, $\mathcal{Y}Sym$, and $\mathfrak{C}Sym$ are cofree as coalgebras. The primitive elements of $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$ are indexed by trees with no nodes off the right branch from the root.* □

The result for $\mathfrak{C}Sym$ is easy. Proposition 2.2 is proven for $\mathfrak{S}Sym$ and $\mathcal{Y}Sym$ in (Aguiar and Sottile (2005, 2006)) by performing a change of basis—from the *fundamental basis* F_w to the *monomial basis* M_w—by means of Möbius inversion in a poset structure placed on $\mathfrak{S}_{\bullet}$ and $\mathcal{Y}_{\bullet}$.

3 Constructing Cofree Compositions of Coalgebras

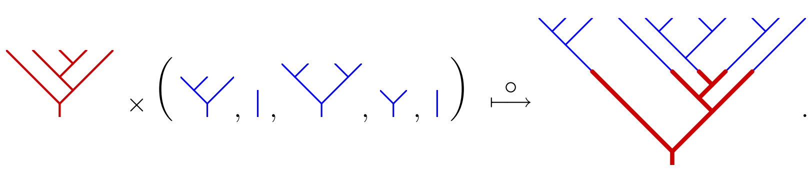
3.1 Cofree compositions of coalgebras

Let $\mathcal{C}$ and $\mathcal{D}$ be graded coalgebras. Form a new coalgebra $\mathcal{E} = \mathcal{D} \circ \mathcal{C}$ on the vector space

$$\mathcal{D} \circ \mathcal{C} \ := \bigoplus_{n \ge 0} \mathcal{D}_n \otimes \mathcal{C}^{\otimes (n+1)}\ . \tag{3.1}$$

When $\mathcal{C}$ and $\mathcal{D}$ are spaces of rooted, planar trees we may interpret $\circ$ in terms of a rule for grafting trees.

Example 3.1 Suppose $\mathcal{C} = \mathcal{D} = \mathcal{Y}Sym$ and let $d \times (c_0, \dots, c_n) \in \mathcal{Y}_n \times \left(\mathcal{Y}_{\bullet}{}^{n+1}\right)$. Define $\circ$ by attaching the forest $(c_0, \dots, c_n)$ to the leaves of d while remembering d, giving a *painted tree*,



We represent an indecomposable tensor in $\mathcal{E} := \mathcal{D} \circ \mathcal{C}$ as

$$d \circ (c_0 \bullet \cdots \bullet c_n) \qquad \text{or} \qquad \frac{c_0 \bullet \cdots \bullet c_n}{d} \, .$$

The *degree* of such an element is $|d| + |c_0| + \cdots + |c_n|$. Write $\mathcal{E}_n$ for the span of elements of degree n.

3.1.1 The coalgebra $\mathcal{D} \circ \mathcal{C}$

We define the *compositional coproduct* Δ for $\mathcal{D} \circ \mathcal{C}$ on indecomposable tensors: if $|d| = n$, put

$$\Delta\left(\frac{c_0 \bullet \cdots \bullet c_n}{d}\right) = \sum_{i=0}^{n} \sum_{\substack{(d) \\ |d'|=i}} \sum_{(c_i)} \frac{c_0 \bullet \cdots \bullet c_{i-1} \bullet c_i'}{d'} \otimes \frac{c_i'' \bullet c_{i+1} \bullet \cdots \bullet c_n}{d''} \, . \tag{3.2}$$

The *counit* $\varepsilon : \mathcal{D} \circ \mathcal{C} \to \mathbb{K}$ is given by $\varepsilon(d \circ (c_0 \bullet \cdots \bullet c_n)) = \varepsilon_{\mathcal{D}}(d) \cdot \prod_j \varepsilon_{\mathcal{C}}(c_j)$.

For the painted trees of Example 3.1, if the c_i and d are elements of the F-basis, then $\Delta(d \circ (c_0 \bullet \cdots \bullet c_n))$ is the sum over all splittings $t \xrightarrow{\curlyvee} (t', t'')$ of t into a pair of painted trees.

Theorem 3.2 *$(\mathcal{D} \circ \mathcal{C}, \Delta, \varepsilon)$ is a coalgebra. This composition is functorial, i.e., if $\varphi \colon \mathcal{C} \to \mathcal{C}'$ and $\psi \colon \mathcal{D} \to \mathcal{D}'$ are morphisms of graded coalgebras, then*

$$\frac{c_0 \bullet \cdots \bullet c_n}{d} \longmapsto \frac{\varphi(c_0) \bullet \cdots \bullet \varphi(c_n)}{\psi(d)}$$

defines a morphism of graded coalgebras $\varphi \circ \psi \colon \mathcal{D} \circ \mathcal{C} \to \mathcal{D}' \circ \mathcal{C}'$.

3.1.2 The cofree coalgebra $\mathcal{D} \circ \mathcal{C}$

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are graded, connected, and cofree. Then $\mathcal{C} = \mathsf{C}(P_{\mathcal{C}})$, where $P_{\mathcal{C}} \subset \mathcal{C}$ is its space of primitive elements. Likewise, $\mathcal{D} = \mathsf{C}(P_{\mathcal{D}})$, where $P_{\mathcal{D}} \subset \mathcal{D}$ is its space of primitive elements.

Theorem 3.3 *If $\mathcal{C}$ and $\mathcal{D}$ are cofree coalgebras then $\mathcal{D} \circ \mathcal{C}$ is also a cofree coalgebra. Its space of primitive elements is spanned by indecomposible tensors of the form*

$$\frac{1 \bullet c_1 \bullet \cdots \bullet c_{n-1} \bullet 1}{\delta} \qquad \text{and} \qquad \frac{\gamma}{1} \, , \tag{3.3}$$

where $\gamma, c_i \in \mathcal{C}$ and $\delta \in \mathcal{D}_n$, with γ and δ primitive.

Example 3.4 The graded Hopf algebras of ordered trees $\mathfrak{S}Sym$, planar trees $\mathcal{Y}Sym$, and divided powers $\mathfrak{C}Sym$ are all cofree, and so their compositions are cofree. We have the surjective Hopf algebra maps

$$\mathfrak{S}Sym \xrightarrow{\boldsymbol{\tau}} \mathcal{Y}Sym \xrightarrow{\boldsymbol{\kappa}} \mathfrak{C}Sym$$

giving the commutative diagram of Figure 1 of nine cofree coalgebras as the composition $\circ$ is functorial.

$$\begin{tikzcd}[column sep=tiny]
 & & \mathfrak{S}Sym \circ \mathfrak{S}Sym \arrow[dl] \arrow[dr] & & \\
 & \mathfrak{S}Sym \circ \mathcal{Y}Sym \arrow[dl] \arrow[dr] & & \mathcal{Y}Sym \circ \mathfrak{S}Sym \arrow[dl] \arrow[dr] & \\
\mathfrak{S}Sym \circ \mathfrak{C}Sym \arrow[dr] & & \mathcal{Y}Sym \circ \mathcal{Y}Sym \arrow[dl] \arrow[dr] & & \mathfrak{C}Sym \circ \mathfrak{S}Sym \arrow[dl] \\
 & \mathcal{Y}Sym \circ \mathfrak{C}Sym \arrow[dr] & & \mathfrak{C}Sym \circ \mathcal{Y}Sym \arrow[dl] & \\
 & & \mathfrak{C}Sym \circ \mathfrak{C}Sym & &
\end{tikzcd}$$

Fig. 1: A commutative diagram of cofree compositions of coalgebras.

3.2 Some enumeration

Set $\mathcal{E} := \mathcal{D} \circ \mathcal{C}$ and let C_n and E_n be the dimensions of $\mathcal{C}_n$ and $\mathcal{E}_n$, respectively.

Theorem 3.5 *When $\mathcal{D}_n$ has a basis indexed by combs with n internal nodes we have the recursion*

$$E_0 \;=\; 1\,, \quad \textit{and for } \; n>0, \quad E_n \;=\; C_n \;+\; \sum_{i=0}^{n-1} C_i E_{n-i-1}\,.$$

Proof: The first term counts elements in $\mathcal{E}_n$ of the form $| \circ c$. Removing the root node of d from $d \circ (c_0 \cdot \cdots \cdot c_k)$ gives a pair $| \circ (c_0)$ and $d' \circ (c_1 \cdot \cdots \cdot c_k)$ with $c_0 \in \mathcal{C}_i$, whose dimensions are enumerated by the terms of the sum. □

For combs over a comb, $E_n = 2^n$, for trees over a comb, E_n are the Catalan numbers, and for permutations over a comb, we have the recursion

$$E_0 \;=\; 1\,, \quad \text{and for } \; n>0, \quad E_n \;=\; n! + \sum_{i=0}^{n-1} i! E_{n-i-1}\,,$$

which begins $1, 2, 5, 15, 54, 235, \ldots$, and is sequence A051295 of the OEIS (Sloane). Similarly,

Theorem 3.6 *When $\mathcal{D}_n$ has a basis indexed by $\mathcal{Y}_n$ then we have the recursion*

$$E_0 \;=\; 1\,, \quad \textit{and for } \; n>0, \quad E_n \;=\; C_n + \sum_{i=0}^{n-1} E_i E_{n-i-1}\,.$$

For example, the combs over a tree are enumerated by the binary transform of the Catalan numbers (Forcey (2008b)). The trees over a tree are enumerated by the Catalan transform of the Catalan numbers (Forcey (2008a)). The permutations over a tree are enumerated by the recursion

$$E_0 \;=\; 1\,, \quad \text{and for } \; n>0, \quad E_n \;=\; n! + \sum_{i=0}^{n-1} E_i E_{n-i-1}\,,$$

which begins $1, 2, 6, 22, 92, 428, \ldots$ and is not a recognized sequence in the OEIS (Sloane). We do not have attractve recursive formulas when $\mathcal{D}_n$ has a basis indexed by $\mathfrak{S}_n$.

4 Composition of Coalgebras and Hopf Modules

We give conditions that imply a composition of coalgebras is a one-sided Hopf algebra, interpret this via operads, and then investigate which compositions of Fig. 1 are one-sided Hopf algebras.

4.1 Module coalgebras

Let $\mathcal{D}$ be a connected graded Hopf algebra with product $m_{\mathcal{D}}$, coproduct $\Delta_{\mathcal{D}}$, and unit element $1_{\mathcal{D}}$.

A map $f : \mathcal{E} \to \mathcal{D}$ of graded coalgebras is a *connection* on $\mathcal{D}$ if $\mathcal{E}$ is a $\mathcal{D}$–module coalgebra, f is a map of $\mathcal{D}$-module coalgebras, and $\mathcal{E}$ is connected. This means that $\mathcal{E}$ is an associative (left or right) $\mathcal{D}$-module whose action (denoted $\star$) commutes with the coproducts, so that $\Delta_{\mathcal{E}}(e \star d) = \Delta_{\mathcal{E}}(e) \star \Delta_{\mathcal{D}}(d)$, for $e \in \mathcal{E}$ and $d \in \mathcal{D}$, *and* the coalgebra map f is also a module map, so that for $e \in \mathcal{E}$ and $d \in \mathcal{D}$ we have

$$(f \otimes f)\, \Delta_{\mathcal{E}}(e) \;=\; \Delta_{\mathcal{D}}\, f(e) \qquad \text{and} \qquad f(e \star d) \;=\; m_{\mathcal{D}}\,(f(e) \otimes d)\,.$$

Theorem 4.1 *If $\mathcal{E}$ is a connection on $\mathcal{D}$, then $\mathcal{E}$ is also a Hopf module and a comodule algebra over $\mathcal{D}$. It is also a one-sided Hopf algebra with right-sided unit $1_{\mathcal{E}} := f^{-1}(1_{\mathcal{D}})$ and left-sided antipode.*

Proof: Suppose $\mathcal{E}$ is a right $\mathcal{D}$-module. Define the product $m_{\mathcal{E}} : \mathcal{E} \otimes \mathcal{E} \to \mathcal{E}$ via the $\mathcal{D}$-action: $m_{\mathcal{E}} := \star \circ (1 \otimes f)$. The one-sided unit is $1_{\mathcal{E}}$. Then $\Delta_{\mathcal{E}}$ is an algebra map. Indeed, for $e, e' \in \mathcal{E}$, we have

$$\Delta_{\mathcal{E}}(e \cdot e') \;=\; \Delta_{\mathcal{E}}(e \star f(e')) \;=\; \Delta_{\mathcal{E}} e \star \Delta_{\mathcal{D}} f(e') \;=\; \Delta_{\mathcal{E}} e \star (f \otimes f)(\Delta_{\mathcal{E}} e') \;=\; \Delta_{\mathcal{E}} e \cdot \Delta_{\mathcal{E}} e'\,.$$

As usual, $\varepsilon_{\mathcal{E}}$ is just projection onto $\mathcal{E}_0$. The unit $1_{\mathcal{E}}$ is one-sided, since

$$e \cdot 1_{\mathcal{E}} \;=\; e \star f(1_{\mathcal{E}}) \;=\; e \star f(f^{-1}(1_{\mathcal{D}})) \;=\; e \star 1_{\mathcal{D}} \;=\; e\,,$$

but $1_{\mathcal{E}} \cdot e = 1_{\mathcal{E}} \star f(e)$ is not necessarily equal to e. The antipode S may be defined recursively to satisfy $m_{\mathcal{E}}(S \otimes \mathbb{1})\Delta_{\mathcal{E}} = \varepsilon_{\mathcal{E}}$, just as for graded bialgebras with two-sided units.

Define $\rho \colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{D}$ by $\rho := (1 \otimes f)\, \Delta_{\mathcal{E}}$, which gives a coaction so that $\mathcal{E}$ is a Hopf module and a comodule algebra over $\mathcal{D}$. $\square$

4.2 Operads and operad modules

Composition of coalgebras is the same product used to define operads internal to a symmetric monoidal category (Aguiar and Mahajan, 2010, App. B). A *monoid* in a category with a product $\bullet$ is an object $\mathcal{D}$ with a morphism $\gamma \colon \mathcal{D} \bullet \mathcal{D} \to \mathcal{D}$ that is associative. An *operad* is a monoid in the category of graded sets with an analog of the composition product $\circ$ defined in Section 3.1.

Connected graded coalgebras form a symmetric monoidal category under the composition $\circ$ of coalgebras. A *graded Hopf operad* $\mathcal{D}$ is a monoid in this monoidal category of connected graded coalgebras and coalgebra maps. That is, $\mathcal{D}$ is equipped with associative composition maps

$$\gamma \colon \mathcal{D} \circ \mathcal{D} \to \mathcal{D}, \;\; \text{obeying} \;\; \Delta_{\mathcal{D}} \gamma(a) \;=\; (\gamma \otimes \gamma)\,(\Delta_{\mathcal{D} \circ \mathcal{D}}(a)) \quad \text{for all} \;\; a \in \mathcal{D} \circ \mathcal{D}\,.$$

A *graded Hopf operad module $\mathcal{E}$* is an operad module over $\mathcal{D}$ and a graded coassociative coalgebra whose module action is compatible with its coproduct. We denote the left and right action maps by $\mu_l : \mathcal{D} \circ \mathcal{E} \to \mathcal{E}$ and $\mu_r : \mathcal{E} \circ \mathcal{D} \to \mathcal{E}$, obeying, e.g., $\Delta_{\mathcal{E}} \mu_r(b) = (\mu_r \otimes \mu_r)\Delta_{\mathcal{E} \circ \mathcal{D}} b$ for all $b \in \mathcal{E} \circ \mathcal{D}$.

Example 4.2 $\mathcal{Y}Sym$ is an operad in the category of vector spaces. The action of γ on $F_t \circ (F_{t_0} \cdot \cdots \cdot F_{t_n})$ grafts the trees $t_0, \ldots, t_n$ onto the tree t and, unlike in Example 3.1, forgets which nodes of the resulting tree came from t. This is associative in the appropriate sense. With the same γ, $\mathcal{Y}Sym$ is an operad in the category of connected graded coalgebras, making it a graded Hopf operad. Finally, operads are operad modules over themselves, so $\mathcal{Y}Sym$ is also a graded Hopf operad module.

Remark 4.3 Our graded Hopf operads differ from those of Getzler and Jones, who defined a Hopf operad to be an operad of *level coalgebras*, where each component $\mathcal{D}_n$ is a coalgebra.

Theorem 4.4 *A graded Hopf operad $\mathcal{D}$ is also a Hopf algebra with product*

$$a \cdot b \ := \ \gamma(b \otimes \Delta^{(n)} a) \tag{4.1}$$

where $b \in \mathcal{D}_n$ and $\Delta^{(n)}$ is the iterated coproduct from $\mathcal{D}$ to $\mathcal{D}^{\otimes(n+1)}$.

It is possible to swap the roles of a and b on the right-hand side of (4.1). Our choice agrees with the product in $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$. In fact, the well-known Hopf algebra structures of $\mathcal{Y}Sym$ and $\mathfrak{C}Sym$ follow from their structure as graded Hopf operads.

Lemma 4.5 *If $\mathcal{C}$ is a graded coalgebra and $\mathcal{D}$ is a graded Hopf operad, then $\mathcal{D} \circ \mathcal{C}$ is a (left) graded Hopf operad module and $\mathcal{C} \circ \mathcal{D}$ is a (right) graded Hopf operad module.*

Lemma 4.6 *A graded Hopf operad module over a graded Hopf operad is also a module coalgebra.*

Theorem 4.7 *Given a coalgebra map $\lambda \colon \mathcal{C} \to \mathcal{D}$ from a connected graded coalgebra $\mathcal{C}$ to a graded Hopf operad $\mathcal{D}$, the maps $\gamma \circ (1 \circ \lambda) \colon \mathcal{D} \circ \mathcal{C} \to \mathcal{D}$ and $\gamma \circ (\lambda \circ 1) \colon \mathcal{C} \circ \mathcal{D} \to \mathcal{D}$ give connections on $\mathcal{D}$.*

4.3 Examples of module coalgebra connections

Eight of the nine compositions of Example 3.4 are connections on one or both of the factors $\mathcal{C}$ and $\mathcal{D}$.

Theorem 4.8 *For $\mathcal{C} \in \{\mathfrak{S}Sym, \mathcal{Y}Sym, \mathfrak{C}Sym\}$, the coalgebra compositions $\mathcal{C} \circ \mathfrak{C}Sym$ and $\mathfrak{C}Sym \circ \mathcal{C}$ are connections on $\mathfrak{C}Sym$. For $\mathcal{C} \in \{\mathfrak{S}Sym, \mathcal{Y}Sym, \mathfrak{C}Sym\}$, the coalgebra compositions $\mathcal{C} \circ \mathcal{Y}Sym$ and $\mathcal{Y}Sym \circ \mathcal{C}$ are connections on $\mathcal{Y}Sym$.*

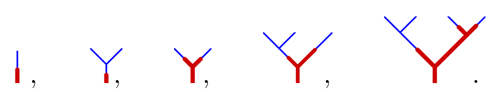
Note that $\mathfrak{C}Sym \circ \mathcal{Y}Sym$ is a connection on both $\mathfrak{C}Sym$ and on $\mathcal{Y}Sym$, which gives two distinct one-sided Hopf algebra structures. Similarly, $\mathcal{Y}Sym \circ \mathcal{Y}Sym$ is a connection on $\mathcal{Y}Sym$ in two distinct ways (again leading to two distinct one-sided Hopf structures). We do not know if $\mathfrak{S}Sym \circ \mathfrak{S}Sym$ is a connection over $\mathfrak{S}Sym$.

5 Three Examples

The three underlined algebras in Example 3.4 arose previously in algebra, topology, and category theory.

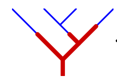
5.1 Painted Trees

A *painted binary tree* is a planar binary tree t, together with a (possibly empty) upper order ideal of its node poset. We indicate this ideal by painting on top of a representation of t, as in Example 3.1 and below,



An A_n*-space* is a topological H-space with a weakly associative multiplication of points (Stasheff (1963)). Stasheff (1970) described these maps using cell complexes called multiplihedra, while Boardman and Vogt (1973) used spaces of painted trees. Both the spaces of trees and the cell complexes are homeomorphic to convex polytope realizations of the multiplihedra as shown in (Forcey (2008a)).

If $f\colon (X, \bullet) \to (Y, *)$ is a map of A_n-spaces, then the different ways to multiply and map n points of X are represented by a painted tree. Unpainted nodes are multiplications in X, painted nodes are multiplications in Y, and the beginning of the painting indicates that f is applied to a given point in X,

$$f(a) * \big(f(b \bullet c) * f(d)\big) \longleftrightarrow$$  .

5.1.1 Algebra structures on painted trees.

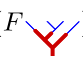
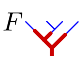
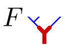
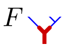
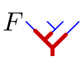
Let $\mathcal{P}_n$ be the poset of painted trees on n internal nodes, with partial order inherited from the identification with the poset $\mathcal{M}_{n+1}$ of bi-leveled trees (i.e., the multiplihedron). Forcey et al. (2010) studied this order.

We describe the key definitions of Section 3.1 and Section 4 for $\mathcal{P}Sym := \mathcal{Y}Sym \circ \mathcal{Y}Sym$. In the fundamental basis $\big\{F_p \mid p \in \mathcal{P}_\bullet\big\}$ of $\mathcal{P}Sym$, the counit is $\varepsilon(F_p) = \delta_{0,|p|}$, and the product is given by

$$\Delta(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0, p_1)} F_{p_0} \otimes F_{p_1}\,,$$

where the painting in $p \in \mathcal{P}_n$ is preserved in the splitting $p \xrightarrow{\curlyvee} (p_0, p_1)$.

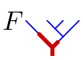
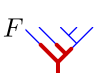
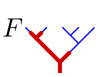
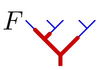
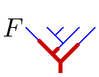
For example, we have

$\Delta(F$$) \;=\; 1 \otimes F$$\;+\; F$$\otimes F$$\;+\; F$$\otimes F$$\;+\; F$$\otimes 1\,.$

The identity map on $\mathcal{Y}Sym$ makes $\mathcal{P}Sym$ into a connection on $\mathcal{Y}Sym$. By Theorem 4.1, $\mathcal{P}Sym$ is thus also a one-sided Hopf algebra, a $\mathcal{Y}Sym$-Hopf module, and a $\mathcal{Y}Sym$-comodule algebra. The product $F_p \cdot F_q$ in $\mathcal{P}Sym$ is

$$F_p \cdot F_q \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0, p_1, \ldots, p_r)} F_{(p_0, p_1, \ldots, p_r)/q^+}\,,$$

where the painting in p is preserved in the splitting $(p_0, p_1, \ldots, p_r)$, and q^+ signifies that q is painted completely before grafting. For example,

F$\cdot F$$\;=\; F$$\;+\; F$$\;+\; F$$\;+\; F$$.$

The painted tree  with 0 nodes is only a right multiplicative identity element,

$F_q \cdot F$$\;=\; F_q$ but F$\cdot F_q \;=\; F_{q^+}$ for $q \in \mathcal{P}_\bullet$.

As $\mathcal{P}Sym$ is graded and connected, it has a one-sided antipode.

Theorem 5.1 *There are unit and antipode maps $\mu\colon \mathbb{K} \to \mathcal{P}Sym$ and $S\colon \mathcal{P}Sym \to \mathcal{P}Sym$ making $\mathcal{P}Sym$ a one-sided Hopf algebra.*

The $\mathcal{Y}Sym$-Hopf module structure on $\mathcal{P}Sym$ from Theorem 4.1 has coaction

$$\rho(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0,p_1)} F_{p_0} \otimes F_{f(p_1)}\,,$$

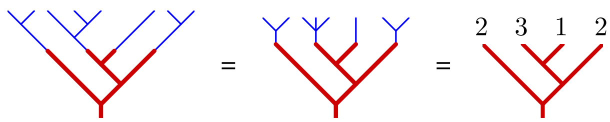
where the painting in p is preserved in p_0 and forgotten (f) in p_1.

Since painted trees and bi-leveled trees both index vertices of the multiplihedra, these structures for $\mathcal{P}Sym$ give structures on the linear span $\mathcal{M}Sym_+$ of bi-leveled trees with at least one node.

Corollary 5.2 *The $\mathcal{Y}Sym$ action and coaction defined in (Forcey et al., 2010, Section 4.1) make $\mathcal{M}Sym_+$ into a Hopf module isomorphic to the Hopf module $\mathcal{P}Sym$.* □

5.2 Composite Trees

In a forest of combs attached to a binary tree, the combs may be replaced by corollae or by a positive *weight* counting the number of leaves in the comb. These all give *composite trees*.



(5.1)

Composite trees with weights summing to $n{+}1$, $\mathcal{CK}_n$, were shown to be the vertices of a n-dimensional polytope, the *composihedron*, $\mathcal{CK}(n)$ (Forcey (2008b)). This sequence of polytopes is used to parameterize homotopy maps between strictly associative and homotopy associative H-spaces. For small values of n, the polytopes $\mathcal{CK}(n)$ appear as the commuting diagrams in enriched bicategories (Forcey (2008b)). These diagrams also appear in the definition of pseudomonoids (Aguiar and Mahajan, 2010, App. C).

5.2.1 Algebra structures on composite trees

We describe the key definitions of Section 3.1 and Section 4 for $\mathcal{CK}Sym := \mathcal{Y}Sym \circ \mathfrak{C}Sym$. In the fundamental basis $\{F_p \mid p \in \mathcal{CK}_\bullet\}$ of $\mathcal{CK}Sym$, the counit is $\varepsilon(F_p) = \delta_{0,|p|}$ and the coproduct is

$$\Delta(F_p) \;=\; \sum_{p \xrightarrow{\curlyvee} (p_0,p_1)} F_{p_0} \otimes F_{p_1}\,,$$

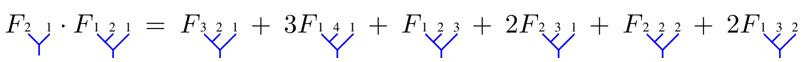
where the painting in $p \in \mathcal{CK}_\bullet$ is preserved in the splitting $p \xrightarrow{\curlyvee} (p_0,p_1)$. In the weighted-leaves representation of composite trees, the effect of Δ is subtle and best illustrated by an example,

$$\Delta(F_{2\,1\,2}) \;=\; F_{1} \otimes F_{2\,1\,2} \;+\; F_{2} \otimes F_{1\,1\,2} \;+\; F_{2\,1} \otimes F_{1\,2} \;+\; F_{2\,1\,1} \otimes F_{2} \;+\; F_{2\,1\,2} \otimes F_{1}\,.$$

For the product, Theorem 4.1, using the left module coalgebra action defined in Lemma 4.6, gives

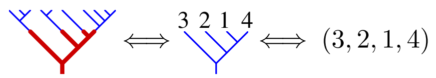
$$F_a \cdot F_b \;:=\; g(F_a) \star F_b\,, \qquad \text{where } a,b \in \mathcal{CK}_\bullet\,,$$

where $g\colon \mathcal{CK}Sym \to \mathfrak{C}Sym$ is the following connection. On the indices, it sends a composite tree a to the unique comb $g(a)$ with the same number of nodes as a. For the action $\star$, $g(a)$ is split in all ways to make a forest of $|b|{+}1$ combs, which are grafted onto the leaves of the forest of combs in b, then each tree in the forest is combed and attached to the binary tree in b. We illustrate one term in the product. Suppose that $a =$ $=$ and $b =$ $=$. Then $g(a) =$. One way to split $g(a)$ gives the forest (, , ,). Graft this onto b to get , then comb the forest to get , which is . Doing this for the other nine splittings of $g(a)$ gives,

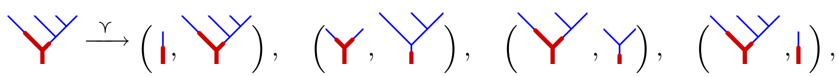


5.3 Composition trees

The simplest composition of Fig. 1 is $\mathfrak{C}Sym \circ \mathfrak{C}Sym$, whose basis is indexed by combs over combs. If we represent these as weighted trees as in (5.1), we see that we may identify combs over combs with n internal nodes as compositions of $n{+}1$. Thus we refer to these as *composition trees*.

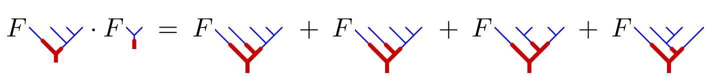


The coproduct is again given by splitting. Since the composition tree $(1,3)$ has the four splittings,


(5.2)

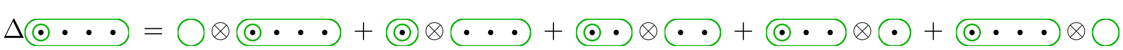
we have $\Delta(F_{1,3}) = F_1 \otimes F_{1,3} + F_{1,1} \otimes F_3 + F_{1,2} \otimes F_2 + F_{1,3} \otimes F_1$.

As we remarked, there are two connections $\mathfrak{C}Sym \circ \mathfrak{C}Sym \to \mathfrak{C}Sym$, using either the right or left action of $\mathfrak{C}Sym$. This gives two new one-sided Hopf algebra structures on compositions. With the right action, we have $F_{1,3} \cdot F_2 = 2F_{1,1,3} + F_{1,2,2} + F_{1,3,1}$, as

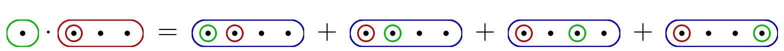

(5.3)

which may be seen by grafting the different splittings (5.2) onto the tree and coloring $\rightsquigarrow$.

Forcey and Springfield (2010) defined a one-sided Hopf algebra ΔSym on the graded vector space spanned by the faces of the simplices. Faces of the simplices correspond to subsets of $[n]$. Here is an example of the coproduct on the basis element corresponding to $\{1\} \subset [4]$, where subsets of $[n]$ are illustrated as circled subsets of the circled edgeless graph on n nodes numbered left to right:



Here is an example of the product



Let φ denote the bijection between subsets $S = \{a, b, \ldots c\} \subset [n]$ and compositions $\varphi(S) = (a, b - a, \ldots, n + 1 - c)$ of $n + 1$. Applying this bijection the indices of their fundamental bases gives a linear isomorphism $\boldsymbol{\varphi}\colon \Delta Sym \xrightarrow{\sim} \mathfrak{C}Sym \circ \mathfrak{C}Sym$, which is nearly an isomorphism of one-sided Hopf algebras, as may be seen by comparing these schematics of operations in ΔSym to formulas (5.2) and (5.3) in $\mathfrak{C}Sym \circ \mathfrak{C}Sym$.

Theorem 5.3 *The map φ is an isomorphism of coalgebras and an anti-isomorphism ($\varphi(a \cdot b) = \varphi(b) \cdot \varphi(a)$) of one-sided algebras.*

Corollary 5.4 *The one-sided Hopf algebra of simplices introduced in (Forcey and Springfield (2010)) is cofree as a coalgebra.*

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Dissimilarity Vectors of Trees and Their Tropical Linear Spaces (Extended Abstract)

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Abstract. We study the combinatorics of weighted trees from the point of view of tropical algebraic geometry and tropical linear spaces. The set of dissimilarity vectors of weighted trees is contained in the tropical Grassmannian, so we describe here the tropical linear space of a dissimilarity vector and its associated family of matroids. This gives a family of complete flags of tropical linear spaces, where each flag is described by a weighted tree.

Résumé. Nous étudions les propriétés combinatoires des arbres pondérés avec le formalisme de la géométrie tropicale et des espaces linéaires tropicaux. L'ensemble de vecteurs de dissimilarité des arbres pondérés est contenu dans la grassmannienne tropicale, donc nous décrivons ici l'espace linéaire tropical d'un vecteur de dissimilarité et sa famille de matroïdes associée. Cela permet d'obtenir une famille de drapeaux complets d'espaces linéaires tropicaux, où chaque drapeau est décrit par un arbre pondéré.

Resumen. Estudiamos la combinatoria de los árboles valuados desde el punto de vista de la geometría algebraica tropical y de los espacios lineales tropicales. El conjunto de los vectores de disimilaridad de un árbol valuado está contenido en el grassmanniano tropical y aquí describimos el espacio lineal tropical de un vector de disimilaridad y su familia asociada de matroides. Se obtiene entonces una familia de banderas completas de espacios lineales tropicales, donde cada bandera se describe mediante un árbol.

Keywords: dissimilarity vector, tropical linear space, tight span, weighted tree, matroid polytope, T-theory

1 Introduction

1.1 Basic Definitions

For every finite set E, define $\binom{E}{m}$ to be the collection of all subsets of E of size m. We adopt the convention that $[n] = \{1, 2, \ldots, n\}$, and we further define $[0] = \emptyset$. We will use the letters $\mathcal{E}$, $\mathcal{V}$ and $\mathcal{L}$ to denote the set of edges, vertices and leaves of a tree, respectively.

Throughout this article, we will consider a tree T with $n \geq 1$ leaves labeled by the set $[n]$, and such that T does not contain internal vertices of degree 2. Moreover, we will assume the existence of a weight function $\mathcal{E}(T) \xrightarrow{\omega} \mathbb{R}$ such that $\omega(e) > 0$ if e is an internal edge of T. We say that S is a **subtree** of T and write $S \subseteq_{\mathrm{st}} T$ whenever $\mathcal{V}(S) \subseteq \mathcal{V}(T)$, S is the restriction of T to $\mathcal{V}(S)$ and S is itself a tree. We will

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also write $S \subsetneq_{\mathrm{st}} T$ if $\mathcal{V}(S) \subsetneq \mathcal{V}(T)$. The function ω will extend to a function from the subtrees of T to the reals in the natural way so that we may speak of total weights of subtrees of T. As a convention, if S is a subtree of T and $\mathcal{E}(S) = \emptyset$, we will let $\omega(S) = 0$.

At all points, we will identify $\mathcal{L}(T)$ with $[n]$.

Now, for $m \in [n]$, we define a vector $d \in \mathbb{R}^{\binom{[n]}{m}}$ associated with the tree T and called the m-**dissimilarity vector** of T. For each $A \in \binom{[n]}{m}$, let d_A be the total weight of the subtree of T spanned by the set of leaves A. The letter d will always denote a dissimilarity vector.

At some points, we will consider instead a tree U with n leaves labeled by the set $[n]$ with $n \geq 1$, with a weight function $\mathcal{E}(U) \xrightarrow{\omega} \mathbb{R}_{\geq 0}$. The function ω will again extend to a function from the subtrees of U to the nonnegative reals in the natural way, so that we may speak of total weights of subtrees of U and define the m-dissimilarity vector of U in a completely analogous way. Furthermore, we will assume that U is trivalent, rooted, ℓ-equidistant (*i.e.* the total weight of the minimal path from every leaf to the root is a constant ℓ), and that ω induces a metric on $\mathcal{L}(U) = [n]$. Such a tree U will be referred to as an **ultrametric tree**. Along with U we consider the poset $(\mathcal{V}(U), \leq_{\mathrm{TO}})$, called the **tree-order** of U, by which $u \leq_{\mathrm{TO}} v$ if the minimal path from the root of U to v contains u.

1.2 Tropical Algebraic Geometry

Consider the field $\mathbb{K} = \mathbb{C}\{\{t\}\}$ of **dual Puiseux series**. Let $\mathbb{K}^* := \mathbb{K}\backslash\{0\}$. Recall that $\mathbb{K}^*$ consists of all formal expressions $\mathfrak{t} = \sum_{-\infty}^{k=p} c_k t^{k/q}$, where $p \in \mathbb{Z}$, $c_p \neq 0$, $q \in \mathbb{Z}^+$ and $c_k \in \mathbb{C}$ for all $k \leq p$. The field comes equipped with a standard valuation val : $\mathbb{K}^* \mapsto \mathbb{Q}$ by which $\mathrm{val}(\mathfrak{t}) = p/q$. We extend this valuation to a function val : $(\mathbb{K}^*)^n \mapsto \mathbb{Q}^n$ by taking pointwise valuation.

Let $X = (X_1, \ldots, X_n)$ be a vector of variables. Consider a polynomial $f \in \mathbb{K}[X]$ and write it explicitly as $f(X) = \sum_{a \in A \subseteq \mathbb{Z}^n_{\geq 0}} \mathfrak{t}_a X^a$, where $\mathfrak{t}_a \in \mathbb{K}^*$ for all $a \in A$.

The **tropicalization** trop $f : \mathbb{R}^n \mapsto \mathbb{R}$ of f is trop $f(x) = \max\limits_{a \in A}\{\mathrm{val}(\mathfrak{t}_a) + a \cdot x\}$, where $x = (x_1, \ldots, x_n)$ is a vector of variables and the product of vectors is the dot product.

Let $\mathcal{T}(\mathrm{trop}\ f) \subseteq \mathbb{R}^n$ be the set of points where trop f attains its maximum twice. In this definition, trop f can be replaced by any convex piecewise linear form $L : \mathbb{R}^n \mapsto \mathbb{R}$. Call $\mathcal{T}(\mathrm{trop}\ f)$ the **tropical hypersurface** of f. More generally, consider an arbitrary set $S \subseteq \mathbb{K}[X]$. Let trop $S := \{\mathrm{trop}\ f \mid f \in S\}$. Define the **tropical variety** of S to be the set $\mathcal{T}(\mathrm{trop}\ S) = \bigcap_{f \in S} \mathcal{T}(\mathrm{trop}\ f)$. Again, trop S can be replaced by a set of convex piecewise linear forms mapping $\mathbb{R}^n$ into $\mathbb{R}$.

Let $V(S) := \{\mathfrak{t} \in (\mathbb{K}^*)^n \mid f(\mathfrak{t}) = 0 \text{ for all } f \in S\}$.

We have:

Theorem 1.1 (Theorem 2.1, Speyer and Sturmfels [SS04]) *Let $I \subseteq \mathbb{K}[X]$ be an ideal. Then, the following subsets of $\mathbb{R}^n$ coincide:*

a) *The closure of the set $\{val(\mathfrak{t}) \mid \mathfrak{t} \in V(I)\}$.*

b) *The tropical variety $\mathcal{T}(trop\ I)$.*

□

Let $Z = (Z_{ij})$ be an $m \times n$ matrix of indeterminates. Consider the polynomial ring: $\mathbb{K}[Y] = \mathbb{K}[Y_A : A \in \binom{[n]}{m}]$. Let $\phi_{m,n} : \mathbb{K}[Y] \mapsto \mathbb{K}[Z]$ be the homomorphism of rings taking Y_A to the maximal minor of Z obtained from the set of columns A.

The **Plücker ideal** or ideal of **Plücker relations** is the homogeneous prime ideal $I_{m,n} = \ker(\phi_{m,n}) \subseteq \mathbb{K}[Y]$. In particular, if $P \in (\mathbb{K}^*)^{\binom{[n]}{m}}$ is a vector of **Plücker coordinates** of an m-dimensional linear subspace V of $\mathbb{K}^n$, then $P \in V(I_{m,n})$.

The tropical variety $\mathcal{T}(\text{trop } I_{m,n})$ receives the special name of **tropical Grassmannian** and is denoted by $\mathcal{G}_{m,n}$. Theorem 1.1 then shows that if P is a vector of Plücker coordinates of an m-dimensional linear subspace V of $\mathbb{K}^n$ and P has non-zero entries, then $\text{val}(P) \in \mathcal{G}_{m,n}$.

1.3 Tropical Linear Spaces

Consider an m-dimensional linear subspace V of $\mathbb{K}^n$, and let P be a vector of Plücker coordinates of V. Consider the family $\mathcal{S}_V$ of linear polynomials in $\mathbb{K}[X]$:

$$\sum_{r=0}^{m+1} (-1)^r P_{i_1 i_2 \ldots \widehat{i_r} \ldots i_m i_{m+1}} X_{i_r} \text{ for all } 1 \leq i_1 < i_2 < \cdots < i_m < i_{m+1} \leq n. \tag{1.1}$$

Then $\mathfrak{t} \in V$ if and only if $f(\mathfrak{t}) = 0$ for all $f \in \mathcal{S}_V$, defining V as a Zariski closed set.

Let $p \in \mathbb{R}^{\binom{[n]}{m}}$. Consider the set of three-term Plücker relations $S_{m,n} := \{Y_{Aij}Y_{Akl} - Y_{Aik}Y_{Ajl} + Y_{Ail}Y_{Ajk} \mid A \in \binom{[n]}{m-2} \text{ and } \{i,j,k,l\} \subseteq \binom{[n]\setminus A}{4}\} \subseteq I_{m,n} \subseteq \mathbb{K}[Y]$, so that $\mathcal{G}_{m,n} \subseteq \mathcal{T}(\text{trop } S_{m,n})$. Say that p satisfies the **tropical Plücker relations** if $p \in \mathcal{T}(\text{trop } S_{m,n})$.

In particular, if $p = \text{val}(P)$ and P is a vector of Plücker coordinates, then $p \in \mathcal{G}_{m,n}$ implies that $p \in \mathcal{T}(\text{trop } S_{m,n})$, but if $p \in \mathcal{T}(\text{trop } S_{m,n})$, we may not necessarily find such P.

Exploiting the definition of linear spaces as closed sets, let us define the tropical analogue of a linear space.

Let $p \in \mathcal{T}(\text{trop } S_{m,n})$. Considering the tropical analogues of the equations 1.1, let

$$L(p) := \bigcap_{A \in \binom{[n]}{m+1}} \mathcal{T}\left(\max_{i \in A}\{p_{A \setminus i} + x_i\}\right).$$

The set $L(p)$ is the m-dimensional **tropical linear space** associated to p.

For all $A \in \binom{[n]}{m}$, define $e^A = \sum_{i \in A} e_i$, where the sum takes place in $\mathbb{R}^n$ with its canonical basis $\{e_1, e_2, \ldots, e_n\}$. In general, for $x \in \mathbb{R}^n$, let $x^A = \sum_{i \in A} x_i$. Let $\mathscr{H}_m$ be the m-**hypersimplex** of $\mathbb{R}^n$, *i.e.* the convex hull of all vectors e^A with $A \in \binom{[n]}{m}$.

Suppose that $p \in \mathbb{R}^{\binom{[n]}{m}}$. Let $\mathscr{H}_m^p \subseteq \mathbb{R}^{n+1}$ be the convex hull of all points (e^A, p_A) with $A \in \binom{[n]}{m}$. Then p induces a regular subdivision $\mathcal{D}_p$ of $\mathscr{H}_m$ by projecting the upper faces of $\mathscr{H}_m^p$ down to $\mathscr{H}_m$, where by upper face we mean that its outer normal vector has a positive last coordinate.

Let $\mathcal{P}$ be a subpolytope of $\mathscr{H}_m$. We say that $\mathcal{P}$ is **matroidal** or that $\mathcal{P}$ is a **matroid polytope** if the collection of sets $A \in \binom{[n]}{m}$ for which $e^A \in \mathcal{P}$ is the set of bases of a rank-m matroid over $[n]$.

The following theorem motivates considering the set $S_{m,n}$.

Theorem 1.2 (Speyer [Spe08]) *The following assertions are equivalent:*

a) *The vector $p \in \mathbb{R}^{\binom{[n]}{m}}$ satisfies the tropical Plücker relations;*

b) *Every face of $\mathcal{D}_p$ is matroidal.*

□

If p satisfies the tropical Plücker relations, let $\mathfrak{M}_p$ be the family of rank-m matroids given by the faces of $\mathcal{D}_p$. Recall that a matroid over a finite set E is said to be **loopless** if every element of E is contained in at least one basis. Let $\mathfrak{M}_p^{loop}$ be the subfamily of loopless matroids of $\mathfrak{M}_p$, and let $\mathcal{D}_p^{loop}$ be the subcomplex of faces of $\mathcal{D}_p$ described by $\mathfrak{M}_p^{loop}$. Similarly, let $\mathcal{D}_p^{int}$ be the subcomplex of internal faces of $\mathcal{D}_p$.

For any fixed $x \in \mathbb{R}^n$, the projection of the face of $\mathscr{H}_m^p$ maximizing the dot product with $(-x, 1)$ gives a matroid M_x with set of bases B_x, and we have:

Theorem 1.3 (Speyer [Spe08]) *$x \in L(p)$ if and only if M_x is a loopless matroid.*

□

Tropical linear spaces can be defined differently. Let $p \in \mathcal{T}(\text{trop } S_{m,n})$. Consider the non-empty unbounded n-dimensional polyhedron $P_p := \{x \in \mathbb{R}^n \mid x^A \geq p_A \text{ for all } A \in \binom{[n]}{m}\}$. Let $\preceq$ denote componentwise inequality for vectors in $\mathbb{R}^n$, and define the **reduced tropical linear space** of p to be the set $P'_p := \{x \in P_p \mid y \preceq x \text{ with } y \in P_p \text{ implies } y = x\}$. The polyhedral complex P'_p is pure $(m-1)$-dimensional [Spe08] and coincides with the dual complex of $\mathcal{D}_p^{loop}$, see Herrmann and Joswig [HJ08, Proposition 2.3].

Take $x \in \partial(P_p)$ and let B'_x be the collection of sets $A \in \binom{[n]}{m}$ for which $x^A = p_A$. Then, $B'_x = \{A \in \binom{[n]}{m} \mid -x \cdot e^A + p_A \geq -x \cdot e^B + p_B \text{ for all } B \in \binom{[n]}{m}\}$, so $B'_x = B_x$ as above. Using the definition of P'_p, we can check that $x \in P'_p$ if and only if M_x is a loopless matroid. In particular, (closed) bounded faces of P_p are faces of P'_p. Hence, we have:

- For $x \in \mathbb{R}^n$, $x \in L(p)$ if and only if M_x is loopless.
- For $x \in \partial(P_p)$, $x \in P'_p$ if and only if M_x is loopless.

For every $x \in \mathbb{R}^n$, there exists a unique t for which $x + te^{[n]} \in \partial(P_p)$. Also, $L(p)$ and the matroid M_x are invariant under translation by $e^{[n]}$, so $P'_p = L(p)/\mathbb{R}(1, \ldots, 1)$, under a natural choice of representative for each class.

Let the **tight span** $\mathscr{T}_p$ of p be the complex of bounded faces of P_p (or of P'_p). The tight span coincides with the dual complex of $\mathcal{D}_p^{int}$ [HJ08].

We now present a lemma on the topology of $L(p)$, P'_p and $\mathscr{T}_p$: all three spaces deformation retract onto any point of $\mathscr{T}_p$.

Lemma 1.4 *Let $P \subseteq \mathbb{R}^n$ be an n-dimensional unbounded polyhedron with no lines. Let $P_\emptyset$ be the complex of bounded faces of P and let $\mathcal{F}_\infty$ be the set of closed unbounded faces of P. For $A \subseteq \mathcal{F}_\infty$, define*

$$P_A = P_\emptyset \cup \left(\bigcup_{F \in A} F \right).$$

Then, P_A deformation retracts onto any point of $P_\emptyset$.

1.4 Results About Weighted Trees

The motivation of this subsection is the following result, known as the **four point condition theorem**. It is a theorem of Buneman [Bun74].

Theorem 1.5 (Pachter and Sturmfels [PS05]) *The set of* 2*-dissimilarity vectors of trees is equal to the tropical Grassmannian* $\mathcal{G}_{2,n}$.

□

From here, Pachter and Speyer [PS03] asked whether the set of m-dissimilarity vectors is contained in $\mathcal{G}_{m,n}$ for all $m \geq 3$. In Section 2, we answer this question affirmatively:

Theorem 1.6 (Iriarte [Iri10]) *Let* T *be a tree with* m*-dissimilarity vector* d *with* $m \geq 2$*. Then,* $d \in \mathcal{G}_{m,n}$.

□

This result is in the direction of solving the more general problem:

Problem 1.7 *Characterize the set of* m*-dissimilarity vectors of trees,* $m \geq 2$.

For the case $m = 2$, this problem is solved in a useful and concrete way by the four point condition theorem and by a result of Dress [Dre84] which relates tropical linear spaces and 2-dissimilarity vectors. The latter is also nicely obtained in Hirai [Hir06, Appendix A], with the additional verification that the face the author finds to prove it is indeed a bounded face of the reduced tropical linear space.

Theorem 1.8 (Dress [Dre84]) *A vector* $d \in \mathbb{R}^{\binom{[n]}{2}}$ *is a* 2*-dissimilarity vector if and only if* $\mathscr{T}_d$ *is a tree.*

□

In the same line of thought, using Part **b)** of Theorem 1.2 we can directly say:

Theorem 1.9 *A vector* $d \in \mathbb{R}^{\binom{[n]}{2}}$ *is a* 2*-dissimilarity vector if and only if every face of* $\mathcal{D}_d$ *is matroidal.*

□

Currently, no analogous results are known for $m \geq 3$.

2 Dissimilarity Vectors are Contained in the Tropical Grassmannian

Consider the tree T. We begin with a key lemma.

Lemma 2.1 (Cools [Coo09]) *Fix a set* $A \in \binom{[n]}{m}$*. Denote by* $\mathfrak{S}_A$ *the group of permutations of* A*. Let* $\mathcal{C}_A \subseteq \mathfrak{S}_A$ *be the set of cycles of length* m*. For any* $j \in [m] \cup \{0\}$*,* $a \in A$ *and* $\sigma \in \mathcal{C}_A$ *consider the following sum, which is independent of the choice of* a*:*

$$s_{A,\sigma} = \sum_{j=1}^{m} d_{\sigma^{j-1}(a)\sigma^{j}(a)}.$$

Then, there exists $\varsigma \in \mathcal{C}_A$ *(depending on* A*) such that* $d_A = \frac{1}{2} s_{A,\varsigma}$ *and* $s_{A,\varsigma} = \min_{\sigma \in \mathcal{C}_A} s_{A,\sigma}$.

□

In order to prove Theorem 1.6, it suffices only to consider the case of a trivalent tree T with rational weights. Moreover, Lemma 2.1 implies that if such T corresponds to an m-linear subspace V of $\mathbb{K}^n$, so that $d = \text{val}(P)$ where P is a vector of Plücker coordinates of V, and T' is obtained from T by assigning new rational weights to the external edges, then the m-dissimilarity vector d' of T' corresponds analogously to a linear subspace V' obtained from V via a torus action on $\mathbb{K}^n$, so there exists a fixed $(t^{q_1}, t^{q_2}, \ldots, t^{q_n}) \in (\mathbb{K}^*)^n$ such that $V' = \{(t^{q_1}v_1, t^{q_2}v_2, \ldots, t^{q_n}v_n) \mid (v_1, v_2, \ldots, v_n) \in V\}$. Therefore, the following theorem implies Theorem 1.6:

Theorem 2.2 *Let U be an ultrametric tree with rational weights and m-dissimilarity vector d, $m \geq 2$. Then, $d \in \mathcal{G}_{m,n}$.*

□

Now, Theorem 2.2 follows from the following technical result proposed by Cools [Coo09]:

Proposition 2.3 (Iriarte [Iri10]) *Suppose that $3 \leq m \leq n$. Let U be an ultrametric tree with m-dissimilarity vector d, all of whose edges have rational weight. For each $e = (u,v) \in \mathcal{E}(U)$ with $u \leq_{\mathrm{TO}} v$, denote by $h(e)$ the total weight of the minimal path from u to any $j \in \mathcal{L}(U) = [n]$ such that $v \leq_{\mathrm{TO}} j$.*

For all $i \in [n-2]$ and $e \in \mathcal{E}(U)$, let $a_{i,e}$ be a generic complex number. Now, for all $i \in [n-2]$ and $j \in \mathcal{L}(U) = [n]$ let

$$\mathfrak{f}_{ij} := \sum_{\substack{e=(u,v)\in\mathcal{E}(U)\\ u\leq_{\mathrm{TO}} v \leq_{\mathrm{TO}} j}} a_{i,e} t^{h(e)},$$

and then define the $n \times n$ matrix:

$$M := \begin{pmatrix} 1 & 1 & \dots & 1 \\ \mathfrak{f}_{11} & \mathfrak{f}_{12} & \dots & \mathfrak{f}_{1n} \\ (\mathfrak{f}_{11})^2 & (\mathfrak{f}_{12})^2 & \dots & (\mathfrak{f}_{1n})^2 \\ \mathfrak{f}_{21} & \mathfrak{f}_{22} & \dots & \mathfrak{f}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathfrak{f}_{(n-2)1} & \mathfrak{f}_{(n-2)2} & \dots & \mathfrak{f}_{(n-2)n} \end{pmatrix}.$$

For all $A \in \binom{[n]}{m}$, let $\mathfrak{m}_A$ be the $m \times m$ upper minor of M coming from the columns A. Then,

$$val(\mathfrak{m}_A) = d_A.$$

□

3 The Tropical Linear Space of a Dissimilarity Vector

In this section, we will further assume that our tree T is trivalent and that $1 < m < n$. Per Theorem 1.6, we can construct the tropical linear space P'_d of its m-dissimilarity vector. The purpose of the section is to present a geometric and combinatorial description of P'_d.

Let $S \subseteq_{\mathrm{st}} T$. Let $\mathtt{in}(S)$ be the subtree of S that consists of the internal edges and vertices of S. If S does not contain internal edges or vertices, as a technical convenience we let $\mathtt{in}(S) = \Lambda$, where Λ is the abstract tree consisting of an empty set of vertices and edges, and which we consider to be a proper subtree of every other tree. We call $\mathtt{in}(S)$ the **internal tree** of S. Let also $\overline{S}$ be the subtree of T consisting of all edges and vertices of T that are in S or adjacent to S. We call $\overline{S}$ the **extended tree** of S. For every leaf $i \in \mathcal{L}(T) = [n]$, there is a unique minimal path in T from i to S, denoted by $i \rightharpoonup S$.

Considering $i \in [m-1] \cup \{0\}$ and $S \subseteq_{\mathrm{st}} T$, we say that $(S, \mathcal{A})$ is an (m,i)-**good pair** of T if furthermore **a)** $i \leq |\mathcal{L}(S)| \leq m-1$, **b)** $m-1+i \leq |\mathcal{V}(S)|$, **c)** $\mathcal{A} \subseteq \mathcal{L}(S)$ with $|\mathcal{A}| = i$, and **d)** $\mathcal{L}(S) \cap \mathcal{L}(T) \subseteq \mathcal{A}$.

3.1 Faces of P'_d and Their Matroids

We describe explicitly the faces of P'_d.

Let $S \subseteq_{\mathrm{st}} T$. For all $l \in \mathcal{L}(S)$, let H_l be the set of leaves of T whose minimal path to S meets l. Analogously, for all $l \in \mathcal{L}(\overline{S})$ let R_l be the set of leaves of T whose minimal path to $\overline{S}$ meets l. Let $\mathcal{R}_S = \{R_l\}_{l \in \mathcal{L}(\overline{S})}$. Then, $\mathcal{R}_S$ is a partition of $[n]$. For any $\mathcal{A} \subseteq \mathcal{L}(S)$, let $\mathcal{H}_{(S,\mathcal{A})} = \{H_l\}_{l \in \mathcal{A}}$. It is worth noting that $\mathcal{H}_{(S,\mathcal{A})}$ is also a collection of pairwise disjoint subsets of $[n]$, but not necessarily a partition of $[n]$. In fact, we will let $H^c_S := \{i \in [n] \mid i \notin H \text{ for all } H \in \mathcal{H}_{(S,\mathcal{L}(S))}\}$. Note also that each set in the collection $\mathcal{H}_{(S,\mathcal{A})}$ can be written as a disjoint union of at least two sets from the collection $\mathcal{R}_S$. If $\mathcal{H}_{(S,\mathcal{L}(S))}$ is a partition of $[n]$, we say that S is a **full subtree** of T.

For all $i \in [m-1] \cup \{0\}$, define

$$\begin{aligned}
\mathfrak{T}_i &:= \{(S,\mathcal{A}) \mid (S,\mathcal{A}) \text{ is an } (m,i)\text{-good pair of } T\},\\
\mathfrak{T}_i^{\mathtt{in}} &:= \{(S,\mathcal{A}) \in \mathfrak{T}_i \mid S \subseteq_{\mathrm{st}} \mathtt{in}(T)\},\\
\mathfrak{T}_* &:= \{S \mid S \text{ is a full subtree of } T \text{ with } m \text{ leaves}\},\\
\mathfrak{T}_*^{\mathtt{in}} &:= \{S \in \mathfrak{T}_* \mid S \subseteq_{\mathrm{st}} \mathtt{in}(T)\}.
\end{aligned}$$

We are now ready to present the main results of this section.

Theorem 3.1 *Let $i \in [m-2] \cup \{0\}$. Then, the set of i-dimensional faces of P'_d is in bijection with the set $\mathfrak{T}_i$, and the set of i-dimensional faces of $\mathscr{T}_d$ is in bijection with the set $\mathfrak{T}_i^{\mathtt{in}}$.*

□

Theorem 3.2 *The set of $(m-1)$-dimensional faces of P'_d is in bijection with the set $\mathfrak{T}_{m-1} \sqcup \mathfrak{T}_*$, and the set of $(m-1)$-dimensional faces of $\mathscr{T}_d$ is in bijection with the set $\mathfrak{T}_{m-1}^{\mathtt{in}} \sqcup \mathfrak{T}_*^{\mathtt{in}}$.*

□

For a tree $S \subseteq_{\mathrm{st}} T$ and sets $\mathcal{A} \subseteq \mathcal{L}(S)$, $\mathcal{B} \subseteq \mathcal{V}(T)$, let $S_{\mathcal{A}}$ be the subtree of S spanned by the set of vertices $\mathcal{V}(S)\backslash\mathcal{A}$, and let $S^{\mathcal{B}}$ be the subtree of T spanned by the set of vertices $\mathcal{V}(S) \cup \mathcal{B}$.

Proposition 3.3 *Let F and F' be faces of $\mathscr{T}_d$. Suppose that F corresponds to an $(m,|\mathcal{A}|)$-good pair $(S,\mathcal{A})$ of T and that F' corresponds to an $(m,|\mathcal{A}'|)$-good pair $(S',\mathcal{A}')$. Then, $F' \subseteq F$ if and only if*

$$S_{\mathcal{A}} \subseteq_{\mathrm{st}} S' \subseteq_{\mathrm{st}} S \text{ and } \mathcal{A}' \subseteq \mathcal{A}.$$

□

Let us now describe P'_d in greater detail.

Theorem 3.4 *Let $i \in [m-1] \cup \{0\}$. The matroid M_F of the face F of P'_d corresponding to an (m,i)-good pair $(S,\mathcal{A})$ of T has a collection of bases*

$$B_F = B^1 \cap B^2 \cap B^3,$$

where:

$B^1 := \{A \in \binom{[n]}{m} \mid |A \cap H| = 1 \text{ for all } H \in \mathcal{H}_{(S,\mathcal{A})}\}$,

$B^2 := \{A \in \binom{[n]}{m} \mid |A \cap H| \geq 1 \text{ for all } H \in \mathcal{H}_{(S,\mathcal{L}(S)\backslash\mathcal{A})}\}$,

$B^3 := \{A \in \binom{[n]}{m} \mid |A \cap R| \leq 1$ *for all* $R \in \mathcal{R}_S\}$.

If F is bounded, then its face lattice is isomorphic to the face lattice of the i-dimensional cube in $\mathbb{R}^n$.

Proof Sketch: We only describe the lattice isomorphism when F is bounded. Using Proposition 3.3, we see that the faces of F are described in a natural way as the set of triples of sets $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$, via the good pair $\left(S_{\mathcal{A}}^{\mathcal{B}}, \mathcal{C}\right)$. Now, consider the map

$$\begin{aligned} \phi : \{\text{Faces of } F\} &\mapsto \{\text{Faces of the 0-1 } i\text{-dimensional cube in } \mathbb{R}^{\mathcal{A}}\}, \\ \mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A} &\mapsto \text{Convex hull of } \textbf{all} \text{ 0-1 vectors with 0's in } \mathcal{A}\backslash\mathcal{B} \text{ and 1's in } \mathcal{B}\backslash C. \end{aligned}$$

Then, ϕ is a lattice isomorphism. □

Consider the $(m-1)$-dimensional **pyrope** $\mathscr{P}_{m-1} :=$ conv $\left([-1,0]^{m-1} \cup [1,0]^{m-1}\right)$, *i.e.* the convex hull of the 0-1 and 0-(-1) cubes in $\mathbb{R}^{m-1}$. See Joswig and Kulas [JK10] for a detailed description of this tropical polytope.

Theorem 3.5 *The matroid M_F of an $(m-1)$-dimensional face F of P'_d corresponding to a full subtree S of T with m leaves has a collection of bases*

$B_F := \{A \in \binom{[n]}{m} \mid |A \cap H| = 1$ *for all* $H \in \mathcal{H}_{(S,\mathcal{L}(S))}\}$, *so it is a transversal matroid.*

If F is bounded, then its face lattice is isomorphic to the face lattice of the $(m-1)$-dimensional pyrope.

Proof Sketch: We only describe the lattice isomorphism when F is bounded. Fix a choice of $l \in \mathcal{L}(S)$. Per Proposition 3.3, the proper faces of F can be described as pairs of sets $(\mathcal{A}, \mathcal{B})$ satisfying $\emptyset \subsetneq \mathcal{A} \subsetneq \mathcal{L}(S)$ and $\mathcal{B} \subsetneq \mathcal{A}$, via the good pair $\left(\mathtt{in}(S)^{\mathcal{A}}, \mathcal{B}\right)$. For one such pair $(\mathcal{A}, \mathcal{B})$, consider the finite set $V_{\mathcal{A},\mathcal{B}} \subseteq \mathbb{R}^{\mathcal{L}(S)\backslash l}$ that we now construct. For each $\mathcal{C} \subseteq \mathcal{B}$:

- If $l \in \mathcal{L}(S)\backslash\left(\mathcal{A}\backslash\mathcal{C}\right)$, define $v \in \mathbb{R}^{\mathcal{L}(S)\backslash l}$ as the vector of 1's in $\mathcal{A}\backslash\mathcal{C}$ and 0's everywhere else. Then, let $v \in V_{\mathcal{A},\mathcal{B}}$.
- If $l \in \mathcal{A}\backslash\mathcal{C}$, define $v \in \mathbb{R}^{\mathcal{L}(S)\backslash l}$ as the vector of 0's in $\mathcal{A}\backslash\mathcal{C}$ and -1's everywhere else. Then, let $v \in V_{\mathcal{A},\mathcal{B}}$.

Consider the map

$$\phi : \{ \text{ Faces of } \partial(F) \ \} \mapsto \{ \text{ Faces of } \partial(\mathscr{P}_{m-1}) \text{ in } \mathbb{R}^{\mathcal{L}(S)\backslash l}\},$$

under which the face of $\partial(F)$ corresponding to $(\mathcal{A}, \mathcal{B})$ gets mapped to the convex hull of the set $V_{\mathcal{A},\mathcal{B}}$.
Then, ϕ is a lattice isomorphism. □

In Figures 1, 2 and 3 we present an example of the tight spans of a tree with 9 leaves for $m = 2, 3, 4$. Notice that Theorems 3.4 and 3.5 describe explicitly the family of matroids

$$\mathfrak{M}_d^{loop} = \{B_F \mid F \text{ is a face of } P'_d\}.$$

Of main interest are the matroids of the $(m-1)$-dimensional faces of P'_d. These turn out to be transversal matroids which encode the combinatorial structure of T in a natural way, as we now explain. Note that an $(m-1)$-dimensional face F of P'_d corresponds to a tree $S \subseteq_{\mathrm{st}} T$ such that:

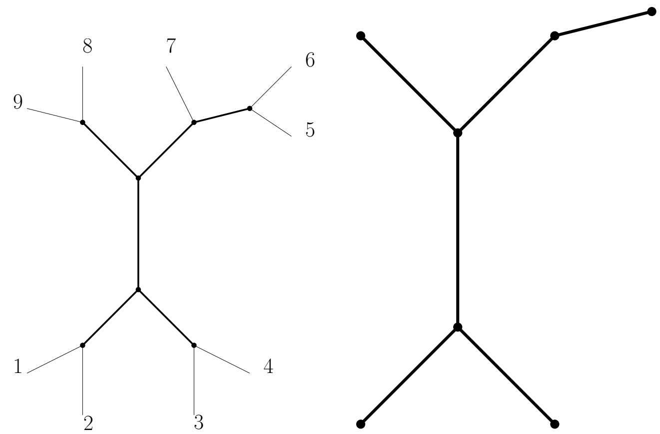

Fig. 1: On the left we have a tree T with 9 leaves. On the right we show its tight span for $m = 2$. Note that it is isomorphic to $\mathtt{in}(T)$.

- **(case 1)** either S has $m-1$ leaves and at least $2(m-1)$ vertices, so $(S, \mathcal{L}(S))$ is an $(m, m-1)$-good pair of T,
- **(case 2)** or S has m leaves and exactly $2(m-1)$ vertices, so S is a full subtree of T with m leaves.

For both cases, M_F is a transversal matroid and B_F is the set of transversals of the collection $\mathcal{H}_{(S,\mathcal{L}(S))} \cup \{H_S^c\}$.

Conjecture 3.6 *Assume $n \geq 2m-1$. Then, the collection of bases of matroids*

$$\{B_F \mid F \textit{ is an } (m-1)\textit{-dimensional face of } P'_d\}$$

recovers uniquely the shape of T.

In particular, the collection $\mathcal{H}_{(S,\mathcal{L}(S))} \cup \{H_S^c\}$ can be recovered by computing the 2-circuits of M_F.

We propose the following problems:

Problem 3.7 *Characterize the family of matroids $\mathfrak{M}_d^{loop}$ when d is the m-dissimilarity vector of a tree T.*

And more ambitiously,

Problem 3.8 *Characterize the family of matroids $\mathfrak{M}_p^{loop}$ when $p \in \mathcal{G}_{m,n}$.*

We now discuss some aspects of these results. From the point of view of $\mathcal{D}_d$, the most relevant faces of P'_d are its vertices. Let us briefly describe how we found these.

Note that an $(m, 0)$-good pair $(S, \emptyset)$ of T is equivalent to a tree $S \subseteq_{\mathrm{st}} \mathtt{in}(T)$ satisfying the inequalities $|\mathcal{L}(S)| \leq m-1 \leq |\mathcal{V}(S)|$.

Now, the following result holds:

Proposition 3.9 *Let $S \subseteq_{\mathrm{st}} \mathtt{in}(T)$ satisfy the inequalities $|\mathcal{L}(S)| \leq m-1 \leq |\mathcal{V}(S)|$, and define $x \in \mathbb{R}^n$ by:*

$$x_i = \omega(i \rightharpoonup S) + \frac{\omega(S)}{m} \textit{ for all } i \in [n].$$

Then, x is a vertex of P'_d and $B_x = B^1 \cap B^2$, where:

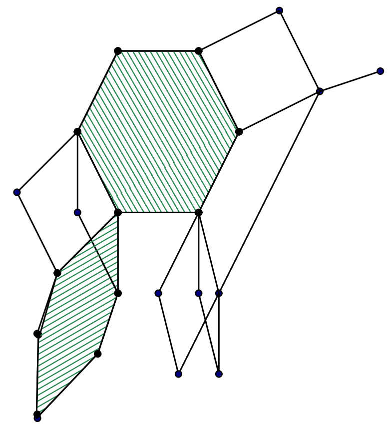

Fig. 2: The tight span of the tree T in Figure 1 for $m = 3$. Hexagons (which correspond to 2-dimensional pyropes) are shown in green. Enclosed regions are 2-dimensional faces and they are 2-dimensional cubes.

$$B^1 := \{A \in \binom{[n]}{m} \mid |A \cap H| \geq 1 \textit{ for all } H \in \mathcal{H}_{(S,\mathcal{L}(S))}\},$$

$$B^2 := \{A \in \binom{[n]}{m} \mid |A \cap R| \leq 1 \textit{ for all } R \in \mathcal{R}_S\}.$$

□

The subpolytope of $\mathscr{H}_m$ corresponding to the vertex x of P'_d found in Proposition 3.9 is then $\mathcal{P}_x := \text{conv}\{e^A \in \mathbb{R}^n \mid A \in B_x\}$. We would like an inequality description of $\mathcal{P}_x$.

Lemma 3.10 *We have:*

$$\mathcal{P}_x = \{y \in \mathscr{H}_m \mid \sum_{j\in H} y_j \geq 1 \textit{ for all } H \in \mathcal{H}_{(S,\mathcal{L}(S))} \textit{ and } \sum_{j \in R} y_j \leq 1 \textit{ for all } R \in \mathcal{R}_S\}.$$

□

We then use Lemma 3.10 to justify why all vertices of P'_d are obtained in this way. This result was a key step in our study, so we present the full proof.

Proposition 3.11 *Let y be a generic point in the interior of $\mathscr{H}_m$. Then, there exists a vertex x of P'_d corresponding to an $(m, 0)$-good pair $(S, \emptyset)$ of T, such that $y \in \mathcal{P}_x$.*

Proof: Let $v \in \mathcal{V}(T)$. The vertex v defines a partition Π_v on the set of leaves $[n]$ of T, under which two different leaves $i, j \in [n]$ belong to the same class if and only if the minimal paths from i to v and from j to v contain a common edge adjacent to v.

For each $\overline{a} \in \Pi_v$, let $\sigma_{\overline{a}} = \sum_{i \in \overline{a}} y_i$, and introduce a subset V_y of the set of internal vertices of T:

$$V_y = \{v \in \mathcal{V}(\mathsf{in}(T)) \mid m - \sigma_{\overline{a}} > 1 \text{ for all } \overline{a} \in \Pi_v\}.$$

We prove that $V_y \neq \emptyset$. To begin, notice that if we take a vertex $v \in \mathcal{V}(\mathsf{in}(T)) \setminus V_y$ and there exist two different classes $\overline{a}, \overline{b} \in \Pi_v$ such that $\sigma_{\overline{a}} > m - 1$ and $\sigma_{\overline{b}} > m - 1$, then $m \geq \sigma_{\overline{a}} + \sigma_{\overline{b}} > 2m - 2$ or $m < 2$.

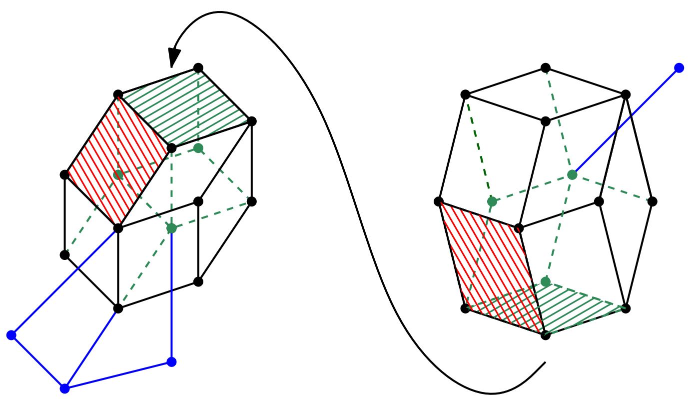

Fig. 3: The tight span of the tree T in Figure 1 for $m = 4$. On the left we have a complex of three 3-dimensional cubes pasted together, and two 2-dimensional cubes (in blue). On top of this, we paste as indicated a rhombic dodecahedron (which corresponds to a 3-dimensional pyrope) with a 1-dimensional cube appended to it (in blue).

This contradicts one of the earliest assumptions made in this section. Thus, for each $v \in \mathcal{V}(\mathfrak{in}(T)) \setminus V_y$ there exists a unique $\overline{a} \in \Pi_v$ such that $\sigma_{\overline{a}} > m-1$.

If $V_y = \emptyset$, among all pairs $(v, \sigma_{\overline{a}})$ with $v \in \mathcal{V}(\mathfrak{in}(T))$ and $\overline{a} \in \Pi_v$ such that $\sigma_{\overline{a}} > m-1$, take the one with $\sigma_{\overline{a}}$ minimal. The class $\overline{a}$ corresponds to a unique edge e of T adjacent to v. Let u be the other vertex adjacent to the edge e. If $u \in \mathcal{V}(\mathfrak{in}(T))$ so that $\deg(u) = 3$, then by the minimality of $\sigma_{\overline{a}}$ and because y is an interior point of $\mathscr{H}_m$, it can only be the case that $\sigma_{\overline{b}} > m-1$, where $\overline{b} \in \Pi_u$ is the class corresponding to the edge e. However, in this latter case we obtain $m = \sigma_{\overline{a}} + \sigma_{\overline{b}} > 2m-2$ or $m < 2$, a contradiction. Finally, if $u \notin \mathcal{V}(\mathfrak{in}(T))$, then u is a leaf of T. But then $1 > y_u = \sigma_{\overline{a}} > m-1$ or $m < 2$, the first inequality coming from the fact that $y \in \mathscr{H}_m$.

Therefore, $V_y \neq \emptyset$.

It is not difficult to check that V_y defines a tree $S \subseteq_{\mathrm{st}} \mathfrak{in}(T)$, so that $V_y = \mathcal{V}(S)$. More precisely, if we have three different vertices $u, v \in V_y$ and $w \in \mathcal{V}(\mathfrak{in}(T))$ such that w lies in the minimal path from u to v, then $w \in V_y$. For example, if $\sigma_{\overline{c}} > m-1$ for some $\overline{c} \in \Pi_w$, then that implies that either $\sigma_{\overline{a}} > m-1$ or $\sigma_{\overline{b}} > m-1$ holds for some $\overline{a} \in \Pi_v$ or some $\overline{b} \in \Pi_u$.

Now, $|\mathcal{L}(S)| \leq m-1$. Otherwise, for every leaf l of S, let $\overline{a_l} \in \Pi_l$ correspond to the edge of S adjacent to l. Then:

$$m \geq \sum_{l \in \mathcal{L}(S)} (m - \sigma_{\overline{a_l}}) > \sum_{l \in \mathcal{L}(S)} 1 = |\mathcal{L}(S)| \geq m.$$

Also, $|\mathcal{V}(S)| = |V_y| \geq m-1$. To see this, note that $\mathcal{L}(\overline{S}) \cap V_y = \emptyset$. For each $l \in \mathcal{L}(\overline{S})$, let $\overline{a_l} \in \Pi_l$ correspond to the edge e adjacent to S. It has to be the case that $\sigma_{\overline{a_l}} > m-1$ or $1 > m - \sigma_{\overline{a_l}}$. Otherwise, if we let v be the vertex of V_y adjacent to e and $\overline{a} \in \Pi_v$ be the class corresponding to the edge e, then we would have $\sigma_{\overline{a}} > m-1$, contradicting the fact that $v \in V_y$. But then, as the set of leaves $\mathcal{L}(\overline{S})$ induces a partition of $[n]$, we see that:

$$m = \sum_{l \in \mathcal{L}(\overline{S})} (m - \sigma_{\overline{a_l}}) < \sum_{l \in \mathcal{L}(\overline{S})} 1 = |\mathcal{L}(\overline{S})| \leq m.$$

Hence, $(S, \emptyset)$ is an $(m, 0)$-good pair of T and corresponds to a vertex x of P'_d.

If $l \in \mathcal{L}(S)$ and $\overline{a} \in \Pi_l$ is the class corresponding to the edge of S adjacent to l, then $m - \sigma_{\overline{a}} > 1$. On the other hand, if $l \in \mathcal{L}(\overline{S})$ and $\overline{a} \in \Pi_l$ is the class corresponding to the edge adjacent to both S and l, then $m - \sigma_{\overline{a}} < 1$. Therefore, $y \in \mathcal{P}_x$ per the description of $\mathcal{P}_x$ in terms of inequalities. □

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Algebraic and combinatorial structures on Baxter permutations

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Abstract. We give a new construction of a Hopf subalgebra of the Hopf algebra of Free quasi-symmetric functions whose bases are indexed by objects belonging to the Baxter combinatorial family (*i.e.* Baxter permutations, pairs of twin binary trees, *etc.*). This construction relies on the definition of the Baxter monoid, analog of the plactic monoid and the sylvester monoid, and on a Robinson-Schensted-like insertion algorithm. The algebraic properties of this Hopf algebra are studied. This Hopf algebra appeared for the first time in the work of Reading [Lattice congruences, fans and Hopf algebras, *Journal of Combinatorial Theory Series A*, 110:237–273, 2005].

Résumé. Nous proposons une nouvelle construction d'une sous-algèbre de Hopf de l'algèbre de Hopf des fonctions quasi-symétriques libres dont les bases sont indexées par les objets de la famille combinatoire de Baxter (*i.e.* permutations de Baxter, couples d'arbres binaires jumeaux, *etc.*). Cette construction repose sur la définition du monoïde de Baxter, analogue du monoïde plaxique et du monoïde sylvestre, et d'un algorithme d'insertion analogue à l'algorithme de Robinson-Schensted. Les propriétés algébriques de cette algèbre de Hopf sont étudiées. Cette algèbre de Hopf est apparue pour la première fois dans le travail de Reading [Lattice congruences, fans and Hopf algebras, *Journal of Combinatorial Theory Series A*, 110:237–273, 2005].

Keywords: Hopf algebras, Robinson-Schensted algorithm, quotient monoid, Baxter permutations

1 Introduction

In the recent years, many combinatorial Hopf algebras, whose bases are indexed by combinatorial objects, have been intensively studied. For example, the Malvenuto-Reutenauer Hopf algebra $\mathbf{FQSym}$ of Free quasi-symmetric functions [19, 7] has bases indexed by permutations. This Hopf algebra admits several Hopf subalgebras: The Hopf algebra of Free symmetric functions $\mathbf{FSym}$ [21, 7], whose bases are indexed by standard Young tableaux, the Hopf algebra $\mathbf{Bell}$ [23] whose bases are indexed by set partitions, the Loday-Ronco Hopf algebra $\mathbf{PBT}$ [18, 12] whose bases are indexed by planar binary trees and the Hopf algebra $\mathbf{Sym}$ of non-commutative symmetric functions [10] whose bases are indexed by integer compositions. An unifying approach to construct all these structures relies on a definition of a congruence on words leading to the definition of monoids on combinatorial objects. Indeed, $\mathbf{FSym}$ is directly obtained from the plactic monoid [15], $\mathbf{Bell}$ from the Bell monoid [23], $\mathbf{PBT}$ from the sylvester monoid [11, 12], and $\mathbf{Sym}$ from the hypoplactic monoid [20]. The richness of these constructions relies on the fact that, in addition to construct Hopf algebras, the definition of such monoids often brings partial orders, combinatorial algorithms and Robinson-Schensted-like algorithms, of independent interest.

In this paper, we propose to enrich this collection of Hopf algebras by providing a construction of a Hopf algebra whose bases are indexed by objects belonging to the Baxter combinatorial family. This combinatorial family admits various representations as Baxter permutations [4], pairs of twin binary trees [8], quadrangulations [1], plane bipolar orientations [5], *etc*. In [22], Reading defines first a Hopf algebra on Baxter permutations in the context of lattice congruences; Moreover, very recently, Law and Reading [16] have studied and detailed their construction of this Hopf algebra. However, even if both points of view lead to the same general theory, their paths are different and provide different ways of understanding this Hopf algebra, one centered, as in Law and Reading's work, on lattice theory, the other, as in our work, centered on combinatorics on words. Moreover, a large part of the results of each paper does not appear in the other.

We begin by recalling in Section 2 the preliminary notions used thereafter. In Section 3, we define the Baxter congruence. This congruence allows to define a quotient of the free monoid, the Baxter monoid, which has a number of properties required for the Hopf algebraic construction which follows. We show that the Baxter monoid is intimately linked to the sylvester monoid. Next, in Section 4, we develop a Robinson-Schensted-like insertion algorithm that allows to decide if two words are equivalent according to the Baxter congruence. Given a word, this algorithm computes a pair of twin binary trees. Section 5 is devoted to the study of some properties of the equivalence classes of permutations under the Baxter congruence. This leads to the definition of a lattice structure on pairs of twin binary trees. Finally, in Section 6, we define the Hopf algebra $\mathbf{Baxter}$ and study it. Using the order structure on pairs of twin binary trees, we provide multiplicative bases and show that $\mathbf{Baxter}$ is free as an algebra. Using the results of Foissy on bidendriform bialgebras [9], we show that $\mathbf{Baxter}$ is also self-dual and that the Lie algebra of its primitive elements is free.

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2 Preliminaries

2.1 Words

In the sequel, $A := \{a_1 < a_2 < \ldots\}$ is a totally ordered infinite alphabet and A^* is the free monoid spanned by A. Let $u \in A^*$. For $S \subseteq A$, we denote by $u_{|S}$ the *restriction* of u on the alphabet S, that is the longest subword of u made of letters of S. The *evaluation* $\mathrm{eval}(u)$ of the word u is the non-negative integer vector such that its i-th entry is the number of occurrences of the letter a_i in u. Let $\max(u)$ be the maximal letter of u. The *Schützenberger transformation* $\#$ is defined by $u^{\#} := \max(u){+}1{-}u_{|u|} \ldots \max(u){+}1{-}u_1$; For example, $(a_5a_3a_1a_1a_5a_2)^{\#} = a_4a_1a_5a_5a_3a_1$. Note that it is an involution if u has an occurrence of a_1. Let $v \in A^*$ and $\mathtt{a}, \mathtt{b} \in A$. The *shuffle product* $\sqcup\!\sqcup$ is defined on $\mathbb{Z}\langle A\rangle$ recursively by $u \sqcup\!\sqcup \epsilon := \epsilon \sqcup\!\sqcup u := u$ and $\mathtt{a}u \sqcup\!\sqcup \mathtt{b}v := \mathtt{a}(u \sqcup\!\sqcup \mathtt{b}v) + \mathtt{b}(\mathtt{a}u \sqcup\!\sqcup v)$.

2.2 Permutations

Denote by $\mathfrak{S}_n$ the set of permutations of size n and $\mathfrak{S} := \cup_{n\geq 0}\mathfrak{S}_n$. We shall call (i, j) a *co-inversion* of $\sigma \in \mathfrak{S}$ if $i < j$ and $\sigma_i^{-1} > \sigma_j^{-1}$. Let us recall that the *(right) permutohedron order* is the partial order $\leq_{\mathrm{P}}$ defined on $\mathfrak{S}_n$ where σ is covered by ν if $\sigma = u\mathtt{ab}v$ and $\nu = u\mathtt{ba}v$ where $\mathtt{a} < \mathtt{b}$. Let $\sigma, \nu \in \mathfrak{S}$. The permutation $\sigma \diagup \nu$ is obtained by concatenating σ and the letters of ν incremented by $|\sigma|$; In the same way, the permutation $\sigma \diagdown \nu$ is obtained by concatenating the letters of ν incremented by $|\sigma|$ and σ; For example, $\mathbf{312} \diagup 2314 = \mathbf{312}5647$ and $\mathbf{312} \diagdown 2314 = 5647\mathbf{312}$. The permutation σ is *connected* if $\sigma = \nu \diagup \pi$ implies $\nu = \sigma$ or $\pi = \sigma$. The *shifted shuffle product* $\overline{\sqcup\!\sqcup}$ of two permutations is defined by $\sigma \,\overline{\sqcup\!\sqcup}\, \nu :=$

$\sigma \sqcup\!\sqcup (\nu_1 + |\sigma| \dots \nu_{|\nu|} + |\sigma|)$; For example, $\mathbf{12}\,\overline{\sqcup\!\sqcup}\,21 = \mathbf{12} \sqcup\!\sqcup 43 = \mathbf{12}43 + \mathbf{1}4\mathbf{2}3 + \mathbf{1}43\mathbf{2} + 4\mathbf{12}3 + 4\mathbf{1}3\mathbf{2} + 43\mathbf{12}$. The *standardized word* $\mathrm{std}(u)$ of $u \in A^*$ is the unique permutation σ satisfying $\sigma_i < \sigma_j$ iff $u_i \leq u_j$ for all $1 \leq i < j \leq |u|$; For example, $\mathrm{std}(a_3 a_1 a_4 a_2 a_5 a_7 a_4 a_2 a_3) = 416289735$.

2.3 Binary trees

Denote by $\mathcal{BT}_n$ the set of binary trees with n internal nodes and $\mathcal{BT} := \cup_{n \geq 0} \mathcal{BT}_n$. We use in the sequel the standard terminology (*i.e.*, *child*, *ancestor*, ...) about binary trees [2]. The only element of $\mathcal{BT}_0$ is the *leaf* or *empty tree*, denoted by $\perp$. Let us recall that the *Tamari order* [14] is the partial order $\leq_{\mathrm{T}}$ defined on $\mathcal{BT}_n$ where $T_0 \in \mathcal{BT}_n$ is covered by $T_1 \in \mathcal{BT}_n$ if it is possible to transform T_0 into T_1 by performing a right rotation (see Figure 1).

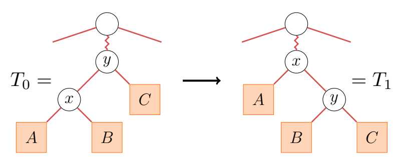


Figure 1: The right rotation of root y.

Let $T_0, T_1 \in \mathcal{BT}$. The binary tree $T_0 \diagup T_1$ is obtained by grafting T_0 from its root on the leftmost leaf of T_1; In the same way, the binary tree $T_0 \diagdown T_1$ is obtained by grafting T_1 from its root on the rightmost leaf of T_0. The *canopy* (see [18] and [26]) $\mathrm{cnp}(T)$ of $T \in \mathcal{BT}$ is the word on the alphabet $\{0, 1\}$ obtained by browsing the leaves of T from left to right except the first and the last one, writing 0 if the considered leaf is oriented to the right, 1 otherwise (see Figure 2). Note that the orientation of the leaves in a binary tree is determined only by its nodes so that we can omit to draw the leaves in our next graphical representations.

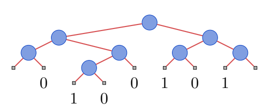


Figure 2: The canopy of this binary tree is 0100101.

An A-labeled binary tree T is a *left* (resp. *right*) *binary search tree* if for any node x labeled by $\mathtt{b}$, each label $\mathtt{a}$ of a node in the left subtree of x and each label $\mathtt{c}$ of a node in the right subtree of x, the inequality $\mathtt{a} < \mathtt{b} \leq \mathtt{c}$ (resp. $\mathtt{a} \leq \mathtt{b} < \mathtt{c}$) holds. A binary tree $T \in \mathcal{BT}_n$ is a *decreasing binary tree* if it is bijectively labeled on $\{1, \dots, n\}$ and, for all node y of T, if x is a child of y, then the label of x is smaller than the label of y. The *shape* of a labeled binary tree is the unlabeled binary tree obtained by forgetting its labels.

2.4 Baxter permutations and pairs of twin binary trees

A permutation σ is a *Baxter permutation* if for any subword $u = u_1 u_2 u_3 u_4$ of σ such that the letters u_2 and u_3 are adjacent in σ, $\mathrm{std}(u) \notin \{2413, 3142\}$. In other words, σ is a Baxter permutation if it avoids the *generalized permutation patterns* $2-41-3$ and $3-14-2$ (see [3] for an introduction on generalized permutation patterns). For example, $\mathbf{4}21\mathbf{73}8\mathbf{5}6$ is not a Baxter permutation; On the other hand 436975128 is a Baxter permutation. Let us denote by $\mathfrak{S}^{\mathrm{B}}_n$ the set of Baxter permutations of size n and $\mathfrak{S}^{\mathrm{B}} := \cup_{n \geq 0} \mathfrak{S}^{\mathrm{B}}_n$.

A *pair of twin binary trees* (T_L, T_R) is made of two binary trees $T_L, T_R \in \mathcal{BT}_n$ such that the canopies of T_L and T_R are complementary, that is $\mathrm{cnp}(T_L)_i \neq \mathrm{cnp}(T_R)_i$ for all $1 \leq i \leq n-1$. Denote by $\mathcal{TBT}_n$

the set of pairs of twin binary trees where each binary tree has n nodes and $\mathcal{TBT} := \cup_{n\geq 0}\mathcal{TBT}_n$. In [8], Dulucq and Guibert have highlighted a bijection between Baxter permutations and pairs of twin binary trees. In the sequel, we shall make use of a very similar bijection.

3 The Baxter monoid

3.1 Definition and first properties

Recall that an equivalence relation $\equiv$ defined on A^* is a *congruence* if for all $u, u', v, v' \in A^*$, $u \equiv u'$ and $v \equiv v'$ imply $u.v \equiv u'.v'$.

Definition 3.1 *The* Baxter monoid *is the quotient of the free monoid A^* by the congruence $\equiv_{\mathrm{B}}$ that is the transitive closure of the* adjacency relations $\leftrightharpoons_{\mathrm{B}}$ *and* $\rightleftharpoons_{\mathrm{B}}$ *defined for $u, v \in A^*$ and* $\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d} \in A$ *by:*

$$\mathtt{c}u\mathtt{ad}v\mathtt{b} \leftrightharpoons_{\mathrm{B}} \mathtt{c}u\mathtt{da}v\mathtt{b} \qquad \textit{where} \quad \mathtt{a} \leq \mathtt{b} < \mathtt{c} \leq \mathtt{d}, \tag{1}$$

$$\mathtt{b}u\mathtt{da}v\mathtt{c} \rightleftharpoons_{\mathrm{B}} \mathtt{b}u\mathtt{ad}v\mathtt{c} \qquad \textit{where} \quad \mathtt{a} < \mathtt{b} \leq \mathtt{c} < \mathtt{d}. \tag{2}$$

For $u \in A^*$, denote by $\widehat{u}$ the $\equiv_{\mathrm{B}}$ -equivalence class of u; For example, the $\equiv_{\mathrm{B}}$ -equivalence class of 5273641 is $\{5237641, 5273641, 5276341, 5723641, 5726341, 5762341\}$.

An equivalence relation $\equiv$ defined on A^* is *compatible with the restriction of alphabet intervals* if for all interval I of A and for all $u, v \in A^*$, $u \equiv v$ implies $u_{|I} \equiv v_{|I}$.

Proposition 3.2 *The Baxter monoid is compatible with the restriction of alphabet intervals.*

Proof: We only have to check the property on adjacency relations. □

An equivalence relation $\equiv$ defined on A^* is *compatible with the destandardization process* if for all $u, v \in A^*$, $u \equiv v$ iff $\mathrm{std}(u) \equiv \mathrm{std}(v)$ and $\mathrm{eval}(u) = \mathrm{eval}(v)$.

Proposition 3.3 *The Baxter monoid is compatible with the destandardization process.*

An equivalence relation $\equiv$ defined on A^* is *compatible with the Schützenberger involution* if for all $u, v \in A^*$, $u \equiv v$ implies $u^{\#} \equiv v^{\#}$.

Proposition 3.4 *The Baxter monoid is compatible with the Schützenberger involution.*

3.2 Connection with the sylvester monoid

The *sylvester monoid* [11, 12] is the quotient of the free monoid A^* by the congruence $\equiv_{\mathrm{S}}$ that is the transitive closure of the adjacency relation $\leftrightharpoons_{\mathrm{S}}$ defined for $u \in A^*$ and $\mathtt{a}, \mathtt{b}, \mathtt{c} \in A$ by:

$$\mathtt{ac}u\mathtt{b} \leftrightharpoons_{\mathrm{S}} \mathtt{ca}u\mathtt{b} \qquad \text{where} \quad \mathtt{a} \leq \mathtt{b} < \mathtt{c}. \tag{3}$$

In the same way, let us define the $\#$*-sylvester monoid* by the congruence $\equiv_{\mathrm{S}\#}$ that is the transitive closure of the adjacency relation $\leftrightharpoons_{\mathrm{S}\#}$ defined for $u \in A^*$ and $\mathtt{a}, \mathtt{b}, \mathtt{c} \in A$ by:

$$\mathtt{b}u\mathtt{ac} \leftrightharpoons_{\mathrm{S}\#} \mathtt{b}u\mathtt{ca} \qquad \text{where} \quad \mathtt{a} < \mathtt{b} \leq \mathtt{c}. \tag{4}$$

Note that this adjacency relation is defined by taking the images by the Schützenberger involution of the sylvester adjacency relation. Indeed, for all $u, v \in A^*$, $u \equiv_{\mathrm{S}\#} v$ iff $u^{\#} \equiv_{\mathrm{S}} v^{\#}$. The Baxter monoid and the sylvester monoid are related in the following way:

Proposition 3.5 *Let $u, v \in A^*$. Then, $u \equiv_{\mathrm{B}} v$ iff $u \equiv_{\mathrm{S}} v$ and $u \equiv_{\mathrm{S}\#} v$.*

Proposition 3.5 shows that the $\equiv_{\mathrm{B}}$ -equivalence classes are the intersection of $\equiv_{\mathrm{S}}$ -equivalence classes and $\equiv_{\mathrm{S}\#}$ -equivalence classes.

4 A Robinson-Schensted-like algorithm

We shall describe here an insertion algorithm $u \mapsto (\mathbb{P}(u), \mathbb{Q}(u))$, such that, given a word $u \in A^*$, it computes its $\mathbb{P}$-symbol, that is a pair of A-labeled twin binary trees (T_L, T_R) where T_L (resp. T_R) is a left (resp. right) binary search tree, and its $\mathbb{Q}$-symbol, a decreasing binary tree.

4.1 Definition of the insertion algorithm

Let T be an A-labeled right binary search tree and $\mathtt{b}$ a letter of A. The *lower restricted binary tree* of T compared to $\mathtt{b}$, namely $T_{\leq \mathtt{b}}$, is the right binary search tree uniquely made of the nodes x of T labeled by a letter $\mathtt{a}$ satisfying $\mathtt{a} \leq \mathtt{b}$ and such that for all nodes x and y of $T_{\leq \mathtt{b}}$, if x is ancestor of y in $T_{\leq \mathtt{b}}$, then x is ancestor of y in T. In the same way, we define the *higher restricted binary tree* of T compared to $\mathtt{b}$, namely $T_{>\mathtt{b}}$ (see Figure 3).

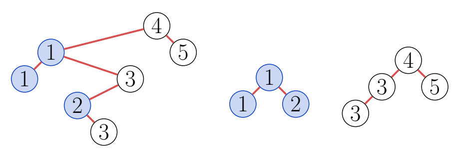


Figure 3: A right binary search tree T, $T_{\leq 2}$ and $T_{>2}$.

Let T be an A-labeled right binary search tree and $\mathtt{a}$ a letter of A. The *root insertion* of $\mathtt{a}$ into T consists in modifying T so that the root of T is a new node labeled by $\mathtt{a}$, its left subtree is $T_{\leq \mathtt{a}}$ and its right subtree is $T_{>\mathtt{a}}$.

Let T be an A-labeled left (resp. right) binary search tree and $\mathtt{a}$ a letter of A. The *leaf insertion* of $\mathtt{a}$ into T is recursively defined by: If $T = \perp$, the result is the one-node binary tree labeled by $\mathtt{a}$; Else, if the label $\mathtt{b}$ of the root of T satisfies $\mathtt{a} < \mathtt{b}$ (resp. $\mathtt{a} \leq \mathtt{b}$), make a leaf insertion of $\mathtt{a}$ into the left subtree of T, else, make a leaf insertion of $\mathtt{a}$ into the right subtree of T.

Given a pair of A-labeled twin binary trees (T_L, T_R) where T_L (resp. T_R) is a left (resp. right) binary search tree, the *insertion* of the letter $\mathtt{a}$ of A into (T_L, T_R) consists in making a leaf insertion of $\mathtt{a}$ into T_L and a root insertion of $\mathtt{a}$ into T_R.

The $\mathbb{P}$-symbol (T_L, T_R) of a word $u \in A^*$ is computed by iteratively inserting the letters of u, from left to right, into the pair of twin binary trees $(\perp, \perp)$. The $\mathbb{Q}$-symbol of u is the decreasing binary tree labeled on $\{1, \dots, |u|\}$, built by recording the dates of creation of each node of T_R (see Figure 4).

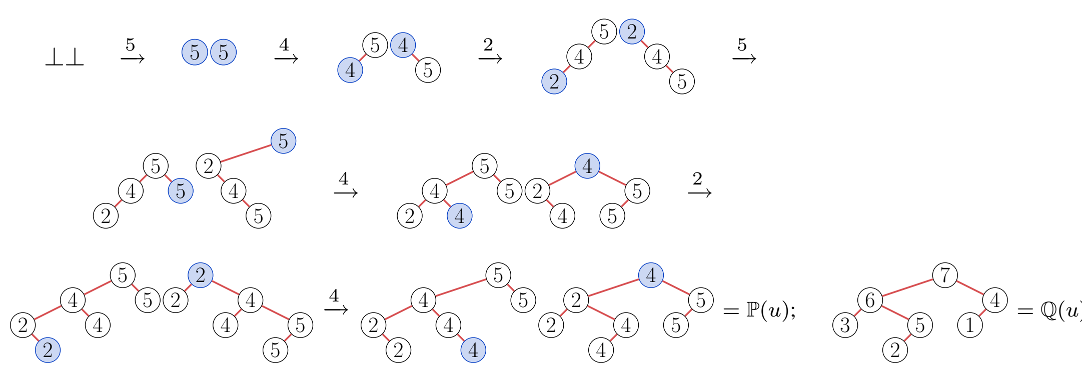


Figure 4: Steps of computation of the $\mathbb{P}$-symbol and the $\mathbb{Q}$-symbol of $u := 5425424$.

4.2 Validity of the insertion algorithm

Lemma 4.1 *Let $u \in A^*$. Let T be the right binary search tree obtained by root insertions of the letters of u, from left to right. Let T' be the right binary search tree obtained by leaf insertions of the letters of u, from right to left. Then, $T = T'$.*

Lemma 4.2 *Let $\sigma \in \mathfrak{S}$ and $T \in \mathcal{BT}_{|\sigma|}$ be the binary search tree obtained by leaf insertions of the letters of σ, from left to right. Then, for $1 \leq i \leq |\sigma|-1$, the $i+1$-st leaf of T is right-oriented iff $(i, i+1)$ is a co-inversion of σ.*

If (T_L, T_R) is a pair of labeled twin binary trees, define its *shape*, that is the pair of unlabeled twin binary trees (T'_L, T'_R) where T'_L (resp. T'_R) is the shape of T_L (resp. T_R).

Proposition 4.3 *For all word $u \in A^*$, the shape of the $\mathbb{P}$-symbol of u is a pair of twin binary trees.*

Proposition 4.4 *Let $u, v \in A^*$. Then, $u \equiv_{\mathrm{B}} v$ iff $\mathbb{P}(u) = \mathbb{P}(v)$.*

In particular, we have $\mathbb{P}(\sigma) = \mathbb{P}(\nu)$ iff the permutations σ and ν are $\equiv_{\mathrm{B}}$-equivalent. Moreover, each $\equiv_{\mathrm{B}}$-equivalence class of permutations can be encoded by a pair of unlabeled twin binary trees because there is one unique way to bijectively label a binary tree with n nodes on $\{1, \ldots, n\}$ such that it is a binary search tree.

Remark 4.5 *Let $u, v \in A^*$ and $(T_L, T_R) := \mathbb{P}(u)$. We have $u \equiv_{\mathrm{B}} v$ iff the following two assertions are satisfied:*

(i) v is a linear extension of T_L seen as a poset in which the smallest element is its root;

(ii) v is a linear extension of T_R seen as a poset in which minimal elements are the nodes with no descents.

5 The Baxter lattice

5.1 Some properties of the $\equiv_{\mathrm{B}}$-equivalence classes of permutations

Theorem 5.1 *For all $n \geq 0$, each equivalence class of $\mathfrak{S}_n/_{\equiv_{\mathrm{B}}}$ contains exactly one Baxter permutation.*

Proposition 5.2 *For all $n \geq 0$, each equivalence class of $\mathfrak{S}_n/_{\equiv_{\mathrm{B}}}$ is an interval of the permutohedron.*

For all permutation σ, let us define $\sigma\uparrow$ (resp. $\sigma\downarrow$) the maximal (resp. minimal) permutation of the $\equiv_{\mathrm{B}}$-equivalence class of σ for the permutohedron order.

Proposition 5.3 *Let $\sigma, \nu \in \mathfrak{S}_n$ such that $\sigma \leq_{\mathrm{P}} \nu$. Then, $\sigma\uparrow \leq_{\mathrm{P}} \nu\uparrow$ and $\sigma\downarrow \leq_{\mathrm{P}} \nu\downarrow$.*

5.2 A lattice structure on the set of pairs of twin binary trees

Definition 5.4 *For all $n \geq 0$, define the order relation $\leq_{\mathrm{B}}$ on the set $\mathcal{TBT}_n$ setting $J_0 \leq_{\mathrm{B}} J_1$, where $J_0, J_1 \in \mathcal{TBT}_n$, if there exists $\sigma_0, \sigma_1 \in \mathfrak{S}_n$ such that $\mathbb{P}(\sigma_0) = J_0$, $\mathbb{P}(\sigma_1) = J_1$ and $\sigma_0 \leq_{\mathrm{P}} \sigma_1$.*

Propositions 5.2 and 5.3 ensure that this order is well-defined, and in particular that the relation $\leq_{\mathrm{B}}$ is transitive and antisymmetric.

The pair of twin binary trees (T_L, T_R) is covered by $(T'_L, T'_R) \in \mathcal{TBT}$ if one of the three following conditions is satisfied:

1. $T'_R = T_R$ and T'_L is obtained from T_L by performing a left rotation into T_L such that $\mathrm{cnp}(T_L) = \mathrm{cnp}(T'_L)$;

2. $T'_L = T_L$ and T'_R is obtained from T_R by performing a right rotation into T_R such that $\mathrm{cnp}(T_R) = \mathrm{cnp}(T'_R)$;

3. T'_L (resp. T'_R) is obtained by performing a left (resp. right) rotation into T_L (resp. T_R) such that $\mathrm{cnp}(T_L) \neq \mathrm{cnp}(T'_L)$ (resp. $\mathrm{cnp}(T_R) \neq \mathrm{cnp}(T'_R)$).

Moreover, it is possible to compare two pairs of twin binary trees $J_0 := (T^0_L, T^0_R)$ and $J_1 := (T^1_L, T^1_R)$ very easily by computing the *Tamari vector* (see [14]) of each binary tree. Indeed, we have $J_0 \leq_{\mathrm{B}} J_1$ iff the Tamari vector of T^0_L (resp. T^0_R) is greater (resp. smaller) component by component than the Tamari vector of T^1_L (resp. T^1_R).

Propositions 5.2 and 5.3 implies that that $\equiv_{\mathrm{B}}$ is also a lattice congruence [6, 22]. Thus, since the permutohedron is a lattice,

Proposition 5.5 *For all $n \geq 0$, the poset $(\mathcal{TBT}_n, \leq_{\mathrm{B}})$ is a lattice.*

6 The Baxter Hopf Algebra

In the sequel, all the algebraic structures have a field of characteristic zero $\mathbb{K}$ as ground field.

6.1 The Hopf algebra $\mathbf{FQSym}$

Recall that the family $\{\mathbf{F}_\sigma\}_{\sigma\in\mathfrak{S}}$ form the *fundamental* basis of $\mathbf{FQSym}$ [7]. Its product and its coproduct are defined by:

$$\mathbf{F}_\sigma \cdot \mathbf{F}_\nu := \sum_{\pi \in \sigma \overline{\sqcup\!\sqcup} \nu} \mathbf{F}_\pi, \qquad \Delta(\mathbf{F}_\sigma) := \sum_{0\leq i\leq|\sigma|} \mathbf{F}_{\mathrm{std}(\sigma_1\ldots\sigma_i)} \otimes \mathbf{F}_{\mathrm{std}(\sigma_{i+1}\ldots\sigma_{|\sigma|})}. \tag{5}$$

The following theorem due to Hivert and Nzeutchap [13] shows that an equivalence relation on A^* satisfying some properties can be used to define Hopf subalgebras of $\mathbf{FQSym}$:

Theorem 6.1 *Let $\equiv$ be an equivalence relation defined on A^*. If $\equiv$ is a congruence, compatible with the restriction of alphabet intervals and compatible with the destandardization process, then, the family $\{\mathbf{P}_{\widehat{\sigma}}\}_{\widehat{\sigma}\in\mathfrak{S}/_\equiv}$ defined by:*

$$\mathbf{P}_{\widehat{\sigma}} := \sum_{\sigma\in\widehat{\sigma}} \mathbf{F}_\sigma \tag{6}$$

spans a Hopf subalgebra of $\mathbf{FQSym}$.

6.2 The Hopf algebra $\mathbf{Baxter}$

By definition, $\equiv_{\mathrm{B}}$ is a congruence, and, by Proposition 3.2 and 3.3, $\equiv_{\mathrm{B}}$ checks the conditions of Theorem 6.1. Moreover, by Proposition 4.4, the $\equiv_{\mathrm{B}}$ -equivalence classes of permutations can be encoded by pairs of unlabeled twin binary trees. Hence, we have the following theorem:

Theorem 6.2 *The family $\{\mathbf{P}_J\}_{J\in\mathcal{TBT}}$ defined by:*

$$\mathbf{P}_J := \sum_{\substack{\sigma\in\mathfrak{S} \\ \mathbb{P}(\sigma)=J}} \mathbf{F}_\sigma \tag{7}$$

spans a Hopf subalgebra of $\mathbf{FQSym}$, namely the Hopf algebra $\mathbf{Baxter}$.

The Hilbert series of $\mathbf{Baxter}$ is $B(z) := 1 + z + 2z^2 + 6z^3 + 22z^4 + 92z^5 + 422z^6 + 2074z^7 + 10754z^8 + 58202z^9 + 326240z^{10} + 1882960z^{11} + \ldots$, the generating series of Baxter permutations (sequence **A001181** of [24]).

One has for example,

$$\mathbf{P} = \mathbf{F}_{12}, \qquad \mathbf{P} = \mathbf{F}_{2143} + \mathbf{F}_{2413}, \qquad \mathbf{P} = \mathbf{F}_{542163} + \mathbf{F}_{542613} + \mathbf{F}_{546213}. \tag{8}$$

By Theorem 6.1, the product of $\mathbf{Baxter}$ is well-defined. We deduce it from the product of $\mathbf{FQSym}$ and obtain

$$\mathbf{P}_{J_0} \cdot \mathbf{P}_{J_1} = \sum_{\substack{\mathbb{P}(\sigma)=J_0,\ \mathbb{P}(\nu)=J_1 \\ \pi \,\in\, \sigma \sqcup\!\sqcup \nu \,\cap\, \mathfrak{S}^{\mathrm{B}}}} \mathbf{P}_{\mathbb{P}(\pi)}. \tag{9}$$

For example,

$$\mathbf{P} \cdot \mathbf{P} = \mathbf{P} + \mathbf{P} + \mathbf{P} + \mathbf{P} + \mathbf{P} + \mathbf{P}. \tag{10}$$

In the same way, we deduce the coproduct of $\mathbf{Baxter}$ from the coproduct of $\mathbf{FQSym}$ and obtain

$$\Delta(\mathbf{P}_J) = \sum_{\substack{\mathbb{P}(\pi)=J \\ \pi=u.v \\ \sigma:=\mathrm{std}(u),\ \nu:=\mathrm{std}(v)\in\mathfrak{S}^{\mathrm{B}}}} \mathbf{P}_{\mathbb{P}(\sigma)} \otimes \mathbf{P}_{\mathbb{P}(\nu)}. \tag{11}$$

For example,

$$\Delta\mathbf{P} = 1 \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes \mathbf{P} + \mathbf{P} \otimes 1. \tag{12}$$

Remark 6.3 *It is well-known that the Hopf algebra* $\mathbf{PBT}$ *[18, 12] is a Hopf subalgebra of* $\mathbf{FQSym}$*. Besides, we have the following sequence of injective Hopf maps:*

$$\mathbf{PBT} \overset{\rho}{\hookrightarrow} \mathbf{Baxter} \hookrightarrow \mathbf{FQSym}. \tag{13}$$

Indeed, by Proposition 3.5, every $\equiv_{\mathrm{S}}$ *-equivalence class is an union of some* $\equiv_{\mathrm{B}}$ *-equivalence classes. Denoting by* $\{\mathbf{P}_T\}_{T\in\mathcal{BT}}$ *the basis of* $\mathbf{PBT}$ *defined in accordance with (6) by the sylvester equivalence relation* $\equiv_{\mathrm{S}}$ *, we have*

$$\rho\,(\mathbf{P}_T) = \sum_{\substack{T'\in\mathcal{BT} \\ J:=(T',T)\in\mathcal{TBT}}} \mathbf{P}_J. \tag{14}$$

For example,

$$\rho\left(\mathbf{P}\right) = \mathbf{P} + \mathbf{P} + \mathbf{P}. \tag{15}$$

6.3 Multiplicative bases

Define the *elementary* family $\{\mathbf{E}_J\}_{J\in\mathcal{TBT}}$ and the *homogeneous* family $\{\mathbf{H}_J\}_{J\in\mathcal{TBT}}$ respectively by:

$$\mathbf{E}_J := \sum_{J \leq_{\mathrm{B}} J'} \mathbf{P}_{J'} \qquad \text{and} \qquad \mathbf{H}_J := \sum_{J' \leq_{\mathrm{B}} J} \mathbf{P}_{J'}. \tag{16}$$

These families are bases of $\mathbf{Baxter}$ since they are defined by triangularity.

Let $J_0 := (T_L^0, T_R^0)$ and $J_1 := (T_L^1, T_R^1)$ be two pairs of twin binary trees. Let us define the pair of twin binary trees $J_0 \diagup J_1$ by $J_0 \diagup J_1 := (T_L^0 \diagdown T_L^1, T_R^0 \diagup T_R^1)$. In the same way, the pair of twin binary trees $J_0 \diagdown J_1$ is defined by $J_0 \diagdown J_1 := (T_L^0 \diagup T_L^1, T_R^0 \diagdown T_R^1)$.

Using the multiplicative bases of $\mathbf{FQSym}$, we establish the following proposition:

Proposition 6.4 *For all $J_0, J_1 \in \mathcal{TBT}$, we have*

$$\mathbf{E}_{J_0} \cdot \mathbf{E}_{J_1} = \mathbf{E}_{J_0 \diagup J_1} \qquad \textit{and} \qquad \mathbf{H}_{J_0} \cdot \mathbf{H}_{J_1} = \mathbf{H}_{J_0 \diagdown J_1}. \tag{17}$$

Lemma 6.5 *Let C be an equivalence class of $\mathfrak{S}_n/_{\equiv_{\mathrm{B}}}$. The Baxter permutation belonging to C is connected iff all the permutations of C are connected.*

Let us say that a pair of twin binary trees J is *connected* if the unique Baxter permutation σ satisfying $\mathbb{P}(\sigma) = J$ is connected.

Proposition 6.6 *The Hopf algebra $\mathbf{Baxter}$ is free on the elements $\mathbf{E}_J$ where J is a connected pair of twin binary trees.*

The generating series $B_C(z)$ of connected Baxter permutations is $B_C(z) = 1 - B(z)^{-1}$. First dimensions of algebraic generators of $\mathbf{Baxter}$ are 1, 1, 1, 3, 11, 47, 221, 1113, 5903, 32607, 186143, 1092015.

6.4 Bidendriform bialgebra structure

A Hopf algebra $(H, \cdot, \Delta)$ can be fitted into a bidendriform bialgebra structure [9] if $(H^+, \prec, \succ)$ is a dendriform algebra [17] and $(H^+, \Delta_\prec, \Delta_\succ)$ a codendriform coalgebra, where H^+ is the augmentation ideal of H. The operators $\prec$, $\succ$, $\Delta_\prec$ and $\Delta_\succ$ have to fulfil some compatibility relations. In particular, for all $x, y \in H^+$, the product $\cdot$ of H is retrieved by $x \cdot y = x \prec y + x \succ y$ and the coproduct Δ of H is retrieved by $\Delta(x) = 1 \otimes x + \Delta_\prec(x) + \Delta_\succ(x) + x \otimes 1$.

The Hopf algebra $\mathbf{FQSym}$ admits a bidendriform bialgebra structure [9]. Indeed, for all $\sigma, \nu \in \mathfrak{S}$ set

$$\mathbf{F}_\sigma \prec \mathbf{F}_\nu := \sum_{\substack{\pi \in \sigma \,\overline{\sqcup\!\sqcup}\, \nu \\ \pi_{|\pi|} = \sigma_{|\sigma|}}} \mathbf{F}_\pi, \qquad \mathbf{F}_\sigma \succ \mathbf{F}_\nu := \sum_{\substack{\pi \in \sigma \,\overline{\sqcup\!\sqcup}\, \nu \\ \pi_{|\pi|} = \nu_{|\nu|} + |\sigma|}} \mathbf{F}_\pi, \tag{18}$$

$$\Delta_\prec(\mathbf{F}_\sigma) := \sum_{\sigma^{-1}_{|\sigma|} \leq i \leq |\sigma|-1} \mathbf{F}_{\mathrm{std}(\sigma_1 \ldots \sigma_i)} \otimes \mathbf{F}_{\mathrm{std}(\sigma_{i+1} \ldots \sigma_{|\sigma|})}, \tag{19}$$

$$\Delta_\succ(\mathbf{F}_\sigma) := \sum_{1 \leq i \leq \sigma^{-1}_{|\sigma|} - 1} \mathbf{F}_{\mathrm{std}(\sigma_1 \ldots \sigma_i)} \otimes \mathbf{F}_{\mathrm{std}(\sigma_{i+1} \ldots \sigma_{|\sigma|})}. \tag{20}$$

Proposition 6.7 *If $\equiv$ is an equivalence relation defined on A^* satisfying the conditions of Theorem 6.1 and additionally, for all $u, v \in A^*$, the relation $u \equiv v$ implies $u_{|u|} = v_{|v|}$, then, the family defined in (6) spans a bidendriform sub-bialgebra of $\mathbf{FQSym}$, and is free as an algebra, cofree as a coalgebra, self-dual, and the Lie algebra of its primitive elements is free.*

The equivalence relation $\equiv_{\mathrm{B}}$ satisfies the premises of Proposition 6.7 so that $\mathbf{Baxter}$ is free as an algebra, cofree as a coalgebra, self-dual, and the Lie algebra of its primitive elements is free.

6.5 The dual Hopf algebra $\mathbf{Baxter}^\star$

Let $\{\mathbf{P}^\star_J\}_{J \in \mathcal{TBT}}$ be the dual basis of the basis $\{\mathbf{P}_J\}_{J \in \mathcal{TBT}}$. The Hopf algebra $\mathbf{Baxter}^\star$, dual of $\mathbf{Baxter}$, is a quotient Hopf algebra of $\mathbf{FQSym}^\star$. More precisely,

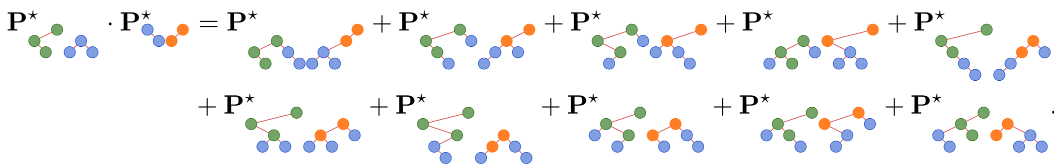
$$\mathbf{Baxter}^\star = \mathbf{FQSym}^\star / I \tag{21}$$

where I is the Hopf ideal of $\mathbf{FQSym}^\star$ spanned by the relations $\mathbf{F}^\star_\sigma = \mathbf{F}^\star_\nu$ whenever $\sigma \equiv_{\mathrm{B}} \nu$.

Let $\phi : \mathbf{FQSym}^\star \to \mathbf{Baxter}^\star$ be the canonical projection, mapping $\mathbf{F}^\star_\sigma$ on $\mathbf{P}^\star_J$ whenever $\mathbb{P}(\sigma) = J$. By definition, the product of $\mathbf{Baxter}^\star$ is

$$\mathbf{P}^\star_{J_0} \cdot \mathbf{P}^\star_{J_1} = \phi\left(\mathbf{F}^\star_\sigma \cdot \mathbf{F}^\star_\nu\right) \tag{22}$$

where σ and ν are any permutations such that $\mathbb{P}(\sigma) = J_0$ and $\mathbb{P}(\nu) = J_1$. For example,

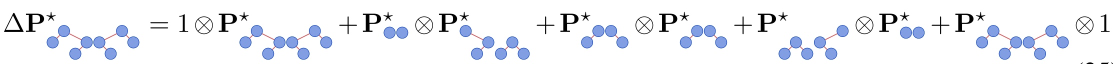


(23)

In the same way, the coproduct of $\mathbf{Baxter}^\star$ is

$$\Delta(\mathbf{P}_J) = (\phi \otimes \phi)(\Delta\left(\mathbf{F}^\star_\sigma\right)) \tag{24}$$

where σ is any permutation such that $\mathbb{P}(\sigma) = J$. For example,



(25)

Remark 6.8 *By Proposition 6.7, the Hopf algebras* $\mathbf{Baxter}$ *and* $\mathbf{Baxter}^\star$ *are isomorphic. However, denoting by* $\theta : \mathbf{Baxter} \hookrightarrow \mathbf{FQSym}$ *the injection from* $\mathbf{Baxter}$ *to* $\mathbf{FQSym}$*,* $\psi : \mathbf{FQSym} \leftrightarrow \mathbf{FQSym}^\star$ *the isomorphism from* $\mathbf{FQSym}$ *to* $\mathbf{FQSym}^\star$ *defined by* $\psi(\mathbf{F}_\sigma) := \mathbf{F}^\star_{\sigma^{-1}}$*, and* $\phi : \mathbf{FQSym}^\star \twoheadrightarrow \mathbf{Baxter}^\star$ *the surjection from* $\mathbf{FQSym}^\star$ *to* $\mathbf{Baxter}^\star$*, the map* $\phi \circ \psi \circ \theta : \mathbf{Baxter} \to \mathbf{Baxter}^\star$ *is not an isomorphism. Indeed:*

$$\phi \circ \psi \circ \theta \mathbf{P} = \phi \circ \psi\left(\mathbf{F}_{2143} + \mathbf{F}_{2413}\right) = \phi\left(\mathbf{F}^\star_{2143} + \mathbf{F}^\star_{3142}\right) = \mathbf{P}^\star + \mathbf{P}^\star, \tag{26}$$

$$\phi \circ \psi \circ \theta \mathbf{P} = \phi \circ \psi\left(\mathbf{F}_{3142} + \mathbf{F}_{3412}\right) = \phi\left(\mathbf{F}^\star_{2413} + \mathbf{F}^\star_{3412}\right) = \mathbf{P}^\star + \mathbf{P}^\star, \tag{27}$$

showing that $\phi \circ \psi \circ \theta$ *is not injective.*

6.6 Primitive and totally primitive elements

6.6.1 Primitive elements

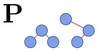
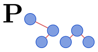
Since the family $\{\mathbf{E}_J\}_{J \in C}$, where C is the set of connected pairs of twin binary trees, are indecomposable elements of $\mathbf{Baxter}$, its dual family $\{\mathbf{E}^\star_J\}_{J \in C}$ forms a basis of the Lie algebra $\mathfrak{p}^\star$ of the primitive elements of $\mathbf{Baxter}^\star$. By Proposition 6.7, the Lie algebra $\mathfrak{p}^\star$ is free.

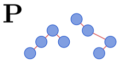
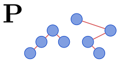
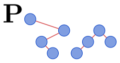
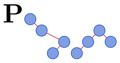
6.6.2 Totally primitive elements

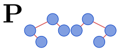
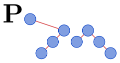
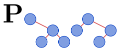
An element x of a bidendriform bialgebra is *totally primitive* if $\Delta_{\prec}(x) = 0 = \Delta_{\succ}(x)$.

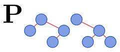
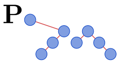
Following [9], the generating series $B_T(z)$ of the totally primitive elements of $\mathbf{Baxter}$ is $B_T(z) = \frac{B(z)-1}{B(z)^2}$. First dimensions of totally primitive elements of $\mathbf{Baxter}$ are 0, 1, 0, 1, 4, 19, 96, 511, 2832, 16215, 95374, 573837. Here follows a basis of the totally primitive elements of $\mathbf{Baxter}$ of order 1, 3 and 4:

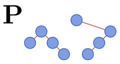
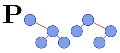
$t_{1,1} = \mathbf{P}$  , (28)

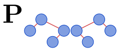
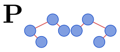
$t_{3,1} = \mathbf{P}$  $- \mathbf{P}$  , (29)

$t_{4,1} = \mathbf{P}$  $+ \mathbf{P}$  $+ \mathbf{P}$  $+ \mathbf{P}$  (30)

$- \mathbf{P}$  $- \mathbf{P}$  $- \mathbf{P}$  ,

$t_{4,2} = \mathbf{P}$  $- \mathbf{P}$  , (31)

$t_{4,3} = \mathbf{P}$  $- \mathbf{P}$  , (32)

$t_{4,4} = \mathbf{P}$  $- \mathbf{P}$  . (33)

$\mathbf{Baxter}$ is free as dendriform algebra on its totally primitive elements.

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The pentagram map and Y-patterns

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Abstract. The pentagram map, introduced by R. Schwartz, is defined by the following construction: given a polygon as input, draw all of its "shortest" diagonals, and output the smaller polygon which they cut out. We employ the machinery of cluster algebras to obtain explicit formulas for the iterates of the pentagram map.

Résumé. L'application pentagramme de R. Schwartz est définie par la construction suivante: on trace les diagonales "les plus courtes" d'un polygone donné en entrée et on retourne en sortie le plus petit polygone que ces diagonales découpent. Nous employons la machinerie des algèbres "clusters" pour obtenir des formules explicites pour les itérations de l'application pentagramme.

Keywords: pentagram map, cluster algebra, Y-pattern, alternating sign matrix

1 Introduction and main formula

The pentagram map, introduced by Richard Schwartz, is a geometric construction which produces one polygon from another. Figure 1 gives an example of this operation. Schwartz [8] uses a collection of cross ratio coordinates to study various properties of the pentagram map. In this paper, we work with a related set of quantities, which we term the y-parameters. A polygon can be reconstructed (up to a projective transformation) from its y-parameters together with one additional quantity. The other quantity transforms in a very simple manner under the pentagram map, so we focus on the y-parameters. Specifically, we derive a formula for the y-parameters of a polygon resulting from repeated applications of the pentagram map.

We show that the transition equations of the y-parameters under the pentagram map coincide with mutations in the Y-pattern associated to a certain cluster algebra. We exploit this connection to prove our formulas for the iterates of the pentagram map. These formulas depend on the F-polynomials of the corresponding cluster algebra, which in general are defined recursively. In this instance, a non-recursive description of these polynomials can be found. Specifically, the F-polynomials are generating functions for the order ideals of a certain sequence of partially ordered sets. These posets were originally defined by N. Elkies, G. Kuperberg, M. Larsen, and J. Propp [1]. It is clear from this description of the F-polynomials that they have positive coefficients, verifying that the Laurent positivity conjecture of S. Fomin and A. Zelevinsky [2] holds in this case.

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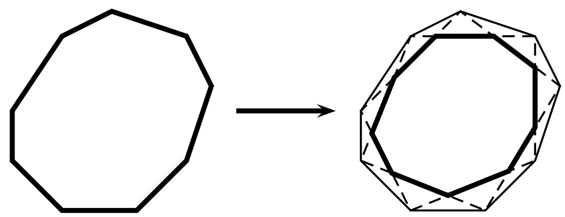

Fig. 1: The pentagram map

This paper is organized as follows. In the remainder of this section we state our main result, the formula for the y-parameters of the iterated pentagram map. This formula is proven in the subsequent sections. Section 2 gives the transition equations of the y-parameters under a single application of the pentagram map. In Section 3, we explain the connection to Y-patterns. This connection is used in Section 4 to derive our main formula in terms of the F-polynomials. Section 4 also provides an analogous formula expressed in the original coordinate system used by Schwartz. Section 5 contains the proof of the formula for the F-polynomials in terms of order ideals. Lastly, Section 6 applies the results of this paper to axis-aligned polygons, expanding on a result of Schwartz. Detailed proofs of all statements given in this paper can be found in the full version [4].

Schwartz [8] studies the pentagram map on a class of objects called twisted polygons. A *twisted polygon* is a sequence $A = (A_i)_{i\in\mathbb{Z}}$ of points in the projective plane that is periodic modulo some projective transformation ϕ, i.e., $A_{i+n} = \phi(A_i)$ for all $i \in \mathbb{Z}$. Two twisted polygons A and B are said to be *projectively equivalent* if there exists a projective transformation ψ such that $\psi(A_i) = B_i$ for all i. Let $\mathcal{P}_n$ denote the space of twisted n-gons modulo projective equivalence.

The *pentagram map*, denoted T, inputs a twisted polygon A and constructs a new twisted polygon $T(A)$ given by the following sequence of points:

$$\ldots, \overleftrightarrow{A_{-1}A_1} \cap \overleftrightarrow{A_0A_2}, \overleftrightarrow{A_0A_2} \cap \overleftrightarrow{A_1A_3}, \overleftrightarrow{A_1A_3} \cap \overleftrightarrow{A_2A_4}, \ldots$$

(we denote by $\overleftrightarrow{AB}$ the line passing through A and B). Note that this operation is only defined for generic twisted polygons. The pentagram map preserves projective equivalence, so it is well defined for generic points of $\mathcal{P}_n$.

If $A \in \mathcal{P}_n$ then the vertices of $B = T(A)$ naturally correspond to edges of A. To reflect this, we use $\frac{1}{2} + \mathbb{Z} = \{\ldots, -0.5, 0.5, 1.5, 2.5 \ldots\}$ to label the vertices of B. Specifically, we let

$$B_i = \overleftrightarrow{A_{i-\frac{3}{2}}A_{i+\frac{1}{2}}} \cap \overleftrightarrow{A_{i-\frac{1}{2}}A_{i+\frac{3}{2}}}$$

for all $i \in (\frac{1}{2} + \mathbb{Z})$. This indexing scheme is illustrated in Figure 2. Similarly, if B is a sequence of points indexed by $\frac{1}{2} + \mathbb{Z}$ then $T(B)$ is defined in the same way and is indexed by $\mathbb{Z}$. Let $\mathcal{P}_n^*$ denote the space of twisted n-gons indexed by $\frac{1}{2} + \mathbb{Z}$, modulo projective equivalence.

The *cross ratio* of 4 real numbers a, b, c, d is defined to be

$$\chi(a, b, c, d) = \frac{(a - b)(c - d)}{(a - c)(b - d)}$$

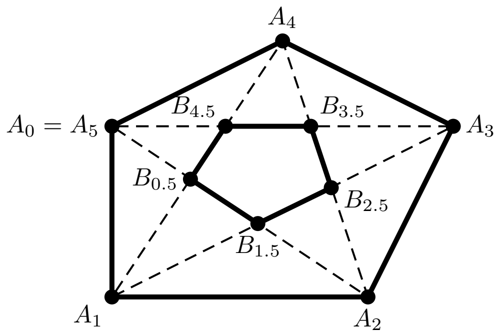

Fig. 2: The pentagon $B = T(A)$ is indexed by $\frac{1}{2} + \mathbb{Z}$.

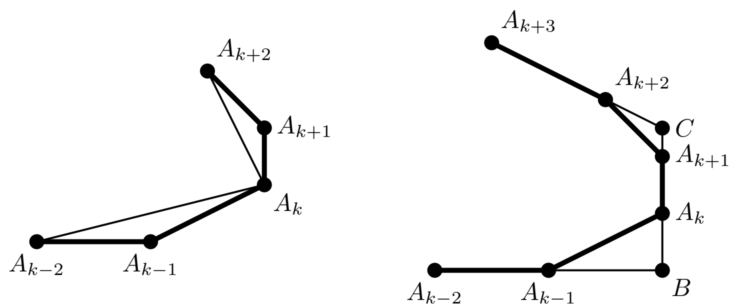

Fig. 3: The cross ratios corresponding to the y-parameters. On the left, $-(y_{2k}(A))^{-1}$ is the cross ratio of the 4 lines through A_k. On the right, $y_{2k+1}(A) = -\chi(B, A_k, A_{k+1}, C)$.

Define similarly the cross ratio of 4 collinear points in the plane, or dually, 4 lines which pass through a common point.

Definition 1.1 *Let A be a twisted polygon indexed either by $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. The y-*parameters *of A are the real numbers $y_j(A)$ for $j \in \mathbb{Z}$ defined as follows. For each index k of A let*

$$y_{2k}(A) = -\left(\chi(\overleftrightarrow{A_kA_{k-2}}, \overleftrightarrow{A_kA_{k-1}}, \overleftrightarrow{A_kA_{k+1}}, \overleftrightarrow{A_kA_{k+2}})\right)^{-1} \tag{1.1}$$

$$y_{2k+1}(A) = -\chi(\overleftrightarrow{A_{k-2}A_{k-1}} \cap L, A_k, A_{k+1}, \overleftrightarrow{A_{k+2}A_{k+3}} \cap L) \tag{1.2}$$

where $L = \overleftrightarrow{A_kA_{k+1}}$.

Note that the 4 lines in (1.1) all pass through the point A_k, and the 4 points in (1.2) all lie on the line L. Therefore the cross ratios are defined. These cross ratios are illustrated in Figure 3.

As will be demonstrated, each y-parameter of $T(A)$ can be expressed as a rational function of the y-parameters of A. It follows that each iterate of T corresponds to a rational map of the y-parameters. Our formulas for these maps involve the F-polynomials of a particular cluster algebra. These can in turn be expressed in terms of certain posets which we define now.

The original definition of the posets, given by Elkies, Kuperberg, Larsen, and Propp [1], involves height functions of domino tilings. For our purposes, the following self-contained definition will suffice. Let Q_k be the set of triples $(r,s,t) \in \mathbb{Z}^3$ such that

$$2|s| - (k-2) \leq t \leq k - 2 - 2|r|$$

and

$$2|s| - (k-2) \equiv t \equiv k - 2 - 2|r| \pmod 4$$

Let $P_k = Q_{k+1} \cup Q_k$. The partial order on P_k is defined by saying that (r', s', t') covers (r,s,t) if and only if $t' = t+1$ and $|r'-r| + |s'-s| = 1$. We denote by $J(P_k)$ the set of *order ideals* in P_k, i.e., subsets $I \subseteq P_k$ such that $x \in I$ and $y < x$ implies $y \in I$. The partial order on P_k restricts to a partial order on Q_k. The Hasse diagram for P_2 is given in Figure 7(a).

Theorem 1.2 *Let $A \in \mathcal{P}_n$ and let $y_j = y_j(A)$ for all $j \in \mathbb{Z}$. If $k \geq 1$ then the y-parameters of $T^k(A)$ are given by*

$$y_j(T^k(A)) = \begin{cases} \left(\prod_{i=-k}^{k} y_{j+3i}\right) \dfrac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}}, & j+k \textit{ even} \\ \left(\prod_{i=-k+1}^{k-1} y_{j+3i}^{-1}\right) \dfrac{F_{j-3,k-1}F_{j+3,k-1}}{F_{j-1,k-1}F_{j+1,k-1}}, & j+k \textit{ odd} \end{cases} \tag{1.3}$$

where

$$F_{j,k} = \sum_{I \in J(P_k)} \prod_{(r,s,t)\in I} y_{3r+s+j} \tag{1.4}$$

A sample computation of $F_{j,k}$ using (1.4) is given at the end of Section 5.

Throughout this paper, we adopt the convention that $\prod_{i=a}^{a-1} z_i = 1$ and $\prod_{i=a}^{b} z_i = \prod_{i=b+1}^{a-1}(1/z_i)$ for $b < a-1$. This will frequently allow a single formula to encompass what otherwise would require several cases. With this convention, the property $\prod_{i=a}^{b} z_i \prod_{i=b+1}^{c} z_i = \prod_{i=a}^{c} z_i$ holds for all $a,b,c \in \mathbb{Z}$.

Acknowledgments. I thank Sergey Fomin for suggesting this problem and providing valuable guidance throughout.

2 The transition equations

Let A be a twisted n-gon. Since the cross ratio is invariant under projective transformations, it follows that $y_{j+2n}(A) = y_j(A)$ for all j. In this section, we show that each y-parameter of $T(A)$ is a rational function of $y_1(A), \ldots, y_{2n}(A)$. The proof of this fact makes use of the cross ratio coordinates $x_1, \ldots, x_{2n}$ introduced by Schwartz [8].

For each index k of A let

$$x_{2k}(A) = \chi(A_{k-2}, A_{k-1}, \overleftrightarrow{A_kA_{k+1}} \cap \overleftrightarrow{A_{k-2}A_{k-1}}, \overleftrightarrow{A_{k+1}A_{k+2}} \cap \overleftrightarrow{A_{k-2}A_{k-1}})$$
$$x_{2k+1}(A) = \chi(A_{k+2}, A_{k+1}, \overleftrightarrow{A_kA_{k-1}} \cap \overleftrightarrow{A_{k+2}A_{k+1}}, \overleftrightarrow{A_{k-1}A_{k-2}} \cap \overleftrightarrow{A_{k+2}A_{k+1}})$$

This definition makes sense as all 4 points in the first cross ratio lie on the line $\overleftrightarrow{A_{k-2}A_{k-1}}$ and those in the second all lie on the line $\overleftrightarrow{A_{k+2}A_{k+1}}$. As with the y_j, we have that the x_j are periodic mod $2n$.

Proposition 2.1 ([8]) *The functions $x_1, \ldots, x_{2n}$ are (generically) a set of coordinates of the space $\mathcal{P}_n$ and of the space $\mathcal{P}_n^*$.*

As observed by V. Ovsienko, R. Schwartz, and S. Tabachnikov in [5], the products $x_j x_{j+1}$ are themselves cross ratios. In fact, $x_j x_{j+1}$ equals the cross ratios used in (1.1)–(1.2) to define y_j. Therefore

$$y_j = -(x_j x_{j+1})^{-1} \tag{2.1}$$

if $j/2$ is an index of A and

$$y_j = -(x_j x_{j+1}) \tag{2.2}$$

otherwise. It follows that $y_1 y_2 y_3 \cdots y_{2n} = 1$ for any twisted polygon, so the y-parameters do not give a complete set of coordinates on $\mathcal{P}_n$. However, the y-parameters of $T(A)$ can be expressed in terms of the y-parameters of A as follows.

Proposition 2.2 *Let $(y_1, \ldots, y_{2n})$ be the y-parameters of A. If A is indexed by $\frac{1}{2}+\mathbb{Z}$ then $y_j(T(A)) = y'_j$ where*

$$y'_j = \begin{cases} y_{j-3}y_jy_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ even} \\ y_j^{-1}, & j \text{ odd} \end{cases} \tag{2.3}$$

If A is indexed by $\mathbb{Z}$ then $y_j(T(A)) = y''_j$ where

$$y''_j = \begin{cases} y_j^{-1}, & j \text{ even} \\ y_{j-3}y_jy_{j+3}\frac{(1+y_{j-1})(1+y_{j+1})}{(1+y_{j-3})(1+y_{j+3})}, & j \text{ odd} \end{cases} \tag{2.4}$$

Let α_1 be the rational map $(y_1, \ldots, y_{2n}) \mapsto (y'_1, \ldots, y'_{2n})$ defined by (2.3). Similarly, let α_2 be the rational map $(y_1, \ldots, y_{2n}) \mapsto (y''_1, \ldots, y''_{2n})$ defined by (2.4). Proposition 2.2 implies that the y-parameters transform under the map T^k according to the rational map $\ldots \circ \alpha_1 \circ \alpha_2 \circ \alpha_1 \circ \alpha_2$ (the composition of k functions), assuming the initial polygon is indexed by integers.

3 The associated Y-pattern

The equations (2.3)–(2.4) can be viewed as transition equations of a certain Y-pattern. Y-patterns represent a part of cluster algebra dynamics; they were introduced by Fomin and Zelevinsky [3]. A simplified (but sufficient for our current purposes) version of the relevant definitions is given below.

Definition 3.1 *A Y-seed is a pair $(\mathbf{y}, B)$ where $\mathbf{y} = (y_1, \ldots, y_n)$ is an n-tuple of quantities and B is an $n \times n$ skew-symmetric, integer matrix. The integer n is called the* rank *of the seed. Given a Y-seed $(\mathbf{y}, B)$ and some $k = 1, \ldots, n$, the* seed mutation *μ_k in direction k results in a new Y-seed $\mu_k(\mathbf{y}, B) = (\mathbf{y}', B')$ where*

$$y'_j = \begin{cases} y_j^{-1}, & j = k \\ y_j y_k^{[b_{kj}]_+}(1+y_k)^{-b_{kj}}, & j \neq k \end{cases}$$

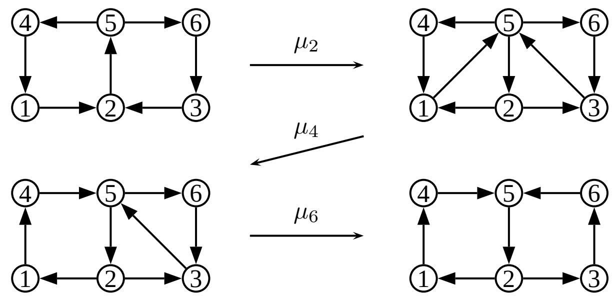

Fig. 4: Some quiver mutations

and B' is the matrix with entries

$$b'_{ij} = \begin{cases} -b_{ij}, & i = k \textit{ or } j = k \\ b_{ij} + \operatorname{sgn}(b_{ik})[b_{ik}b_{kj}]_+, & \textit{otherwise} \end{cases}$$

In these formulas, $[x]_+$ is shorthand for $\max(x, 0)$.

The data of the exchange matrix B can alternately be represented by a *quiver*. This is a directed graph on vertex set $\{1, \ldots, n\}$. For each i and j, there are $|b_{ij}|$ arcs connecting vertex i and vertex j. Each such arc is oriented from i to j if $b_{ij} > 0$ and from j to i if $b_{ij} < 0$. In terms of quivers, the mutation μ_k consists of the following three steps

1. For every length 2 path $i \to k \to j$, add an arc from i to j.
2. Reverse the orientation of all arcs incident to k.
3. Remove all oriented 2-cycles.

Figure 4 illustrates a sequence of quiver mutations. Note that in this example the mutated quiver is the same as the initial one except that all of the arrows have been reversed. The is an instance of a more general phenomenon described by the following lemma.

Lemma 3.2 *Suppose that $(\mathbf{y}, B)$ is a Y-seed of rank $2n$ such that $b_{ij} = 0$ whenever i, j have the same parity. Assume also that for all i and j the number of length 2 paths in the quiver from i to j equals the number of length 2 paths from j to i. Then the μ_i for i odd pairwise commute as do the μ_i for i even. Moreover, $\mu_{2n-1} \circ \cdots \circ \mu_3 \circ \mu_1(\mathbf{y}, B) = (\mathbf{y}', -B)$ and $\mu_{2n} \circ \cdots \circ \mu_4 \circ \mu_2(\mathbf{y}, B) = (\mathbf{y}'', -B)$ where*

$$y'_j = \begin{cases} y_j \prod_k y_k^{[b_{kj}]_+}(1 + y_k)^{-b_{kj}}, & j \textit{ even} \\ y_j^{-1}, & j \textit{ odd} \end{cases} \tag{3.1}$$

$$y''_j = \begin{cases} y_j^{-1}, & j \textit{ even} \\ y_j \prod_k y_k^{[b_{kj}]_+}(1 + y_k)^{-b_{kj}}, & j \textit{ odd} \end{cases} \tag{3.2}$$

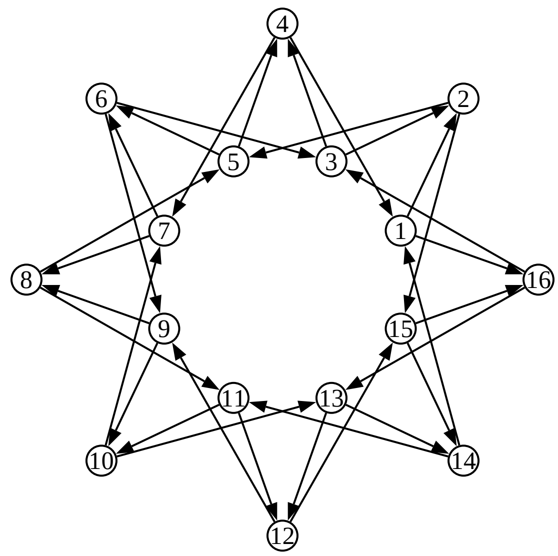

Fig. 5: The quiver associated with the exchange matrix B_0 for $n = 8$

Let μ_{even} be the compound mutation $\mu_{\text{even}} = \mu_{2n} \circ \ldots \circ \mu_4 \circ \mu_2$ and let $\mu_{\text{odd}} = \mu_{2n-1} \circ \ldots \circ \mu_3 \circ \mu_1$. Equations (2.3)–(2.4) and (3.1)–(3.2) suggest that α_1 and α_2 are instances of μ_{odd} and μ_{even}, respectively. Indeed, let B_0 be the matrix with entries

$$b^0_{ij} = \begin{cases} (-1)^j, & i - j \equiv \pm 1 \pmod{2n} \\ (-1)^{j+1}, & i - j \equiv \pm 3 \pmod{2n} \\ 0, & \text{otherwise} \end{cases}$$

The corresponding quiver in the case $n = 8$ is shown in Figure 5.

Proposition 3.3 $\mu_{\textit{even}}(\mathbf{y}, B_0) = (\alpha_2(\mathbf{y}), -B_0)$ *and* $\mu_{\textit{odd}}(\mathbf{y}, -B_0) = (\alpha_1(\mathbf{y}), B_0)$.

4 The formula for an iterate of the pentagram map

Let A be a twisted n-gon indexed by $\mathbb{Z}$, and let $\mathbf{y} = (y_1, \ldots, y_{2n})$ be its y-parameters. For $k \geq 0$ let $\mathbf{y}_k = (y_{1,k}, \ldots, y_{2n,k})$ be the y-parameters of $T^k(A)$. In other words, $\mathbf{y}_0 = \mathbf{y}$, $\mathbf{y}_{2m+1} = \alpha_2(\mathbf{y}_{2m})$, and $\mathbf{y}_{2m} = \alpha_1(\mathbf{y}_{2m-1})$. The results of the previous section show that the $\mathbf{y}_k$ are related by seed mutations:

$$(\mathbf{y}_0, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_1, -B_0) \xrightarrow{\mu_{\text{odd}}} (\mathbf{y}_2, B_0) \xrightarrow{\mu_{\text{even}}} (\mathbf{y}_3, -B_0) \xrightarrow{\mu_{\text{odd}}} \cdots$$

Note that each $y_{j,k}$ is a rational function of $y_1, \ldots, y_{2n}$. In the language of cluster algebras, this rational function is denoted $Y_{j,k} \in \mathbb{Q}(y_1, \ldots, y_{2n})$. By (3.1) and (3.2) we have that $Y_{j,k} = 1/Y_{j,k-1}$ for $j + k$

odd, so it suffices to compute the $Y_{j,k}$ for $j+k$ even. Proposition 3.13 of [3], specialized to the present context, says that if $j+k$ is even then $Y_{j,k}$ can be written in the form

$$Y_{j,k} = M_{j,k}\frac{F_{j-1,k}F_{j+1,k}}{F_{j-3,k}F_{j+3,k}} \tag{4.1}$$

Here, $M_{j,k}$ is a Laurent monomial in $y_1,\dots,y_{2n}$ and the $F_{i,k}$ are certain polynomials over $y_1,\dots,y_{2n}$. A description of these component pieces follows.

The monomial $M_{j,k}$ is given by the evaluation of the rational expressions $Y_{j,k}$ in the tropical semifield $\mathbb{P} = Trop(y_1,\dots,y_{2n})$. This is carried out as follows. First of all, $Y_{j,k}$ is expressed in such a manner that no minus signs appear (that this is possible is clear from transition equations of the Y-pattern.) Next, each plus sign is replaced by the auxiliary addition $\oplus$ symbol. This is a binary operation on Laurent monomials defined by $\prod_i y_i^{a_i} \oplus \prod_i y_i^{a'_i} = \prod_i y_i^{\min(a_i,a'_i)}$. Finally, this operation together with multiplication and division of monomials is used to compute a result. As an example, by (2.4) we know

$$Y_{3,1} = y_0y_3y_6\frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}$$

so

$$M_{3,1} = y_0y_3y_6\frac{(1+y_2)(1+y_4)}{(1+y_0)(1+y_6)}\bigg|_{\mathbb{P}} = y_0y_3y_6\frac{(1\oplus y_2)(1\oplus y_4)}{(1\oplus y_0)(1\oplus y_6)} = y_0y_3y_6$$

Proposition 4.1

$$M_{j,k} = \prod_{i=-k}^{k} y_{j+3i} \tag{4.2}$$

for $j+k$ even.

The $F_{j,k}$ for $j+k$ odd are defined recursively as follows. Put $F_{j,-1}=1$ for j even, $F_{j,0}=1$ for j odd, and

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + M_{j,k}F_{j-1,k}F_{j+1,k}}{(1\oplus M_{j,k})F_{j,k-1}}$$

for $j+k$ even and $k \geq 0$. Recall, $M_{j,k} = \prod_{i=-k}^{k} y_{j+3i}$ so the formula simplifies to

$$F_{j,k+1} = \frac{F_{j-3,k}F_{j+3,k} + (\prod_{i=-k}^{k} y_{j+3i})F_{j-1,k}F_{j+1,k}}{F_{j,k-1}} \tag{4.3}$$

For example, $F_{j,1} = 1 + y_j$ and

$$F_{j,2} = (1+y_{j-3})(1+y_{j+3}) + y_{j-3}y_jy_{j+3}(1+y_{j-1})(1+y_{j+1}) \tag{4.4}$$

Although it is not clear from this definition, the $F_{j,k}$ are indeed polynomials. This is a consequence of general cluster algebra theory.

Equations (4.1)–(4.2) and the fact that $Y_{j,k} = 1/Y_{j,k-1}$ for $j+k$ odd combine to prove that the formula given in Theorem 1.2 is of the right form. What remains is to prove (1.4), which expresses the F-polynomials in terms of order ideals. This proof will be outlined in the next section. Before moving on, we point out that Theorem 1.2 can be used to prove an analogous formula for the x-coordinates of $T^k(A)$.

Theorem 4.2 *Let $A \in \mathcal{P}_n$, $x_j = x_j(A)$, and $y_j = y_j(A)$. Then $x_{j,k} = x_j(T^k(A))$ is given by*

$$x_{j,k} = \begin{cases} x_{j-3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j+2,k-1} F_{j-3,k}}{F_{j-2,k-1} F_{j+1,k}}, & j+k \text{ even} \\ x_{j+3k} \left(\prod_{i=-k}^{k-1} y_{j+1+3i} \right) \dfrac{F_{j-3,k-1} F_{j+2,k}}{F_{j+1,k-1} F_{j-2,k}}, & j+k \text{ odd} \end{cases} \tag{4.5}$$

It will be convenient in the following section to define $M_{j,k}$ and $F_{j,k}$ for all j, k (as opposed to just for $j+k$ even or, respectively, odd). This is done by asserting that (4.2) and (4.3) hold for all j, k.

5 Computation of the F-polynomials

This section proves the formula for the F-polynomials given in (1.4).

Define Laurent monomials $m_{i,j,k}$ for $k \geq -1$ recursively as follows. Let

$$m_{i,j,0} = \prod_{l=0}^{j-1} \prod_{m=0}^{l} y_{3i+j-4l+6m-1} \tag{5.1}$$

and $m_{i,j,-1} = 1/m_{i,j,0}$ for all $i, j \in \mathbb{Z}$. For $k \geq 1$, put

$$m_{i,j,k} = \frac{m_{i-1,j,k-1} m_{i+1,j,k-1}}{m_{i,j,k-2}} \tag{5.2}$$

Note that in (5.1), if $j \leq 0$ the conventions for products mentioned in the introduction are needed. Applying these conventions and simplifying yields $m_{i,-1,0} = m_{i,0,0} = 1$ and

$$m_{i,j,0} = \prod_{l=j}^{-2} \prod_{m=l+1}^{-1} y_{3i+j-4l+6m-1}$$

for $j \leq -2$. A portion of the array $m_{i,j,0}$ is given in Figure 6.

Proposition 5.1 *Let $f_{i,j,k} = m_{i,j,k} F_{3i+j,k}$ for all i, j, k with $k \geq -1$. Then*

$$f_{i,j,k-1} f_{i,j,k+1} = f_{i-1,j,k} f_{i+1,j,k} + f_{i,j-1,k} f_{i,j+1,k} \tag{5.3}$$

for all i, j, k with $k \geq 0$.

The difference equation (5.3) is known as the octahedron recurrence. Applied recursively, it can be used to express $f_{0,0,k} = F_{0,k}$ as a rational function of the $f_{i,j,0} = m_{i,j,0}$ and the $f_{i,j,-1} = 1/m_{i,j,0}$. D. Robbins and H. Rumsey proved [6] that this rational function is in fact a Laurent polynomial whose terms are indexed by pairs of alternating sign matrices. After reviewing the necessary terminology, we will apply this result to write a formula for $F_{0,k}$.

An *alternating sign matrix* is a square matrix of 1's, 0's, and -1's such that

- the non-zero entries of each row and column alternate in sign and

$$\begin{array}{cccccc}
 & & \vdots & \vdots & \vdots & \\
(j=2) & \cdots & y_{-6}y_{-2}y_0 & y_{-3}y_1y_3 & y_0y_4y_6 & \cdots \\
(j=1) & \cdots & y_{-3} & y_0 & y_3 & \cdots \\
(j=0) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-1) & \cdots & 1 & 1 & 1 & \cdots \\
(j=-2) & \cdots & y_{-4} & y_{-1} & y_2 & \cdots \\
(j=-3) & \cdots & y_{-7}y_{-5}y_{-1} & y_{-4}y_{-2}y_2 & y_{-1}y_1y_5 & \cdots \\
 & & \vdots & \vdots & \vdots & \\
 & & (i=-1) & (i=0) & (i=1) &
\end{array}$$

Fig. 6: The monomials $m_{i,j,0}$

- the sum of the entries of each row and column is 1.

Let $ASM(k)$ denote the set of k by k alternating sign matrices.

A bijection is given by Elkies, Kuperberg, Larsen, and Propp in [1] between $ASM(k)$ and the set of order ideals of Q_k, where Q_k is the poset defined in Section 1. Call order ideals $I \subseteq Q_{k+1}$, $J \subseteq Q_k$ compatible if $I \cup J$ is an order ideal of $P_k = Q_{k+1} \cup Q_k$. Call alternating sign matrices $A \in ASM(k+1)$ and $B \in ASM(k)$ compatible if they correspond under the bijection to compatible order ideals.

The initial data of the recurrence (5.3) can be gathered into matrices of the form

$$X_k = \begin{bmatrix} m_{-k+1,0,0} & m_{-k+2,1,0} & \cdots & m_{0,k-1,0} \\ m_{-k+2,-1,0} & m_{-k+3,0,0} & \cdots & m_{1,k-2,0} \\ \vdots & \vdots & & \vdots \\ m_{0,-k+1,0} & m_{1,-k+2,0} & \cdots & m_{k-1,0,0} \end{bmatrix}$$

In the following, the notation X^A, with X and A matrices of the same dimensions, represents the product $\prod_i \prod_j x_{ij}^{a_{ij}}$.

Proposition 5.2

$$F_{0,k} = \sum_{A,B} (X_{k+1})^A (X_k)^B \tag{5.4}$$

where the sum is over all compatible pairs $A \in ASM(k+1)$, $B \in ASM(k)$.

Alternatively, $F_{0,k}$ can be expressed as a sum over compatible pairs of order ideals of Q_{k+1} and Q_k. Such pairs are in turn in bijection with order ideals of P_k. The following proposition indicates how to translate the individual terms of (5.4) to the language of order ideals.

Proposition 5.3 *If* $A \in ASM(k)$ *and* $I \subseteq Q_k$ *is the associated order ideal then*

$$X_k^A = \prod_{(r,s,t) \in I} y_{3r+s}$$

(a) (b)

Fig. 7: (a) The poset $P_2 = Q_3 \cup Q_2$. Here $Q_3 = \{(-1,0,-1),(1,0,-1),(0,-1,1),(0,1,1)\}$, $Q_2 = \{(0,0,0)\}$. (b) The poset P_2 with each element (r,s,t) labeled by y_{3r+s}.

Theorem 5.4

$$F_{j,k} = \sum_{I \in J(P_k)} \prod_{(r,s,t) \in I} y_{3r+s+j}$$

where $J(P_k)$ denotes the set of order ideals of P_k.

As an example, let $j = 0$ and $k = 2$. The poset P_2 (see Figure 7(b)) has eight order ideals. The four which do not include y_0 have weights which sum to $(1 + y_{-3})(1 + y_3)$. The other four have weights summing to $y_{-3}y_0y_3(1 + y_{-1})(1 + y_1)$. Adding these yields a formula for $F_{0,2}$ which agrees with (4.4).

6 Axis-aligned polygons

The remainder of this paper is devoted to axis-aligned polygons, i.e. polygons whose sides are alternately parallel to the x and y axes.

Lemma 6.1 *Let A be a twisted polygon indexed either by $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$. Suppose that no 3 consecutive points of A are collinear. Then for each index i of A:*

1. *A_{i-2}, A_i, A_{i+2} are collinear if and only if $y_{2i}(A) = -1$.*
2. *$\overleftrightarrow{A_{i-2}A_{i-1}}, \overleftrightarrow{A_iA_{i+1}}, \overleftrightarrow{A_{i+2}A_{i+3}}$ are concurrent if and only if $y_{2i+1}(A) = -1$.*

Let $A \in \mathcal{P}_{2n}$ be an axis-aligned polygon. Suppose in addition that A is closed, i.e. $A_{i+2n} = A_i$ for all $i \in \mathbb{Z}$. Let s_{2j+1} denote the signed length of the side joining A_j and A_{j+1}, where the sign is taken to be positive if and only if A_{j+1} is to the right of or above A_j. An example of an axis-aligned octagon is given in Figure 8. It follows from the second statement in Lemma 6.1 that $y_{2j+1}(A) = -1$ for all $j \in \mathbb{Z}$. On the other hand, the even y-parameters can be expressed directly in terms of the side lengths.

Lemma 6.2 *For all $j \in \mathbb{Z}$*

$$y_{2j}(A) = -\frac{s_{2j-1}s_{2j+1}}{s_{2j-3}s_{2j+3}} \tag{6.1}$$

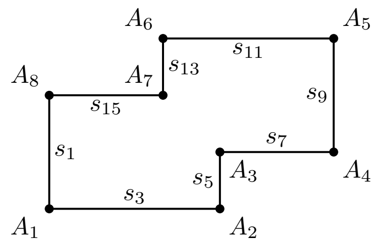

Fig. 8: An axis-aligned octagon. The side lengths s_3, s_5, s_7, and s_9 are positive and the others are negative.

Theorem 6.3 (Schwartz) *Let A be a closed, axis-aligned $2n$-gon. Then the odd vertices of $T^{n-2}(A)$ are collinear, as are its even vertices.*

Theorem 6.3 is stated for all n in [7] and proven for n even (i.e. the number of sides of A divisible by 4) in [8]. The results of this paper lead to a new proof which works for all n. By the above lemmas, $y_{2j+1}(A) = -1$ and the $y_{2j}(A)$ are given by (6.1). A calculation using Theorem 1.2 shows that half of the $y_j(T^{n-2}(A))$ also equal -1 in this case. The first statement of Lemma 6.1 then shows that the vertices of $T^{n-2}(A)$ lie alternately on 2 lines, as claimed.

As an extension of this theorem, suppose that A is not closed but twisted with $A_{i+2n} = \phi(A_i)$. Amazingly, under certain assumptions on ϕ a result similar to Theorem 6.3 still holds.

Theorem 6.4 *Let A be a twisted, axis-aligned $2n$-gon with $A_{i+2n} = \phi(A_i)$ and suppose that ϕ fixes every point at infinity. Then the odd vertices of $T^{n-1}(A)$ are collinear, as are its even vertices.*

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On the evaluation of the Tutte polynomial at the points $(1, -1)$ *and* $(2, -1)$

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Abstract. C. Merino [Electron. J. Combin. 15 (2008)] showed that the Tutte polynomial of a complete graph satisfies $t(K_{n+2}; 2, -1) = t(K_n; 1, -1)$. We first give a bijective proof of this identity based on the relationship between the Tutte polynomial and the inversion polynomial for trees. Next we move to our main result, a sufficient condition for a graph G to have two vertices u and v such that $t(G; 2, -1) = t(G - \{u, v\}; 1, -1)$; the condition is satisfied in particular by the class of threshold graphs. Finally, we give a formula for the evaluation of $t(K_{n,m}; 2, -1)$ involving up-down permutations.

Résumé. C. Merino [Electron. J. Combin. 15 (2008)] a montré que le polynôme de Tutte du graphe complet satisfait $t(K_{n+2}; 2, -1) = t(K_n; 1, -1)$. Le rapport entre le polynôme de Tutte et le polynôme d'inversions d'un arbre nous permet de donner une preuve bijective de cette identité. Le résultat principal du travail est une condition suffisante pour qu'un graphe ait deux sommets u et v tels que $t(G; 2, -1) = t(G - \{u, v\}; 1, -1)$; en particulier, les graphes "threshold" satisfont cette condition. Finalement, nous donnons une formule pour $t(K_{n,m}; 2, -1)$ qui fait intervenir les permutations alternées.

Keywords: Tutte polynomial, increasing tree, threshold graph, generating function, up-down permutation

1 Introduction

The Tutte polynomial is one of the most studied polynomial graph invariants. For a graph $G = (V, E)$, it is given by

$$t(G; x, y) = \sum_{A \subseteq E} (x - 1)^{r(G) - r(A)} (y - 1)^{|A| - r(A)},$$

where $r(A)$ is the *rank* of A, defined as $|V| - c(G_A)$, where $c(G_A)$ is the number of connected components of the spanning subgraph $G_A = (V, A)$ induced by A. (Although the definition of the Tutte polynomial allows multiple edges and loops, all graphs in this paper are simple.)

We refer to (7) for details about the many combinatorial interpretations of evaluations of the Tutte polynomial at various points of the plane and also along several algebraic curves. For example, $t(G; 1, 1)$

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is the number of spanning trees of G when G is connected and $t(G; 2, 1)$ is the number of spanning forests of G. A pair of interpretations especially relevant to the context of this paper are that $t(G; 2, 0)$ is the number of acyclic orientations of G and that $t(G; 1, 0)$ is the number of acyclic orientations of G with a unique fixed source. With this in mind, it follows that $t(K_{n+1}; 1, 0) = t(K_n; 2, 0)$ (in fact, the same is true of any graph G with a universal vertex). As for curves, the hyperbolae $H_q = \{(x, y) : (x-1)(y-1) = q\}$ play a significant role in the theory of the Tutte polynomial. In particular, for $q \in \mathbb{N}$ the Tutte polynomial specializes on H_q to the partition function of the q-state Potts model.

In this paper we shall be concerned with evaluations of the Tutte polynomial at the points $(1, -1)$ and $(2, -1)$. Merino (6) proved the following identity, which is the starting point for our work:

$$t(K_{n+2}; 1, -1) = t(K_n; 2, -1).$$

Non-trivial relationships between evaluations of the Tutte polynomial at points on different hyperbolae are uncommon. Here, the point $(2, -1)$ lies on the hyperbola H_{-2} and $(1, -1)$ on the hyperbola H_0.

We are interested in whether there are other graphs G with the property that $t(G; 1, -1) = t(G - \{u, v\}; 2, -1)$ for some pair of vertices u and v. Merino's proof when G is a complete graph uses generating functions. It is not very difficult to adapt his proof to show the property holds for complete bipartite graphs and for graphs that are the join of a clique and a coclique. (By a clique we mean a complete graph, K_n, and by a coclique a graph with no edges, $\overline{K}_n$; the join $G + H$ of two graphs G and H is formed by taking their disjoint union and adding an edge between each vertex of G and each vertex of H.) Our main result (Theorem 1 below) generalizes these examples by giving sufficient conditions for a graph to have this property; moreover, it describes graphs for which the property still holds when each vertex is replaced by an arbitrary clique or coclique of twin vertices.

The rest of the paper is organized as follows. Section 2 is devoted to giving a bijective proof of Merino's theorem by using an interpretation of the Tutte polynomial given by Gessel and Sagan (2). After introducing the necessary notation, in Section 3 we state the main theorem and discuss its consequences. Section 4 is devoted to its proof, including some intermediate results. Finally, in Section 5 we give a formula for $t(G; 2, -1)$ when G is a complete bipartite graph, $K_{n,m}$, or a graph of the form $K_n + \overline{K}_m$.

2 Bijective proof

We start by giving a bijective proof of Merino's identity

$$t(K_{n+2}; 1, -1) = t(K_n; 2, -1). \tag{1}$$

To translate this identity into combinatorial terms, we use an interpretation of the Tutte polynomial due to Gessel and Sagan (2). They express $t(G; 1 + x, y)$ as a generating function of spanning forests of G according to the number of connected components and an "external activity" $\epsilon(F)$ (it is not the usual external activity for trees as defined by Tutte). More concretely, let $\mathcal{T}(G)$ and $\mathcal{F}(G)$ be the set of spanning trees and spanning forests of a graph G, respectively (assume G is connected from now on). The evaluations we are interested in are

$$t(G; 1, y) = \sum_{T \in \mathcal{T}(G)} y^{\epsilon(T)}, \quad t(G; 2, y) = \sum_{F \in \mathcal{F}(G)} y^{\epsilon(F)}. \tag{2}$$

Moreover, we want first to look at these expressions when G is a complete graph. We recall the facts from (2) needed for this. Consider the usual order on $[n]$ and root each tree in a forest in $\mathcal{F}(K_n)$ at its smallest vertex. We say that a vertex u *precedes* a vertex v if u and v are in the same component and u lies on the unique path from the root to v. Then the external activity $\epsilon(F)$ of a forest F is equal to the number of inversions of F, where an *inversion* is a pair (u,v) such that u precedes v in F and v is smaller than u. Therefore, $t(K_n;1,y)$ is the generating function for inversions in trees with n vertices (rooted at 1) and $t(K_n;2,y)$ is the generating function for inversions in forests with n vertices. Henceforth we use the notation $\mathrm{inv}(F)$ instead of $\epsilon(F)$ for referring to the external activity of a spanning forest of K_n.

Remark. The fact that the Tutte polynomial of K_n at $x=1$ is the inversion polynomial is well known. Gessel and Wang (3) prove that the inversion polynomial is the generating function of connected subgraphs of K_n counted by number of edges, and Beissinger (1) gives a bijection between trees counted by numbers of inversions and by (Tutte's) external activity. Kuznetsov, Pak and Postnikov (4) prove that $t(K_n;1,y)$ is the inversion polynomial by showing they satisfy the same recurrence relation.

Let $\mathcal{T}_n$ denote the set of labelled trees on $[n]$ rooted at 1, and similarly let $\mathcal{F}_n$ denote the set of labelled forests on $[n]$ where each component is rooted at its minimum vertex. Identity (1) can be then rephrased as

$$\sum_{T\in\mathcal{T}_{n+2}}(-1)^{\mathrm{inv}(T)}=\sum_{F\in\mathcal{F}_n}(-1)^{\mathrm{inv}(F)}.$$

To prove this identity, we first cancel out some terms in the sums, so that all remaining terms are positive. A forest $F\in\mathcal{F}_n$ is *increasing* if it has no inversions and it is *even* if all non-root vertices have an even number of children.

Lemma 1 *(i)* $\sum_{T\in\mathcal{T}_n}(-1)^{\mathrm{inv}(T)}$ *equals the number of even increasing trees in* $\mathcal{T}_n$.

(ii) $\sum_{F\in\mathcal{F}_n}(-1)^{\mathrm{inv}(F)}$ *equals the number of even increasing forests of* $\mathcal{F}_n$.

Proof: The second statement follows directly from the first, which appears in (4). The proof given there proceeds by showing that even increasing trees are counted by up-down (or alternating) permutations, which in turn satisfy the same recurrence as the inversion polynomial evaluated at $y=-1$. Alternatively, it is not very difficult to define an involution on trees that fixes even increasing trees and reverses the parity of the number of inversions in the remaining trees (see the end of Section 3.3 of (4)). $\square$

To complete the proof of identity (1), we give a bijection between even increasing trees with $n+2$ vertices and even increasing forests with n vertices. The core of the bijection is the following lemma, whose easy proof is omitted.

Lemma 2 *Let T be an even increasing tree and let u be any vertex of T. Then the forest F obtained from T by removing all edges in the unique path from the root to u is even and increasing.*

Now, given an even increasing tree T with $n+2$ vertices we construct an even increasing forest F with n vertices. First, remove the edges of the path that goes from 1 to $n+2$, resulting in an even increasing forest with $n+2$ vertices. Now remove vertices 1 and $n+2$ (the latter being an isolated vertex), obtaining an even increasing forest with n vertices labelled from 2 to $n+1$. Relabel them from 1 to n to obtain the desired forest. See Figure 1 for an illustration of the process.

Conversely, we show how to recover T if F is given. First, increase all the labels by 1, so that they run from 2 to $n+1$. Of the components of F, let $T_1,\dots,T_k$ be those where the root has even degree and let $U_1,\dots,U_l$ be those with odd root-degree; let the roots of these components be $r_1,\dots,r_k$ and $s_1,\dots,s_l$, respectively, and assume also that $s_1<s_2<\cdots<s_l$. Construct T by adding vertices 1 and $n+2$ and edges $\{1,r_1\},\dots,\{1,r_k\},\{1,s_1\},\{s_1,s_2\},\dots,\{s_l,n+2\}$. It is clear that this procedure recovers T.

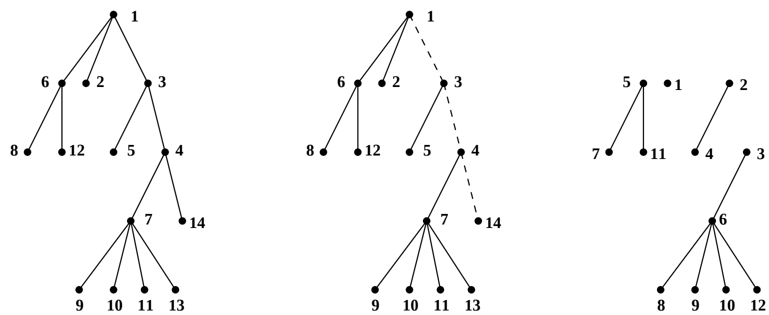

Fig. 1: Obtaining an even increasing forest from an even increasing tree.

3 Statement of the main result

As mentioned before, it is not difficult to adapt Merino's generating function proof to show that complete bipartite graphs $K_{n,m}$ and the graphs $K_n + \overline{K}_m$ satisfy an identity analogous to (1). Since both these graphs can be seen as a graph K_2 where each vertex has been substituted by a clique or a coclique, it seems natural to consider graphs constructed from a fixed graph by replacing vertices by cliques or cocliques.

Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V, E)$, $\mathbf{n} \in \mathbb{N}^V$ and $\mathbf{c} \in \{0, 1\}^V$, define $G(\mathbf{c}; \mathbf{n})$ to be the graph obtained from G by replacing each vertex $k \in V$ by K_{n_k} if $c_k = 1$ or by $\overline{K}_{n_k}$ if $c_k = 0$; then, for each edge $kl \in E$ join the (co)clique on n_k vertices to the (co)clique on n_l vertices by adding an edge for each of the $n_k n_l$ pairs of vertices. For example, $K_1(1; n) = K_n$, $K_1(0; n) = \overline{K}_n$ and $K_2((0, 0); (m, n)) = K_{m,n}$. Note that $K_r((1, 1, \dots, 1); (n_1, \dots, n_r)) = K_1(1; n_1 + \cdots + n_r) = K_{n_1+n_2+\cdots+n_r}$.

We are looking for parameters $G, \mathbf{c}$ with the property that for all $\mathbf{n} \in \mathbb{N}^V$ there exist vertices u, v of $G(\mathbf{c}; \mathbf{n})$ such that $t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}, \mathbf{n}) - \{u, v\}; 2, -1)$, where $n_i, n_j \geq 1$ if u, v belong to the (co)cliques at vertices i, j of G. In fact, we shall find $i, j \in V$ such that for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$ we have

$$t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}; \mathbf{n}'); 2, -1) \tag{3}$$

where $\mathbf{n}'$ is obtained from $\mathbf{n}$ by subtracting 1 from the ith and jth components. In other words, the vertices u, v of $G(\mathbf{c}; \mathbf{n})$ are taken from the fixed (co)cliques that replace the vertices i and j of G in making the graph $G(\mathbf{c}; \mathbf{n})$.

Theorem 2 in Section 4 characterizes pairs $(G, \mathbf{c})$ for which this holds. The following theorem rewrites this characterization in terms of induced subgraphs. (See Figure 2 for an illustration of the statement.) For a subset of vertices $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by the vertices in U.

Theorem 1 *Let $G = (V, E)$ be a simple graph and i and j distinct vertices of G such that $\{i, j\}$ is a vertex cover of G. Let $A = \{v \in V \setminus \{i, j\} : vi \in E, vj \notin E\}$, $B = \{v \in V \setminus \{i, j\} : vi \notin E, vj \in E\}$ and $C = \{v \in V \setminus \{i, j\} : vi \in E, vj \in E\}$.*

Then $t(G; 1, -1) = t(G - \{i, j\}; 2, -1)$ if the following conditions hold:

(i) $G[A]$ and $G[B]$ are cocliques, and $G[C \cup \{i, j\}]$ is a clique (in particular, $ij \in E$);

(ii) there is no induced pair of disjoint edges $2P_2$ with endpoints in $A \cup B$, nor an induced path of length three P_4 with both endpoints in A or both endpoints in B;

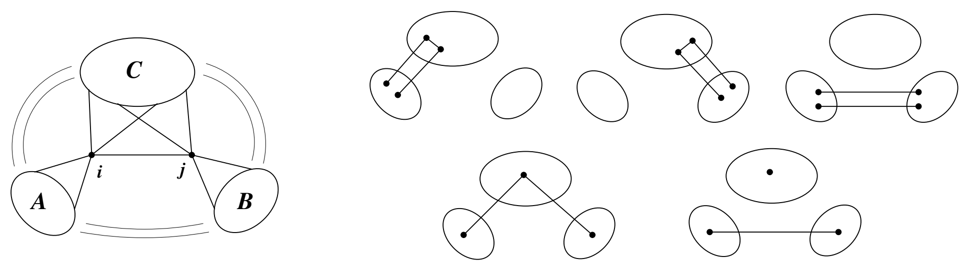

Fig. 2: On the left, structure of the graph described in Theorem 1; A and B induce cocliques, and $C \cup \{i,j\}$ induces a clique. On the right, the five forbidden induced subgraphs.

(iii) there is no induced path of length two P_3 *with one endpoint in* A *and the other in* B*, nor the complement of such a path.*

Furthermore, if G *satisfies these conditions then so does any graph obtained from* G *by replacing a vertex of* $A \cup B \cup \{i,j\}$ *by a coclique of twin vertices, or a vertex of* $C \cup \{i,j\}$ *by a clique of twin vertices.*

Since K_2 satisfies the conditions of the theorem (it is the simplest case $A = B = C = \emptyset$), we recover complete graphs, complete bipartite graphs, and the join of a clique and a coclique. If we take $G = K_3$, we have $A = B = \emptyset$ and $|C| = 1$. This means that we cannot replace the three vertices of a K_3 by cocliques, but all the other possibilities are fine.

The case $B = \emptyset$ gives a much richer class of graphs, threshold graphs. These are the graphs for which the vertices can be ordered so that each one is adjacent to either all or none of the previous ones. They are also the graphs with no induced P_4, C_4 or $2P_2$. (See (5) for a wealth of characterizations and applications.)

Corollary 1 *Let* G *be a threshold graph and let* u *and* v *be the first and the last vertex in an ordering as above. Then* $t(G;1,-1) = t(G - \{u,v\};2,-1)$.

It is by no means the case that all graphs G for which there exist two vertices $\{u,v\}$ such that $t(G;1,-1) = t(G - \{u,v\};2,-1)$ arise from Theorem 1. For instance, taking G to be a cycle of length 6 and u, v two vertices at distance two in the cycle yields such a graph.

4 Proof of the main result

This section is devoted to proving Theorem 1. We begin by finding the generating function for the Tutte polynomials of the family $G(\mathbf{c},\mathbf{n})$ and then we express the relationship between the evaluations at $(1,-1)$ and $(2,-1)$ as a differential equation. The statement of Theorem 2 is read from the solutions of this equation, and finally Theorem 1 is deduced.

Let us fix a connected graph G with two distinguished vertices i, j and a $\{0,1\}$-labelling of the vertices, that is, $\mathbf{c} \in \{0,1\}^V$. We look for conditions so that $G(\mathbf{c};\mathbf{n})$ satisfies (3) for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$.

The following are well-known facts: $t(K_2;x,y) = x$, $t(K_3;x,y) = x^2 + x + y$, $t(\overline{K}_n;x,y) = 1$, and if G has blocks $G_1, \ldots, G_k$, then $t(G;x,y) = t(G_1;x,y) \cdots t(G_k;x,y)$. From this it follows that every vertex $k \in V \backslash \{i,j\}$ is adjacent to either i or j. Indeed, suppose k is not adjacent to either i or j, and choose a neighbour l of k. Then it is easy to check that equation (3) does not hold if we take $\mathbf{n}$ to be zero everywhere except $n_i = n_j = n_k = n_l = 1$. So from now on we assume that i and j together cover V.

The proof relies on the use of generating functions. Let $\mathbf{u} = (u_k : k \in V)$ and define

$$T(x,y;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} t(G(\mathbf{c};\mathbf{n});x,y)\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \qquad \mathbf{u}^{\mathbf{n}} = \prod_k u_k^{n_k}, \quad \mathbf{n}! = \prod_k n_k!,$$

taking $t(G(\mathbf{c},\mathbf{0});x,y) = t(\emptyset;x,y) = 1$. Equation (3) holds if and only if

$$\frac{\partial^2 T(1,-1;\mathbf{u})}{\partial u_i \partial u_j} = T(2,-1;\mathbf{u}). \tag{4}$$

The next lemma follows by a change of variables.

Lemma 3 *Let $G = (V,E)$ be a connected graph containing vertices i and j such that $ki \in E$ or $kj \in E$ for every $k \in V \setminus \{i,j\}$. Define*

$$S(z,w;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} z^{|A|}w^{c(A)}.$$

Then

$$\frac{\partial^2 T(x,y;\mathbf{u})}{\partial u_i \partial u_j} = \frac{1}{x-1}\frac{\partial^2 S(y-1,(x-1)(y-1);\frac{\mathbf{u}}{y-1})}{\partial u_i \partial u_j} \tag{5}$$

and

$$T(2,y;\mathbf{u}) = S(y-1,y-1;\frac{\mathbf{u}}{y-1}). \tag{6}$$

As an induced subgraph of $G(\mathbf{c};\mathbf{n})$ is of the form $G(\mathbf{c};\mathbf{m})$ for some $\mathbf{m}$, we deduce that $S(z,w;\mathbf{u}) = e^{C(z;\mathbf{u})w}$, where $C(z;\mathbf{u})$ is the exponential generating function (EGF) for connected spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ (counted by number of edges). The term $F(z;\mathbf{u}) = e^{C(z;\mathbf{u})}$ is the EGF for spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ and it is given by

$$F(z;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (1+z)^{q(\mathbf{n})}\frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \quad \text{with} \quad q(\mathbf{n}) = \sum_{kl\in E} n_k n_l + \sum_{k\in V} c_k\binom{n_k}{2}. \tag{7}$$

Let $f(\mathbf{u}) = F(-2;\mathbf{u})$. By combining Lemma 3 and Equation (7), Equation (4) becomes

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = 2. \tag{8}$$

Solving the differential equation (8) will put conditions on the quadratic form $q(\mathbf{n})$ that translate to structural conditions on the graph G and the clique/coclique parameter $\mathbf{c}$ that together specify the graph $G(\mathbf{c};\mathbf{n})$. This is Theorem 2 below.

We use $\mathbb{I}(P)$ to denote the indicator function, equal to 1 when the statement P is true and 0 otherwise.

Theorem 2 *A pair G and $\mathbf{c}$ satisfies equation (3) for all $\mathbf{n}$ if and only if the following conditions hold:*

(i) $ij \in E$;

(ii) for each $k \in V \setminus \{i,j\}$, $\mathbb{I}(ki \in E) + \mathbb{I}(kj \in E) = c_k + 1$;

(iii) for all $U \subseteq V \setminus \{j\}$, *either* j *has odd degree in* $G[U \cup \{j\}]$ *or there is a vertex* $k \in U$ *whose degree in the induced subgraph* $G[U]$ *has the same parity as* c_k.

Proof: We have already observed that each $k \in V \setminus \{i,j\}$ must be adjacent to at least one of i and j. We now wish to find all f that solve equation (8). We differentiate the expression for $f(\mathbf{u})$ in terms of $q(\mathbf{n})$, obtaining

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_i q(\mathbf{n})} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \qquad \frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = \sum_{\mathbf{n}\in\mathbb{N}^V} (-1)^{q(\mathbf{n})+\Delta_{i,j} q(\mathbf{n})} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!},$$

where $\Delta_i q(\mathbf{n}) = q(\ldots,n_i+1,\ldots) - q(\ldots,n_i,\ldots)$ and $\Delta_{i,j} q(\mathbf{n}) = q(\ldots,n_i+1,\ldots,n_j+1,\ldots) - q(\ldots,n_i,\ldots,n_j,\ldots)$.

Multiplying power series we find that

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} =$$
$$\sum_{\mathbf{n}\in\mathbb{N}^V} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\mathbf{m}\le\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})} \left((-1)^{\Delta_i q(\mathbf{m})+\Delta_j q(\mathbf{n}-\mathbf{m})} - (-1)^{\Delta_{i,j} q(\mathbf{m})}\right) \prod_k \binom{n_k}{m_k}. \quad (9)$$

(Here we write $\mathbf{m} \le \mathbf{n}$ to mean $m_k \le n_k$ for each $k \in V$.)

After some manipulation, we find that the relative parity of $\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m})$ and $\Delta_{i,j} q(\mathbf{m})$ is given by

$$\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n}-\mathbf{m}) + \Delta_{i,j} q(\mathbf{m}) \equiv \sum_{k\sim j} n_k + \mathbb{I}(i \sim j) \pmod 2, \quad (10)$$

where two vertices a, b satisfy $a \sim b$ either if $ab \in E$ or if $a = b$ and $c_a = 1$. If the right-hand side of equation (10) is zero then the coefficient of $\mathbf{u}^{\mathbf{n}}$ in equation (9) is equal to zero. Since the constant term ($\mathbf{n} = \mathbf{0}$) should be equal to 2 we must have $i \sim j$. Therefore, we need to focus only on the coefficients of $\mathbf{u}^{\mathbf{n}}$ where $\sum_{k\sim j} n_k \equiv 0 \pmod 2$, which are the ones we still do not know are equal to zero. For them we find the expression

$$\frac{1}{\mathbf{n}!}[\mathbf{u}^{\mathbf{n}}]\left(\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j}\right) = 2\sum_{\mathbf{m}\le\mathbf{n}} (-1)^{q(\mathbf{m})+q(\mathbf{n}-\mathbf{m})+\Delta_{i,j} q(\mathbf{m})} \prod_k \binom{n_k}{m_k}.$$

So we wish to find necessary and sufficient conditions for this coefficient of $\frac{1}{\mathbf{n}!}\mathbf{u}^{\mathbf{n}}$ to equal zero for all $\mathbf{n} \ne \mathbf{0}$ subject to $\sum_{k\sim j} n_j \equiv 0 \pmod 2$ and $i \sim j$. After some easy manipulation, we find that the coefficient we are interested in can be rewritten as:

$$\begin{aligned} 0 &= \sum_{\mathbf{m}\le\mathbf{n}} (-1)^{\sum_k m_k \sum_{l\sim k}[n_l+\mathbb{I}(l=i)+\mathbb{I}(l=j)+\mathbb{I}(l=k)]} \prod_k \binom{n_k}{m_k} \\ &= \prod_k \sum_{m_k\le n_k} (-1)^{\left[\sum_{l\sim k} n_l+\mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)\right]m_k} \binom{n_k}{m_k} \\ &= \prod_k \left[1+(-1)^{\sum_{l\sim k} n_l+\mathbb{I}(i\sim k)+\mathbb{I}(j\sim k)+\mathbb{I}(k\sim k)}\right]^{n_k}. \end{aligned} \quad (11)$$

By taking each n_k to be even, for the expression (11) to be zero it is necessary that, for each $k \in V$,

$$\mathbb{I}(i \sim k) + \mathbb{I}(j \sim k) + \mathbb{I}(k \sim k) \equiv 1 \pmod 2. \tag{12}$$

Thus if $c_k = 1$ in $G(\mathbf{c}; \mathbf{n})$ (a clique) the vertex k must be adjacent to both i and j, whereas if $c_k = 0$ (a coclique) then the vertex k must be adjacent to exactly one of i, j. Since by assumption $i \sim j$ and $\sum_{l \sim j} n_l \equiv 0 \pmod 2$ we can assume $n_j = 0$, otherwise we have a zero factor and we are done.

Since expression (11) depends only on the parity of each n_k, it is enough to look at $n_k \in \{0, 1\}$. In terms of the graph G, this is to say we may assume each vertex k is either deleted or is present as a single vertex; if this graph satisfies the required conditions then so does $G(\mathbf{c}; \mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^V$.

Define $U \subseteq V \setminus \{j\}$ by $U = \{k \in V : n_k \neq 0\}$. Since we assume $\sum_{k \sim j} n_k \equiv 0 \pmod 2$ we restrict attention to U such that the induced subgraph $G[U]$ of G has the property that the number of vertices $k \in U$ such that $kj \in E$ is even. A necessary and sufficient condition that expression (11) is zero (under the assumption that $i \sim j$, $n_j = 0$ and $\sum_{k \sim j} n_k \equiv 0 \pmod 2$) is that for any such choice of U there is a vertex k of $G[U]$ of odd degree if $k \sim k$ or of even degree if $k \not\sim k$ (i.e., there is a vertex k of degree the same parity as c_k in the induced subgraph on U). $\square$

From this theorem we wish now to deduce the induced subgraph characterization of Theorem 1. First we need to give some properties of the pairs $(G, \mathbf{c})$ that satisfy the conditions of Theorem 2. Condition (ii) implies that the following sets partition $V \setminus \{i, j\}$ (recall Figure 2):

$$\begin{aligned} A &= \{k \in V \setminus \{i, j\} : ki \in E, c_k = 0\}, \\ B &= \{k \in V \setminus \{i, j\} : kj \in E, c_k = 0\}, \\ C &= \{k \in V \setminus \{i, j\} : ki, kj \in E, c_k = 1\}. \end{aligned}$$

The next lemma is equivalent to saying that the values of c_i and c_j can be chosen freely.

Lemma 4 *If $G = (V, E)$, $i, j \in V$ and $\mathbf{c} \in \{0, 1\}^V$ satisfy the conditions of Theorem 2, then so do G and $\mathbf{c}'$ where $\mathbf{c}'$ is $\mathbf{c}$ with c_i replaced by $1 - c_i$ or with c_j replaced by $1 - c_j$ (or both).*

Proof: The conditions of Theorem 2 are clearly independent of the value of c_j. To see they do not depend on the value of c_i either, suppose on the contrary that there is an induced subgraph $G[U]$ with $i \in U \subseteq V \setminus \{j\}$ such that j has even degree in $G[U \cup \{j\}]$ and where vertex i is the only one in $G[U]$ with degree congruent to $c_i \pmod 2$, as required by condition (iii) of Theorem 2.

Suppose first that $c_i = 0$ and set $A' = A \cap U$, $B' = B \cap U$ and $C' = C \cap U$. The degree of i in $G[U]$ is $|A'| + |C'|$ and the degree of j in $G[U \cup \{j\}]$ is $1 + |B'| + |C'|$; since both degrees are even, we conclude that $|A'| + |B'|$ is odd. Since the vertices in $A' \cup B'$ are by assumption the ones that have odd degree in $G[U]$, we reach a contradiction because no graph has an odd number of vertices of odd degree.

The case $c_i = 1$ is treated by an analogous parity argument. $\square$

Corollary 2 *The induced subgraphs $G[A]$ and $G[B]$ are cocliques and the induced subgraph $G[C \cup \{i, j\}]$ is a clique.*

Lemma 4 and Corollary 2 imply that condition (iii) of Theorem 2 is satisfied if and only if:

$(\star)$ for all $U \subseteq V \setminus \{i, j\}$ such that $|U \cap (B \cup C)|$ is even, the induced subgraph $G[U]$ contains either a vertex in $A \cup B$ of even degree or a vertex in C of odd degree.

Proof of Theorem 1. We prove that if G contains none of the five induced subgraphs described in the statement of Theorem 1 (and depicted in Figure 2), then condition ($\star$) holds.

Suppose for a contradiction that there is $U \subseteq V \setminus \{i,j\}$ that contains none of the five induced subgraphs and for which condition ($\star$) fails, that is, $|U \cap (B \cup C)|$ is even, all vertices in $U \cap (A \cup B)$ have odd degree and all vertices in $U \cap C$ have even degree. We lose no generality by assuming that $U \cap A = A, U \cap B = B, U \cap C = C$. For any vertex $x \in U$, let A_x (respectively, B_x, C_x) be its set of neighbours in A (resp., in B, C). The following two claims hold because otherwise we could find one of the forbidden induced subgraphs.

Claim 1. Let D be one of A, B, or C and let E be one of $\{A,B,C\} \setminus \{D\}$. If $x, y \in D$, then E_x and E_y are comparable sets.

Claim 2. If x is a vertex in C, then $A_x \cup B_x$ induces a complete bipartite graph.

Claim 1 with $D = C$ and $E = A$ implies that there is a vertex $a_0 \in A$ adjacent to all vertices of C that have at least one neighbour in A.

Now let $B' \subseteq B$ be the set of those vertices that are not adjacent to any vertex of C. If B' is non-empty, each of its vertices must be adjacent to at least one vertex in A, because vertices in B have odd degree. Suppose $b \in B'$ is adjacent to $a \in A$. If a is not adjacent to every vertex in C, then we find the fifth graph in Figure 2 as an induced subgraph, therefore a must be adjacent to all vertices in C. Now, since the neighbourhoods of vertices of B in A are nested (Claim 1), we conclude that there is some vertex in A adjacent to all vertices in B and hence to all vertices in C. But that makes this vertex have degree equal to $|B \cup C|$, which is even and hence contradicts our assumption. Therefore, B' must be empty.

Hence, every vertex in B (if any) must be adjacent to at least one vertex in C. By Claim 1 again, there is a vertex in C adjacent to all vertices in B. Any vertex with this property must be adjacent to some vertex in A, and hence to a_0 as well (otherwise it would have degree $|B| + |C| - 1$, which is odd). Also, a_0 is adjacent to all vertices of B by Claim 2. Now, let C' be the vertices in C that are not adjacent to a_0. Since a_0 has odd degree and $|B \cup C|$ is even, $|C'|$ is odd. Now, any $c' \in C'$ cannot be adjacent to all of B, because we just showed that in this case it would be adjacent to a_0 as well. But then, if c' is not adjacent to, say, $b' \in B$, then the edge $a_0 b'$ and vertex c' form one of the forbidden induced subgraphs.

Therefore, we are forced to have $B = \emptyset$. Then either there is a vertex in C not adjacent to a_0, and hence to no vertex in A, or a_0 is adjacent to every vertex in C. But in the former case there is a vertex in C of odd degree and in the latter case a_0 has even degree. □

5 Evaluating $t(K_{n,m}; 2, -1)$ and $t(K_n + \overline{K}_m; 2, -1)$

As mentioned in the proof of Lemma 1, it is known that $t(K_n; 2, -1)$ is the number of up-down permutations of $[n+1]$. The corresponding exponential generating function is $\sec(t)(\tan(t) + \sec(t))$ (recall that the EGF for up-down permutations is $\tan(t) + \sec(t)$). In this section we focus on the evaluations $t(K_{n,m}; 2, -1)$ and $t(K_n + \overline{K}_m; 2, -1)$.

Let $B(u,v)$ be the bivariate EGF for $t(K_{m,n}; 2, -1)$. Here are the first few values of $t(K_{m,n}; 2, -1)$ for $1 \le m \le n$. (That the first column is given by 2^n and the second by $(3^{n+1} - 1)/2$ is easy to prove from the definition and properties of the Tutte polynomial.)

$n\backslash m$	1	2	3	4	5	6
1	2					
2	4	13				
3	8	40	176			
4	16	121	736	4081		
5	32	364	3008	21616	144512	
6	64	1093	12160	111721	927424	7256173

By Lemma 3 and Equations (6) and (7)

$$B(u,v) = F(-2;-u/2,-v/2)^{-2} = \left(\sum_{m,n\geq 0}(-1)^{mn}\frac{u^m}{(-2)^m m!}\frac{v^n}{(-2)^n n!}\right)^{-2}.$$

Since the EGF for $(-1)^{nm}$ is $e^u\cosh(v)+e^{-u}\sinh(v)$, using some hyperbolic function identities, we obtain $B(u,v) = (\cosh(u)\cosh(v)-\sinh(u)-\sinh(v))^{-1}$.

We would like to extract from $B(u,v)$ the coefficient of $u^m v^n$. Let D^m denote the operation of taking the derivative m times with respect to u. Then

$$D^m(B(u,v))\big|_{u=0} = \sum_{n\geq 0} t(K_{m,n};2,-1)\frac{v^n}{n!}.$$

Let $g = \cosh(u)\cosh(v)-\sinh(u)-\sinh(v)$. Applying the rule for the derivative of a product to the equality $D^m(g\cdot g^{-1}) = 0$ we obtain the following recursion

$$gD^m(g^{-1}) = -\sum_{k=0}^{m-1}\binom{m}{k}D^{m-k}(g)D^k(g^{-1}).$$

It is easy to show by induction that, for $i\geq 1$, $D^{2i}(g) = \cosh(u)\cosh(v)-\sinh(u)$ and $D^{2i-1}(g) = \sinh(u)\cosh(v)-\cosh(u)$. Evaluating at $u=0$ and using the above recurrence, we arrive at

$$D^m(g^{-1})\big|_{u=0} = -e^v\left(\sum_{k=0}^{m-1}\binom{m}{k}D^k(g^{-1})\big|_{u=0}\left(\delta^0_{k,m}\cosh(v)-\delta^1_{k,m}\right)\right),$$

where $\delta^0_{k,m}$ (respectively, $\delta^1_{k,m}$) is equal to 1 if m and k have the same parity (resp., different parity), and zero otherwise.

Writing b_m for $D^m(g^{-1})\big|_{u=0}$, we have

$$b_m = \sum_{k=0}^{m-1}\binom{m}{k}b_k\left(e^v\delta^1_{k,m}-\frac{1}{2}(1+e^{2v})\delta^0_{k,m}\right).$$

Since $b_0 = e^v$, it follows that b_k is a linear combination of exponentials e^{lv}, the first ones being

$$e^v, e^{2v}, \frac{1}{2}(3e^{3v}-e^v), \frac{1}{2}(6e^{4v}-4e^{2v}), \frac{1}{2}(2e^v-15e^{3v}+15e^{5v}).$$

Let $b_{m,j}$ be the coefficient of e^{jv} in b_m, so that $t(K_{m,n};2,-1) = \sum_{j=1}^{m+1} b_{m,j} j^n$. The $b_{m,j}$ satisfy the recurrence

$$b_{m,j} = \sum_{k=0}^{m-1} \binom{m}{k} \left(b_{k,j-1}\delta^1_{k,m} - \frac{1}{2}(b_{k,j} + b_{k,j-2})\delta^0_{k,m} \right). \tag{13}$$

The first values of $b_{m,j}$ are given in the table below. From them one guesses the Pascal-like recurrence stated in Theorem 3, which can be proved by induction.

$m\backslash j$	1	2	3	4	5	6	7	8
0	1							
1	0	1						
2	-$\frac{1}{2}$	0	$\frac{3}{2}$					
3	0	-2	0	3				
4	1	0	-$\frac{15}{2}$	0	$\frac{15}{2}$			
5	0	$\frac{17}{2}$	0	-30	0	$\frac{45}{2}$		
6	-$\frac{17}{4}$	0	$\frac{231}{4}$	0	-$\frac{525}{4}$	0	$\frac{315}{4}$	
7	0	-62	0	378	0	-630	0	315

Theorem 3 *For* $m \geq 0$, $t(K_{m,n};2,-1) = \sum_{j=1}^{m+1} b_{m,j} j^n$, *where* $b_{m,j}$ *is given by*

$$b_{0,1} = 1,\ b_{m,0} = 0,\ b_{m,m+1} = 0,\ b_{m,j} = \frac{j}{2}(b_{m-1,j-1} - b_{m-1,j+1}) \ \textit{for}\ 1 \leq j \leq m.$$

In particular, $b_{m,j} = 0$ *if* m *and* j *are of the same parity.*

The generating function for the numbers $b_{m,j}$ is related to that of up-down permutations, as we now explain. Let $B_j(u) = \sum_{m\geq 0} b_{m,j} \frac{u^m}{m!}$. From the expression we have for $B(u,v)$, we obtain

$$B_1(u) = \frac{2}{1+\cosh(u)}.$$

The recurrence for the $b_{m,j}$ in Theorem 3 gives

$$B_{j+1}(u) = B_{j-1}(u) - \frac{2}{j} B'_j(u).$$

Finally, solving this equation yields

$$B_j(u) = 2(\tanh^j(\frac{u}{2}))'.$$

Recall that $\tan(x)$ is the EGF of up-down permutations of $[n]$ for odd n (odd up-down permutations). Then $\tanh(x)$ is the EGF for *signed* odd up-down permutations, where the sign depends only on n and is given by $(-1)^{(n-1)/2}$. So $\tanh(x)^j$ is, up to signs, the EGF for permutations that can be decomposed as a sequence of j odd up-down permutations.

For instance, consider $b_{3,2}$. There is one odd up-down permutation of $[1]$ and two odd up-down permutations of $[3]$. There are thus 16 permutations of $[4]$ that can be split as a sequence of two odd up-down permutations. Then the coefficient of u^3 in $2(\tanh^2(u/2))'$ is $2 \cdot 16/(3! \cdot 2^4) = 2/3!$, which agrees with $|b_{3,2}| = 2$.

We can take similar steps to evaluate $t(K_m+\overline{K}_n;2,-1)$ and obtain the following unexpected relationship.

Theorem 4 *For $m\geq 0$, $t(K_m+\overline{K}_n;2,-1)=\sum_{j=1}^{m+1}c_{m,j}j^n$, where $c_{m,j}=b_{m,j}(-1)^{(m-j-1)/2}$.*

The EGF for the sequence $\{c_{m,j}\}_m$ follows immediately from the one for $\{b_{m,j}\}_m$:

$$\sum_{m\geq 0}c_{m,j}\frac{u^m}{m!}=(\tan(u)^j)'.$$

Let us conclude with the open problem of proving the identity $t(K_{n+1,m+1};1,-1)=t(K_{n,m};2,-1)$ bijectively. The interpretation of Gessel and Sagan (2) of the Tutte polynomial allows us again to recast this identity into combinatorial terms. We take $[n]\cup[m]'=[n]\cup\{1',2',\ldots,m'\}$ as the vertex set of $K_{n,m}$. We call the vertices in $[n]$ *black* and the ones in $[m]'$ *white*. Black vertices among themselves are ordered by the usual order; the same applies to white vertices. A black vertex is smaller than a white one. A *white inversion* (respectively, *black inversion*) is an inversion where the two vertices involved are white (resp., black). Their union is the set of *monochromatic inversions of* F and its cardinality is $\mathrm{binv}(F)$.

Let $\mathcal{T}_{n,m}$ be the set of spanning trees of $K_{n,m}$ and let $\mathcal{F}_{n,m}$ be the set of spanning forests of $K_{n,m}$, with all trees rooted at its smallest element. A forest in $\mathcal{F}_{n,m}$ is χ*-increasing* if it has no monochromatic inversions, and it is *bi-even* if each non-root vertex has an even number of grandchildren (descendants at distance two). Then the identity $t(K_{n+1,m+1};1,-1)=t(K_{n,m};2,-1)$ is equivalent to the equality of the numbers of bi-even χ-increasing trees of $\mathcal{T}_{n+1,m+1}$ and of bi-even χ-increasing forests of $\mathcal{F}_{n,m}$.

Problem 1 *Find a bijection between bi-even χ-increasing trees of $\mathcal{T}_{n+1,m+1}$ and bi-even χ-increasing forests of $\mathcal{F}_{n,m}$.*

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Enumeration of minimal $3D$ polyominoes inscribed in a rectangular prism

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Abstract. We consider the family of $3D$ minimal polyominoes inscribed in a rectanglar prism. These objects are polyominos and so they are connected sets of unitary cubic cells inscribed in a given rectangular prism of size $b \times k \times h$ and of minimal volume equal to $b + k + h - 2$. They extend the concept of minimal $2D$ polyominoes inscribed in a rectangle studied in a previous work. Using their geometric structure and elementary combinatorial principles, we construct rational generating functions of minimal $3D$ polyominoes. We also obtain a number of exact formulas and recurrences for sub-families of these polyominoes.

Résumé. Nous considérons la famille des polyominos $3D$ de volume minimal inscrits dans un prisme rectangulaire. Ces objets sont des polyominos et sont donc des ensembles connexes de cubes unitaires. De plus ils sont inscrits dans un prisme rectangulaire de format $b \times k \times h$ donné et ont un volume minimal égal à $b + k + h - 2$. Ces polyominos généralisent le concept de polyomino $2D$ étudié dans un travail précédent. Nous construisons des séries génératrices rationnelles de polyominos $3D$ minimaux et nous obtenons des formules exactes et des récurrences pour des sous-familles de ces polyominos.

Keywords: polycube, inscribed polyomino, enumeration, rectangular prism, generating function, minimal volume.

1 Introduction

Since the rise of modern combinatorics in the early 1960's, most combinatorial objects are visualized and investigated with pencil and paper and therefore, are 2-dimensional. Despite this natural inclination, a number of extensions from $2D$ combinatorial objects to $3D$ objects were introduced: Ferrers diagrams were extended to plane partitions, permutations were extended to maps on a surface and to braids, $2D$ fractals were extended to $3D$ fractals and a short list of exact enumerative results for $3D$ objects have been produced so far (see (1),(7)). Behind these efforts lay a fundamental question: Is $3D$ combinatorics a natural extension of notions and concepts already known in $2D$ combinatorics or does it introduce new concepts unknown in $2D$ combinatorics ? This question was part of our motivation to begin a study of $3D$ inscribed polyominoes.

A $2D$-polyomino is a 4-connected set of unit square cells in the discrete plane. That is, the cells are connected by their edges. A polyomino is inscribed in a $b \times k$ rectangle when it is contained in this

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(a) Generic $2D$ minimal polyomino

(b) $3D$ diagonal polyomino

Fig. 1: $2D$ and $3D$ inscribed minimal polyominoes

rectangle and touches each of its four sides. Inscribed $2D$ polyominoes with minimal area were introduced in a previous work (see (4)) where an elementary geometric characterization was given that permitted their enumeration and the construction of their generating functions. The geometry of an inscribed minimal $2D$ polyomino can be described in simple terms as a *hook-stair-hook* structure where a hook is formed with two mutually perpendicular rows of cells starting on an edge of the rectangle and meeting at their corner end to end in the opposite corner of the rectangle (see fig. 3(a) red cells and (4) for more details). A $2D$ *stair* is a path of connected cells beginning on one corner of a rectangle, say north-west, and moving along the corresponding diagonal in the east - south direction (see fig. 3(a), black cells and their circumscribed rectangle).

$3D$ polyominoes, sometimes called polycubes, are known in the litterature and in recreational mathematics in the context of packing problems (see (2)) and their enumeration according to their volume is known up to volume 16 in (3) as the result of a computer program. However combinatorial enumeration of inscribed $3D$ polyominoes does not seem to have been considered so far.

We define $3D$ polyominoes inscribed in a rectangular $b \times k \times h$ prism as collections of unit 6-connected cubic cells contained in the prism and touching each of its six faces. We give a geometric description of a complete collection of families of inscribed $3D$ polyominoes with minimal volume. This allows us to present generating functions, recurrences and exact formulas these families. One fundamental principle used throughout this work to enumerate inscribed $3D$ polyominoes is the fact that they can be broken in elementary parts easier to describe and used as building blocks with the multiplication principle.

We will introduce three disjoint families of minimal $3D$ inscribed polyominoes and show that their union forms a complete set of $3D$ inscribed polyominoes with minimal volume. These three families will be called respectively $3D$ *diagonal poliominoes*, $2D \times 2D$ polyominoes and *skew cross* polyominoes.

We will use the orthogonal projection of inscribed $3D$ polyominoes on the upper face of the prism in view of the fact that an inscribed $3D$ polyomino is of minimal volume if and only if its orthogonal projection on each face of the circumscribed prism is a $2D$ polyomino of minimal area. This is easily proved by contradiction for if a $3D$ inscribed polyomino is not minimal, then one of its projections is not $2D$ minimal. Similarly, if one projection is not minimal, then the $3D$ polyomino cannot be minimal.

Notations We will use capital letters for sets and generating functions and their corresponding lower case letters will be used for set cardinalities. For example $P_{3D,min}(b,k,h)$ will denote the set of $3D$ polyminoes inscribed in a $b \times k \times h$ rectangular prism with minimal volume, $p_{3D,min}(b,k,h)$ will be their number and $P_{3D,min}(x,y,z) = \sum_{b,k,h} p_{3D,min}(b,k,h)x^b y^k z^h$ will be their generating function. We will use the convention that the edge of length b of the prism is along the x axis and similarly the lengths k, h are along the y and z axis respectively.

The *degree* of a $3D$ cell c in a polyomino, denoted $deg(c)$, is the number of cells having a face in contact with c and the degree of a $2D$ cell c is the number of cells with an edge contact with c. All polyominoes considered in this paper are $2D$ or $3D$, always inscribed in a rectangle or a rectangular prism and of minimal area or volume. Therefore we will often omit to specify these constraints on polyominoes. We will use trinomial coefficients in their standard notation $\binom{a+b+c}{a,b,c}$. We refer the reader to (4) for results and definitions on $2D$ polyominoes.

The paper is organized as follow. In section 2, we introduce diagonal $3D$ polyominoes and the subfamilies needed for their geometric description. We give generating functions, recurrences and exact formulas for these subfamilies. In section 3, we define two families of non diagonal polyominoes: $2D \times 2D$ polyominoes and skew cross polyominoes. We give their generating functions and some exact formulas. In section 4, we sketch the proof of the main result of the paper which states that these three families of polyominoes form a complete set of $3D$ minimal inscribed polyominoes.

2 Diagonal polyominoes

In similarity with $2D$ stairs, we define a $3D$ *stair* as an inscribed polyomino of minimal volume forming a path starting in a given corner of the prism, say the north-west-back corner, and moving with unit steps in the south, east or forward direction until it reaches the opposite $3D$ diagonal corner as in figure 2(d). In what follows, we will use $3D$ stairs as components of polyominoes.

A $2D$ corner-polyomino is a $2D$ minimal polyomino inscribed in a rectangle with a cell in a given corner of the rectangle. The number $P_c(b,k)$ of $2D$ corner-polyominoes inscribed in a $b \times k$ rectangle satisfies the following recurrence and exact formula:

$$P_c(b,k) = 1 + P_c(b,k-1) + P_c(b-1,k) = 2\binom{b+k-2}{b-1} - 1 \tag{1}$$

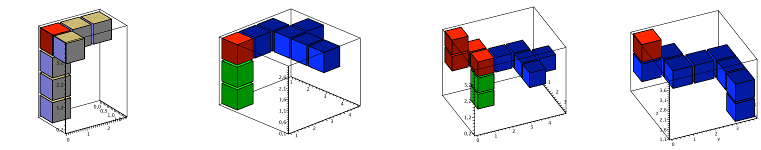
with initial conditions $P_c(b,1) = P_c(1,k) = 1$. Its generating function has the rational form

$$P_c(x,y) = \sum_{b,k\geq 1} P_c(b,k)x^b y^k = \frac{2xy}{(1-x-y)} - \frac{xy}{(1-x)(1-y)} \tag{2}$$

Recall also (see (4)) that the total number of polyominoes of minimal area inscribed in a rectangle $p_{2D,min}(b,k)$ of size $b \times k$ is given by the formula

$$p_{2D,min}(b,k) = 8\binom{b+k-2}{b-1} + 2(b+k) - 3bk - 8$$

We first define and investiguate $3D$ *corner-polyominoes*. A $3D$ *corner-polyomino* is a minimal polyomino inscribed in a prism with one cell in a given corner of the prism, say the north-west-back corner. Let $P_c(b,k,h)$ be the set of corner-polyominoes inscribed in a $b \times k \times h$ prism.



(a) deg3: tripod (b) Degree two: $3D$ hook (c) Degree one (d) 3D stair

Fig. 2: Corner polyominoes

Theorem 1 *For all positive integers b, k, h, the number $p_c(b,k,h)$ of $3D$ corner-polyominoes inscribed in a prism of size $b \times k \times h$ with minimal volume satisfies the following recurrence :*

$$p_c(b,k,h) \;=\; \begin{cases} 2\binom{b+k+h-3}{b-1,k-1,h-1} - 1 & \textit{if b=1 or k=1 or h=1} \\ 1 + 2\binom{b+k-2}{b-1} + 2\binom{b+h-2}{b-1} + 2\binom{k+h-2}{k-1} - 6 & \\ + p_c(b-1,k,h) + p_c(b,k-1,h) + p_c(b,k,h-1) & \textit{otherwise} \end{cases}$$

Proof: The first case is the $2D$ case. It provides the initial conditions for the $3D$ case and is obtained from equations (1). In the second case, observe that a corner cell has degree one, two or three. There is exactly one $3D$ corner-polyomino with corner of degree three inscribed in a $b \times k \times h$ prism and we call this polyomino a *tripod*. This explains the term 1 in the recurrence. When the corner cell c is of degree two, then c is the corner cell of a $2D$ corner-polyomino different from a $2D$ hook that is inscribed in a face of the prism and attached to a perpendicular row of cells along an edge of the prism. A row of cells connecting the polyomino to a face of the prism will often be considered and we will call these components *pilars*. Figure 2(b) illustrates this situation: the corner cell of degree two is the red cell, the $2D$ corner-polyomino is made of the red and blue cells and the set of green cells forms a pilar. The next four terms in the recurrence are thus deduced from equation (1) Now if the corner c has degree one, as in figure 2(c), then the polyomino starts with a $3D$ stair giving the last three terms of the recurrence. □

Observe that the separation according to the degree of the corner cell also gives the following equivalent formulation for the recurrence:

$$\begin{aligned} P_c(b,k,h) &= \textit{tripod} + (2D\textit{-corner} - 2D\textit{-hook}) + \textit{deg}1 \\ &= 1 + (P_c(b,k,1) + P_c(b,1,h) + P_c(1,k,h) - 3) + (P_c(b,k,h-1) + P_c(b,k-1,h) + P_c(b-1,k,h) \end{aligned}$$

Generating functions To establish the generating function for the set of $3D$ corner-polyominoes, we will first give the generating functions $Stair(x,y,z)$, $Tripod(x,y,z)$, $2dhook(x,y,z)$ and $Deg2(x,y,z)$ which are respectively $3D$ stairs, tripods, $2D$ hooks and $3D$ corner-polyominoes of degree two:

$$Tripod(x,y,z) = \sum_{i,j,k \geq 2} x^i y^j z^k = \frac{x^2 y^2 z^2}{(1-x)(1-y)(1-z)}$$

$$Stair(x,y,z) = \sum_{i,j,k\geq 1} \binom{i+j+k-3}{i-1,j-1,k-1} x^i y^j z^k = xyz \sum_{n\geq 0} (x+y+z)^n = \frac{xyz}{(1-x-y-z)}$$

$$2Dhook(x,y,z) = \frac{x^2y^2z}{(1-x)(1-y)} + \frac{x^2yz^2}{(1-x)(1-z)} + \frac{xy^2z^2}{(1-z)(1-y)}$$

$$Deg2(x,y,z) = \left[\frac{2yz}{(1-y-z)} - \frac{2yz}{(1-y)(1-z)}\right]\frac{x^2}{(1-x)} + \left[\frac{2xz}{(1-x-z)} - \frac{2xz}{(1-x)(1-z)}\right]\frac{y^2}{(1-y)} + \left[\frac{2xy}{(1-x-y)} - \frac{2xy}{(1-x)(1-y)}\right]\frac{z^2}{(1-z)} \quad (3)$$

The proof for the rational form of these generating functions is straightforward once we understand the geometric nature of the corresponding objects: there is one tripod per prism because, by definition, their corner cell is in a given corner of the prism. The number of stairs from one corner to its diagonal opposite corner in a prism of size $b \times k \times h$ is equal to the trinomial coefficient $\binom{b+k+h-3}{b-1,k-1,h-1}$. $2D$-hooks appear on a slice parallel to one of the faces so we have three terms, one for each coordinate plane. The generating function for corner-polyominoes of degree two (equation (3)) is directly obtained from its definition: a $2D$ corner of degree one perpendicular to a $pilar$.

Now we are ready to use these building blocs. For instance a $3D$ corner of degree one always begins as a $3D$ stair of length at least two connected to a $3D$ corner of any degree. The generating function $Deg1(x,y,z)$ of $3D$ corners of degree one is thus

$$Deg1(x,y,z) = (Stair(x,y,z) - xyz)\,(1 + Tripod + Deg2 + 2Dhook)$$

Since we now have the generating functions for corner-polyominoes of degree one, two and three, we deduce the following result.

Proposition 1 *The generating function $P_c(x,y,z)$ for $3D$ corner-polyominoes is the following:*

$$\begin{aligned} P_c(x,y,z) &= \sum_{b,k,h\geq 1} p_c(b,k,h) x^b y^k z^h \\ &= Stair(x,y,z)\left[1 + \frac{Tripod(x,y,z) + Deg2(x,y,z) + 2Dhook(x,y,z)}{xyz}\right] \end{aligned} \quad (4)$$

Proof: This is an immediate consequence of the fact that a $3D$ corner-polyomino is the connection of a $3D$ stair with a $3D$ corner-polyomino of arbitrary degree. □

Theorem 2 *For all positive integers b, k, h, we have*

$$\begin{aligned} p_c(b,k,h) &= 4\binom{b+h-2}{h-1}\binom{b+k+h-3}{b+h-2} + \sum_{i=0}^{h-2}(-1)^i \binom{b+h-4-2i}{h-2-i}\binom{b+k+h-4-i}{b+h-3-2i} \\ &\quad -2\left[\binom{b+h-2}{b-1} + \binom{b+k-2}{k-1} + \binom{k+h-2}{h-1}\right] + 3 - \frac{(1+(-1)^h)}{2} \end{aligned} \quad (5)$$

Proof: (Sketched) By induction on $b+k+h$. If $h=1$ then the prism is reduced to a rectangle in the xy plane and formula (5) gives $p_c(b,k,1) = 2\binom{b+k-2}{b-1} - 1$ which agrees with equation (1). The same argument is true for $b=1$ and $k=1$. Suppose that formula (2) is true for a prism of size $b \times k \times h$ with $b,k,h \geq 2$. We have

$$\begin{aligned} p_c(b,k,h+1) \qquad &= 1 + 2\binom{b+k-2}{b-1} + 2\binom{b+h-1}{b-1} + 2\binom{k+h-1}{k-1} - 6 \\ &+ p_c(b-1,k,h+1) + p_c(b,k-1,h+1) + p_c(b,k,h) \end{aligned}$$

by theorem 1 and by induction hypothesis we obtain expression (5). □

It is now possible to construct formulas for the set of polyominoes along one given diagonal of the prism. We define *diagonal polyominoes* as inscribed polyominoes of minimal volume formed with three pieces: two hooks on each end of a diagonal of the prism connected by a stair in contact with their corner cell (see figure 1(b)). By a hook we mean either a $3D$ corner-polyomino with corner of degree two or three or a $2D$ hook. From this definition, we deduce the rational form of the generating function of diagonal polyominoes.

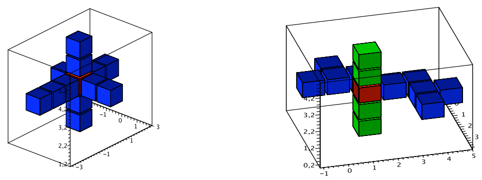
Proposition 2 *The generating function $1Diag(x,y,z)$ of diagonal polyominoes along one given diagonal of a prism is the following*

$$1Diag(x,y,z) = Stair(x,y,z)\left[1 + \frac{Tripod(x,y,z) + Deg2(x,y,z) + 2Dhook(x,y,z)}{xyz}\right]^2 \tag{6}$$

Proof: This is a direct consequence of the definition of diagonal polyominoes, tripods, stairs, corner-polyominoes and $2D$ hooks. The number 1 inside the brackets of equation (6) stands for the fact that $3D$ hooks could be absent and we divide by xyz the next term because we arbitrarily decide that the cell common to a hook and a stair belongs to the stair so that we remove it from the hook with this division. □

In the next step, we count the total number of diagonal polyominoes in a prism. There are four $3D$ diagonals in a prism. If a polyomino belongs to exactly two diagonals, then the two diagonals always define a plane perpendicular to two parallel faces of the prism. The orthogonal projection of the polyomino on these faces must be a $2D$ minimal polyomino and therefore this projection has the generic form *hook-stair-hook* of a $2D$ minimal polyomino. The projection of the two $3D$ diagonals on any other face are the two diagonal of these rectangles. Since the only $2D$ polyomino that belongs to two diagonals of a rectangle is a $2D$ cross, the projection of the polyomino on the other faces is always a $2D$ cross. This has consequences on the form of any $3D$ polyomino along two diagonals which must be made of a *full pilar*, i.e. a pilar connecting two opposite faces, connected to a perpendicular $2D$ generic polyomino inscribed in a full $2D$ slice of the prism (see the blue part in figure 3(b). Moreover the full pilar must meet the orthogonal $2D$ polyomino on its stair part. Now if a diagonal polyomino belongs to three diagonals, then it also belong to the four diagonals (see figure 3(a)).

Polyominoes along two diagonals The generic form of polyominoes on two diagonals can be described as two $2D$ corner-polyominoes sharing their corner cell which also belongs to a full pilar perpendicular to the corner-polyominoes. Since we already know the generating function for $2D$ corner-polyominoes, it is easy to deduce the generating function for diagonal polyominoes belonging to two and three diagonals.



(a) $3D$ cross (b) A polyomino on two diagonals

Fig. 3: Diagonal polyominoes on more than one diagonal

Proposition 3 *The number $2diag_z(b,k,h)$ of $3D$ diagonal polyominoes belonging to the two diagonals perpendicular to the xy face of a prism such that the projection of these two diagonals has a vertex in the upper left corner of the face of size $b \times k$ has the following generating function*

$$2diag_z(x,y,z) = \sum_{b,k,h\geq 1} 2diag_z(b,k,h)x^b y^k z^h = \frac{1}{xy}\left(\frac{2xy}{(1-x-y)} - \frac{xy}{(1-x)(1-y)}\right)^2 \frac{z}{(1-z)^2}$$

Proof: This is immediate from equation (2) and the fact that these polyominoes have the geometric structure $2D\ corner \times (2D\ corner - corner\ cell) \times pilar$. □

$3D$ **crosses** Next we need the generating function $3Dcross(x,y,z)$ of $3D$ crosses which are the $3D$ minimal polyominoes made only of pilars, at least three, meeting on one common cell c (see figure 3(a)). Observe that for a prism of size $b \times k \times h$ with $b,k,h \geq 2$, there are bkh cross polyominoes inscribed in that prism and only one if any two of these three parameters equals one. We will only consider crosses in a box of size at least $2 \times 2 \times 2$. We thus have:

$$3Dcross(x,y,z) = \sum_{b,k,h\geq 2} bkhx^b y^k z^h = \frac{x^2(2-x)\,y^2(2-y)\,z^2(2-z)}{(1-x)^2(1-y)^2(1-z)^2}$$

Proposition 4 *The generating function $Diag(x,y,z)$ of the total number of diagonal polyominoes is the following*

$$Diag(x,y,z) = 4 \cdot 1Diag(x,y,z) - 2\left(2diag_z(x,y,z) + 2diag_y(x,y,z) + 2diag_x(x,y,z)\right) + 3 \cdot 3Dcross(x,y,z) \qquad (7)$$

Proof: In order to count all $3D$ diagonal polyominoes, we use inclusion-exclusion. Here are the steps: 1- Count polyominoes along one diagonal and multiply by four. 2- The polyominoes that belong to two diagonals or more were counted twice or more so for each pair of $3D$ diagonals, remove the polyominoes belonging to those two diagonals. 3- The polyominoes belonging to three diagonals, and thus to four, were counted four times in the first step, removed six times in the second step and so must be added three times to be counted once. Notice that this inclusion-exclusion argument is not valid for degenerate prisms that have one side of length one and for their corresponding terms in the generating function (7). □

3 Non diagonal polyominoes

Does there exists minimal $3D$ polyominoes that are not diagonals ? The answer is yes and figure 4 shows a sample of these objects. For instance the polyomino in figure 4(a) is not diagonal because it has no corner-polyomino as component. This polyomino can be seen as the juxtaposition of two perpendicular $2D$ polyominoes each with contact cell that is not a corner cell. This is our definition for the family of non diagonal minimal polyominoes that we call $2D \times 2D$ polyominoes.

3.1 $2D \times 2D$ polyominoes

In what follows, we establish the generating function for $2D \times 2D$ polyominoes. For that purpose, we split these polyominoes in three parts, each part corresponding to one color in figure 4(a). The central part, made of green cells with red corners, will be called a *skew hook*. It consists of three mutually orthogonal segments of cells. The two end segments touch a face of the prism and so are pilars with at least one cell. They touch the middle segment on its end cells. These two end cells are the contact cells of the two other parts (one in blue and one in yellow in figure 4(a)). If we discard the two pilars, each end cell of the middle segment can be seen as the corner cell of a $2D$ corner-polyominoe. The two $2D$ corner-polyominoes with their associated pilars are perpendicular and each one goes from one face to its opposite face. Notice that the smallest prism that contains a $2D \times 2D$ polyominoe has size $2 \times 3 \times 3$ and in that case, the polyominoes are made of two perpendicular full pilars that are the discrete version of euclidian skew lines.

We begin with the generating function of skew hooks. This is quite elementary when we consider that each pilar contains at least one green cell and the central segment contains two red end cells but not necessarily green cells. In order to fix ideas, we agree that the yellow $2D$ polyomino is in the yz plane with z length at least two if we count the red corner cell. The blue $2D$ polyomino is in the xy plane. If we decide that we do not count the contribution in x and z of the central segment and the contribution in y of the red corner cells. We have the following generating function $SH(x, y, z)$ for skew hooks :

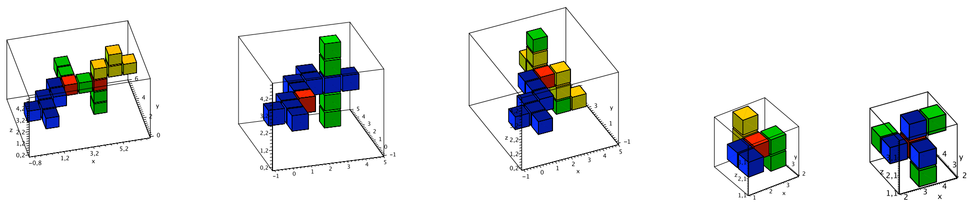
$$SH(x,y,z) = \frac{x}{(1-x)} \times \frac{1}{(1-y)} \times \frac{z}{(1-z)}$$

The yellow $2D$ corner polyominoes in the yz plane of z height at least 2 and the blue $2D$ corner polyominoes in the xy plane of x length at least 2 : are obtained from (2):

$$2D_{c,z\geq 2}(y,z) = yz\left(\frac{2}{(1-y-z)} - \frac{2-z}{(1-y)(1-z)}\right), 2D_{c,x\geq 2}(x,y) = xy\left(\frac{2}{(1-x-y)} - \frac{2-x}{(1-x)(1-y)}\right).$$

In order to assemble these three components, observe that if we fix the vertical pilar and the yellow $2D$ corner, then the horizontal green pilar may take two directions that determines the direction of the blue $2D$ polyomino which is equivalent to multiply by two the number of blue $2D$ corner-polyominoes and remove the $2D$ crosses which would be counted twice otherwise. We do the same for the yellow $2D$ corner-polyominoes. Finally, observe that the yellow polyomino could be on the left rather than on the right of the central part which multiplies by two again the number of polyominoes and we obtain the following generating function for $2D \times 2D$ polyominoes with orthogonal planes xy and yz.

$$P_{xy\times yz}(x,y,z) = 2\left(2\cdot 2D_{c,x\geq 2}(x,y) - \frac{x^2y}{(1-x)(1-y)}\right)\cdot SH\cdot\left(2\cdot 2D_{c,z\geq 2}(y,z) - \frac{yz^2}{(1-y)(1-z)}\right) \quad (8)$$



(a) case 1 (b) case 2 (c) case 3 (d) type a) (e) type b)

Fig. 4: Non diagonal Polyominoes

Finally, observing that two pairs of orthogonal planes determine two disjoint sets of $2D \times 2D$ polyominoes, we obtain the generating function $P_{2D\times 2D}(x, y, z)$ for the total number of non diagonal $2D \times 2D$ polyominoes by adding the three generating functions corresponding to each pair of orthogonal planes:

$$P_{2D\times 2D}(x, y, z) = P_{xy\times yz} + P_{xy\times xz} + P_{xz\times yz}. \tag{9}$$

3.2 Skew cross polyominoes

We define our second family of non diagonal polyominoes as follow: a *skew cross* polyomino starts with a central cell c of degree three which is the corner cell of three $2D$ corner-polyominoes mutually perpendicular. We partition this family in two types. **Type a)** The cell c has two parallel contact faces. **Type b)** The three contact faces of the central cell c are incident to a vertex of c. These two families are illustrated in figures 4(d) and 4(e).

Type a) We start by establishing the generating function for each of the three $2D$ corner-polyominoes needed to obtain a skew cross polyomino of type a). To fix the ideas, suppose that the three contact faces of the cell c have already been chosen and that the $2D$ corner-polyomino red and green is in the yz plane, the yellow part is in the xz plane and the blue part is in the xy plane as illustrated in figure 4(d). We have the choice between the red central cell c and the cell in contact with it as the corner cell of the $2D$ corner-polyomino. We choose the cell in contact with c. For the $2D$ corner-polyomino in the yz plane, the z length must be at least 2 and the generating function is

$$P_{c,z\geq 2}(y, z) = yz\left(\frac{2}{(1-y-z)} - \frac{1}{(1-y)(1-z)} - \frac{1}{(1-y)}\right) \tag{10}$$

Similarly we obtain generating functions $P_{c,x\geq 2}(x, y), P_{c,z\geq 2}(x, z)$ for the $2D$ corner-polyominoes in the xy and xz planes. The product of these three series gives the generating function of skew cross polyominoes of type a with preselected faces of the central cell provided we adjust with the fact that the z length of the cell c was counted twice and its y lenght was not counted. Now once the faces of c are chosen, there is some freedom for the direction of the $2D$ corner-polyominoes. Indeed, if we choose first one of the two directions of the corner-polyomino coming from the face between opposite faces, then we still have to choose between two directions for another $2D$ corner-polyomino. For two faces in the xz plane, we have two choices for a face yz. Thus the generating function $SC_{a1}(x, y, z)$ of skew crosses of type a when two faces in plane xz and one face in the plane yz are chosen is :

$$SC_{a1}(x,y,z) \qquad = \frac{4y}{z} P_{c,z\geq 2}(y,z) P_{c,x\geq 2}(x,y) P_{c,z\geq 2}(x,z)$$

There are 12 triplets of faces of type a on a cell c, we sum six generating functions similar to equation (11) and obtain the generating function $SC_a(x,y,z)$ for skew crosses of type a which simplifies to:

$$SC_a(x,y,z) = \frac{-16x^3y^3z^3((1-x+y)(1-x+z)+(1-y+x)(1-y+z)+(1-z+x)(1-z+y))}{(1-x)^2(1-y)^2(1-z)^2(1-y-z)(1-x-y)(1-x-z)}$$

Type b To establish the generating function of skew crosses of type b, we choose three faces of the cell c incident to one vertex of c. We choose each cell in contact with a face of c to be the corner cell of a $2D$ corner-polyomino. There are two possibilities once the three corner cells are chosen. Here is the generating function for a given set of three faces corresponding to one vertex of c :

$$P_{c,z\geq 2}(x,z) \times P_{c,x\geq 2}(x,y) \times P_{c,y\geq 2}(y,z) + P_{c,y\geq 2}(x,y) \times P_{c,z\geq 2}(y,z) \times P_{c,x\geq 2}(x,z)$$

There are 8 sets of three faces of c incident to one vertex and for each set, we obtain the same generating function which means that the generating function for skew crosses of type b is the following:

$$SC_b(x,y,z) \;= 8(P_{c,z\geq 2}(x,z) \times P_{c,x\geq 2}(x,y) \times P_{c,y\geq 2}(y,z) + P_{c,y\geq 2}(x,y) \times P_{c,z\geq 2}(y,z) \times P_{c,x\geq 2}(x,z))$$

The generating function for all skew crosses $SC(x,y,z)$ is the sum of the generating functions for types a and b so that we obtain

$$SC(x,y,z) = \frac{64x^3y^3z^3}{(1-x-y)(1-x-z)(1-y-z)(1-x)^2(1-y)^2(1-z)^2}. \qquad (11)$$

4 Main result

So far we have established three disjoint classes of $3D$ polyominoes. We claim that the union of these three classes forms the whole set of $3D$ inscribed polyominoes with minimal volume.

Theorem 3 *The total number $p_{3D,min}(b,k,h)$ of polyominoes inscribed in a $b \times k \times h$ rectangular prism and minimal volume $b+k+h-2$ is the sum of diagonal polyominoes and non diagonal polyominoes of type $2D \times 2D$ and skew crosses:*

$$p_{3D,min}(b,k,h) = diag(b,k,h) + p_{2D\times 2D}(b,k,h) + sc(b,k,h).$$

Proof: In order to prove this result, we introduce a second classification of $3D$ polyominoes and we show that every set of polyominoes forming this classification belongs to one of our three families of polyominoes.

Consider the orthogonal projection $\Pi(P)$ of an inscribed $3D$ polyomino P on the upper face of the prism. $\Pi(P)$ is a $2D$ inscribed polyomino of minimal area and therefore possesses the geometric structure *hook-stair-hook* of minimal $2D$ polyominoes. Two cells of the $3D$ polyomino play a special role in our classification. We call them *contact cells* and define them as follow. For every polyomino $P \in$

$P_{3D,min}(b,k,h)$ there is a unique $3D$ stair connecting the lower and upper faces of the prism which forms a non decreasing path from floor to ceiling. The two contact cells c_1, c_2 are respectively, the last cell touching the floor and the first cell touching the ceiling in this path. We use the positions of the projections $\Pi(c_1), \Pi(c_2)$ in our classification. If, without loss of generality, we fix a $2D$ diagonal in the upper face to give a direction to the *hook-stair-hook* structure, there are ten positions of the pair $\Pi(c_1), \Pi(c_2)$ with respect to the upper hook, each pair giving a class in this classification of $P_{3D,min}(b,k,h)$. The ten positions can be seen in figure 4 where $\Pi(c_1)$ and $\Pi(c_2)$ are black. Observe that these ten cases do not form a complete partition of the set $P \in P_{3D,min}(b,k,h)$ but our goal is to provide a complete set of representatives up to symmetry so that every other case is similar to one of the cases considered.

The remaining part of the proof shows that the polyominoes in each of the 10 cases also belong to one of the three families of polyominoes, namely diagonal, $2D \times 2D$ and skew cross polyominoes.

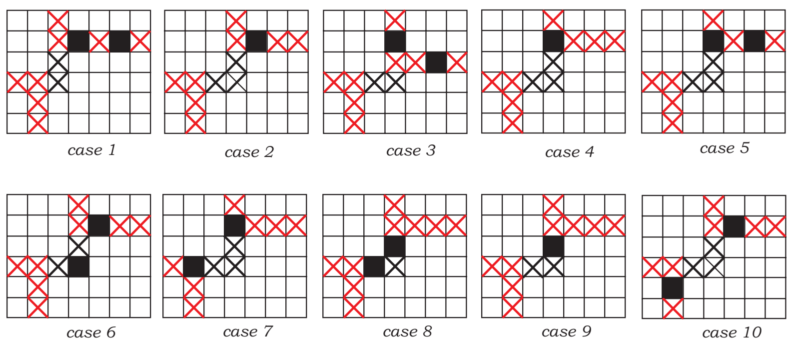


Fig. 5: Classification of the projection of $3D$ polyominoes on the upper face of the prism

□

5 Exact formulas

We did not find exact formulas for all the generating functions produced: the exact expressions are not always reducible. For example, here is an exact expression for the number $sc(b,k,h)$ of skew crosses inscribed in a $b \times k \times h$ prism that we could not reduce:

$$sc(b,k,h) = 64 \sum_{i=0}^{b+k-6} \sum_{r=0}^{i} \sum_{j=0}^{b-3-r} \binom{b}{3+r+j} \binom{k}{3+i+j-r} \binom{h}{3+i} \quad b \geq 3,\ k \geq 3,\ h \geq 3 \tag{12}$$

but if we turn our interest to the number of all minimum inscribed polyominoes of a given volume n, we obtain interesting exact formulas and generating functions. In what follows, we obtain exact formulas by setting $x = y = z$ for each of the three families of $3D$ polyominoes to obtain one for the set $P_{3D,min}(n)$.

$$sc(n-2) = 2^{n+2}\left(n^2 - 27n + 194\right) - 8\left(\frac{n^5}{15} + \frac{11n^3}{3} + 12n^2 + \frac{844n}{15} + 96\right) \tag{13}$$

$$p_{2D\times 2D}(n-2) = 3\cdot 2^{n+2}\,(n-15) + \left(\frac{3}{40}n^6 - \frac{33}{40}n^5 + \frac{65}{8}n^4 - \frac{183}{8}n^3 + \frac{544}{5}n^2 + \frac{147}{10}n + 234\right) \quad (14)$$

$$diag(n-2) = \frac{121}{48}3^n - 2^n(45n-411) - \left(\frac{53}{120}n^5 - \frac{15}{8}n^4 + \frac{823}{24}n^3 - 6n^2 + \frac{22711}{60}n + \frac{4995}{16}\right) \quad (15)$$

Adding equations (13), (14), (15),we finally obtain an exact formula for $p_{3D,min}(n)$.

$$\begin{aligned} P_{3D,min}(x) &= \sum_n p_{3D,min}(n)x^{n+2} \\ &= \frac{x^3(72x^{10}+36x^9+510x^8-1117x^7+1276x^6-1155x^5+710x^4-293x^3+81x^2-13x+1)}{(1-3x)(1-2x)^3(1-x)^7} \\ p_{3D,min}(n) &= \frac{11^2\cdot 3^{n+1}}{16} + 2^{n+2}(4n^2-125n+741) + \frac{3n^6}{40} - \frac{9n^5}{10} - \frac{7n^4}{2} - \frac{133n^3}{2} - \frac{1931n^2}{5} - \frac{31727n}{20} \\ &\quad - \frac{47739}{16} \end{aligned}$$

Remarks

1. One of the authors (H. Cloutier), wrote two programs to count minimal inscribed $3D$ polyominoes. One program uses formulas obtained from the projection $\Pi(P)$ of the polyominoes on the ceiling of the prism. We used the datas obtained from these programs to validate our results.
2. An exact formula for $p_{3D,min}(b,k,h)$ is for the moment out of reach but we have produced exact formulas for $p_{3D,min}(b,k,2)$ and $p_{3D,min}(b,k,3)$.
3. The diagonal subseries $DP_{3D,min}(t) = \sum_n p_{3D,min}(n,n,n)t^n$ obtained from $P_{3D,min}(x,y,z)$ by setting equals the exponents of x, y, z satisfies a functional equation of degree six with coefficients polynoms in t (6). But no exact expression for $p_{3D,min}(n,n,n)$ could be found.

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Cyclic sieving phenomenon in non-crossing connected graphs

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Abstract. A non-crossing connected graph is a connected graph on vertices arranged in a circle such that its edges do not cross. The count for such graphs can be made naturally into a q-binomial generating function. We prove that this generating function exhibits the cyclic sieving phenomenon, as conjectured by S.-P. Eu.

Résumé. Un graphe connexe dont les sommets sont disposés sur un cercle est sans croisement si ses arêtes ne se croisent pas. Nous démontrons une conjecture de S.-P. Eu affirmant que la fonction génératrice q-binomiale dénombrant de tels graphes exhibe le phénomène du crible cyclique.

Keywords: cyclic sieving phenomenon, non-crossing connected graphs, Lagrange inversion

1 Introduction

A *non-crossing graph* on a finite set S is a graph with vertices indexed by S arranged in a circle such that no edges cross. When we say a graph on n vertices, we will mean $S = \{1, \ldots, n\}$. In [3], Flajolet and Noy showed that the number $c_{n,k}$ of non-crossing connected graphs (see Figure 1) on n vertices with k edges, $n-1 \le k \le 2n-3$, is

$$c_{n,k} = \frac{1}{n-1}\binom{3n-3}{n+k}\binom{k-1}{n-2}. \tag{1}$$

Define

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

where $[n]!_q = [n]_q [n-1]_q \cdots [1]_q$ and $[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$. The formula in (1) admits a natural q-analogue:

$$c(n,k;q) = \frac{1}{[n-1]_q}\begin{bmatrix} 3n-3 \\ n+k \end{bmatrix}_q \begin{bmatrix} k-1 \\ n-2 \end{bmatrix}_q. \tag{2}$$

It is straightforward to verity fhat $c(n,k;q)$ is a polynomial in q with nonnegative integer coefficients.

The main result of this paper is the following, which was conjectured by S.-P. Eu [1].

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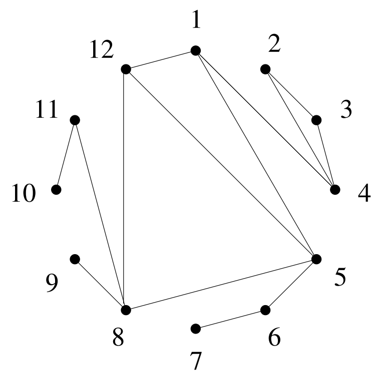

Fig. 1: A non-crossing connected graph on 12 vertices with 14 edges

Theorem 1.1. *Let $n \geq 1$ and $n-1 \leq k \leq 2n-3$, and let X be the set of non-crossing connected graphs on n vertices with k edges. If $d \geq 1$ divides n and ω is a primitive d-th root of unity, then*

$$c(n,k;\omega) = s_d(n,k)$$

where we define

$$s_d(n,k) = \#\left\{x \in X : x \text{ is fixed under rotation by } \frac{2\pi}{d}\right\}.$$

In [5], Reiner, Stanton, and White introduced the notion of the cyclic sieving phenomenon. A triple $(X, X(q), C)$ consisting of a finite set X, a polynomial $X(q) \in \mathbb{N}[q]$ satisfying $X(1) = |X|$, and a cyclic group C acting on X exhibits the *cyclic sieving phenomenon* if, for every $c \in C$, if ω is a primitive root of unity of the same multiplicative order as c, then

$$X(\omega) = \#\{x \in X : c(x) = x\}.$$

In (1), the two extreme cases, $k = n-1$ and $k = 2n-3$, correspond to non-crossing spanning trees and n-gon triangulations respectively. In the former case, Eu and Fu showed in [2] that quadrangulations of a polygon exhibit the cyclic sieving phenomenon, where the cyclic action is cyclic rotation of the polygon, and they showed a bijection between quadrangulations of a $2n$-gon with non-crossing spanning trees on n vertices. The bijection mapping is as follows: given a non-crossing spanning tree on n vertices, for each edge connecting i to j, draw a dotted line from $2i-1$ to $2j-1$ in a $2n$-gon. Then the quadrangulation of this $2n$-gon is defined by quadrangles whose diagonals are the dotted lines; conversely, given a $2n$-gon, every quadrangle has a diagonal whose endpoints are odd numbers, so we may perform the reverse procedure to get an inverse mapping (see Figure 2). This bijection preserves the cyclic sieving phenomenon, since rotation by $\frac{2\pi}{n}$ in the tree corresponds to rotation by $\frac{\pi}{n}$ in the $2n$-gon.

In the latter case, Reiner, Stanton, and White showed in [5] that polygon dissections of a polygon exhibit the cyclic sieving phenomenon where the cyclic action is also rotation. In particular, triangulations acted upon by rotations exhibit the cyclic sieving phenomenon. These results inspired Eu to conjecture Theorem 1.1, which we prove in the following sections. The case $d = 1$ in Theorem 1.1 follows from (1).

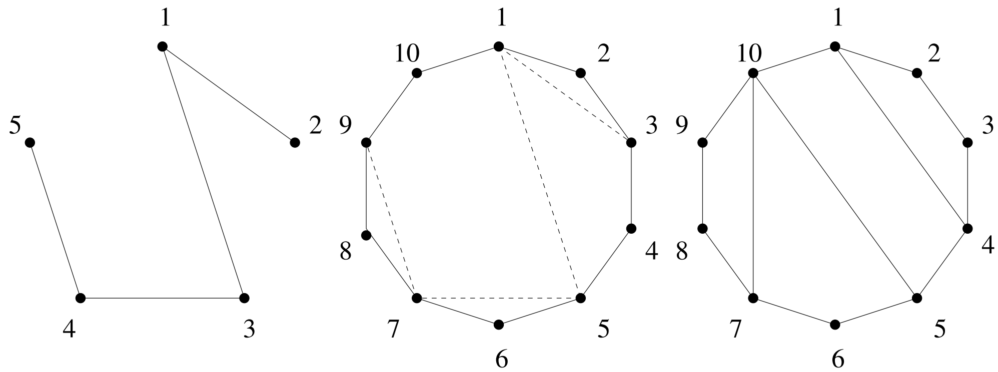

Fig. 2: Bijection between a spanning tree on a 5 vertices and a quadrangulation of a 10-gon.

We therefore consider the following three cases: $d = 2$ and k is odd, $d = 2$ and k is even, and $d \geq 3$. The majority of the work is done in the proofs of the case where $d = 2$, and we show that the case where $d \geq 3$ reduces to this case.

2 Lagrange Inversion Theorem

In the following sections, we will use the Lagrange Inversion Theorem to extract coefficients of certain generating functions. If $\phi(z) \in \mathbb{Q}[[z]]$, then we define $[z^n]\phi(z)$ to be the coefficient of z^n in $\phi(z)$.

Lagrange inversion. *Let $\phi(u) \in \mathbb{Q}[[u]]$ be a formal power series with $\phi(0) \neq 0$, and let $y(z) \in \mathbb{Q}[[z]]$ satisfy $y = z\phi(y)$. Then, for an arbitrary series ψ, the coefficient of z^n in $\phi(y)$ is given by*

$$[z^n]\psi(y(z)) = \frac{1}{n}[u^{n-1}]\phi(u)^n\psi'(u).$$

Lagrange inversion may be applied to bivariate generating functions by treating the second variable as a parameter.

We begin by illustrating how Flajolet and Noy used Lagrange inversion to find (1). Let $C(z, w)$ be the generating function for $c_{n,k}$, that is,

$$C(z, w) = \sum_{n,k} c_{n,k} z^n w^k.$$

Then it can be shown using a combinatorial argument that C satisfies

$$wC^3 + wC^2 - z(1 + 2w)C + z^2(1 + w) = 0.$$

Setting $C = z + zy$, this becomes

$$wz(1 + y)^3 = y(1 - wy)$$

which can be put in the Lagrange form

$$y = z\frac{w(1 + y)^3}{1 - wy}. \tag{3}$$

The result (1) then follows upon application of Lagrange inversion on y. We will in fact use this same function y multiple times in our proofs.

3 The case where $d = 2$ and k is odd

In this section, we prove that Theorem 1.1 holds when $d = 2$ and k is odd. Recall that d divides n, so n must be even in this case. The case where $n = 2$ is trivial since there is only 1 non-crossing connected graph on 2 vertices, so we may assume that $n > 2$. For this section, define $n' = \frac{n}{2}$ and $k' = \frac{k+1}{2}$. It is a straightforward computation to verify that

$$c(n, k; -1) = \binom{3n' - 2}{n' + k' - 1}\binom{k' - 1}{n' - 1}. \tag{4}$$

The goal of this section is to show that $s_2(n, k) = c(n, k, -1)$, and we do this by showing that both sides satisfy the same recurrence and initial conditions.

Recall that $c_{n,k} = |X|$. Define $d_{n,k}$ to be the number of non-crossing graphs on $\{1, \ldots, n\}$ with k edges and exactly two connected components such that 1 and n are in different components.

Lemma 3.1. *With $d_{n,k}$ defined above, we have*

$$d_{n,k} = \frac{2}{n-2}\binom{3n-5}{n+k}\binom{k-1}{n-3}.$$

Proof. Let $D(z, w) = \sum d_{n,k} z^n w^k$ and let $C(z, w) = \sum c_{n,k} z^n w^k$. Since $d_{n,k}$ counts graphs with two connected components, which are each counted by $c_{n,k}$, we therefore have $D = C^2$. To find the coefficient of $z^n w^k$, we use Lagrange inversion. Recall from (3) that $y = z\frac{w(1+y)^3}{1-wy}$. But $D = C^2 = z^2 + z^2(y^2 + 2y)$. Therefore

$$[z^n w^k]D = [z^{n-2} w^k]y^2 + 2[z^{n-2} w^k]y.$$

Computing each of these separately, we have

$$\begin{aligned}
[z^{n-2} w^k]y &= \frac{1}{n-2}[u^{n-3} w^k]\frac{w^{n-2}(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{1}{n-2}\binom{3n-6}{n+k-1}\binom{k-1}{n-3}
\end{aligned}$$

and

$$\begin{aligned}
[z^{n-2} w^k]y^2 &= \frac{2}{n-2}[u^{n-4} w^k]\frac{w^{n-2}(1+u)^{3n-6}}{(1-uw)^{n-2}} \\
&= \frac{2}{n-2}\binom{3n-6}{n+k}\binom{k-1}{n-3}.
\end{aligned}$$

The result then follows from Pascal's identity. □

We define some more notation. Define

$$f_{n,k} = \#\{x \in X : x \text{ has an edge from 1 to } n\}.$$

Lemma 3.2. *With $f_{n,k}$ defined as above, we have*

$$s_2(n,k) = n' \cdot f_{n'+1,k'}.$$

Proof. Given a centrally symmetric graph with an odd number of edges, exactly one of the edges must be a diameter. There are n' choices for the diameter. Once a diameter has been fixed, the remaining $k-1$ edges are determined by the $k'-1$ edges on either side of the diameter. Without loss of generality, assume

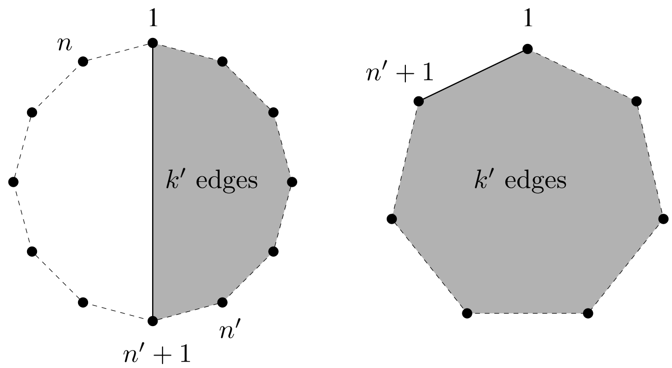

Fig. 3: The bijection between centrally symmetric n-vertex, k-edge graph with fixed diameter and $(\frac{n}{2}+1)$-vertex, $\frac{k+1}{2}$-edge graph with edge between 1 and $\frac{n}{2}+1$.

the diameter has endpoints 1 and $n'+1$. Then we have a bijection (see Figure 3) between the graphs we wish to count and graphs on $\{1,\ldots,n'+1\}$ with k' edges including the edge from 1 to $n'+1$. This is counted by $f_{n'+1,k'}$. □

Lemma 3.3. *The sequence $f_{n,k}$ satisfies the recurrence*

$$f_{n,k} + f_{n,k+1} = c_{n,k} + d_{n,k}$$

with the base case

$$f_{n,2n-3} = c_{n,2n-3} = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

Proof. The base case follows from the fact that every triangulation must contain the edge from 1 to n. Now consider a non-crossing connected graph with $k+1$ edges on $\{1,\ldots,n\}$ with the edge 1 to n. We have two cases. When we remove this edge, either the remaining graph is connected or not. If the remaining graph is connected, then we have a non-crossing connected graph with k edges without the edge from 1 to n. This is counted by $c_{n,k} - f_{n,k}$. If the remaining graph is not connected, then there are exactly two connected components, and 1 and n lie in separate components. This is counted by $d_{n,k}$. Hence

$$f_{n,k+1} = c_{n,k} + d_{n,k} - f_{n,k}.$$

□

As a corollary to this lemma, it follows that one has the recurrence

$$s_2(2n-2,2k-1)+s_2(2n-2,2k+1)=(n-1)c_{n,k}+(n-1)d_{n,k} \tag{5}$$

with base case

$$s_2(2n-2,4n-7)=(n-1)c_{n,2n-3}=\binom{2n-4}{n-2}.$$

To show that $c(n,k;-1)=s_2(n,k)$ for even n and odd k or, equivalently, $c(2n-2,2k-1;-1)=s_2(2n-2,2k-1)$ for any positive integers $n>2$ and $n-1\le k\le 2n-3$, it suffices to show that $c(2n-2,2k-1;-1)$ satisfies the same recurrence (5) as $s_2(2n-2,2k-1)$. The base case is immediate:

$$c(2n-2,4n-7;-1)=\binom{3n-5}{3n-5}\binom{2n-4}{n-2}=\binom{2n-4}{n-2}.$$

We now show that $c(2n-2,2k-1;-1)$ satisfies the recurrence relation as well, which completes the proof that the theorem holds for $d=2$ and odd k.

Proposition 3.4. $c(2n-2,2k-1;-1)$ *satisfies*

$$c(2n-2,2k-1;-1)+c(2n-2,2k+1;-1)=(n-1)c_{n,k}+(n-1)d_{n,k}.$$

Proof. From (4), we see that all we need to verify is

$$\binom{3n-5}{n+k-2}\binom{k-1}{n-2}+\binom{3n-5}{n+k-1}\binom{k}{n-2}=\binom{3n-3}{n+k}\binom{k-1}{n-2}+\frac{2n-2}{n-2}\binom{3n-5}{n+k}\binom{k-1}{n-3},$$

which we leave as a straightforward exercise for the reader. □

4 The case where $d=2$ and k is even

In this section, we prove that Theorem 1.1 holds when $d=2$ and k is even. As in the previous case, it is again a straightforward computation to verify that

$$c(n,k;-1)=\binom{\frac{3n-4}{2}}{\frac{n+k}{2}}\binom{\frac{k-2}{2}}{\frac{n-2}{2}}.$$

Let $a_{2n,k}$ denote the number of non-crossing connected graphs with $2n$ vertices and k pairs of antipodal edges, where a diameter counts as one pair. When counting $a_{2n,k}$, we have two cases. In one case, there is a diameter, and in the second case, there is not. This gives us the sum

$$a_{2n,k}=s_2(2n,2k-1)+s_2(2n,2k)=c(2n,2k-1;-1)+s_2(2n,2k).$$

where the second equality follows from our results in the previous section. Our goal in this section is to show that $s_2(2n,2k)=c(2n,2k;-1)$, so it suffices to show that

$$a_{2n,k}=c(2n,2k-1;-1)+c(2n,2k;-1)=\binom{3n-1}{n+k}\binom{k-1}{n-1}. \tag{6}$$

Let F be the generating function for $f_{n,k}$, i.e. $F(z,w)=\sum f_{n,k}z^nw^k$. Similarly, let $A(z,w)=\sum a_{2n,k}z^nw^k$. Our strategy in this section is to use the Lagrange Inversion Theorem on $A(z,w)$ to obtain (6).

Lemma 4.1.

$$a_{2n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots+k_m=k} \sum_{1\le v_1<\cdots<v_m\le n} \prod_{i=1}^{m} f_{v_{i+1}-v_i+1,k_i}$$

where $v_{m+1} = v_1 + n$.

Proof. Consider a non-crossing connected graph with $2n$ vertices and k pairs of antipodal edges. There exists a unique positive integer m such that the center of the $2n$-gon lies inside a $2m$-gon formed by edges of the graph and such that no other edges lie inside the $2m$-gon. This m is at most n. Now, exactly m of the vertices of this $2m$-gon, call them $v_1 < \cdots < v_m$, lie in the set $\{1, \dots, n\}$ due to the antipodal condition on the edges. All edges not used in the $2m$-gon lie outside of it (see Figure 4). The $(m+1)$-th

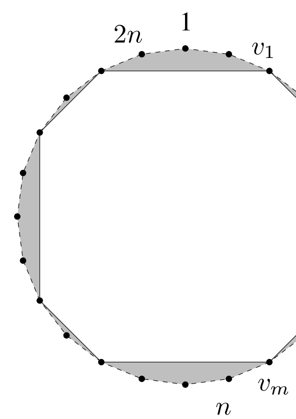

Fig. 4: A graph with an inner $2m$-gon, where $m = 4$.

vertex is antipodal to v_1, hence $v_{m+1} = v_1 + n$. For each i, there is an edge from v_i to v_{i+1} and $k_i - 1$ other edges on the vertices $\{v_i, v_i + 1, \dots, v_{i+1}\}$, such that $k_1 + \cdots + k_m = k$. Such a graph is counted by $f_{v_{i+1}-v_i+1,k_i}$. Thus we get the corresponding sum. □

Lemma 4.2. *With A and F as defined above, we have*

$$\frac{A}{z} = \frac{\partial(F/z)/\partial z}{1 - F/z}.$$

Proof. We show that

$$a_{2n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots k_m=k} \sum_{n_1+\cdots+n_m=n+m} (n_m - 1) f_{n_m,k_m} \prod_{i=1}^{m-1} f_{n_i,k_i}.$$

In the sum in the previous lemma, the term $\prod_{i=1}^{m} f_{v_{i+1}-v_i+1,k_i}$ is counted multiple times with the product written in this order. We show that it is counted exactly $n + v_1 - v_m$ times. Consider any m-element subset $\{v_1,\dots,v_m\} \subseteq \{1,\dots,n\}$ with $v_1 < \cdots < v_m$. For $j = 1,\dots,v_1-1$, this subset yields the same summand as $\{v_1-j,\dots,v_m-j\}$. Therefore, we can identify any subset $\{v_1,\dots,v_m\}$ with $\{1,\dots,v_m-v_1+1\}$. There are exactly $n+v_1-v_m$ subsets corresponding to this one, each with largest element $v_m-v_1+1, v_m-v_1+2,\dots,n$. This proves the sum identity above.

For the equality of generating functions, we insert variables into the above identity:

$$a_{2n,k}z^n w^k = \frac{1}{z^{m-2}}\sum_{m=1}^{n}\sum_{k_1+\cdots k_m=k}\sum_{n_1+\cdots+n_m=n+m}(n_m-1)f_{n_m,k_m}z^{n_m-2}w^{k_m}\prod_{i=1}^{m-1}f_{n_i,k_i}z^{n_i}w^{k_i}. \tag{7}$$

We note that

$$\frac{\partial(F/z)}{\partial z} = \sum_{n,k}(n-1)f_{n,k}z^{n-2}w^k$$

so, summing over all n and k in (7), we get

$$A = \frac{\partial(F/z)}{\partial z}\left(z+F+\frac{F^2}{z}+\frac{F^3}{z^2}+\cdots\right) = z\frac{\partial(F/z)}{\partial z}\left(\frac{1}{1-F/z}\right).$$

□

Proposition 4.3.

$$a_{2n,k} = \binom{3n-1}{n+k}\binom{k-1}{n-1}.$$

Proof. Let $H = F/z$ and let C be the generating function for $c_{n,k}$ as in the previous section and let $C = z+zy$. From the recurrence $f_{n,k}+f_{n,k+1} = d_{n,k}+c_{n,k},\ \ n\geq 2,\quad$ and $f_{1,k}=0$, we have

$$\left(1+\frac{1}{w}\right)F = D+C-z = z^2(1+y)^2+zy.$$

Therefore, after some substitution and simplification, applying the identity in (3), we get

$$1-H = \frac{1}{1+y}.$$

From

$$\frac{A}{z} = \frac{\partial H/\partial z}{1-H}$$

we get

$$\int\frac{A}{z}dz = \int\frac{dH}{1-H}$$

or equivalently

$$\sum_{n,k}\frac{1}{n}a_{2n,k}z^n w^k = -\log(1-H) = \log(1+y).$$

By the Lagrange inversion formula,

$$\begin{aligned}\frac{1}{n}a_{2n,k} &= [z^n w^k]\int \frac{A}{z}\,dz\\ &= [z^n w^k]\log(1+y)\\ &= \frac{1}{n}[u^{n-1}w^k]\frac{w^n(1+u)^{3n}}{(1-uw)^n}\frac{1}{1+u}\\ &= \frac{1}{n}\binom{3n-1}{n+k}\binom{k-1}{n-1}\end{aligned}$$

whence our desired result. □

Comparing with (6) shows that Theorem 1.1 holds when $d=2$ and k is even.

5 The case where $d \geq 3$

Finally, in this section, we prove that Theorem 1.1 holds when $d \geq 3$. For this section, define $n'' = \frac{n}{d}$ and $k'' = \frac{k}{d}$. Again, it is a straightforward computation to verify that if $d|k$, then

$$c(n,k;\omega) = \binom{3n''-1}{n''+k''}\binom{k''-1}{n''-1}.$$

Lemma 5.1. *If $d \geq 3$ does not divide k, then $c(n,k;\omega)=0$, where ω is a primitive d-th root of unity.*

If d does not divide k, then in fact there are no graphs with k edges that are fixed under rotation by $\frac{2\pi}{d}$, since each edge lies in a free orbit under the action of rotation. We henceforth assume that $d|k$.

Lemma 5.2.

$$s_d(n,k) = n''\cdot f_{n''+1,k''} + s_2(2n'',2k'').$$

Proof. For a non-crossing connected graph on $\{1,\ldots,n\}$ fixed under rotation by $\frac{2\pi}{d}$, then there are two cases: either the edges form a central d-gon or not. In the former case, every edge is purely determined by the edges on the first $n''+1$ vertices. In fact, there is bijection between such graphs and non-crossing connected graphs on $n''+1$ vertices with the edge from 1 to $n''+1$. There are $f_{n''+1,k''}$ such graphs, and there are n'' possible d-gons. In the latter case, the edges are determined by edges on the first $2n''$ vertices. We construct a bijection between such graphs and centrally symmetric non-crossing connected graphs on $2n''$ vertices with $2k''$ edges as follows (see Figure 5): Going around clockwise in the graph, label the first m vertices $1_1, 2_1, \ldots, n''_1$, label the next set of vertices $1_2, 2_2, \ldots, n''_2$, and so on. Construct a non-crossing graph with $2n''$ vertices labeled $1_1, 2_1, \ldots, n''_1, 1_2, 2_2, \ldots, n''_2$. For each edge from i to j in the original graph, we put an edge with the same endpoints in the new graph. Finally, if there is an edge from some i_2 to some j_3, we put an edge from i_2 to j_1 in the new graph. This new graph therefore has $2k''$ edges. It is straightforward to check that this is a bijection. □

Proposition 5.3. *For $d \geq 3$ and ω a primitive dth root of unity,*

$$c(n,k;\omega) = s_d(n,k).$$

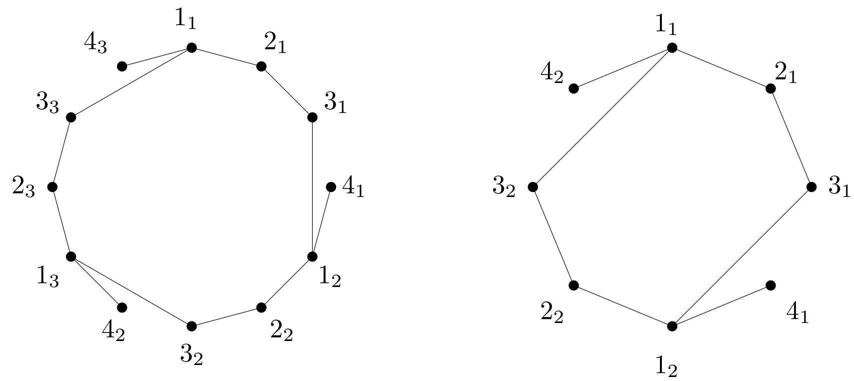

Fig. 5: The bijective construction when $(n, k, d) = (12, 12, 3)$

Proof. This follows by the previous lemma and our results from the case where $d = 2$, after applying Pascal's rule. $\square$

This completes the proof of Theorem 1.1.

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A polynomial expression for the Hilbert series of the quotient ring of diagonal coinvariants (condensed version)

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Abstract. A special case of Haiman's identity [Invent. Math. **149** (2002), pp. 371–407] for the character of the quotient ring of diagonal coinvariants under the diagonal action of the symmetric group yields a formula for the bigraded Hilbert series as a sum of rational functions in q, t. In this paper we show how a summation identity of Garsia and Zabrocki for Macdonald polynomial Pieri coefficients can be used to transform Haiman's formula for the Hilbert series into an explicit polynomial in q, t with integer coefficients. We also provide an equivalent formula for the Hilbert series as the constant term in a multivariate Laurent series.

Résumé. Un cas spécial de l'identité de Haiman [Invent. Math. **149** (2002), pp. 371–407] pour le caractère de l'anneau quotient des coinvariants diagonaux sous l'action du groupe symétrique fournit une formule pour la série de Hilbert bigraduée comme somme de fonctions rationnelles en q, t. Dans cet article nous montrons comment une identité de sommation de Garsia et Zabrocki pour les coefficients de Pieri des polynômes de Macdonald peut être utilisée pour transformer la formule de Haiman pour la série de Hilbert en un polynôme explicite en q, t à coefficients entiers. Nous présentons également une formule équivalente pour la série de Hilbert comme terme constant d'une série de Laurent multivariée.

Keywords: Hilbert series, diagonal coinvariants

1 Introduction

Let $X_n = \{x_1, \ldots, x_n\}, Y_n = \{y_1, \ldots, y_n\}$ be two sets of variables and let

$$\mathrm{DR}_n = \mathbb{C}[X_n, Y_n] / \big\langle \{ \sum_i x_i^h y_i^k, \forall h, k \geq 0, h + k > 0 \} \big\rangle \tag{1}$$

be the quotient ring of diagonal coinvariants. Let ∇ be the linear operator defined on the modified Macdonald polynomial basis $\{\tilde{H}_\mu(X_n; q, t)\}$, where $\mu \vdash n$ (i.e. μ is a partition of n), by

$$\nabla \tilde{H}_\mu(X_n; q, t) = T_\mu \tilde{H}_\mu(X_n; q, t), \tag{2}$$

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where $T_\mu = t^{n(\mu)}q^{n(\mu')}$ and $n(\mu) = \sum_i (i-1)\mu_i$. The symmetric group acts "diagonally" on a polynomial $f(x_1,\ldots,x_n,y_1,\ldots,y_n)$ by $\sigma f = f(x_{\sigma(1)},\ldots,x_{\sigma(n)},y_{\sigma(1)},\ldots,y_{\sigma(n)})$ and this action extends to DR_n. Haiman (Hai02) proved an earlier conjecture of Garsia and Haiman (GH96) that the Frobenius series of this action is given by $\nabla e_n(X_n)$, where e_n is the nth elementary symmetric function in a set of variables. (The Frobenius series is obtained by starting with the character and mapping the irreducible S_n-character χ^λ to the Schur function s_λ.) Since the Frobenius series of DR_n is given by ∇e_n, the Hilbert series $\mathrm{Hilb}(\mathrm{DR}_n)$ is given by $\langle \nabla e_n, h_1^n\rangle$ (See (Hag08, p. 24) for an explanation of why. Here $\langle , \rangle$ is the Hall scalar product, with respect to which the Schur functions are orthonormal, and $h_1(X) = \sum_i x_i$.) This results in a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ as an explicit sum of rational functions in q, t, described in detail in the next section. A corollary of this formula is that $\dim(\mathrm{DR}_n) = (n+1)^{n-1}$. See also (Hai94) and (Ber09) for background on this problem. We mention that many articles in the literature refer to the space of diagonal harmonics DH_n, which is known to be isomorphic to DR_n, and so $\mathrm{Hilb}(\mathrm{DH}_n) = \mathrm{Hilb}(\mathrm{DR}_n)$.

A Dyck path is a lattice path in the first quadrant of the xy-plane from $(0,0)$ to (n,n) consisting of unit north N and east E steps which never goes below the diagonal $x=y$. A parking function is a placement of the integers $1,2,\ldots,n$ (called "cars") just to the right of the N steps of a Dyck path, so there is strict decrease down columns. An open conjecture of Loehr and the author (HL05) expresses $\mathrm{Hilb}(\mathrm{DR}_n)$ as a positive sum of monomials, one for each parking function. In a recent preprint, Armstrong (Arm10) introduces a hyperplane arrangement model for $\mathrm{Hilb}(\mathrm{DR}_n)$ involving a pair of hyperplane arrangements with a statistic associated to each one. See also (AR). He gives a bijection with parking functions which sends his pair of hyperplane arrangement statistics to the pair of statistics on parking functions introduced by Haglund and Loehr.

In this article we use a plethystic summation formula of Garsia and Zabrocki for Macdonald Pieri coefficients to show how $\langle \nabla e_n, h_1^n\rangle$ can be expressed as an element of $\mathbb{Z}[q,t]$. The most elegant way of expressing our result is to say that $\mathrm{Hilb}(\mathrm{DR}_n)$ is the coefficient of $z_1z_2\cdots z_n$ in a certain multivariate Laurent series (see (41)). We are currently unable to see how our result implies a positive formula such as the conjecture of Haglund and Loehr, but are hopeful that further work will lead to such applications.

2 Background Material

For $\mu \vdash n$, and s a square of the Ferrers diagram of μ, let $l(s), a(s), l'(s), a'(s)$ denote the leg, arm, coleg, coarm, respectively, of s, i.e. the number of squares above s, to the right of s, below s, and to the left of s, as in Figure 1. Furthermore let

$$M = (1-q)(1-t), \quad B_\mu = \sum_{s\in\mu} t^{l'}q^{a'}, \quad \Pi_\mu = \prod_{\substack{s\in\mu\\ s\neq(0,0)}} (1-t^{l'}q^{a'}), \quad w_\mu = \prod_{s\in\mu}(q^a - t^{l+1})(t^l - q^{a+1}). \tag{3}$$

The known expansion

$$e_n(X) = \sum_{\mu\vdash n} \frac{\tilde{H}_\mu(X;q,t)M\Pi_\mu B_\mu}{w_\mu} \tag{4}$$

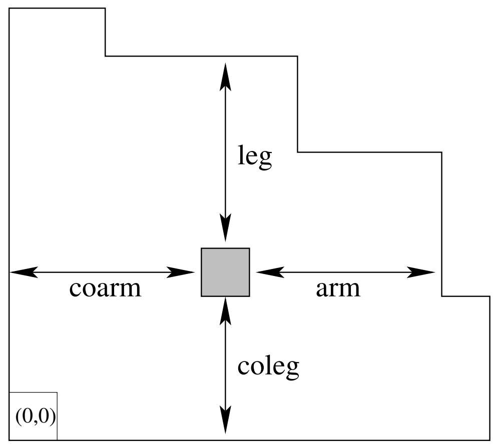

Fig. 1: The leg, coleg, arm, and coarm of a square

then implies

$$\nabla e_n(X) = \sum_{\mu \vdash n} \frac{T_\mu \tilde{H}_\mu(X;q,t) M \Pi_\mu B_\mu}{w_\mu}. \tag{5}$$

Letting $F_\mu = \langle \tilde{H}_\mu, h_1^n \rangle$, by taking the scalar product of both sides of (5) with respect to h_1^n we get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\mu \vdash n} \frac{T_\mu F_\mu M \Pi_\mu B_\mu}{w_\mu}. \tag{6}$$

Let $^\perp$ be the operation on symmetric functions which is adjoint to multiplication with respect to the Hall scalar product, i.e. for any symmetric functions f, g, h,

$$\langle f^\perp g, h \rangle = \langle g, fh \rangle. \tag{7}$$

If $\mu \vdash n$ and $\nu \vdash n-1$, then $\nu \to \mu$ means ν is obtained from μ by removing some corner square of μ, and $\mu \leftarrow \nu$ means μ is obtained from ν by adding a single square to the Ferrers shape of ν. Define generalized skew Pieri coefficients $c^{f^\perp}_{\mu,\nu}(q,t)$ and Pieri coefficients $d^f_{\mu,\nu}(q,t)$ by the formulas

$$f^\perp \tilde{H}_\mu(X;q,t) = \sum_{\substack{\nu \\ \nu \to \mu}} c^{f^\perp}_{\mu,\nu}(q,t) \tilde{H}_\nu(X;q,t) \tag{8}$$

$$f \tilde{H}_\nu(X;q,t) = \sum_{\substack{\mu \\ \mu \leftarrow \nu}} d^f_{\mu,\nu}(q,t) \tilde{H}_\mu(X;q,t). \tag{9}$$

Many of the identities in this paper are expressed using plethystic notation, defined as follows. If $p_k(X) = \sum_i x_i^k$ is the kth power sum, then for any expression E, the plethystic substitution of E into

p_k is obtained by replacing all indeterminates in E by their kth powers. We denote this by $p_k[E]$, so for example

$$p_k[X(1-t)] = p_k(X)(1-t^k). \tag{10}$$

For any symmetric function $f(X)$, we define $f[E]$ by first expressing f as a polynomial in the p_k, then replacing each p_k by $p_k[E]$.

The $c^{f^\perp}_{\mu,\nu}$ and the $d^f_{\mu,\nu}$ are related via (GH02, (3.5))

$$c^{f^\perp}_{\mu,\nu} w_\nu = d^{\omega f[X/M]}_{\mu,\nu} w_\mu, \tag{11}$$

where ω is the linear operator on symmetric functions satisfying $\omega s_\lambda = s_{\lambda'}$. Note $d^{\omega h_1[X/M]}_{\mu,\nu} = d^{h_1[X]}_{\mu,\nu}/M$. We abbreviate $c^{h_1\perp}_{\mu,\nu}(q,t)$ by $c_{\mu,\nu}$ and $d^{h_1[X/M]}_{\mu,\nu}(q,t)$ by $d_{\mu,\nu}$.

A special case of Macdonald's Pieri formulas (Mac95, Section 6.6) gives an expression for $d_{\mu,\nu}$ as a quotient of factors of the form $(t^a q^b - t^c q^d)$, where a, b, c, d have simple combinatorial descriptions. Garsia found a simplification in this formula, which Garsia and Zabrocki used to obtain the $k = 1$ case of the following summation formula (GZ05). The proof of the result for general k appears in (BGHT99) and (Gar10).

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu} T^k = \begin{cases} 1/M & \text{if } k = 0, \\ (-1)^{k-1} e_{k-1}[MB_\nu - 1]/M & \text{if } k \geq 1, \end{cases} \tag{12}$$

where throughout this article T is an abbreviation for T_μ/T_ν. Eq. (12) is closely related to a corresponding summation formula involving the $c_{\mu,\nu}$ (GT96, Theorem 2.2).

Identity (12) can be recast in the following form.

Lemma 1

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu}(1-T)T^k = \begin{cases} 0 & \textit{if } k = 0, \\ (-1)^{k-1} e_k[MB_\nu]/M & \textit{if } k \geq 1 \end{cases}. \tag{13}$$

The following simple fact will be useful later.

Lemma 2

$$(-1)^{k-1} e_k[M]/M = \frac{t^k - q^k}{t - q} \qquad k \geq 1. \tag{14}$$

3 A New Recursive Procedure to Generate the Hilbert Series

By definition we have

$$e_1^\perp \tilde{H}_\mu(X;q,t) = \sum_{\substack{\nu \\ \nu \to \mu}} c_{\mu,\nu} \tilde{H}_\nu(X;q,t). \tag{15}$$

Taking the scalar product of both sides with respect to h_1^{n-1} we get

$$\langle e_1^{\perp}\tilde{H}_\mu, h_1^{n-1}\rangle = \langle \tilde{H}_\mu, e_1 h_1^{n-1}\rangle = F_\mu = \sum_{\substack{\nu \\ \nu\to\mu}} c_{\mu,\nu}F_\nu. \tag{16}$$

Plugging this recurrence for the F_μ into (6) yields

$$\begin{aligned} \mathrm{Hilb}(\mathrm{DR}_n) &= \sum_{\mu\vdash n} \frac{T_\mu M \Pi_\mu B_\mu}{w_\mu} \sum_{\substack{\nu \\ \nu\to\mu}} c_{\mu,\nu}F_\nu & (17)\\ &= \sum_{\nu\vdash n-1} F_\nu M \sum_{\substack{\mu \\ \mu\leftarrow\nu}} \frac{B_\mu \Pi_\mu c_{\mu,\nu} T_\mu}{w_\mu}. & (18)\end{aligned}$$

Now from (3) we see

$$B_\mu = B_\nu + T, \qquad \Pi_\mu = \Pi_\nu(1-T). \tag{19}$$

Using this and the $f = e_1$ case of (11) in (18) we get

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu\vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu}(B_\nu + T)(1-T)T. \tag{20}$$

By (13) this implies

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\nu\vdash n-1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} \left(\frac{e_1[MB_\nu]}{M}\frac{e_1[MB_\nu]}{M} - \frac{e_2[MB_\nu]}{M} \right). \tag{21}$$

(Although $e_1[MB_\nu]/M$ can be expressed more simply as $e_1[B_\nu]$, leaving (21) in the above form will prove more useful in the sequel.)

We now iterate the argument; first re-index the sum in (21) as a sum over $\mu \vdash n-1$, and replace F_μ by $\sum_{\nu\to\mu} c_{\mu,\nu}F_\nu$. Then write B_μ as $B_\nu + T$ as before, and reverse summation to get

$$\begin{aligned} \mathrm{Hilb}(\mathrm{DR}_n) = &\sum_{\nu\vdash n-2} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} & (22)\\ &\times \sum_{\substack{\mu \\ \mu\leftarrow\nu}} d_{\mu,\nu}(1-T)T \left(\frac{e_1[M(B_\nu+T)]}{M}\frac{e_1[M(B_\nu+T)]}{M} - \frac{e_2[M(B_\nu+T)]}{M} \right). \end{aligned}$$

Now for any alphabets X, Y we have

$$e_k\left[X - Y\right] = \sum_{j=0}^{k} e_j\left[X\right] e_{k-j}\left[-Y\right]. \tag{23}$$

Hence for $k \geq 1$

$$(-1)^{k-1}\frac{e_k[M(B_\nu+T)]}{M} = b_k + T^k a_k + \sum_{j=1}^{k-1} -Mb_jT^{k-j}a_{k-j}, \tag{24}$$

where we have abbreviated $(-1)^{j-1}e_j[M]/M$ by a_j and $(-1)^{j-1}e_j[MB_\nu]/M$ by $b_j = b_j(\nu)$. Here we have used the fact that $e_k[MT]/M = T^k e_k[M]/M$ (since for any expression $p_j[XT] = T^j p_j[X]$). Note also that $a_1 = 1$. The inner sum in (22) thus becomes

$$\sum_{\substack{\mu \\ \mu \leftarrow \nu}} d_{\mu,\nu}(1-T)T\left((b_1+Ta_1)^2 + b_2a_1 + T^2a_2 - Mb_1Ta_1\right) \tag{25}$$

$$= b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3 \tag{26}$$

by (13).

Let

$$A_1 = b_1 \tag{27}$$
$$A_2 = b_1^2 + b_2a_1 \tag{28}$$
$$A_3 = b_1^3 + 2b_1a_1b_2 + a_1^2b_3 + a_1b_2b_1 - Mb_1a_1^2b_2 + a_1a_2b_3. \tag{29}$$

The above discussion implies

Theorem 1 *For* $p \in \mathbb{N}$, $1 \le p \le n$,

$$\mathit{Hilb}(DR_n) = \sum_{\nu \vdash n-p+1} \frac{T_\nu F_\nu M \Pi_\nu}{w_\nu} A_p, \tag{30}$$

where $A_p = A_p(\nu)$ *is a certain polynomial in the* a_i, b_i*. Moreover,* A_p *can be calculated recursively from* A_{p-1} *by the following procedure. First replace each* b_k *in* A_{p-1} *by* $b_k + T^k a_k - \sum_{j=1}^{k-1} Mb_jT^{k-j}a_{k-j}$*. Then multiply the resulting expression out to form a polynomial in* T*, say*

$$\sum_j c_jT^j. \tag{31}$$

Finally, replace T^j *by* b_{j+1}*, i.e.*

$$A_p = \sum_j c_jb_{j+1}. \tag{32}$$

(We replace T^j *by* b_{j+1} *since, after multiplying the expression above out to get* $\sum c_jT^j$*, we still have another factor of* T *coming from the outer sum. Applying (13) replaces* T^{j+1} *by* b_{j+1}*.)*

We now give a non-recursive expression for A_p. Let Q_n denote the set of all $n \times n$ upper-triangular matrices C of nonnegative integers which satisfy

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = 1, \quad \text{for each } j,\, 1 \le j \le n. \tag{33}$$

For example,

$$Q_1 = \{ \begin{bmatrix} 1 \end{bmatrix} \} \tag{34}$$

$$Q_2 = \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix} \} \tag{35}$$

$$Q_3 = \{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 3 \end{bmatrix} \}. \tag{36}$$

Geometrically, the condition (33) says that for all j, if we add all the entries of C in the jth row together, and then subtract all the entries in the jth column above the diagonal, we get 1. Note that these conditions imply that each row of C must have at least one positive entry.

For $C \in Q_n$, let $\mathrm{Pos}(C)$ denote the multiset of positive entries in C, and $\mathrm{pos}(C)$ its cardinality. Matrices of this kind can be generated recursively, in a manner similar to the recursion generating the polynomials A_p, and using this one can prove the following.

Theorem 2 *For $1 \le p \le n$ and A_p, b_j, a_j as above,*

$$A_p = \sum_{C \in Q_p} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C) \\ i<j}} a_{c_{ij}}. \tag{37}$$

Corollary 1

$$Hilb(DR_n) = \sum_{C \in Q_n} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C) \\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}, \tag{38}$$

where $[k]_{q,t} = (t^k - q^k)/(t-q)$ is the q,t-analog of the integer k.

Example 1 *The weights associated to the elements of Q_3, listed in the same left-to-right order as in (36) are*

$$1, \quad t+q, \quad t+q, \quad -M(t+q), \quad (t+q)(t^2+qt+q^2), \quad t+q, \quad t^2+qt+q^2. \tag{39}$$

Thus Hilb(DR_3) is the sum of these terms, namely

$$1 + 2q + 2t + 2q^2 + 3qt + 2t^2 + q^3 + q^2t + qt^2 + t^3. \tag{40}$$

The sequence $1, 2, 7, 40, 357, 4820, \ldots$ consisting of the cardinalities of the sets $Q_1, Q_2, Q_3, \ldots$ form entry $A008608$ in Sloane's on-line encyclopedia of integer sequences. In fact, it was comparing the number of monomials in A_n for small n with sequences in Sloane's encyclopedia that led the author to the discovery of the non-recursive expression for the A_n in terms of the elements of Q_n. The sequence was introduced to Sloane's list by Glenn Tesler, who in a private conversation with the author said they arose in unpublished work of Tesler's from the late 1990's on plethystic expressions for Macdonald's $D_{n,r}$ operators. Although Tesler doesn't recall any further details about this work, we will refer to elements of Q_n as "Tesler matrices".

The explicit formula (38) for $\mathrm{Hilb}(\mathrm{DR}_n)$ can be formulated as a constant term identity.

Corollary 2 *For $n \geq 1$, Hilb(DR_n) is the coefficient of $z_1 z_2 \cdots z_n$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \leq i < j \leq n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{41}$$

4 The m-parameter

The formula ∇e_n for the Frobenius series of DR_n is a special case of a more general result (also due to Haiman (Hai02)) which says that for any positive integer m, $\nabla^m e_n$ is the Frobenius series of a certain S_n-module $\mathrm{DR}_n^{(m)}$. Hence, from (5) we have

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \langle \nabla^m e_n, h_1^n \rangle \tag{42}$$

$$= \sum_{\substack{\mu \\ \mu \vdash n}} \frac{T_\mu^m F_\mu M \Pi_\mu B_\mu}{w_\mu}. \tag{43}$$

The methods of the previous section can be generalized to show that for any $1 \leq p \leq n$,

$$\mathrm{Hilb}(\mathrm{DR}_n^{(m)}) = \sum_{\mu \vdash n-p+1} \frac{T_\nu^m F_\nu M \Pi_\nu}{w_\nu} A_p^{(m)}, \tag{44}$$

where $A_p^{(m)} = A_p^{(m)}(\mu)$ is a polynomial in the b_j, a_j as before. We have $A_1^{(m)}(\mu) = b_1$, and for $p > 1$, we can construct $A_p^{(m)}$ recursively by the following procedure. First, replace each b_k in $A_{p-1}^{(m)}$ by $b_k + T^k a_k + \sum_{j=1}^{k-1} -M b_j T^{k-j} a_{k-j}$. Then, multiply the resulting expression out to form a polynomial in T say

$$\sum_j c_j T^j. \tag{45}$$

Finally, replace T^j by b_{j+m}, i.e.

$$A_p^{(m)} = \sum_j c_j b_{j+m}. \tag{46}$$

In terms of the Tesler matrices, we want the "hook sums" to be equal to $(1, m, m, \ldots, m)$ instead of $(1, 1, \ldots, 1)$. To be precise, define $Q_n^{(m)}$ to be the set of upper-triangular matrices C of nonnegative integers satisfying

$$-\sum_{i=1}^{j-1} c_{ij} + \sum_{i=j}^{n} c_{ji} = \begin{cases} 1 & \text{if } j = 1, \\ m & \text{if } 2 \leq j \leq n \end{cases}. \tag{47}$$

We get the following extensions of the earlier results.

Theorem 3 *For $1 \le p \le n$, $m \ge 1$, and $A_p^{(m)}$, b_j, a_j as above,*

$$A_p^{(m)} = \sum_{C \in Q_p^{(m)}} (-M)^{pos(C)-n} \prod_{c_{ii} \in Pos(C)} b_{c_{ii}} \prod_{\substack{c_{ij} \in Pos(C) \\ i<j}} a_{c_{ij}}. \tag{48}$$

Furthermore, the special case $p = n$ of (44) reduces to

$$Hilb(DR_n^{(m)}) = \sum_{C \in Q_n^{(m)}} (-M)^{pos(C)-n} \prod_{\substack{c_{ij} \in Pos(C) \\ 1 \le i \le j \le n}} [c_{ij}]_{q,t}. \tag{49}$$

Corollary 3 *For $n \ge 1$, $Hilb(DR_n^{(m)})$ is the coefficient of $z_1 z_2^m z_3^m \cdots z_n^m$ in*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \le i < j \le n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)}. \tag{50}$$

5 Conjectures and Open Questions

5.1 Tesler matrices with more general hook sums

In general the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (50) is not a positive polynomial in q, t, but Maple calculations suggest it is positive if the α_i are positive and nondecreasing.

Conjecture 1 *For $n \ge 1$ and α the reverse of a partition (so $1 \le \alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$)*

$$\frac{1}{(-M)^n} \prod_{i=1}^{n} \frac{(1-z_i)(1-qtz_i)}{(1-qz_i)(1-tz_i)} \prod_{1 \le i < j \le n} \frac{(1-z_i/z_j)(1-qtz_i/z_j)}{(1-qz_i/z_j)(1-tz_i/z_j)} \Big|_{z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}} \in \mathbb{N}[q,t]. \tag{51}$$

Equivalently, the weighted sum over Tesler matrices with hook sums $\alpha_1, \ldots, \alpha_n$ is in $\mathbb{N}[q,t]$.

Remark 1 *The argument proving Theorem 3 shows that if $\alpha_1 = 1$, the coefficient of $z_1^{\alpha_1} z_2^{\alpha_2} \cdots z_n^{\alpha_n}$ in (51) can be obtained by starting with $\nabla^{\alpha_2} e_n$, applying $e_1^{\perp}$, then applying $\nabla^{\alpha_3 - \alpha_2}$, applying $e_1^{\perp}$ again, then applying $\nabla^{\alpha_4 - \alpha_3}$, etc. The author doesn't know if these polynomials have a representation-theoretic interpretation for $\alpha \neq (1, m, m, \ldots, m)$.*

5.2 A refinement of the q, t-positivity

Note that $[k]_{q,t}$ can be expressed as $s_{k-1}(\{q,t\})$, i.e. the $(k-1)$st complete homogeneous symmetric function evaluated in the set of variables $\{q,t\}$. Also, $-M = t + q - 1 - qt$ equals $s_1 - 1 - s_{1,1}$, also in the set of variables $\{q,t\}$. In (38) we can substitute in these Schur function formulations for $[k]_{q,t}$ and $-M$, multiply everything out using the Pieri rule for Schur function multiplication, and thereby obtain a formula for $\mathrm{Hilb}(\mathrm{DR}_n)$ in terms of Schur functions in the set of variables $\{q,t\}$. If we then cancel terms of the form s_λ where λ has more than two parts (which becomes zero since our set of variables has only two elements) it appears that the resulting expression is Schur-positive. For example, for $n = 3$ the terms from (39) become

$$1, \quad s_1, \quad s_1, \quad (s_1 - 1 - s_{1,1})s_1, \quad s_1 s_2, \quad s_1, \quad s_2, \tag{52}$$

and the sum of these equals $1 + 2s_1 + 2s_2 + s_{1,1} + s_2 - s_{1,1,1} + s_3$. Since $s_{1,1,1}(\{q,t\}) = 0$, we can remove this leaving

$$\mathrm{Hilb}(\mathrm{DR}_3) = 1 + 2s_1 + 2s_2 + s_{1,1} + s_2 + s_3. \tag{53}$$

F. Bergeron (Ber09, p.196) has previously conjectured a stronger statement, namely that

$$\mathrm{Hilb}(\mathrm{DR}_n) = \sum_{\sigma \in S_n} h_{\lambda(\sigma)}(\{q,t\}), \tag{54}$$

i.e. that for each permutation on n elements, there is some way of defining a partition $\lambda(\sigma)$ such that the sum of the $h_{\lambda(\sigma)}$ gives $\mathrm{Hilb}(\mathrm{DR}_n)$. Here $h_\lambda = \prod_i s_{\lambda_i}$ as before. When $n = 3$, the expansion is

$$\mathrm{Hilb}(\mathrm{DR}_3) = 1 + 2h_1 + h_2 + h_{1,1} + h_3, \tag{55}$$

in agreement with (53). Bergeron further conjectures that these sums have the remarkable property that if we evaluate them in the set of variables $\{q_1, q_2, \ldots, q_k\}$ we get the Hilbert series of diagonal coinvariants in k sets of variables, for any $k \geq 1$. We hope that further study of how the cancellation in identity (38) results in positivity will lead to progress on the $k = 2$ case of Bergeron's conjecture.

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The enumeration of fully commutative affine permutations

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Abstract. We give a generating function for the fully commutative affine permutations enumerated by rank and Coxeter length, extending formulas due to Stembridge and Barcucci–Del Lungo–Pergola–Pinzani. For fixed rank, the length generating functions have coefficients that are periodic with period dividing the rank. In the course of proving these formulas, we obtain results that elucidate the structure of the fully commutative affine permutations. This is a summary of the results; the full version appears elsewhere.

Résumé. Nous présentons une fonction génératrice qui énumère les permutations affines totalement commutatives par leur rang et par leur longueur de Coxeter, généralisant les formules dues à Stembridge et à Barcucci–Del Lungo–Pergola–Pinzani. Pour un rang précis, les fonctions génératrices ont des coefficients qui sont périodiques de période divisant leur rang. Nous obtenons des résultats qui expliquent la structure des permutations affines totalement commutatives. L'article dessous est un aperçu des résultats; la version complète apparaît ailleurs.

Keywords: affine Coxeter group, abacus diagram, window notation, complete notation, fully commutative

1 Introduction

Let W be a Coxeter group. An element w of W is *fully commutative* if any reduced expression for w can be obtained from any other using only commutation relations among the generators. For example, if W is simply laced then the fully commutative elements of W are those with no $s_is_js_i$ factor in any reduced expression, where s_i and s_j are any noncommuting generators.

The fully commutative elements form an interesting class of Coxeter group elements with many special properties. Stembridge (1996) classified the Coxeter groups having finitely many fully commutative elements, and subsequently enumerated these elements in Stembridge (1998). The type A fully commutative elements were enumerated by Coxeter length by Barcucci et al. (2001), obtaining a q-analogue of the Catalan numbers. Our main result in Theorem 3.2 is an analogue of this result for the affine symmetric group.

In Section 2, we introduce the necessary definitions and background information. In Section 3, we enumerate the fully commutative affine permutations by decomposing them into several subsets. The

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formula that we obtain turns out to involve a ratio of q-Bessel functions as described in Barcucci et al. (1998) arising as the solution obtained by Bousquet-Mélou (1996) of a certain recurrence relation on the generating function. A similar phenomenon occurred in Barcucci et al. (2001), and our work can be viewed as a description of how to lift this formula to the affine case. It turns out that the only additional ingredients that we need for our formula are certain sums and products of q-binomial coefficients.

In Section 4, we prove that for fixed rank, the coefficients of the length generating functions are periodic with period dividing the rank. This result gives another way to determine the generating functions by computing the finite initial sequence of coefficients until the periodicity takes over. We mention some further questions in Section 5.

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2 Background

In this section, we introduce the affine symmetric group, abacus diagrams for minimal length coset representatives, and q-binomial coefficients.

2.1 The affine symmetric group

We view the symmetric group S_n as the Coxeter group of type A with generating set $S = \{s_1, \ldots, s_{n-1}\}$ and relations of the form $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ together with $s_i s_j = s_j s_i$ for $|i - j| \geq 2$ and ${s_i}^2 = 1$. We denote $\bigcup_{n\geq 1} S_n$ by S_∞ and call $n(w)$ the minimal rank n of $w \in S_n \subset S_\infty$. The affine symmetric group $\widetilde{S}_n$ is also a Coxeter group; it is generated by $\widetilde{S} = S \cup \{s_0\}$ with the same relations as in the symmetric group together with ${s_0}^2 = 1$, $s_{n-1}s_0s_{n-1} = s_0s_{n-1}s_0$, $s_0s_1s_0 = s_1s_0s_1$, and $s_0s_j = s_js_0$ for $2 \leq j \leq n - 2$.

Recall that the products of generators from S or $\widetilde{S}$ with a minimal number of factors are called *reduced expressions*, and $\ell(w)$ is the length of such an expression for an (affine) permutation w. Given an (affine) permutation w, we represent reduced expressions for w in sans serif font, say $\mathsf{w} = \mathsf{w}_1\mathsf{w}_2 \cdots \mathsf{w}_p$ where each $\mathsf{w}_i \in S$ or $\widetilde{S}$. We call any expression of the form $s_i s_{i\pm 1} s_i$ a *short braid*, where the indices $i, i \pm 1$ are taken mod n if we are working in $\widetilde{S}_n$. We say that s_i is a *left descent* for $w \in \widetilde{S}_n$ if $\ell(s_i w) < \ell(w)$ and we say that s_i is a *right descent* for $w \in \widetilde{S}_n$ if $\ell(w s_i) < \ell(w)$.

As in Stembridge (1996), we define an equivalence relation on the set of reduced expressions for an (affine) permutation by saying that two reduced expressions are in the same *commutativity class* if one can be obtained from the other by a sequence of *commuting moves* of the form $s_i s_j \mapsto s_j s_i$ where $|i-j| \geq 2$. If the reduced expressions for a permutation w form a single commutativity class, then we say w is *fully commutative*.

We also refer to elements in the symmetric group by the *one-line notation* $w = [w_1 w_2 \cdots w_n]$, where w is the bijection mapping i to w_i. Then the generators s_i are the adjacent transpositions interchanging the entries i and $i + 1$ in the one-line notation. Let $w = [w_1 \cdots w_n]$, and suppose that $p = [p_1 \cdots p_k]$ is another permutation in S_k for $k \leq n$. We say w *contains the permutation pattern* p or w *contains* p *as a one-line pattern* whenever there exists a subsequence $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that

$$w_{i_a} < w_{i_b} \text{ if and only if } p_a < p_b$$

for all $1 \leq a < b \leq k$. We call $(i_1, i_2, \ldots, i_k)$ the *pattern instance*. For example, $[53241]$ contains the pattern $[321]$ in several ways, including the subsequence 541. If w does not contain the pattern p, we say that w *avoids* p.

The *affine symmetric group* $\widetilde{S}_n$ is realized as the group of bijections $w : \mathbb{Z} \to \mathbb{Z}$ satisfying $w(i+n) = w(i) + n$ and $\sum_{i=1}^{n} w(i) = \sum_{i=1}^{n} i = \binom{n+1}{2}$. (See for example Chapter 8 of Björner and Brenti (2005)) We call the infinite sequence

$$(\ldots, w(-1), w(0), w(1), w(2), \ldots, w(n), w(n+1), \ldots)$$

the *complete notation* for w and

$$[w(1), w(2), \ldots, w(n)]$$

the *base window* for w. By definition, the entries of the base window determine w and its complete notation. Moreover, the entries of the base window can be any set of integers that are normalized to sum to $\binom{n+1}{2}$ and such that the entries form a permutation of the residue classes in $\mathbb{Z}/(n\mathbb{Z})$ when reduced mod n. That is, no two entries of the base window have the same residue mod n. With these considerations in mind, we will represent an affine permutation using an abacus diagram together with a finite permutation.

To describe this, observe that S_n acts on the base window by permuting the entries, which induces an action of S_n on $\mathbb{Z}$. In this action, the Coxeter generator s_i simultaneously interchanges $w(i+kn)$ with $w(i+1+kn)$ for all $k \in \mathbb{Z}$. Moreover, the affine generator s_0 interchanges all $w(kn)$ with $w(kn+1)$. Hence, S_n is a parabolic subgroup of $\widetilde{S}_n$. We form the parabolic quotient

$$\widetilde{S}_n/S_n = \{w \in \widetilde{S}_n : \ell(ws_i) > \ell(w) \text{ for all } s_i \text{ where } 1 \leq i \leq n-1\}.$$

By a standard result in the theory of Coxeter groups, this set gives a unique representative of minimal length from each coset wS_n of $\widetilde{S}_n$. For more on this construction, see (Björner and Brenti, 2005, Section 2.4). In our case, the base window of the minimal length coset representative of an element is obtained by ordering the entries that appear in the base window increasingly. This construction implies that, as sets, the affine symmetric group can be identified with the set $(\widetilde{S}_n/S_n) \times S_n$. The minimal length coset representative determines which entries appear in the base window, and the finite permutation orders these entries in the base window.

We say that w has a *descent* at i whenever $w(i) > w(i+1)$. Note that if w has a descent at i, then $s_{(i \bmod n)}$ is a right descent in the usual Coxeter theoretic sense that $\ell(ws_i) < \ell(w)$.

2.2 Abacus diagrams

The abacus diagrams of James and Kerber (1981) give a combinatorial model for the minimal length coset representatives in $\widetilde{S}_n/S_n$. Other combinatorial models and references for these are given in Berg et al. (2009).

An *abacus diagram* is a diagram containing n columns labeled $1, 2, \ldots, n$, called *runners*. The horizontal rows are called *levels* and runner i contains entries labeled by $rn + i$ on each level r where $-\infty < r < \infty$. We draw the abacus so that each runner is vertical, oriented with $-\infty$ at the top and ∞ at the bottom, with runner 1 in the leftmost position, increasing to runner n in the rightmost position. Entries in the abacus diagram may be circled; such circled elements are called *beads*. Entries that are not circled are called *gaps*. The linear ordering of the entries given by the labels $rn + i$ is called the *reading order* of the abacus which corresponds to scanning left to right, top to bottom.

We associate an abacus to each minimal length coset representative $w \in \widetilde{S}_n/S_n$ by drawing beads down to level w_i in runner i for each $1 \leq i \leq n$ where $\{w_1, w_2, \ldots, w_n\}$ is the set of integers in the base window of w, with no two having the same residue mod n. Since the entries w_i sum to $\binom{n+1}{2}$, we call the abacus constructed in this way *balanced*. It follows from the construction that the Coxeter length of the minimal length coset representative can be determined from the abacus.

Proposition 2.1 *Let $w \in \widetilde{S}_n/S_n$ and form the abacus for w as described above. Let m_i denote the number of gaps preceding the lowest bead of runner i in reading order, for each $1 \leq i \leq n$. Then, the Coxeter length $\ell(w)$ is $\sum_{i=1}^{n} m_i$.*

Proof: This result is part of the folklore of the subject. One proof can be obtained by combining Propositions 3.2.5 and 3.2.8 of Berg et al. (2009). □

Example 2.2 *The affine permutation $\widetilde{w} = [-1, -4, 14, 1]$ is identified with the pair (w^0, w) where w is the finite permutation $s_1 s_3 = [2143]$ which sorts the elements of the minimal length coset representative $w^0 = [-4, -1, 1, 14]$. Note that the entries of w^0 sum to $\binom{5}{2} = 10$. The abacus of w^0 is shown below.*

−7	−6	−5	−4
−3	−2	−1	0
1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20

From the abacus, we see that w^0 has Coxeter length $1 + 10 + 0 + 0 = 11$. For example, the ten gaps preceding the lowest bead in runner 2 are $13, 12, 11, 9, 8, 7, 5, 4, 3,$ and 0. Hence, $\widetilde{w}$ has length $\ell(w^0) + \ell(w) = 13$.

In this work, we are primarily concerned with the fully commutative affine permutations. Green (2002) has given a criterion for these in terms of the complete notation for w. His result is a generalization of a theorem from Billey et al. (1993) which states that $w \in S_n$ is fully commutative if and only if w avoids [321] as a permutation pattern.

Theorem 2.3 *Green (2002) Let $w \in \widetilde{S}_n$. Then, w is fully commutative if and only if there do not exist integers $i < j < k$ such that $w(i) > w(j) > w(k)$.*

Observe that even though the entries in the base window of a minimal length coset representative are sorted, the element may not be fully commutative by Theorem 2.3. For example, if we write the element $w^0 = [-4, -1, 1, 14]$ in complete notation

$$w^0 = (\ldots, -8, -5, -3, \mathbf{10}, -4, -1, \mathbf{1}, 14, \mathbf{0}, 3, 5, 18, \ldots)$$

we obtain a [321]-instance as indicated in boldface.

In order to more easily exploit this phenomenon, we slightly modify the construction of the abacus. Observe that the length formula in Proposition 2.1 depends only on the relative positions of the beads in the abacus, and is unchanged if we shift every bead in the abacus exactly k positions to the right in reading order. Moreover, each time we shift the beads one unit to the right, we change the sum of the entries occurring on the lowest bead in each runner by exactly n. In fact, this shifting corresponds to shifting the base window inside the complete notation. Therefore, we may define an abacus in which all of the beads are shifted so that position $n+1$ becomes the first gap in reading order. We call such abaci *normalized*. Although the entries of the lowest beads in each runner will no longer sum to $\binom{n+1}{2}$, we can reverse the shifting to recover the balanced abacus. Hence, this process is a bijection on abaci, and we may assume from now on that our abaci are normalized.

Proposition 2.4 *Let A be a normalized abacus for $w^0 \in \widetilde{S}_n/S_n$, and suppose the last bead occurs at entry i. Then, w^0 is fully commutative if and only if the lowest beads on runners of A occur only in positions that are a subset of $\{1, 2, \ldots, n\} \cup \{i-n+1, i-n+2, \ldots, i\}$.*

We distinguish between two types of fully commutative elements through the position of the last bead in its normalized abacus A. If the last bead occurs in a position $i > 2n$, then we call the element a *long* element. Otherwise, the last bead occurs in a position $n \leq i \leq 2n$, and we call the element a *short* element. As evidenced in Section 3, the long fully commutative elements have a nice structure that allows for an elegant enumeration; the short elements lack this structure.

2.3 q-analogs of binomial coefficients

Calculations involving q-analogs of combinatorial objects often involve q-analogs of counting functions. Define $(a,q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(q)_n = (q,q)_n$. The *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}_q$ (also called the *Gaussian polynomial*) is a q-analog of the binomial coefficient $\binom{n}{k}$. To calculate a q-binomial coefficient directly, we use the formula

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q^k)(1-q^{k-1})\cdots(1-q^1)} = \frac{(q)_n}{(q)_k (q)_{n-k}}. \tag{1}$$

Just as with ordinary binomial coefficients, q-binomial coefficients have multiple combinatorial interpretations and satisfy many identities, a few of which are highlighted below.

Interpretation 1 *(Stanley, 1997, Proposition 1.3.17) Let M be the multiset $M = \{1^k, 2^{n-k}\}$. For an ordering π of the n elements of M, the number of inversions of π, denoted $\mathrm{inv}(\pi)$, is the number of instances of two entries i and j such that $i < j$ and $\pi(i) > \pi(j)$. Then $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_\pi q^{\mathrm{inv}(\pi)}$.*

Interpretation 2 *Let $\begin{bmatrix} n \\ k \end{bmatrix}$ be the set of subsets of $[n] = \{1, 2, \ldots, n\}$ of size k. Given $\mathcal{R} = \{r_1, \ldots, r_k\} \in \begin{bmatrix} n \\ k \end{bmatrix}$, define $|\mathcal{R}| = \sum_{j=1}^{k}(r_j - j)$. Then $\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{\mathcal{R} \in \begin{bmatrix} n \\ k \end{bmatrix}} q^{|\mathcal{R}|}$.*

Interpretation 1 is used most frequently in this article. Interpretation 2 is used in Section 3.1 when counting long fully commutative elements.

3 Decomposition and enumeration of fully commutative elements

Let S_n^{FC} denote the set of fully commutative permutations in S_n. In the following result, Barcucci et al. enumerate these elements by Coxeter length.

Theorem 3.1 *Barcucci et al. (2001) Let $C(x,q) = \sum_{n\geq 0}\sum_{w\in S_n^{FC}} x^n q^{\ell(w)}$. Then,*

$$C(x,q) = \frac{\sum_{n\geq 0}(-1)^n x^{n+1} q^{(n(n+3))/2}/(x,q)_{n+1}(q,q)_n}{\sum_{n\geq 0}(-1)^n x^n q^{(n(n+1))/2}/(x,q)_n(q,q)_n}$$

We enumerate the fully commutative elements $\widetilde{w} \in \widetilde{S}_n$ by identifying each as the product of its minimal length coset representative $w^0 \in \widetilde{S}_n/S_n$ and a finite permutation $w \in S_n$ as described in Section 2.2. Recall that we decompose the set of fully commutative elements into long and short elements. The elements with a short abacus structure break down into those where certain entries intertwine and those in which there is no intertwining. When we assemble these cases, we obtain our main theorem.

Theorem 3.2 *Let $\widetilde{S}_n^{FC}$ denote the set of fully commutative affine permutations in $\widetilde{S}_n$, and let $G(x,q) = \sum_{n\geq 0}\sum_{w\in \widetilde{S}_n^{FC}} x^n q^{\ell(w)}$, where $\ell(w)$ denotes the Coxeter length of w. Then,*

$$G(x,q) = \left(\sum_{n\geq 0}\frac{x^n q^n}{1-q^n}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2\right) + C(x,q) + \left(\sum_{R,L\geq 1} q^{R+L-1}\begin{bmatrix} L+R-2 \\ L-1 \end{bmatrix}_q S(x,q)\right),$$

where $C(x,q)$ is given by Theorem 3.1, and the component parts of $S(x,q) = S_I(x,q) + S_0(x,q) + S_1(x,q) + S_2(x,q)$ are given in Lemmas 3.8, 3.9, 3.10, and 3.11, respectively.

The first summand of $G(x,q)$ counts the long elements, while the remaining summands count the short elements. This theorem will be proved in Section 3.3 below.

3.1 Long elements

In this section, we enumerate the long elements. Recall that the last bead in the normalized abacus for these elements occurs in position $> 2n$.

Lemma 3.3 *For fixed $n \geq 0$, we have*

$$\sum_{w\in \widetilde{S}_n^{FC} \text{ such that } w \text{ is long}} q^{\ell(w)} = \frac{q^n}{1-q^n}\sum_{k=1}^{n-1}\begin{bmatrix} n \\ k \end{bmatrix}_q^2.$$

Proof: Fix a long fully commutative element, and define the set of *long runners* $\mathcal{R}$ of its normalized abacus A to be the set of runners $\{r_1,\dots,r_k\} \subset [n]\setminus\{1\}$ in which there exists a bead in position $n+r_j$ for $1 \leq j \leq k$. We will enumerate the long fully commutative elements by conditioning on $k = |\mathcal{R}|$, the size of the set of long runners of its normalized abacus. Note that by Proposition 2.4, all subsets $\mathcal{R} \subset [n]\setminus\{1\}$ are indeed the set of long runners for some fully commutative element.

For a fixed $\mathcal{R}$, there is an infinite family of abaci $\{A_i^{\mathcal{R}}\}_{i\geq 1}$, each having beads in positions $n+r_j$ for $r_j \in \mathcal{R}$, together with i additional beads that are placed sequentially in the long runners in positions larger than $2n$.

By Proposition 2.1, the Coxeter length of the minimal length coset representative $w^0 \in \widetilde{S}_n/S_n$ having $A_i^{\mathcal{R}}$ as its abacus is $i(n-k)+\sum_{j=1}^{k}(r_j-j)$. In addition, w^0 has base window $[aa\cdots abb\cdots b]$, where the

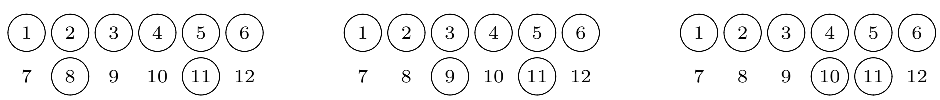
$(n-k)$ numbers a are all at most n, and the k numbers b are all at least $n+2$. The finite permutations w that can be applied to this standard window may not invert any of the larger numbers (b's) without creating a [321]-pattern with $n+1$ in the window following the standard window. Similarly, none of the a's can be inverted. All that remains is to intersperse the a's and the b's, keeping track of how many transpositions are used. This contributes exactly $\begin{bmatrix} n \\ k \end{bmatrix}_q$ to the Coxeter length, by Interpretation 1 of $\begin{bmatrix} n \\ k \end{bmatrix}_q$.

Algebraic simplification and application of Interpretation 2 gives the result. □

3.2 Short elements

The normalized abacus of every short fully commutative element has a particular structure. There must be a gap in position $n+1$, and for runners $2 \leq i \leq n$, the lowest bead is either in position i or $n+i$. In the following arguments, we will assign a status to each runner, depending on the position of the lowest bead in that runner.

Definition 3.4 *An* R-entry *is a bead lying in some position $> n$. Let $n+j$ be the position of the last* R*-entry, or set $j = n$ if there are no* R*-entries; an* M-entry *is a lowest bead lying in position i where $j+1 \leq i \leq n$. Note that it is possible that there do not exist any* M*-entries. The* L-entries *are the remaining lowest beads in position i for $i \leq j$. This assigns a status* left*,* middle*, or* right *to each entry of the base window, depending on the position of the lowest bead in the corresponding runner. We will call an abacus containing L* L*-entries, M* M*-entries, and R* R*-entries an $(L)(M)(R)$* abacus.



$\ell([1,3,4,6,8,11]) = 4$ $\quad$ $\ell([1,2,4,6,9,11]) = 5$ $\quad$ $\ell([1,2,3,6,10,11]) = 6$

Fig. 1: The three $(3)(1)(2)$ abaci and Coxeter length of their corresponding minimal length coset representatives.

Example 3.5 *Figure 1 shows the three* $(3)(1)(2)$ *abaci. In each case,* 6 *is the unique* M*-entry and* 11 *is an* R*-entry. In the first abacus, the* L*-entries are* $\{1,3,4\}$ *and the* R*-entries are* $\{8,11\}$.

The rationale for this assignment is that in the base window of a fully commutative element, not of type $(n)(0)(0)$, neither the L-entries nor the R-entries can have a descent amongst themselves, respectively. To see this, consider the contrary where two R-entries have a descent. These two entries, along with the $n+1$ entry in the window following the standard window, form a [321]-instance. Similarly, the last R-entry in the window previous to the standard window together with two L-entries that have a descent in the standard window would form a [321]-instance.

When the normalized abacus of a short fully commutative element has no R-entries (and therefore no M-entries), the base window for its minimal length coset representative is $[12 \cdots n]$. That is, the fully commutative elements of $\widetilde{S}_n$ having this abacus are in one-to-one correspondence with fully commutative elements of finite S_n. These elements have been enumerated in Theorem 3.1.

From now on, we only concern ourselves with $(L)(M)(R)$ abaci where $R > 0$. Proposition 3.6 proves that it is solely the parameters L, M, and R that determine the set of finite permutations that we can apply to the minimal length coset representative, and not the exact abacus. In Proposition 3.7 we determine the cumulative contribution to the Coxeter length of the minimal length coset representative from all $(L)(M)(R)$ abaci for fixed L, M, and R.

Proposition 3.6 *Let $w_1^0, w_2^0 \in \widetilde{S}_n/S_n$, each corresponding to an $(L)(M)(R)$ abacus for the same L, M, and R with $R > 0$. For any finite permutation $w \in S_n$, $w_1^0 w$ is fully commutative in $\widetilde{S}_n$ if and only if $w_2^0 w$ is fully commutative in $\widetilde{S}_n$.*

Proposition 3.7 *Let L, M and $R > 0$ be fixed. Then, we have*

$$\sum_w q^{\ell(w)} = q^{L+R-1} \begin{bmatrix} L+R-2 \\ L-1 \end{bmatrix}_q .$$

where the sum on the left is over all minimal length coset representatives w having an $(L)(M)(R)$ abacus.

Proof: Every $(L)(M)(R)$ abacus contains beads in all positions through n and in position $2n - M$ as well as gaps in position $n + 1$ and all positions starting with $2n - M + 1$. Depending on the positions of the $L - 1$ remaining gaps (and $R - 1$ remaining beads), the Coxeter length of the minimal length coset representative changes as illustrated by example in Figure 1.

The minimal length coset representative corresponding to an $(L)(M)(R)$ abacus having beads in positions i for $n + 2 \leq i \leq n + R$ together with a bead at position $2n - M$, and gaps in positions i for $n+R+1 \leq i \leq 2n-M-1$ has Coxeter length $L+R-1$. Notice that every time we move a bead from one of the positions between $n+2$ and $2n-M-1$ into a gap in the position directly to its right, the Coxeter length increases by exactly one. In essence, we are intertwining one sequence of length $L-1$ and one sequence of length $R-1$ and keeping track of the number of inversions we apply. By q-binomial Interpretation 1, the contribution to the Coxeter length of the minimal length coset representatives corresponding to the $(L)(M)(R)$ abaci is $q^{L+R-1}\left[\begin{smallmatrix} L+R-2 \\ L-1 \end{smallmatrix}\right]_q$. □

For the remaining arguments, we ignore the exact entries in the base window and simply fix both some positive number L of L-entries and some positive number R of R-entries, and then enumerate the permutations $w \in S_n$ that we can apply to a minimal length coset representative w^0 with base window of the form $[\text{L} \cdots \text{LM} \cdots \text{MR} \cdots \text{R}]$. In Theorem 3.2, we sum the contributions over all possible values of L and R.

3.2.1 Short elements with intertwining

One possibility is that after $w \in S_n$ is applied to our minimal length coset representative w^0 with base window of the form $[\text{L} \cdots \text{LM} \cdots \text{MR} \cdots \text{R}]$, an R-entry lies to the left of an L-entry. In this case, we say that w is *intertwining*, the L-entries are *intertwining* with the R-entries, and that the interval between the leftmost R and the rightmost L inclusive is the *intertwining zone*.

Lemma 3.8 *Fix L and $R > 0$. Then, we have $S_I(x,q) = \sum_w x^{n(w)} q^{\ell(w)} =$*

$$\sum_{M \geq 0} x^{L+M+R} \sum_{\rho=0}^{R-1} \sum_{\lambda=0}^{L-1} \sum_{\mu=0}^{M} q^Q \begin{bmatrix} M \\ \mu \end{bmatrix}_q \begin{bmatrix} L-\lambda-1+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} \lambda+\rho \\ \lambda \end{bmatrix}_q \begin{bmatrix} M-\mu+R-\rho-1 \\ M-\mu \end{bmatrix}_q ,$$

where the sum on the left is over all $w \in S_\infty$ that are intertwining and apply to a short $(L)(M)(R)$ abacus for some M, and $Q = (\lambda+1)(\mu+1) + (\rho+1)(M-\mu+1) - 1$.

3.2.2 Short elements without intertwining

If the L-entries and R-entries are not intertwined, there may be M-entries lying between the rightmost L and the leftmost R. There can be no descents in the M-entries to the left of the rightmost L nor to the right of the leftmost R by the same reasoning as above. However, multiple descents may now occur among the M-entries. We enumerate these short elements without intertwining by conditioning on the number of descents that occur among the M-entries. Lemmas 3.9, 3.10, and 3.11 enumerate the short elements in which there are zero, one, or two or more descents among the M-entries, respectively.

Lemma 3.9 *Fix L and $R > 0$. Then, we have*

$$S_0(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = \sum_{M \geq 0} x^{L+M+R} \sum_{\mu=0}^{M} q^{\mu} \begin{bmatrix} L-1+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q,$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have no descents among the M*-entries, and apply to a short $(L)(M)(R)$ abacus for some M.*

Proof: Let μ be the number of M-entries lying to the left of the rightmost L. Then, the μ M-entries can be intertwined with the remaining $(L-1)$ L-entries, and the remaining $(M-\mu)$ M-entries can be intertwined with the R R-entries.

We compute the Coxeter length offset by counting the inversions among the entries in the base window in the minimal length configuration of this type. In this case, there are simply μ M-entries that are inverted with the rightmost L. Summing over all valid values of μ gives the formula. □

A similar method of proof applies to the following Lemma.

Lemma 3.10 *Fix L and $R > 0$. Then, we have*

$$S_1(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = \sum_{M \geq 0} x^{L+M+R} \sum_{\mu=1}^{M-1} \left(\begin{bmatrix} M \\ \mu \end{bmatrix}_q - 1 \right) \begin{bmatrix} L+\mu \\ \mu \end{bmatrix}_q \begin{bmatrix} R+M-\mu \\ M-\mu \end{bmatrix}_q,$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have exactly one descent among the M*-entries, and apply to a short $(L)(M)(R)$ abacus for some M.*

The proof of our next result, relies on Lemma 2.3 of Bousquet-Mélou (1996) which solves certain functional recurrences.

Lemma 3.11 *Fix L and $R > 0$. Then, we have*

$$S_2(x,q) = \sum_{w} x^{n(w)} q^{\ell(w)} = x^{L+R} \sum_{i,j \geq 1} \begin{bmatrix} L+i \\ L \end{bmatrix}_q \begin{bmatrix} R+j \\ R \end{bmatrix}_q d_{i,j}(x,q),$$

where the sum on the left is over all $w \in S_\infty$ that are not intertwining, have at least two descents among the M*-entries, and apply to a short $(L)(M)(R)$ abacus for some M. Here, $d_{i,j}(x,q)$ is the coefficient of $z^i s^j$ in the generating function that satisfies the functional equation*

$$D(x,q,z,s) = \frac{\sum_{n \geq 0} x^{n+1} \sum_{i=1}^{n-1} \left(\begin{bmatrix} n \\ i \end{bmatrix}_q - 1 \right) z^i \big((qs) - (qs)^{n-i} \big)}{(1-qs)(1-xs)} + \frac{xqs\big(D(x,q,z,1) - D(x,q,z,qs)\big)}{(1-qs)(1-xs)}$$

A nonrecursive form for $D(x,q,z,1)$ is obtained in the proof of Lemma 3.11.

3.3 Proof of the main theorem

Proof of Theorem 3.2: Partition the set of fully commutative elements $\widetilde{w}$ into long elements and short elements. The long elements in $\widetilde{S}_n$ are enumerated by Lemma 3.3; we must sum over all n.

Each short element $\widetilde{w}$ has a normalized abacus of type $(L)(M)(R)$ for some L, M, and R. When this abacus is of type $(n)(0)(0)$ for some n, the base window for the corresponding minimal length coset representative is $[1\,2\,\ldots\,n]$. These elements $\widetilde{w} \in \widetilde{S}_n^{FC}$ are therefore in one-to-one correspondence with elements of S_n^{FC}. Therefore, the generating function $C(x,q)$ enumerates these elements for all n.

The elements that remain to be enumerated are short elements with normalized abacus of type $(L)(M)(R)$ for $R > 0$. We enumerate these elements by grouping these elements into families based on the values of L, M, and R. Decompose each element $\widetilde{w}$ into the product of its minimal length coset representative w^0 and a finite permutation w. Proposition 3.6 proves that for two minimal length coset representatives w_1^0 and w_2^0 of the same abacus type, the set of finite permutations w that multiply to form a fully commutative element is the same. Proposition 3.7 proves that in an $(L)(M)(R)$-family of fully commutative elements, the contribution to the length from the minimal length coset representatives is $q^{L+R-1}\begin{bmatrix} L+R-2 \\ L-1 \end{bmatrix}_q$. What remains to be determined is the generating function for the contributions of the finite permutations w.

In an $(L)(M)(R)$-family of fully commutative elements, the finite permutations w might intermingle the L entries and the R entries of the base window in which case there is at most one descent among the M entries at a prescribed position; the contribution of such w is given by S_I in Lemma 3.8. Otherwise, there is no intermingling and the finite permutations w may induce zero, one, or two or more descents among the M entries; these cases are enumerated by generating functions S_0, S_1, and S_2 in Lemmas 3.9, 3.10, and 3.11, respectively. In each of these lemmas, the values for L and R are held constant as M varies. Summing the product of the contributions of the minimal length coset representatives and the finite permutations over all possible values of L and R completes the enumeration. $\square$

4 Numerical Conclusions

Theorem 3.2 allows us to determine the length generating function $f_n(q)$ for the fully commutative elements of $\widetilde{S}_n$ as n varies. The first few series $f_n(q)$ are presented below.

$$\begin{aligned}
f_3(q) &= 1 + 3q + \mathbf{6q^2} + \mathbf{6q^3} + \mathbf{6q^4} + \cdots \\
f_4(q) &= 1 + 4q + 10q^2 + \mathbf{16q^3} + \mathbf{18q^4} + \mathbf{16q^5} + \mathbf{18q^6} + \cdots \\
f_5(q) &= 1 + 5q + 15q^2 + 30q^3 + 45q^4 + \mathbf{50q^5} + \mathbf{50q^6} + \mathbf{50q^7} + \mathbf{50q^8} + \mathbf{50q^9} + \cdots \\
f_6(q) &= 1 + 6q + 21q^2 + 50q^3 + 90q^4 + 126q^5 + 146q^6 + \mathbf{150q^7} + \mathbf{156q^8} + \mathbf{152q^9} + \\
&\quad \mathbf{156q^{10}} + \mathbf{150q^{11}} + \mathbf{158q^{12}} + \mathbf{150q^{13}} + \mathbf{156q^{14}} + \mathbf{152q^{15}} + \cdots \\
f_7(q) &= 1 + 7q + 28q^2 + 77q^3 + 161q^4 + 266q^5 + 364q^6 + 427q^7 + 462q^8 + 483q^9 + \mathbf{490q^{10}} + \\
&\quad \mathbf{490q^{11}} + \mathbf{490q^{12}} + \mathbf{490q^{13}} + \mathbf{490q^{14}} + \mathbf{490q^{15}} + \cdots \\
f_8(q) &= 1 + 8q + 36q^2 + 112q^3 + 266q^4 + 504q^5 + 792q^6 + 1064q^7 + 1274q^8 + 1416q^9 + \\
&\quad 1520q^{10} + 1568q^{11} + 1602q^{12} + \mathbf{1600q^{13}} + \mathbf{1616q^{14}} + \mathbf{1600q^{15}} + \mathbf{1618q^{16}} + \\
&\quad \mathbf{1600q^{17}} + \mathbf{1616q^{18}} + \mathbf{1600q^{19}} + \mathbf{1618q^{20}} + \cdots \\
f_9(q) &= 1 + 9q + 45q^2 + 156q^3 + 414q^4 + 882q^5 + 1563q^6 + 2367q^7 + 3159q^8 + 3831q^9 + \\
&\quad 4365q^{10} + 4770q^{11} + 5046q^{12} + 5220q^{13} + 5319q^{14} + 5370q^{15} + 5391q^{16} + \mathbf{5400q^{17}} + \\
&\quad \mathbf{5406q^{18}} + \mathbf{5400q^{19}} + \mathbf{5400q^{20}} + \mathbf{5406q^{21}} + \mathbf{5400q^{22}} + \mathbf{5400q^{23}} + \cdots
\end{aligned}$$

One remarkable quality of these series is their periodicity, given by the bold-faced terms. This behavior is explained by the following corollary to Lemma 3.3.

Corollary 4.1 *The coefficients a_i of $f_n(q) = \sum_{w \in \widetilde{S}_n^{FC}} q^{\ell(w)} = \sum_{i \geq 0} a_i q^i$ are periodic with period $m|n$ for sufficiently large i. When $n = p$ is prime, the period m is 1 and in this case there are precisely*

$$\frac{1}{p}\left(\binom{2p}{p} - 2\right)$$

fully commutative elements of length i in $\widetilde{S}_p$, when i is sufficiently large.

Proof: For a given n, the number of short fully commutative elements is finite. The formula for long elements in Lemma 3.3 is a polynomial divided by $1 - q^n$. Hence, the coefficients of this generating function satisfy $a_{i+n} = a_i$, by a fundamental result on rational generating functions.

When n is prime, the formula in Lemma 3.3 can be shown to be a polynomial divided by $1 - q$. $\square$

The distinction between long and short elements allows us to enumerate the fully-commutative elements efficiently. In some respects, this division is not the most natural in that the periodicity of the above series begins before there exist no more short elements. Experimentally, it appears that the periodicity begins at $1 + \lfloor (n-1)/2 \rfloor \lceil (n-1)/2 \rceil$; whereas, we can prove that the longest short element has length $2\lfloor n/2 \rfloor \lceil n/2 \rceil$.

We can bound the Coxeter length of finite fully commutative permutations.

Proposition 4.2 *Let $w \in \widetilde{S}_n^{FC}$ be a short element. Then $\ell(w) \leq 2\lfloor n/2 \rfloor \lceil n/2 \rceil$. In addition, there exists a $w \in \widetilde{S}_n^{FC}$ with length $2\lfloor n/2 \rfloor \lceil n/2 \rceil$.*

Corollary 4.1 and Proposition 4.2 give another way to compute the series $f_n(q)$, without invoking Theorem 3.2. Using a computer program, one needs simply to count the fully-commutative elements of $\widetilde{S}_n$ of length up to $n + 2\lfloor n/2 \rfloor \lceil n/2 \rceil$.

5 Further questions

In this work, we have studied the length generating function for the fully commutative affine permutations. It would be interesting to explore the ramifications of the periodic structure of these elements in terms of the affine Temperley–Lieb algebra. Also, all of our work should have natural extensions to the other Coxeter groups. In fact, we know of no analogue of Barcucci et al. (2001) enumerating the fully commutative elements by length for finite types beyond type A. It is a natural open problem to establish the periodicity of the length generating functions for the other affine types. It would also be interesting to determine the analogues for other types of the q-binomial coefficients and q-Bessel functions that played prominent roles in our enumerative formulas.

Finally, it remains an open problem to prove that the periodicity of the length generating function coefficients for fixed rank begins at length $1 + \lfloor (n-1)/2 \rfloor \lceil (n-1)/2 \rceil$, as indicated by the data. By examining the structure of the heap diagrams associated to the fully commutative affine permutations, we have discovered some plausible reasoning indicating this tighter bound, but a proof remains elusive.

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Meander Graphs

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Abstract. We consider a Markov chain Monte Carlo approach to the uniform sampling of meanders. Combinatorially, a meander $M = [A : B]$ is formed by two noncrossing perfect matchings, above A and below B the same endpoints, which form a single closed loop. We prove that meanders are connected under appropriate pairs of balanced local moves, one operating on A and the other on B. We also prove that the subset of meanders with a fixed B is connected under a suitable local move operating on an appropriately defined meandric triple in A. We provide diameter bounds under such moves, tight up to a (worst case) factor of two. The mixing times of the Markov chains remain open.

Résumé. Nous considérons une approche de Monte Carlo par chaîne de Markov pour l'échantillonnage uniforme des méandres. Combinatoirement, un méandre $M = [A : B]$ est constitué par deux couplages (matchings) parfaits sans intersection A et B, définis sur le même ensemble de points alignés, et qui forment une boucle fermée simple lorsqu'on dessine A "vers le haut" et B "vers le bas". Nous montrons que les méandres sont connectés sous l'action de paires appropriées de mouvements locaux équilibrés, l'un opérant sur A et l'autre sur B. Nous montrons également que le sous-ensemble de méandres avec un B fixe est connecté sous l'action de mouvements locaux définis sur des "triplets méandriques" de A. Nous fournissons des bornes sur les diamètres pour de tels mouvements, exactes à un facteur 2 près (dans le pire des cas). Les temps de mélange des chaînes de Markov demeurent une question ouverte.

Keywords: Markov chain Monte Carlo, combinatorial enumeration, noncrossing partitions, perfect matchings

1 Introduction

A closed meander of order n is a non-self-intersecting closed curve in the plane which crosses a horizontal line at $2n$ points, up to homeomorphisms in the plane. Meanders are easy to define and occur in a variety of mathematical settings, ranging from combinatorics to algebra, geometry, and topology to statistical physics and mathematical biology. Yet, despite this simplicity and ubiquity, how to enumerate meanders exactly is still unknown, and even sampling uniformly from this set is a tantalizing open problem.

The study of meanders is traceable back to Poincaré's work on differential geometry, and has subsequently arisen in different contexts such as the classification of 3-manifolds [KS91] and the Temperley-Lieb algebra [CJ03]. Meanders can be viewed combinatorially as suitable pairs of noncrossing partitions [Sav09, Sim00]. In Section 2, we give an equivalent combinatorial interpretation of meanders as two maximally different noncrossing perfect matchings under an appropriate local move operation, motivated in part by the biological "RNA folding" problem. Since meanders are an abstraction of polymer

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folding [Lun68, Tou50], the problems of counting and sampling meanders are of interest in more applied fields as well, such as statistical physics [DF00b] and mathematical biology [Hei].

Since the early 90's, techniques from statistical physics, such as [DF00a, DFGG00, Jen00, LZ93], have provided increasingly precise conjectures about the size of $\mathcal{M}_n$, the set of meanders of order n, and the formula $|\mathcal{M}_n| \approx \gamma\beta^n n^\alpha$ is believed to hold asymptotically. The most successful combinatorial attack on the enumeration problem [AP05] gives the current best bounds on the exponential growth rate β, obtained using the Goulden-Jackson cluster method for an appropriate meandric language. In terms of sampling, there have been Monte-Carlo approaches [Gol00] to producing a nearly uniform meander, however bounding the bias is a challenging statistical task.

The results given here are motivated by the uniform sampling problem. If a random walk on $\mathcal{M}_n$ (provably) converges rapidly to its stationary distribution, then the number of meanders of order n could be estimated via sampling methods [Jer03]. Hence, in Section 2, we prove that $\mathcal{M}_n$ is connected under suitable pairs of "balanced" local move operations on two noncrossing perfect matchings. These results yield a Markov chain Monte Carlo (MCMC) approach to sampling uniformly from $\mathcal{M}_n$.

The difficulty now lies in proving that the chain mixes rapidly, since analyzing this random walk is not directly amenable to the standard techniques [Jer03, MT06] of path coupling, canonical paths, or conductance. In this respect, meanders seem to resemble other combinatorial objects — such as contingency tables [DS98], Latin squares [JM96], or Eulerian tours [AK80] — where proofs of rapid mixing remain elusive. In these cases, the uniform sampling problem is regarded as hard, and local moves acting on these combinatorial objects are of interest as potential rapidly-mixing Markov chains.

Given this, in Section 3, we introduce a local move operation on the set of meanders with a fixed "bottom" B below the line. The central result in Theorem 6 states that such a subset of meanders is connected under our new "meandric triple" move. Hence, we define the meander graphs $\gamma(B)$, and investigate some structural characteristics in Section 4. The structure of $\gamma(B)$ clearly depends on B in some (as yet to be determined) way. Hence, these meander graphs may be of interest beyond the uniform sampling problem, since elucidating the dependencies on B might shed new light on the challenging exact enumeration problem.

2 Balancing local moves on meanders

We begin with $\mathrm{NC}(2n, \mathrm{match})$, the set of noncrossing perfect matchings of $2n$ points on a line, here often referred to simply as matchings. The points are labeled in increasing index order from 1 to $2n$, and the matching of point i with j for $1 \le i < j \le 2n$, referred to as the arc with endpoints i and j, will be denoted (i, j). If (i, j) is an arc in a (noncrossing perfect) matching, then $j - i$ is odd.

Although usually drawn above the line, we consider the single closed loop of a meander M to be a particular pair of matchings, A above and B below. Hence, let $\mathcal{A}_n$ denote $\mathrm{NC}(2n, \mathrm{match})$ with arcs above the line, respectively $\mathcal{B}_n$ with arcs below. Let $(A : B)$ denote the set of closed curves in the plane formed by drawing arbitrary $A \in \mathcal{A}_n$, $B \in \mathcal{B}_n$ on the same endpoints. In general, there are $1 \le k \le n$ closed loops in $(A : B)$, denoted by $c(A, B) = k$. When $c(A, B) = 1$, then the single closed curve $(A : B)$ is a meander. In this case, we say that A and B form a meander, or are meandric, and write

$$[A : B] = M \in \mathcal{M}_n.$$

Otherwise, the $k > 1$ loops of $(A : B)$ are called a system of meanders.

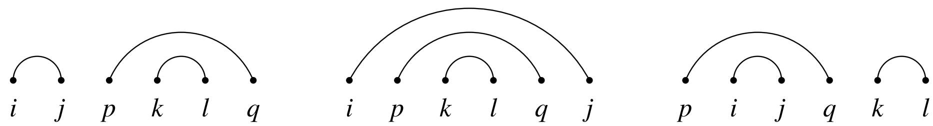

Fig. 1: The three cases where arcs (i, j) and (k, l) are obstructed by (p, q). Only the relevant arcs are drawn.

Noncrossing perfect matchings are a combinatorial model of the biological "RNA folding" problem. As such, it is natural to consider exchanging the matching between two different arcs, which corresponds to an alternate base pairing of the corresponding RNA helices. However, this "helix exchange" operation is well-defined on matchings only when there is no obstructing arc.

Let $(i, j), (k, l) \in A \in \mathcal{A}_n$ with $i < k$. Since there are no crossings $i < k < j < l$, then either

$$i < j < k < l \text{ or } i < k < l < j.$$

As illustrated in Figure 1, the arcs (i, j) and (k, l) are *obstructed* if there is a third arc $(p, q) \in A$ with

$$\text{either } i < j < p < k < l < q \text{ or } i < p < k < l < q < j \text{ or } p < i < j < q < k < l.$$

Otherwise, they are unobstructed. Note that (i, j) and (k, l) are necessarily unobstructed if there exists $a \in \{i, j\}, b \in \{k, l\}$ with $|a - b| = 1 \pmod{2n}$.

Let $P = \{(i, j), (k, l)\}$ be a pair of unobstructed arcs from A. In this case, define a *matching exchange* on $P \subset A$ as the (reversible) local move operation given by

$$\sigma_P(A) = \begin{cases} (A \setminus P) \cup \{(i, k), (l, j)\} & \text{if } i < k < l < j \\ (A \setminus P) \cup \{(i, l), (j, k)\} & \text{if } i < j < k < l \end{cases} .$$

The operation is not defined on obstructed arcs since a crossing would be introduced. Adopting the familial terminology from rooted trees, a matching exchange on $i < k < l < j$ converts "parent" and "child" arcs into two "siblings," and vice versa on $i < j < k < l$. Figure 2 on page 472 illustrates one of each kind. The explicit subscript P may be suppressed for notational simplicity in some circumstances.

This operation is analogous to previously considered local moves on chord diagrams [MT99] and plane trees [Hei]. Via the former, it is known to give a rapidly-mixing Markov chain on $\mathrm{NC}(2n, \mathrm{match})$. The connected graph induced on $\mathrm{NC}(2n, \mathrm{match})$ by this operation is connected and has diameter $n - 1$. Moreover, two matchings A, B form a meander exactly when the geodesic path between them is diameter achieving.

More generally, the partial order on $\mathrm{NC}(2n, \mathrm{match})$ induced by this operation is isomorphic to the lattice of noncrossing partitions under refinement, denoted $\mathrm{NC}(n)$. Two matchings are diameter achieving if and only if their corresponding partitions are complements in the lattice. With only the exceptions from Example 1 below, each $B \in B_n$ has at least two distinct $A, A' \in \mathcal{A}_n$, known as its Kreweras complements [Kre72], such that $[A : B], [A' : B] \in \mathcal{M}_n$. Of course, there are frequently many more.

Example 1 *For* $1 \le i \le n$*, let* $U_n = \{(2i-1, 2i)\}$ *and* $L_n = \{(1, 2n), (2i, 2i+1)\}$*. They correspond to the top and bottom elements in* $\mathrm{NC}(n)$*, and hence form exactly two meanders:* $[U_n : L_n]$ *and* $[L_n : U_n]$.

We will now define a local move operation on $M = [A : B] \in \mathcal{M}_n$ by operating on suitable pairs of unobstructed arcs $P \subset A$, $Q \subset B$ so that $[\sigma_P(A) : \sigma_Q(B)]$ is again a meander. Since this is not true for arbitrary P and Q, we first consider a matching exchange's effect on any system of meanders $(A : B)$.

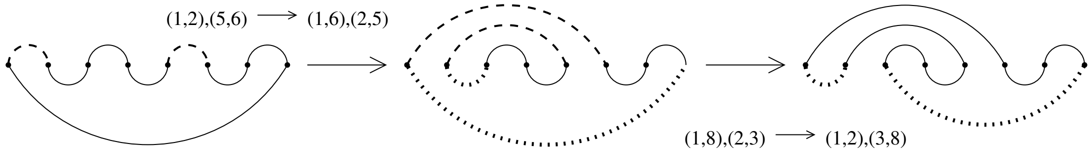

Fig. 2: Moving from the meander $[U_4 : L_4]$ with the balanced matching exchanges described in Example 2.

Lemma 1 *Let $A \in \mathcal{A}_n, B \in \mathcal{B}_n$. Then $|c(A,B) - c(\sigma(A), B)| = 1$.*

Proof: Suppose $A \setminus \sigma(A) = \{(i,j),(i',j')\}$. If the unobstructed arcs $(i,j),(i',j')$ lie on the same curve from $(A : B)$, then $c(\sigma(A), B) = c(A,B) + 1$. Otherwise, $c(\sigma(A), B) = c(A,B) - 1$. □

By symmetry, the result holds for $\sigma(B)$. Since a single exchange $\sigma(A)$ breaks the meander $[A : B]$, we show in Theorem 1 below that there always exists a compensating exchange $\sigma(B)$ which rejoins the two closed loops. Any pair of matching exchanges on A and B will be called *balanced* whenever $c(\sigma(A), \sigma(B)) = 1$. Thus, we can move between meanders connected by balanced pairs of local moves.

To prove this, we introduce another notation for (a system of) meanders. Recall that $j - i$ is odd for $(i,j) \in A$. If i is odd, denote this arc as $i \overset{A}{\rightharpoonup} j$ and as $j \overset{A}{\rightharpoonup} i$ otherwise. Similarly, but with reversed parity, every $(2i, 2j-1) \in B$ is written as $2i \overset{B}{\rightharpoondown} 2j-1$ and $(2j-1, 2i) \in B$ as $2i \overset{B}{\rightharpoondown} 2j-1$. In this way, any system of meanders is can be written as a set of ordered, alternating sequences of arcs from A and B. Typically, we drop the A and B designation and simply write a meander (single closed loop) as:

$$1 \rightharpoonup 2i_1 \rightharpoondown 2j_2 - 1 \rightharpoonup 2i_2 \rightharpoondown \ldots \rightharpoondown 2j_n - 1 \rightharpoonup 2i_n \rightharpoondown 1.$$

Theorem 1 *Let $M = [A : B] \in \mathcal{M}_n$. For every pair of unobstructed arcs P in A there exists a pair of unobstructed arcs Q in B such that $c(\sigma_P(A), \sigma_Q(B)) = 1$.*

Proof: Suppose $i \rightharpoonup j$ and $i' \rightharpoonup j'$ are two unobstructed arcs from A. It suffices to show there exist unobstructed arcs $k \rightharpoondown l$ and $k' \rightharpoondown l'$ in B which occur in the sequence of ordered, alternating arcs as:

$$i \rightharpoonup j \ldots k \rightharpoondown l \ldots i' \rightharpoonup j' \ldots k' \rightharpoondown l' \ldots \rightharpoondown i.$$

Let S be the set of integers in $j \rightharpoondown \ldots \rightharpoondown i'$, respectively S' in $j' \rightharpoondown \ldots \rightharpoondown i$. Then S and S' are a partition of the integers $\{1, 2, \ldots, 2n\}$ and, without loss of generality, there exists $k \in S$ and $l' \in S'$ such that $|k - l'| = 1 \pmod{2n}$. Thus the arcs $k \rightharpoondown l$ and $k' \rightharpoondown l'$ are unobstructed in B. □

We make the previous discussion concrete by considering the following example, pictured in Figure 2.

Example 2 *The meander $[U_4 : L_4]$ is $1 \rightharpoonup 2 \rightharpoondown 3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 1$. A matching exchange on $P = \{(1,2),(5,6)\}$ in U_4 results in two closed loops in $(\sigma_P(U_4) : L_4)$: $1 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 1$ and $3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 2 \rightharpoondown 3$. There exist a compensating exchange on $Q = \{(1,8),(2,3)\}$ in L_4 which yields the meander $[\sigma_P(U_4) : \sigma_Q(L_4)]$: $1 \rightharpoonup 6 \rightharpoondown 7 \rightharpoonup 8 \rightharpoondown 3 \rightharpoonup 4 \rightharpoondown 5 \rightharpoonup 2 \rightharpoondown 1$.*

The proof of Theorem 1 guarantees at least one compensating exchange $\sigma(B)$ for each $\sigma(A)$. In general, there may be many balanced pairs of exchanges for a meander $[A : B]$. For instance, there are three other matching exchanges on B which would rejoin the closed loops of $(\sigma_P(U_4) : L_4)$ in Figure 2.

Definition 1 *Let $\mathcal{G}_n$ be the graph whose vertices are $M, M' \in \mathcal{M}_n$ and whose edges connect meanders $M = [A : B]$ and $M' = [\sigma(A) : \sigma(B)]$ with balanced pairs of matching exchanges.*

Theorem 2 *The graph $\mathcal{G}_n$ is connected.*

Proof: The only meander with arcs U_n above the line is $[U_n : L_n]$. For $[A : B] \in \mathcal{M}_n$, there exists a sequence of matching exchanges on A such that $\sigma(\ldots\sigma(A)) = U_n$. By Theorem 1, for each local move on the upper arcs, there is a compensating matching exchange on the bottom. $\square$

Alternatively, Theorem 2 follows from the connection between meanders and pairs of noncrossing partitions, see [Fra98, FE02] as well as [Hal06, Sav09]. In that context, the graph $\mathcal{G}_n$ is the Hasse diagram of the induced partial order.

Theorem 2 suggests a natural ergodic Markov chain on the state space $\mathcal{M}_n$ with transition probability matrix $\mathbb{P}$. We will define $\mathbb{P}(M, M')$ to be positive if there is a pair of balanced matching exchanges connecting $M, M' \in \mathcal{M}_n$. It is also technically convenient to assume the self-loop probability is positive; $\mathbb{P}(M, M) > 0$ for every $M \in \mathcal{M}_n$. Both sets of probabilities are specified (implicitly) below.

The fact that $\mathcal{G}_n$ is connected implies that such a Markov chain is ergodic; for every pair of states, there is a time by which the probability of visiting one state from the other is positive. The self-loop probability further guarantees aperiodicity; a high enough power of $\mathbb{P}$ has *all* entries positive, which in turn implies that the Markov chain converges to its so-called stationary distribution on $\mathcal{M}_n$. Finally, recall that a *symmetric Markov chain* has the uniform distribution as its stationary distribution. This suggests specifying the off-diagonal transition probabilities so as to make $\mathbb{P}$ symmetric.

One fairly standard way in MCMC methods of achieving a symmetric chain is to consider the so-called *maximum-degree random walk*. Let $\Delta(\mathcal{G}_n)$ be the maximum vertex degree in the meander graph $\mathcal{G}_n$. We have $\Delta(\mathcal{G}_n) = \Theta(n^4)$, based on a degree of $n^2(n^2 - 1)/12$ for $[U_n : L_n]$ and the naive bound of $\binom{n}{2}^2$ on all pairs of two arcs. Defining $\mathbb{P}(M, M') := 1/\Delta(\mathcal{G}_n)$ for every adjacent pair M, M', and $\mathbb{P}(M, M) := 1 - \sum_{M' \neq M} \mathbb{P}(M, M')$, makes $\mathbb{P}$ symmetric and (row, hence column) stochastic.

There are several other ways to define $\mathbb{P}$ so that it is row and column stochastic, which is sufficient to guarantee uniformity of stationary probabilities. However, the seemingly challenging open question we raise here is whether the above Markov chain (or an analogous one) is "rapidly mixing" on $\mathcal{M}_n$? In other words, irrespective of the starting state at time $t = 0$, is the first time the chain is within $1/4$ (say) in total variation distance of the uniform distribution at most polynomial in $\log|\mathcal{M}_n|$?

A second question in the same vein would be to sample uniformly from the subset of meanders with a fixed bottom B. Our main result (Theorem 6 below) provides, once again, a natural way to define an appropriate Markov chain which converges to the correct (uniform) distribution. However, the rate of mixing of this "meandric triple" chain also remains open.

3 Graphing meandric triple moves

Since the matching exchange operation gives a rapidly-mixing Markov chain [MT99] on $\mathrm{NC}(2n, \mathrm{match})$, one direction of attack on the problem of analyzing the mixing time of the Markov chain on $\mathcal{G}_n$ restricts to analyzing a random walk on the set of meanders $[A : B]$ with a fixed $B \in \mathcal{B}_n$.

We introduce such a random walk as follows. We prove that the two closed loops in $(\sigma(A) : B)$ can be rejoined by a move on $\sigma(A)$ when the exchange operation is applied twice to an appropriate triple of arcs in A. This yields a new "meandric triple" move where $[A : B] \to [\sigma(\sigma(A)) : B]$, which provably connects the subset of meanders with fixed B. Hence, we get meander graphs $\gamma(B)$ with differing structures depending on $B \in \mathcal{B}_n$. The study of these meander graphs may well be of interest beyond the uniform sampling problem, since elucidating the (still unknown) dependencies on B might shed new light on the challenging exact enumeration problem.

Two arbitrary exchange operations on A do not yield another meander since $c(\sigma(\sigma(A)), B)$ may be either 1 or 3. Hence, we begin by defining an appropriate triple of arcs in A on which to act.

Definition 2 *Let $i \rightharpoonup j$, $k \rightharpoonup l$, and $q \rightharpoonup p$ be three arcs from $A \in \mathcal{A}_n$. They are a meandric triple if exactly two of the three pairs of arcs are unobstructed.*

Figure 1 on page 471 illustrates the three possible configurations for a meandric triple, assuming no other obstructing arcs. Observe the equivalence of the configurations under the endpoint operations of *rotation*, that is $i \to i-1 \pmod{2n}$, and *reversal*, that is $i \to 2n+1-i$. We will use the fact that these operations preserve the single closed loop forming a meander in subsequent arguments.

Excepting only U_n and L_n, any matching $A \in A_n$ with $n \geq 3$ contains at least one meandric triple. The maximum in any A, and hence the maximum degree over all meander graphs $\gamma(B)$ defined below, is $\Theta(n^2)$. The naive bound of $\binom{n}{3}$ reduces to $\mathcal{O}(n^2)$ by observing that any meandric triple has a unique "youngest" arc. Hence, there are only $\mathcal{O}(n)$ such parent/child combinations, with an additional factor of n for the third arc (either grandparent or parent's sibling). This is best possible, over all bottom arcs $B \in \mathcal{B}_n$, since there is a matching $A \in \mathcal{A}_n$ with $\lceil (n-1)/2 \rceil \lfloor (n-1)/2 \rfloor$ meandric triples.

Theorem 3 *Let $M = [A : B] \in \mathcal{M}_n$. There exists a sequence of two matching exchange operations on a meandric triple in A such that $c(\sigma(\sigma(A)), B) = 1$.*

Proof: Let $i \rightharpoonup j$, $k \rightharpoonup l$, and $q \rightharpoonup p$ be a meandric triple from A where $i \rightharpoonup j$ and $q \rightharpoonup p$ are unobstructed, $q \rightharpoonup p$ and $k \rightharpoonup l$ are unobstructed, and

$$i \rightharpoonup j \overbrace{\ldots}^{R} q \rightharpoonup p \underbrace{\ldots}_{S} k \rightharpoonup l \overbrace{\ldots}^{T}.$$

The six different cases for the ordering of $i, j, \ldots$ along the horizontal line are equivalent under rotation and reversal. Hence, suppose that

$$i < p < k < l < q < j.$$

By assumption, the exchange $\sigma_P(A)$ is defined for $P = \{(p,q), (k,l)\}$. Furthermore, (p,k), (l,q) and (i,j) are all unobstructed in $\sigma_P(A)$. The two closed loops of $\sigma_P(A)$ are rejoined into the meander

$$i \rightharpoonup p \underbrace{\ldots}_{S} k \rightharpoonup j \overbrace{\ldots}^{R} q \rightharpoonup l \overbrace{\ldots}^{T}$$

by operating on $Q = \{(i,j), (p,k)\}$ which results in $(i,p), (k,j) \in \sigma_Q(\sigma_P(A))$. □

Definition 3 *Let $i \rightharpoonup j$, $k \rightharpoonup l$, and $q \rightharpoonup p$ be a meandric triple in A for $M = [A : B] \in \mathcal{M}_n$ with $i \rightharpoonup j \ldots q \rightharpoonup p \ldots k \rightharpoonup l \ldots$. Define a meandric move on M, denoted $\tau(M) = [\sigma(\sigma(A)) : B]$, as the pair of matching exchanges which replaces the meandric triple in A with $i \rightharpoonup p$, $k \rightharpoonup j$, and $q \rightharpoonup l$.*

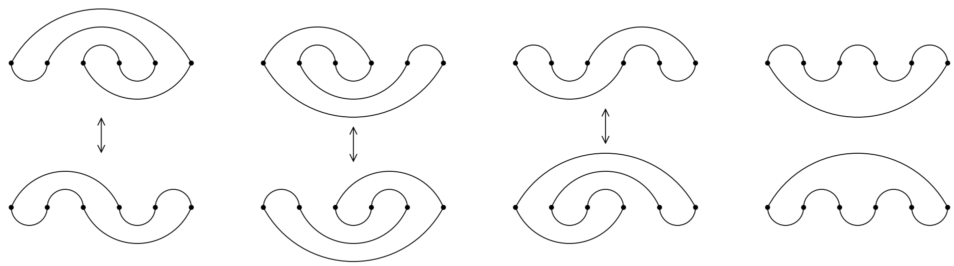

Fig. 3: The eight meanders from $\mathcal{M}_3$ with the three connections under a meandric move.

Figure 3 illustrates that, for a given meandric triple in A, there is exactly one way of preserving a single closed loop while exchanging the matchings among the six endpoints. Also note the equivalence under rotation and reversal, as well as the isolated points $[U_n : L_n]$, $[L_n : U_n]$.

Definition 4 *Let $\gamma(B)$ be the graph with vertices $M = [A : B] \in \mathcal{M}_n$ and edges connecting M, $\tau(M)$.*

As stated in Theorem 6 below, the graph $\gamma(B)$ is connected for any $B \in \mathcal{B}_n$. The proof is by induction, and follows from Theorems 4 and 5 and related definitions.

Observe that there are at least two (i, j) with $|j - i| = 1 \pmod{2n}$ in any noncrossing perfect matching, and the two connecting arcs on the other side of such a "bump" in a meander are necessarily unobstructed.

Definition 5 *Let β_t be the arc $(t, t+1)$ for $1 \leq t < 2n$ and $\beta_{2n} = (1, 2n)$.*

Theorem 4 *For $M = [A : B] \in \mathcal{M}_n$ with $\beta_t \in B$, $\beta_{t-1 \pmod{2n}} \notin A$, there exists a meandric move $\tau(M) = [A' : B]$ such that $\beta_{t-1 \pmod{2n}} \in A'$.*

Proof: Assume without loss of generality that $(2n-1, 2n) \in B$, $(2n-2, 2n-1) \notin A$. We claim there is a meandric move $\tau(M) = [A' : B]$ such that $(2n-2, 2n-1) \in A'$. The arcs

$$(i, 2n), (p, 2n-1), (k, 2n-2) \in A \text{ with } 1 \leq i < p < k < 2n-2 < 2n-1 < 2n$$

are a meandric triple with

$$i \rightharpoonup 2n \rightharpoondown 2n-1 \rightharpoonup p \ldots k \rightharpoonup 2n-2 \ldots.$$

A meandric move on these arcs yields $(i, p), (k, 2n), (2n-1, 2n) \in A' = \sigma(\sigma(A))$. $\square$

There is an immediate dual result for $\beta_{t+1 \pmod{2n}} \notin A$ under reversals. Next, Definition 6 makes precise the intuitive notion of contracting a bump β_{2n} from A and the two connecting arcs in B to produce a reduced meander of order $n-1$. In other words, the meander with arcs

$$i \rightharpoondown 1 \rightharpoonup 2n \rightharpoondown j \rightharpoonup \overbrace{\cdots}^{R} \rightharpoonup i \text{ reduces to } i \rightharpoondown j \rightharpoonup \overbrace{\cdots}^{S} \rightharpoonup i,$$

where the remaining arcs in R stay the same in S except for relabeling the endpoints to account for the removal of the arc $(1, 2n)$ from A and the replacement of arcs $(1, i), (j, 2n)$ in B with (i, j).

Definition 6 *For $M = [A : B] \in \mathcal{M}_n$ and $\beta_{2n} \in A$, define $\rho(M, 2n)$ to be $[A' : B'] \in \mathcal{M}_{n-1}$ with*

$$(i, j) \in B' \quad \textit{for} \quad (1, i), (j, 2n) \in B \quad \textit{and} \quad 1 < i < j < 2n$$

and, for $X = A, B$, with

$$(k, l) \in X' \quad \textit{for} \quad (k + 1, l + 1) \in X \quad \textit{and} \quad 1 < k < l < 2n.$$

For $\beta_t \in A$ with $1 \leq t < 2n$, the definition of $\rho(M, t)$ is fundamentally the same, although a precise statement of the replacement and relabeling is more complicated. If $\beta_t \notin A$, then $\rho(M, t)$ is not defined.

Theorem 5 *The operation $\rho : \mathcal{M}_n \times \{1, 2, \ldots, 2n\} \to \mathcal{M}_{n-1}$ is well-defined.*

Proof: Suppose without loss of generality that $(1, 2n) \in A$. The arcs of A and B form the meander M:

$$i \rightharpoondown 1 \rightharpoonup 2n \rightharpoondown j \rightharpoonup \overbrace{\ldots}^{R} \rightharpoonup i.$$

Consider the exchange $\sigma_P(B)$ on $P = \{(1, i), (j, 2n)\}$. Then $(A : \sigma_P(B))$ has two closed loops:

$$1 \rightharpoonup 2n \rightharpoondown 1 \text{ and } i \rightharpoondown j \rightharpoonup \overbrace{\ldots}^{R} \rightharpoonup i.$$

Under the appropriate endpoint relabeling, the second closed loop is the meander $\rho(M, 2n)$. $\square$

Theorem 6 *For $B \in \mathcal{B}_n$, the graph $\gamma(B)$ is connected.*

Proof: Consider $M = [A : B]$ and $N = [C : B]$ for $B \in \mathcal{B}_n$ with $n > 3$. Let $\beta_t \in B$. Suppose that $\beta_s \notin A \cup C$ for $s = t - 1 \pmod{2n}$. By Theorem 4, there exist meandric moves $\tau(M) = [A' : B] = M'$ and $\tau(N) = [C' : B] = N'$ such that $\beta_s \in A' \cap C'$.

Observe that β_s obstructs no arcs in either A' or C'. By induction, $\rho(M', s)$ and $\rho(N', s)$ are connected by a sequence of meandric moves. Hence, there exists a sequence of meandric moves on the $n - 1$ upper arcs of M' leaving the arc $\beta_s \in A' \cap C'$ fixed and connecting M' to N' in $\gamma(B)$. $\square$

4 Some characteristics of meander graphs

The proof of Theorem 6 implies that the diameter of $\gamma(B)$ is at most $2n$ for $B \in \mathcal{B}_n$. This upper bound is never achieved since for $3 \leq n \leq 8$, the maximum diameter of $\gamma(B)$ is $n - 2$.

Example 3 *When $n = 9$, there is one (nonisomorphic) pair of meanders $[A : B]$ and $[A' : B]$ whose geodesic has 8 meandric moves in $\gamma(B)$:*

$$\begin{array}{rcl} B & = & \{(1,10),(2,9),(3,8),(4,5),(6,7),(11,18),(12,17),(13,14),(15,16)\} \\ A & = & \{(1,16),(2,15),(3,14),(4,13),(5,12),(6,11),(7,10),(8,9),(17,18)\} \\ A' & = & \{(1,4),(2,3),(5,18),(6,17),(7,16),(8,15),(9,14),(10,13),(11,12).\} \end{array}$$

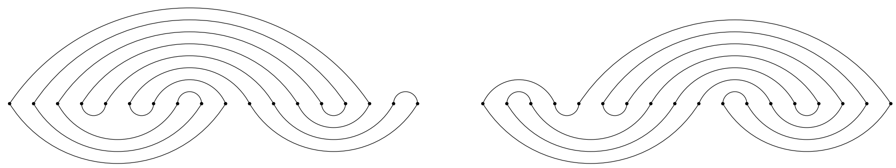

Fig. 4: The smallest instance of meanders $[A : B]$ and $[A' : B]$ with geodesic length $\neq n-2$ from Example 3.

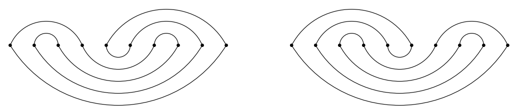

Fig. 5: The smallest instance of interlocking meanders $[A : B]$ and $[A' : B]$ from Example 4.

This is the only pair of meanders, up to rotation and reversal, whose geodesic has length greater than $n-2$ for $n = 9$. When $n = 10$, there are three nonisomorphic pairs with length 9.

We contrast this with $\mathcal{G}_n$ which inherits a diameter of $n-1$ from $\mathrm{NC}(2n, \mathrm{match})$ under matching exchange. If $\beta_t \notin A$, then there is always an exchange such that $\beta_t \in \sigma(A)$. This is not the case for meandric moves; the smallest example is the following.

Example 4 *When $n = 5$, there is one (nonisomorphic) pair of meanders $M = [A : B]$ and $M' = [A' : B]$ such that for every $\beta_t \in A$ there exists no $\tau(M') = [A'' : B]$ with $\beta_t \in A''$, and vice versa:*

$$\begin{array}{rcl} B & = & \{(1,10),(2,9),(3,8),(4,7),(5,6)\} \\ A & = & \{(1,4),(2,3),(5,10),(6,9),(7,8)\} \\ A' & = & \{(1,6),(2,5),(3,4),(7,10),(8,9).\} \end{array}$$

We say that such a pair of meanders is interlocking. There are no interlocking pairs when $n = 6$, eight when $n = 7$, seven when $n = 8$, and 198 when $n = 9$.

Yet, any interlocking pair is still connected in $\gamma(B)$. Hence, for $\beta_t \in A'$ and $\beta_t \notin A$, there is always a sequence of meandric moves $\tau(\ldots\tau(M)) = [A^* : B]$ such that $(t, t+1) \in A^*$.

Theorem 7 *Let $B \in \mathcal{B}_n$ and $\beta_t \notin B$. Then there exists $M = [A : B]$ such that $\beta_t \in A$.*

Proof: The proof essentially inverts the map ρ in Definition 6. Assume $t = 2n$. Let B' be the arcs with

$$\begin{array}{llll} (i,j) \in B' & \text{for} & (1,i),(j,2n) \in B & \text{and} \quad 1 < i < j < 2n \\ (k,l) \in B' & \text{for} & (k+1,l+1) \in B & \text{and} \quad 1 < k < l < 2n. \end{array}$$

Then $B' \in \mathcal{B}_{n-1}$ and there exists $A' \in \mathcal{A}_{n-1}$ such that $[A' : B'] \in \mathcal{M}_{n-1}$. Let A be the set of arcs with

$$(k,l) \in A \quad \text{for} \quad (k-1,l-1) \in A' \quad \text{and} \quad 1 < k < l < 2n.$$

Then by construction $[A : B] \in \mathcal{M}_n$. □

Consequently, for every $\beta_t \notin B$, there is a subgraph of $\gamma(B)$ isomorphic to $\gamma(B')$ as in the proof of Theorem 7. By the proof of Theorem 4, every $M \in \gamma(B)$ is at most distance one from the subgraphs containing $\rho(M, t-1 \pmod{2n})$ and $\rho(M, t+1 \pmod{2n})$ for each $\beta_t \in B$.

We also have the following result. Although it is an immediate corollary to Theorems 6 and 7, we give here a constructive proof to illustrate some of the challenges in working with meandric triples.

Theorem 8 *Let $M = [A : B] \in \mathcal{M}_n$. For $\beta_t \notin B$, there exists a sequence of meandric moves $\tau(\dots\tau(M)) = [A^* : B]$ such that $\beta_t \in A^*$.*

Proof: Assume t is odd. For $k, l \neq t+1$ and $k', l' \neq t$, the meander M has arcs

$$k \rightharpoondown t \rightharpoonup \overbrace{l \dots l'}^{R} \rightharpoonup t+1 \rightharpoondown \overbrace{k' \dots k}^{R'}.$$

Since $\beta_t \notin B$, there is at least one arc from A in the sequence of arcs R'. Suppose there exists $i \rightharpoonup j$ in R' which forms a meandric triple with $t \rightharpoonup l$, and $l' \rightharpoonup t+1$. Then $(t, t+1) \in \tau(M)$.

If not, then consider $i \rightharpoonup j$ from R' having d arcs from R which obstruct it from forming a meandric triple with $t \rightharpoonup l$, $l' \rightharpoonup t+1$. If $d > 2$, then a meandric move on three of the d arcs, which must be a meandric triple, yields $\tau(M)$ which now has $d-2$ obstructing arcs. Hence, the relevant cases are when there are 1 or 2 obstructing arcs.

The three cases for a linear ordering of the points from $t \rightharpoonup l$ and $l' \rightharpoonup t+1$ are equivalent under rotations and reversals. Suppose that $t < t+1 < l' < l$. The endpoints $1, \dots, 2n$ are divided into three sets by the two arcs: $S_1 = \{i \mid 1 \leq i < t,\ l < i \leq 2n\}$, $S_2 = \{i \mid t+1 < i < l'\}$, and $S_3 = \{i \mid l' < i < l\}$. Then i, j and the endpoints of the obstructing arcs must all be in one of the three sets. Moreover, the case when they lie in S_1 is equivalent to S_3.

Suppose there is a single obstructing arc $a \rightharpoonup b$:

$$t \rightharpoonup l \overbrace{\dots a \rightharpoonup b \dots}^{R} l' \rightharpoonup t+1 \overbrace{\dots i \rightharpoonup j \dots}^{R'}.$$

We explicitly consider the two situations when either

$$a < j < i < b < t < t+1 < l' < l \text{ or } t < t+1 < l' < b < i < j < a < l.$$

In the second case when the arcs lie in S_2, operating on M by a meandric move on $i \rightharpoonup j$, $a \rightharpoonup b$, and $t \rightharpoonup l$ followed by a move on the new meandric triple $i \rightharpoonup l$, $t \rightharpoonup b$, $l' \rightharpoonup t+1$ results in $\beta_t \in \tau(\tau(M))$. We claim the first case, when the arcs lie in S_1, results in a contradiction.

Consider $n = 4$. Then the closed loop would be

$$t \rightharpoonup l \rightharpoondown a \rightharpoonup b \rightharpoondown l' \rightharpoonup t+1 \rightharpoondown i \rightharpoonup j \rightharpoondown t.$$

However, it is not possible to have the three arcs $t+1 \rightharpoondown i$, $j \rightharpoondown t$ and $b \rightharpoondown l'$ lying below the horizontal line without intersections. Suppose $n > 4$ and there is a meander $M \in \mathcal{M}_n$ containing the arrangement of four arcs. There exists an additional arc $i'' \rightharpoonup j''$ where $|i'' - j''| = 1$. Without loss of generality, $j'' = i'' + 1$ and $\rho(M, i'')$ has $n-1$ arcs. Inductively, though, the arcs in $\rho(M, i'')$ corresponding to $t+1 \rightharpoondown i$, $j \rightharpoondown t$ and $b \rightharpoondown l'$ intersect.

Suppose now that there are two obstructing arcs $a \rightharpoonup b$, $a' \rightharpoonup b'$ between $i \rightharpoonup j$ and $t \rightharpoonup l$, $l' \rightharpoonup t+1$. There are two distinct orderings for a, b and a', b' along the horizontal line with respect to the other arcs. When the obstructing arcs lie in S_1, one ordering results in a contradiction like the one above while the other yields $\beta_t \in \tau(\tau(M))$. When the obstructing arcs lie in S_2, then both orderings result in a contradiction. $\square$

5 Concluding remarks

The present work raises several interesting directions for further study. For instance, it would be very helpful to have an appropriate statistic to measure how close a given meander is to a random one. Such a statistic would offer ways to measure the "autocorrelation time" of the Markov chain as well as help in bounding its approach to equilibrium. Simulating the Markov chains proposed here and observing "random" meanders after a large number of steps might be one way to come up with some interesting statistics on meanders. Given the simplicity of the chains, this should be a relatively straight-forward task, which we hope to undertake in the near future.

There are also unanswered questions about the structure of meander graphs $\mathcal{G}_n$ and $\gamma(B)$ for $B \in \mathcal{B}_n$. For instance, we give a tight upper bound of $\mathcal{O}(n^4)$ on the maximum vertex degree in $\mathcal{G}_n$, but have not yet fully investigated the amount of variation in vertex degrees over the graph. Similarly, our bounds on the diameter and maximum vertex degree hold for all $\gamma(B)$ over $B \in \mathcal{B}_n$. Yet, it is clear from the graphs for small n that these characteristics depend in some unknown way on the particular B.

Finally, this MCMC approach to counting and sampling closed meanders extends to other types as well. In particular, semi-meanders of order n are in bijection with the subset of closed meanders of order n with the "rainbow" matching on the bottom: $[A : R]$ where $R = \{(i, 2n - i + 1) \mid 1 \leq i \leq n\}$. Likewise, there is a correspondence between closed meanders and open meanders of odd order, and a many-to-one mapping from closed meanders to open meander of even order.

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The short toric polynomial

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Abstract. We introduce the short toric polynomial associated to a graded Eulerian poset. This polynomial contains the same information as Stanley's pair of toric polynomials, but allows different algebraic manipulations. Stanley's intertwined recurrence may be replaced by a single recurrence, in which the degree of the discarded terms is independent of the rank. A short toric variant of the formula by Bayer and Ehrenborg, expressing the toric h-vector in terms of the cd-index, may be stated in a rank-independent form, and it may be shown using weighted lattice path enumeration and the reflection principle. We use our techniques to derive a formula expressing the toric h-vector of a dual simplicial Eulerian poset in terms of its f-vector. This formula implies Gessel's formula for the toric h-vector of a cube, and may be used to prove that the nonnegativity of the toric h-vector of a simple polytope is a consequence of the Generalized Lower Bound Theorem holding for simplicial polytopes.

Résumé. Nous introduisons le polynôme torique court associé à un ensemble ordonné Eulérien. Ce polynôme contient la même information que le couple de polynômes toriques de Stanley, mais il permet des manipulations algébriques différentes. La récurrence entrecroisée de Stanley peut être remplacée par une seule récurrence dans laquelle le degré des termes écartés est indépendant du rang. La variante torique courte de la formule de Bayer et Ehrenborg, qui exprime le vecteur torique d'un ensemble ordonné Eulérien en termes de son cd-index, est énoncée sous une forme qui ne dépend pas du rang et qui peut être démontrée en utilisant une énumération des chemins pondérés et le principe de réflexion. Nous utilisons nos techniques pour dériver une formule exprimant le vecteur h-torique d'un ensemble ordonné Eulérien dont le dual est simplicial, en termes de son f-vecteur. Cette formule implique la formule de Gessel pour le vecteur h-torique d'un cube, et elle peut être utilisée pour démontrer que la positivité du vecteur h-torique d'un polytope simple est une conséquence du Théorème de la Borne Inférieure Généralisé appliqué aux polytopes simpliciaux.

Keywords: Eulerian poset, toric h-vector, Narayana numbers, reflection principle, Morgan-Voyce polynomial

Introduction

We often look for a "magic" simplification that makes known results easier to state, and provides the language to state new results. For Eulerian posets such a wonderful simplification was the introduction of the cd-index by Fine (see [6]) allowing to restate the Bayer-Billera formulas [2] in a simpler form and to formulate Stanley's famous nonnegativity conjecture [18], shown many years later by Karu [14].

A similar "magic" moment has yet to arrive regarding the toric polynomials $f(P,x)$ and $g(P,x)$ associated to an Eulerian poset $\widehat{P} = P \uplus \{\widehat{1}\}$ by Stanley [19]. The new invariant proposed here may not be the desired "magic simplification" yet, but it represents an improvement in some cases. The idea on which it is based is very simple and widely useful. There is a bijective way to associate each *multiplicatively symmetric* polynomial $p(x)$ (satisfying $p(x) = x^{\deg(p(x))}p(1/x)$) to an *additively symmetric polynomial* $q(x)$ (satisfying $q(-x) = (-1)^{\deg(q(x))}q(x)$) of the same degree, having the same set of coefficients (see Section 2). There is no change when we want to extract the coefficients of the individual polynomials only, but when we consider a sequence $\{p_n(x)\}_{n\geq 0}$ of multiplicatively symmetric polynomials, given by some rule, switching to the additively symmetric variant $\{q_n(x)\}_{n\geq 0}$ greatly changes the appearance of the rules, making them sometimes easier to manipulate.

The short toric polynomial $\mathrm{t}(P,x)$, associated to a graded Eulerian poset $\widehat{P}$ is defined in Section 3 as the multiplicatively symmetric variant of $f(P,x)$. The intertwined recurrence defining $f(P,x)$ and $g(P,x)$ is equivalent to a single recurrence for $\mathrm{t}(P,x)$. It is a tempting thought to use this recurrence to generalize the short toric polynomial to all ranked posets having a unique minimum element, even if in the cases of some lower Eulerian posets, "severe loss of information" may occur. We state the short toric variant of Fine's formula (see [1] and [3, Theorem 7.14]) expressing the toric h-vector in terms of the flag f-vector. By inspecting this formula, it is easy to observe that the generalization of $\mathrm{t}(P,x)$ makes most sense when the reduced Euler characteristic of the order complex of $P \setminus \{\widehat{0}\}$ is not zero.

Arguably our nicest result is Theorem 4.6, expressing $\mathrm{t}(P,x)$ associated to a graded Eulerian poset $\widehat{P}$ by defining two linear operators $\mathcal{C}, \mathcal{D} : \mathbb{Q}[x] \to \mathbb{Q}[x]$ that need to be substituted into the reverse of the cd-index and applied to the constant polynomial 1. The fact that the toric h-vector may be computed by replacing the letters c and d in the reverse of the cd-index by some linear operators and applying the resulting linear operator to a specific vector is a direct consequence of the famous result by Bayer and Ehrenborg [3, Theorem 4.2], expressing the toric h-vector in terms of the cd-index. In applications, the use of this result may be facilitated by finding a linearly equivalent presentation that is easier to manipulate. Our Theorem 4.6 is analogous to Lee's result [15, Theorem 5] and it is the first result offering a rank-independent substitution rule. Theorem 4.2, the reason behind Theorem 4.6, also implies the short toric variant of [3, Theorem 4.2], and has a proof using weighted lattice path enumeration and the reflection principle. The use of weighted lattice paths is already present in the work Bayer and Ehrenborg [3, Section 7.4]. By finding the cd-index via calculating the ce-index first, and by using the short toric form, the applicability of the reflection principle becomes apparent.

Theorem 4.6 highlights the importance of the sequence of polynomials $\{\widetilde{Q}_n(x)\}_{n\geq 0}$, a variant of the sequence $\{Q_n(x)\}_{n\geq 0}$ in [3]. In Section 5 we take a closer look at this sequence, alongside the sequence of short toric polynomials $\{t_n(x)\}_{n\geq 0}$ associated to Boolean algebras. The polynomials $\{\widetilde{Q}_n(x)\}_{n\geq 0}$ turn out to be the dual basis to the *Morgan-Voyce polynomials*, whereas the polynomials $\{t_n(x)\}_{n\geq 0}$ may be used to provide a simple formula connecting $\mathrm{t}(P,x)$ to $g(P,x)$.

An application showing the usefulness of our invariant may be found in Section 6, where we express the toric h-vector of an Eulerian dual simplicial poset in terms of its f-vector. This question was raised by Kalai, see [19]. Besides using Theorem 4.6, the proof of the formula depends on a formula conjectured by Stanley [18, Conjecture 3.1] and shown in [10, Theorem 2], expressing the contribution of the h-vector entries of an Eulerian simplicial poset to its cd-index as weights of certain André permutations. An equivalent form of our formula implies that the nonnegativity of the toric h-vector of simple polytope is an elementary consequence of the Generalized Lower Bound Theorem (GLBT) holding for simplicial polytopes [20]. The word elementary has to be stressed since Karu [13] has shown that the GLBT holds for all polytopes.

1 Preliminaries

A poset P is *graded* if it has a unique minimal element $\widehat{0}$, a unique maximal element $\widehat{1}$ and a *rank function* $\rho : P \rightarrow \mathbb{N}$ satisfying $\rho(\widehat{0}) = 0$ and $\rho(y) = \rho(x) + 1$ whenever y covers x. The rank of P is $\rho(\widehat{1})$. The *flag f-vector* of a graded poset P of rank $n+1$ is $(f_S \ : \ S \subseteq [1,n])$ where $f_S = f_S(P)$ is the number of maximal chains in the set $P_S := \{u \in P \ : \ \rho(u) \in S\}$. A graded poset is *Eulerian* if every interval $[u,v] \subseteq P$ with $u < v$ satisfies $\sum_{z\in[u,v]}(-1)^{\rho(z)} = 0$. All linear relations satisfied by the flag f-vector of an Eulerian poset were given by Bayer and Billera [2]. It was observed by Fine and proved by Bayer and Klapper [6] that the Bayer-Billera relations may be restated as the existence of the cd*-index*, as follows. Introducing the *flag h-vector* $(h_S \ : \ S \subseteq [1,n])$ of a graded poset of rank $(n+1)$ by setting $h_S := \sum_{T\subseteq S}(-1)^{|S|-|T|} f_T$, we define the ab*-index* as the polynomial $\Psi_P(a,b) = \sum_{S\subseteq[1,n]} h_S u_S$ in noncommuting variables a and b where the letter u_i in $u_S = u_1 \cdots u_n$ is a if $i \notin S$ and b if $i \in S$. The ab-index of an Eulerian poset is then a polynomial of $c = a + b$ and $d = ab + ba$. This polynomial $\Phi_P(c,d)$ is the cd*-index* of P. The existence of the cd index is equivalent to stating that the ce*-index*, obtained by rewriting the ab-index as a polynomial of $c = a+b$ and $e = a-b$, is a polynomial of c and e^2, see [18]. Let us denote by L_S the coefficient of the ce word $v_1 \cdots v_n$, where S is the set of indices i such that $v_i = e$. It was shown in [4] that the resulting *flag L-vector* $(L_S \ : \ S \subseteq [1,n])$ of a graded poset of rank $(n+1)$ is connected to the flag f-vector by the formulas

$$L_S = (-1)^{n-|S|} \sum_{T\supseteq[1,n]\setminus S} \left(-\frac{1}{2}\right)^{|T|} f_T \quad \text{and} \quad f_S = 2^{|S|} \sum_{T\subseteq[1,n]\setminus S} L_T. \tag{1}$$

The Bayer-Billera relations are thus also equivalent to stating that, for an Eulerian poset, $L_S = 0$ unless S is an *even set*, i.e., a disjoint union of intervals of even cardinality, see [5].

The *toric h-vector* associated to a graded Eulerian poset $[\widehat{0},\widehat{1}]$ was defined by Stanley [19] by introducing the polynomials $f([\widehat{0},\widehat{1}),x)$ and $g([\widehat{0},\widehat{1}),x)$ by the intertwined recurrences

$$f([\widehat{0},\widehat{1}),x) = \sum_{t\in[\widehat{0},\widehat{1})} g([0,p),x)(x-1)^{\rho(\widehat{1})-1-\rho(t)} \quad \text{and} \tag{2}$$

$$g([\widehat{0},\widehat{1}),x) = \sum_{i=0}^{\lfloor(\rho(\widehat{1})-1)/2\rfloor} ([x^i]f([\widehat{0},\widehat{1}),x) - [x^{i-1}]f([\widehat{0},\widehat{1}),x))x^i \tag{3}$$

and by the initial condition $f(\emptyset, x) = g(\emptyset, x) = 1$. Here the operator $[x^i]$ extracts the coefficient of x^i. The toric h-vector associated to $[\widehat{0}, \widehat{1})$ is then the vector of coefficients of the polynomial $x^{\rho(\widehat{1})-1} f([\widehat{0}, \widehat{1}), 1/x)$. The first formula expressing $f(P, x)$ in terms of the flag f-vector was found by Fine (see [1] and [3, Theorem 7.14]). Here we state it in an equivalent form that appears in the paper of Bayer and Ehrenborg [3, Section 7]:

$$f([\widehat{0}, \widehat{1}), x) = \sum_{S \subseteq [1,n]} f_S \sum_{\lambda \in \{-1,1\}^n : S(\lambda) \supseteq S} (-1)^{|S|+n-i_\lambda} x^{i_\lambda}, \tag{4}$$

where $S(\lambda) = \{s \in \{1, \ldots, n\} : \lambda_1 + \cdots + \lambda_s > 0\}$ and i_λ is the number of -1's in λ. Bayer and Ehrenborg [3, Theorem 4.2] also expressed the toric h-vector of an Eulerian poset in terms of its cd-index.

2 Additive and multiplicative symmetry of polynomials

Definition 2.1 *Let K be a field. We say that a polynomial $p(x) \in K[x]$ of degree n is* multiplicatively symmetric *if $x^n p(x^{-1}) = p(x)$ and it is* additively symmetric *if $p(x) = (-1)^n p(-x)$.*

Theorem 2.2 *A polynomial $p(x) \in K[x]$ of degree n is multiplicatively symmetric if and only if there is an additively symmetric polynomial $q(x) \in K[x]$ of degree n satisfying*

$$p(x) = x^{\frac{n}{2}} \left(q(\sqrt{x}) + q\left(\frac{1}{\sqrt{x}}\right) - q(0) \right). \tag{5}$$

Moreover, $q(x)$ is uniquely determined.

Definition 2.3 *Given a multiplicatively symmetric polynomial $p(x)$ we call the* additively symmetric variant of $p(x)$ *the additively symmetric polynomial $q(x)$ associated to $p(x)$ via* (5)*. Conversely, given an additively symmetric polynomial $q(x)$ we call the* multiplicatively symmetric variant of $q(x)$ *the polynomial $p(x)$ defined by* (5)*.*

To express the additively symmetric variant of a multiplicatively symmetric polynomial, we may use the following *truncation operators*.

Definition 2.4 *Let K be a fixed field and $K[x, x^{-1}]$ the ring of Laurent polynomials. For any $z \in \mathbb{Z}$, the* truncation operator $U_{\geq z} : K[x, x^{-1}] \to K[x, x^{-1}]$ *is the linear operator defined by discarding all terms of degree less than z. Similarly $U_{\leq z} : K[x, x^{-1}] \to K[x, x^{-1}]$ is defined by discarding all terms of degree more than z.*

The notation $U_{\geq z}$ and $U_{\leq z}$ is consistent with the notation used in [3], where (3) is rewritten as

$$g([\widehat{0}, \widehat{1}), x) = U_{\leq \lfloor n/2 \rfloor}((1 - x) f([\widehat{0}, \widehat{1}), x). \tag{6}$$

Lemma 2.5 *Let $p(x)$ be a multiplicatively symmetric polynomial of degree n. Then the additively symmetric variant $q(x)$ of $p(x)$ satisfies*

$$q(x) = U_{\geq 0}(x^{-n} p(x^2)) = U_{\geq 0}(x^n p(x^{-2})).$$

3 The short toric polynomial of an arbitrary graded poset

Stanley's generalization [19, Theorem 2.4] of the Dehn-Sommerville equations may be restated as follows.

Theorem 3.1 (Stanley) *For an Eulerian poset $[\widehat{0}, \widehat{1}]$ of rank $n+1$, the polynomial $f([\widehat{0}, \widehat{1}), x)$ is multiplicatively symmetric of degree n.*

Definition 3.2 *The short toric polynomial $\mathrm{t}([\widehat{0}, \widehat{1}), x)$ associated to an Eulerian poset $[\widehat{0}, \widehat{1}]$ is the additively symmetric variant of the toric polynomial $f([\widehat{0}, \widehat{1}), x)$.*

Note that the interval $[\widehat{0}, \widehat{1})$ is half open and $f(\emptyset, x) = 1$ implies $\mathrm{t}(\emptyset, x) = 1$.

Lemma 3.3 *If $[\widehat{0}, \widehat{1}]$ is an Eulerian poset of rank $n+1$ then we have*

$$U_{\geq 1}\left(\mathrm{t}([\widehat{0}, \widehat{1}), x) \cdot \left(x - \frac{1}{x}\right)\right) = x^{n+1} g\left([\widehat{0}, \widehat{1}), x^{-2}\right).$$

Using Lemmas 2.5 and 3.3 we may show the following *fundamental recurrence*.

Theorem 3.4 *The short toric polynomial satisfies the recurrence*

$$\mathrm{t}([\widehat{0}, \widehat{1}), x) = U_{\geq 0}\left((x^{-1} - x)^{\rho(\widehat{1})-1} + \sum_{\widehat{0}<p<\widehat{1}} U_{\geq 1}\left(\mathrm{t}([\widehat{0}, p), x)(x - x^{-1})\right)(x^{-1} - x)^{\rho(\widehat{1})-\rho(p)-1}\right).$$

We may use this fundamental recurrence to extend the definition of $\mathrm{t}([\widehat{0}, \widehat{1}), x)$ to all finite posets P having a unique minimal element $\widehat{0}$ and a rank function. This generalization is not equivalent to Stanley's generalization of $f(P, x)$ to *lower Eulerian posets* in [19, §4]. Recall that a finite poset is lower Eulerian if it has a unique minimal element $\widehat{0}$ and, for each $p \in P$, the interval $[\widehat{0}, p]$ is an Eulerian poset.

Proposition 3.5 *Let P be a lower Eulerian poset and let n be the length of the longest chain in P. Then $\mathrm{t}(P, x) = U_{\geq 0}(x^n f(P, x^{-2}))$.*

Remark 3.6 For a lower Eulerian poset P, the polynomial $\mathrm{t}(P,x)$ may not contain sufficient information to recover $f(P,x)$. If $P=[\widehat{0},\widehat{1}]$ is a graded Eulerian poset of rank $n+1$ then, using [19, (19)] and Proposition 3.5, we can show $\mathrm{t}([\widehat{0},\widehat{1}],x)=0$, yet $f([\widehat{0},\widehat{1}],x)$ is usually not zero.

Proposition 3.7 (Fine's formula)

$$\mathrm{t}(P,x)=\sum_{S\subseteq[1,n]} f_S(P)\cdot \sum_{\lambda\in\{-1,1\}^n\,:\,S(\lambda)\supseteq S, n-2i_\lambda\geq 0} (-1)^{n-i_\lambda+|S|}x^{n-2i_\lambda} \tag{7}$$

holds for all finite posets P having a unique minimal element $\widehat{0}$ and a rank function $\rho: P\to\mathbb{N}$, satisfying $\rho(\widehat{0})=0$ and $n=\max\{\rho(p)\,:\,p\in P\}$. Here, for any $S\subseteq[1,n]$, $f_S=f_S(P)$ is the number of maximal chains in $P_S=\{u\in P\,:\,\rho(u)\in S\}$.

The statement may be shown in a very similar fashion to Fine's original formula. The role of equations (2) and (3) is taken over by the single recurrence given in Theorem 3.4.

Corollary 3.8 *The degree of* $\mathrm{t}(P,x)$ *equals* $\max\{\rho(p)\,:\,p\in P\}$ *if and only if* $\sum_{S\subseteq[1,n]}(-1)^{|S|}f_S(P)\neq 0$.

Note that $\sum_{S\subseteq[1,n]}(-1)^{|S|}f_S(P)$ is the reduced Euler characteristic of the *order complex* of $P\setminus\{\widehat{0}\}$.

4 The short toric polynomial and the cd-index of an Eulerian poset

Using (1) and the binomial theorem we may rewrite (7) as

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\sum_{T\subseteq[1,n]} L_S \sum_{\lambda\in\{-1,1\}^n\,:\,n-2i_\lambda\geq 0} x^{n-2i_\lambda}(-1)^{n-i_\lambda+|S(\lambda)\setminus S|}. \tag{8}$$

Just like in [3, Section 7.4], we represent each $\lambda\in\{-1,1\}^n$ by a lattice path starting at $(0,0)$ and containing $(1,\lambda_i)$ as step i for $i=1,\ldots,n$. The condition $n-2i_\lambda\geq 0$ restricts us to lattice paths whose right endpoint is on or above the horizontal axis. We introduce $R(\lambda):=\{i\in[1,n]\,:\,\lambda_1+\cdots+\lambda_i=0\}$ and say that a set S *evenly contains* the set R if $R\subseteq S$ and $S\setminus R$ is the disjoint union of intervals of even cardinality. We may use the "reflection principle" to match canceling terms, and obtain the following.

Theorem 4.1 *Let $[\widehat{0},\widehat{1}]$ be a graded Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\sum_{S\subseteq[1,n]} L_S\cdot \mathrm{t}_{ce}(S,x).$$

Here $\mathrm{t}_{ce}(S,x)$ is the total weight of all $\lambda\in\{-1,1\}^n$ such that S evenly contains $R(\lambda)\cup(R(\lambda)-1)$ and $\lambda_1+\cdots+\lambda_i\geq 0$ holds for all $i\in\{1,\ldots,n\}$. The weight of each such λ is defined as follows: each $\lambda_i=-1$ contributes a factor of $-1/x$, each $\lambda_i=1$ contributes a factor of x and each element of $R(\lambda)$ contributes an additional factor of 2.

Theorem 4.1 gains an even simpler form when we rephrase it in terms of the cd-index.

Theorem 4.2 *Let $[\widehat{0},\widehat{1}]$ be a graded Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\sum_{w}[w]\Phi_{[\widehat{0},\widehat{1}]}(c,d)\cdot \mathrm{t}(w,x).$$

Here the summation runs over all cd-words w of degree n. The polynomial $\mathrm{t}(w,x)$ is the total weight of all $\lambda\in\{-1,1\}^n$ such that the set of positions covered by letters d equals $R(\lambda)\cup(R(\lambda)-1)$ and $\lambda_1+\cdots+\lambda_i\geq 0$ holds for all $i\in\{1,\ldots,n\}$. The weight of each such λ is defined as follows: each $\lambda_i=-1$ contributes a factor of $-1/x$, each $\lambda_i=1$ contributes a factor of x, and each element of $R(\lambda)$ contributes an additional factor of -1.

Theorem 4.2 allows us to explicitly compute the contribution $\mathrm{t}(w,x)$. Thus we obtain the short toric equivalent of [3, Theorem 4.2], expressing $f([\widehat{0},\widehat{1}),x)$ in terms of the cd-index.

Proposition 4.3 *The polynomial $\mathrm{t}(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k},x)$ is zero if at least one of k_1, k_2, …, k_r is odd. If k_1, k_2, …, k_r are all even then*

$$t(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k},x)=(-1)^{\frac{k_1+\cdots+k_r}{2}}C_{\frac{k_1}{2}}\cdots C_{\frac{k_r}{2}}\widetilde{Q}_k(x).$$

Here $C_k=\frac{1}{k+1}\binom{2k}{k}$ is a Catalan number, and the polynomials $\widetilde{Q}_n(x)$ are given by $\widetilde{Q}_0(x)=1$ and

$$\widetilde{Q}_n(x):=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}(-1)^k\left(\binom{n-1}{k}-\binom{n-1}{k-1}\right)x^{n-2k}\quad \textit{for } n\geq 1.$$

Remark 4.4 The polynomials $\widetilde{Q}_n(x)$ are closely related to the polynomials $Q_n(x)$ introduced by Bayer and Ehrenborg [3]. They may be given by $\widetilde{Q}_n(x)=x^nQ_n(x^{-2})$.

Theorem 4.2 also allows us to introduce two linear maps $\mathcal{C}:\mathbb{Q}[x]\to\mathbb{Q}[x]$ and $\mathcal{D}:\mathbb{Q}[x]\to\mathbb{Q}[x]$ in such a way that, for any graded Eulerian poset $[\widehat{0},\widehat{1}]$, the polynomial $\mathrm{t}([\widehat{0},\widehat{1}),x)$ may be computed by substituting $\mathcal{C}$ into c and $\mathcal{D}$ into d in the *reverse* of $\Phi_P(c,d)$ and applying the resulting linear operator to 1. Note that the definitions and the result below are *independent of the rank of P*.

Definition 4.5 *We define $\mathcal{C}:\mathbb{Q}[x]\to\mathbb{Q}[x]$ by setting $\mathcal{C}(1)=x$, $\mathcal{C}(x)=x^2$ and $\mathcal{C}(x^n)=x^{n+1}-x^{n-1}$ for $n\geq 2$. We define $\mathcal{D}:\mathbb{Q}[x]\to\mathbb{Q}[x]$ by setting $\mathcal{D}(1)=1$, $\mathcal{D}(x^2)=-1$ and $\mathcal{D}(x^n)=0$ for $n\notin\{0,2\}$.*

Theorem 4.6 *For any Eulerian poset $P=[\widehat{0},\widehat{1}]$ we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x)=\Phi_P^{\mathrm{rev}}(\mathcal{C},\mathcal{D})(1)$$

Here $\Phi_P^{\mathrm{rev}}(\mathcal{C},\mathcal{D})$ is obtained from $\Phi_P(c,d)$ by first taking the reverse of each cd-monomial and then replacing each c with $\mathcal{C}$ and each d with $\mathcal{D}$.

Proof: By Theorem 4.2, we only need to show that

$$\mathrm{t}(c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k},x)=\mathcal{C}^{k}\,\mathcal{D}\,\mathcal{C}^{k_r}\,\mathcal{D}\,\mathcal{C}^{k_{r-1}}\,\mathcal{D}\cdots\mathcal{D}\,\mathcal{C}^{k_1}(1) \tag{9}$$

holds for any cd-word $w=c^{k_1}dc^{k_2}\cdots c^{k_r}dc^{k}$. This may be shown by induction on the degree of w, the basis being $\mathrm{t}(\varepsilon,t)=1$ where ε is the empty word. □

5 Two useful bases

Proposition 4.3 highlights the importance of the basis $\{\widetilde{Q}_n(x)\}_{n\geq 0}$ of the vector space $\mathbb{Q}[x]$. In this section we express the elements of the basis $\{x^n\}_{n\geq 0}$, as well as the operators $\mathcal{C}$ and $\mathcal{D}$, in this new basis. We also find the analogous results for the basis $\{t_n(x)\}_{n\geq 0}$ where $t_n(x)=\mathrm{t}(B_{n+1},x)$ for the Boolean algebra $\widehat{B_{n+1}}$ of rank $n+1$. This basis is useful in proving the main result of Section 6, as well as in finding a very simple formula connecting $\mathrm{t}(P,x)$ with $g(P,x)$.

Proposition 5.1 *For $n>0$ we have*

$$x^n=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}\widetilde{Q}_{n-2k}(x).$$

We may rewrite Proposition 5.1 as $x^{2n}=\sum_{k=1}^{n}\binom{n-1+k}{n-k}\widetilde{Q}_{2k}(x)$ for even powers of x and as $x^{2n+1}=\sum_{k=0}^{n}\binom{n+k}{n-k}\widetilde{Q}_{2k+1}(x)$ for odd powers of x. The coefficients appearing in these equations are exactly the coefficients of the *Morgan-Voyce polynomials* $B_n(x)$ and $b_n(x)$ respectively, see [16, 23, 24]. Another connection between the toric g-polynomials of cubes and the Morgan-Voyce polynomials was noted in [11].

Corollary 5.2 *The linear transformation $\mathbb{Q}[x]\to\mathbb{Q}[x]$ given by $x^n\mapsto\widetilde{Q}_n(x)$ takes $B_n(x^2)$ into x^{2n} and $xb_n(x^2)$ into x^{2n+1}.*

Comparing Proposition 4.3 with (9) yields the following consequence.

Corollary 5.3 *The operators $\mathcal{C}$ and $\mathcal{D}$ are equivalently given by*

$$\mathcal{C}(\widetilde{Q}_n(x)):=\widetilde{Q}_{n+1}(x)\quad\textit{and}\quad\mathcal{D}(\widetilde{Q}_n(x))=\begin{cases}0 & \textit{for odd } n,\\ (-1)^{n/2}C_{n/2} & \textit{for even } n.\end{cases}$$

We now turn to the polynomials $t_n(x):=\mathrm{t}(B_{n+1},x)$. Stanley's result [19, Proposition 2.1] may be rewritten as

$$t_n(x)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}x^{n-2k}\quad\text{for } n\geq 0. \tag{10}$$

Inverting the summation given in (10) yields

$$x^n = \begin{cases} t_n(x) - t_{n-2}(x) & \text{if } n \geq 2, \\ t_n(x) & \text{if } n \in \{0, 1\}. \end{cases} \tag{11}$$

As an immediate consequence of Definition 4.5 and (11) we obtain

$$\mathcal{C}(t_n(x)) = t_{n+1}(x) - t_{n-1}(x) \quad \text{and} \quad \mathcal{D}(t_n(x)) = \delta_{n,0} \quad \text{for } n \geq 0. \tag{12}$$

Here we set $t_{-1}(x) := 0$ and $\delta_{n,0}$ is the Kronecker delta function. Finally, the most remarkable property of the basis $\{t_n(x)\}_{n\geq 0}$ is its role in the following result connecting the polynomials $g(P, x)$ and $\mathrm{t}(P, x)$.

Proposition 5.4 *Let $[\widehat{0}, \widehat{1}]$ be any Eulerian poset of rank $n + 1$. Then $\mathrm{t}([\widehat{0}, \widehat{1}], x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_k t_{n-2k}(x)$ holds for some integers $c_0, c_1, \ldots, c_{\lfloor n/2 \rfloor}$ if and only if the same integers satisfy $g([\widehat{0}, \widehat{1}], x) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} c_k x^k$.*

6 The toric h-vector associated to an Eulerian dual simplicial poset

Given any graded poset P of rank $n+1$, let f_i denote number of elements of rank $i+1$ in P. The resulting vector $(f_{-1}, f_0, \ldots, f_n)$ is the *f-vector* of P. A graded poset P is *simplicial* if for all $t \in P \setminus \{\widehat{1}\}$, the interval $[\widehat{0}, t]$ is a Boolean algebra. A graded poset P is *dual simplicial* if its dual P^* is a simplicial poset. It was first observed by Kalai that the toric h-polynomials coefficients of a dual simplicial graded Eulerian poset P depend only on the entries f_i in the f-vector P. This linear combination is not unique, and a simple explicit formula was not known before. We will express the toric h-polynomial coefficients of P in terms of its *h-vector* $(h_0, \ldots, h_n)$, given by

$$h_k = \sum_{i=k}^{n} (-1)^{i-k} \binom{i}{k} f_i,$$

Since $f_i(P) = f_{n-1-i}(P^*)$, by [19, Corollary 2.2] this h-vector coincides with the toric h-vector of the simplicial poset P^*. Our main result is the following:

Theorem 6.1 *The short toric polynomial $\mathrm{t}([\widehat{0}, \widehat{1}], x)$ associated to a graded dual simplicial Eulerian poset $P = [\widehat{0}, \widehat{1}]$ of rank $n + 1$ may be written as*

$$\begin{aligned} \mathrm{t}([\widehat{0}, \widehat{1}], x) &= h_0(t_n(x) - (n-1)t_{n-2}(x)) \\ &\quad + \sum_{i=1}^{n-1} h_i \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{i-1}{k-1} - \binom{n-i-1}{k}\binom{i}{k-1} \right) t_{n-2k}(x) \end{aligned}$$

The proof uses Stanley's description [18, Theorem 3.1] of the cd-index of an Eulerian simplicial poset in terms of its h-vector as a combination $\Phi_P(c, d) = \sum_{i=0}^{n-1} h_i \cdot \check{\Phi}_i^n$ and the combinatorial description of the polynomials $\check{\Phi}_i^n$ stated in [10, Theorem 2], originally conjectured by Stanley [18, Conjecture 3.1]. These results are combined with (9) to obtain recurrence formulas that allow proving the result by induction. See [12] for details. An important equivalent form of Theorem 6.1 is the following statement.

Proposition 6.2 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then we have*

$$\mathrm{t}([\widehat{0},\widehat{1}),x) = h_0 t_n(x) + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(h_i - h_{i-1}) \sum_{k=1}^{\min\{i,n-i\}} \frac{n+1-2i}{k}\binom{n-i}{k-1}\binom{i-1}{k-1} t_{n-2k}(x).$$

Corollary 6.3 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then*

$$\begin{aligned} g([\widehat{0},\widehat{1}),x) &= h_0(1-(n-1)x) \\ &\quad + \sum_{i=1}^{n-1} h_i \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{i-1}{k-1} - \binom{n-i-1}{k}\binom{i}{k-1}\right) x^k. \end{aligned}$$

Corollary 6.4 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then*

$$g([\widehat{0},\widehat{1}),x) = 1 + \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor}(h_i - h_{i-1}) \sum_{k=1}^{\min\{i,n-i\}} \frac{n+1-2i}{k}\binom{n-i}{k-1}\binom{i-1}{k-1} x^k.$$

The most important consequence of Corollary 6.4 is the following.

Corollary 6.5 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. If the h-vector $(h_0,\dots,h_n)$ satisfies $h_0 \leq h_1 \leq \cdots \leq h_{\lfloor n/2 \rfloor}$, then $f([\widehat{0},\widehat{1}],x)$ has nonnegative coefficients.*

Indeed, by Corollary 6.4 above, $g([\widehat{0},\widehat{1}),x)$ has nonnegative coefficients and the statement follows from [19, (19)].

Example 6.6 Let $[\widehat{0},\widehat{1}]$ be the face lattice of an n-dimensional simple polytope $\mathcal{P}$. By Corollary 6.5, the fact that the toric h-vector of $\mathcal{P}$ has nonnegative entries is a consequence of the Generalized Lower Bound Theorem [20] for simplicial polytopes.

Remark 6.7 In the case when $n = 2i$, the coefficients $N(i,k) = \binom{i-1}{k-1}\binom{i}{k-1}/k$, contributed by $h_{\lfloor n/2 \rfloor} - h_{\lfloor n/2 \rfloor - 1}$ in Corollary 6.4 are the *Narayana numbers*, see sequence A001263 in [17]. The same numbers appear also as the coefficients of the contributions of $h_{\lfloor n/2 \rfloor}$ and $h_{\lceil n/2 \rceil}$ for any n in Corollary 6.3.

Motivated by Example 6.9 below, we rewrite Corollary 6.3 in the basis $\{(x-1)^k\}_{k \geq 0}$.

Proposition 6.8 *Let $[\widehat{0},\widehat{1}]$ be a graded dual simplicial Eulerian poset of rank $n+1$. Then we have*

$$g([\widehat{0},\widehat{1}],x) = h_0(n-(n-1)(x-1)) + \sum_{i=1}^{n-1} h_i \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n-i}{k}\binom{n-k-1}{i-k} - \binom{n-i-1}{k}\binom{n-k-1}{i+1-k} \right)(x-1)^k.$$

Example 6.9 Let $[\widehat{0},\widehat{1}]$ be the face lattice $[\widehat{0},\widehat{1}]$ of an n-dimensional cube. Starting with Proposition 6.8, after repeated use of Pascal's identity and the Chu-Vandermonde identity, one can show

$$g([\widehat{0},\widehat{1}],x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} C_{n-k}(x-1)^k. \tag{13}$$

It was noted in [11, Lemma 3.3] that (13) is equivalent to Gessel's result [19, Proposition 2.6], stating

$$g([\widehat{0},\widehat{1}],x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{n-k+1}\binom{n}{k}\binom{2n-2k}{n}(x-1)^k. \tag{14}$$

The first combinatorial interpretation of the right hand side of (14) is due to Shapiro [21, Ex. 3.71g] the proof of which was published by Chan [9, Proposition 2].

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Combinatorics of k-shapes and Genocchi numbers

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Abstract. In this paper we present a work in progress on a conjectural new combinatorial model for the Genocchi numbers. This new model called *irreducible k-shapes* has a strong algebraic background in the theory of symmetric functions and leads to seemingly new features on the theory of Genocchi numbers. In particular, the natural q-analogue coming from the degree of symmetric functions seems to be unknown so far.

Résumé. Dans cet article, nous présentons un travail en cours sur un nouveau modèle combinatoire conjectural pour les nombres de Genocchi. Ce nouveau modèle est celui des *k-formes irréductibles*, qui repose sur de solides bases algébriques en lien avec la théorie des fonctions symétriques et qui conduit à des aspects apparemment nouveaux de la théorie des nombres de Genocchi. En particulier, le q-analogue naturel venant du degré des fonctions symétriques semble inconnu jusqu'ici.

Keywords: partitions, cores, symmetric functions

Contents

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1 Introduction

The goal of this paper is to present some purely combinatorial results on certain partitions; these results are strongly motivated by the theory of symmetric functions. We define a still conjectural new model for the Genocchi numbers and Gandhi polynomials. We begin by explaining the root of this work, namely the k-Schur functions of Lapointe-Lascoux-Morse; however, the reader who is not familiar with the theory of symmetric functions can skip the following paragraphs without harm.

The fundamental theorem of symmetric functions states that their ring Sym is freely generated by the homogeneous symmetric functions $(h_i)_{i>0}$. It therefore makes sense to study the subring $\mathrm{Sym}^{(k)}$ generated by the first k homogeneous functions $\mathrm{Sym}^{(k)} := \mathbb{Q}[h_1, h_2, \dots h_k]$. Say that a partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ is k-bounded if it has no part exceeding k (*i.e.*, $k \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l > 0$). Then, if we define $h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_l}$, then $(h_\lambda)_\lambda$ where λ is k-bounded is a natural basis for $\mathrm{Sym}^{(k)}$. However, there exists another basis which plays the role of Schur functions; this basis is called atoms in [LLM03] and, with a different definition, k-Schur functions in [LM05, LM07]. It should be noted that those papers actually deal with a more general setting where homogeneous symmetric functions are replaced with their natural q-analogues (Hall-Littlewood functions) and q, t-analogues (Macdonald functions). From the combinatorial point of view, it seems that indexing k-Schur functions by k-bounded partitions is not the right approach. In [LM05], Lapointe and Morse showed that k-bounded partitions are in bijection with $k+1$-cores and that there is a natural tableau-like definition of k-Schur functions involving paths in the analogue of Young's lattice on $k+1$-cores.

An important question is to find a combinatorial way to expand the k-Schur function $s_\lambda^{(k)}$ in terms of the usual Schur functions s_λ. The strategy of [LLMS] is to inductively use a combinatorial way to expand k-Schur functions on $k+1$-Schur functions. In order to do that, they need to interpolate between k-cores and $k+1$-cores. They indeed define a certain partially ordered set of partitions they call k-shapes (see Definition 2.1) such that $k+1$-cores are the minimal elements and k-cores the maximal ones. The expansion of $s_\lambda^{(k)}$ is essentially described by counting paths from k-cores to $k+1$-cores in this poset up to some equivalence relation. The combinatorial study of these k-shapes is the main goal of our paper.

Our main result is the following theorem:

Theorem 1.1 *For $k \in \mathbb{N}$, the generating function for k-shapes is given by*

$$f_k(t) = \frac{P_k(t)}{\prod_{u,v}(1 - t^{uv})} \tag{1}$$

where the product is over the set of all couples (u, v) of positive integers such that $u + v \in \{k, k+1\}$ and $P_k(t) \in \mathbb{N}[t]$ is a polynomial with nonnegative integer coefficients.

Using computer, we evaluated the numerator $P_k(t)$ up to $k = 9$ (see Section 5). Setting $t = 1$, one gets the following table

k	1	2	3	4	5	6	7	8	9
$P_k(1)$	1	1	3	17	155	2073	38227	929569	28820619

which seems to be the sequence of Genocchi numbers (sequence A110501 of [Slo]):

Conjecture 1.2 *The polynomials appearing in Equation* (1) *are a q-analogue of the Genocchi numbers: $P_k(1)$ is the k^{th} Genocchi number.*

Outline. The paper is structured as follows: The first background section (Section 2) recalls the necessary definitions on Genocchi numbers (Subsection 2.1), partitions and skew partitions (Subsection 2.2), as well as k-cores and k-shapes (Subsection 2.3). Then, Section 3 is devoted to the operation of adding a rectangle to a k-shape and the construction of the main object of this paper, namely irreducible k-shapes. We finally conclude by stating our main conjecture (Subsection 4.1), as well as some algebraic perspectives (Subsection 4.2).

2 Background

2.1 Genocchi numbers

The unsigned Genocchi numbers G_{2n} can be defined using the Bernoulli numbers B_n by

$$G_{2n} = 2(2^{2n} - 1)|B_{2n}|$$

or via the generating function

$$\frac{2t}{1+e^t} = t + \sum_{n\geq 1} (-1)^n G_{2n} \frac{t^{2n}}{(2n)!}.$$

These numbers were named, apparently by Lucas [Luc91], after the 19th century Italian mathematician Angelo Genocchi; they appear in the latter's papers on the Bernoulli numbers [Gen52, Gen86]. However, Euler had already studied them in [Eul55].

The first combinatorial interpretation of the Genocchi numbers was given by Dumont in [Dum74]: G_{2n+2} is the number of permutations $\sigma \in \mathfrak{S}_{2n}$ such that σ_{2n} is odd and, for i such that $1 \leq i \leq 2n-1$, σ_i is followed by a smaller number if it is even and by a greater number if it is odd. Such permutations are called *Dumont permutations of the first kind*. In the same paper, Dumont defined another family of permutations enumerated by the Genocchi numbers, which are now known as *Dumont permutations of the second kind*. Other families of Genocchi-enumerated permutations were subsequently introduced by Kitaev and Remmel [KR07] and Burstein and Stromquist [BS07].

2.2 Integer partitions and skew partitions

A *partition* $\lambda = (\lambda_1, \ldots, \lambda_m)$ is a nonincreasing sequence of positive integers which are called the *parts* of the partition. We will sometimes also use the *exponential notation* for partitions: if λ is a partition and f_i is the number of occurrences of i in λ, then we write $\lambda = (1^{f_1} 2^{f_2} \ldots)$. The number of parts of λ is denoted by $\ell(\lambda)$. The *weight* of λ is $|\lambda| := \sum_{i=1}^{m} \lambda_i$. If the weight of λ is n, we say that λ is a *partition of* n. A sequence $c = (c_1, \ldots, c_m)$ (not necessarily nonincreasing) of positive integers whose sum is n is called a *composition* of n. We denote by $c \cdot c'$ the concatenation of the two compositions c and c'.

A partition λ is classically represented by a Ferrers diagram, which we denote by $[\lambda]$, where the i^{th} row has λ_i squares (also called *cells*). We use the French notation for Ferrers diagrams: the rows are counted from bottom to top. More precisely, $[\lambda]$ is formed by the unit cells (x, y) with upper right corner at the point (x, y), where $1 \leq y \leq \ell(\lambda)$ and $1 \leq x \leq \lambda_y$. An example is shown on Figure 1.

The *union* of two partitions λ and μ, which we denote by $\lambda \cup \mu$, is a partition that is obtained by gathering all the parts of λ and μ and sorting them in nonincreasing order. For example, if $\lambda = (5, 4, 3, 3)$ and $\mu = (7, 6, 5, 3, 2, 1)$, we have $\lambda \cup \mu = (7, 6, 5, 5, 4, 3, 3, 3, 2, 1)$.

The notion of skew partition will be needed throughout the paper. Given two partitions λ and μ, we write $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If λ and μ satisfy $\mu \subseteq \lambda$, we identify the *skew partition* λ/μ with its diagram, which is the set-theoretic difference $[\lambda] \setminus [\mu]$. The *weight* of λ/μ is $|\lambda/\mu| := |\lambda| - |\mu|$. For example, $\lambda/\mu = (4, 3, 2, 2, 1)/(2, 1, 1)$ has weight 8 and its diagram is shown on Figure 2.

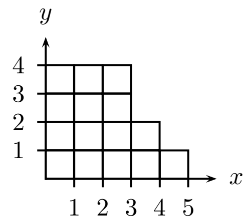


Figure 1: The Ferrers diagram of $(5, 4, 3, 3)$.

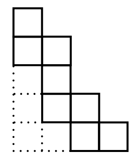


Figure 2: The diagram of the skew partition $\lambda/\mu = (4, 3, 2, 2, 1)/(2, 1, 1)$. The dotted lines indicate the cells of $[\mu]$.

2.3 k-cores and k-shapes

The *hook* of a cell c in the Ferrers diagram of a partition is the set formed by c and the cells that are located to its right in the same row or above it in the same column. The *hook length* of a cell is the number of cells in its hook. An example is shown on Figure 3, where we have encircled the hook of the dotted cell $(2, 2)$; the hook length of this cell is 7.

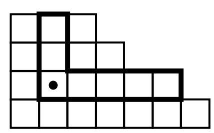


Figure 3: The hook of the cell $(2, 2)$ in the diagram of $\lambda = (7, 6, 4, 3)$.

A partition is a k-*core* if its diagram contains no cells with hook length k [JK81]. See Figure 4 for an example.

It is not difficult to see that in every Ferrers diagram, the hook lengths decrease from left to right and from bottom to top. Consequently, the cells of a diagram $[\lambda]$ whose hook length is (strictly) larger than k form the diagram of a partition. Following [LLMS], we call this partition the k-*interior* of λ and denote it by $\mathrm{Int}^k(\lambda)$. As for the cells whose hook length does not exceed k, they form the diagram of a skew partition, which we call the k-*boundary* of λ and denote by $\partial^k(\lambda)$. We also define the k-*rim* of λ as the line that starts at the upper left corner of the diagram, goes down vertically until reaching the k-interior, follows the limit between the k-interior and the k-boundary and then goes horizontally to the lower right corner of the diagram. See Figure 5 for an example.

We call *row shape* of a skew partition λ/μ and denote by $\mathrm{rs}(\lambda/\mu)$ the sequence of the lengths of the rows (from bottom to top) of λ/μ. Likewise, we call *column shape* of λ/μ and denote by $\mathrm{cs}(\lambda/\mu)$ the heights of the columns (from left to right) of λ/μ. Both $\mathrm{rs}(\lambda/\mu)$ and $\mathrm{cs}(\lambda/\mu)$ are compositions of $|\lambda/\mu|$. For example, if we call λ/μ the 6-interior of the partition represented on Figure 5, we have $\mathrm{rs}(\lambda/\mu) = (5, 4, 2, 1)$ and $\mathrm{cs}(\lambda/\mu) = (2, 2, 1, 1, 2, 1, 1, 1, 1)$. We also define $\mathrm{rs}^k(\lambda) := \mathrm{rs}(\partial^k(\lambda))$ and $\mathrm{cs}^k(\lambda) := \mathrm{cs}(\partial^k(\lambda))$.

Definition 2.1 (Lam *et al.* [LLMS]) *A partition λ is a k-*shape *if $\mathrm{rs}^k(\lambda)$ and $\mathrm{cs}^k(\lambda)$ are both partitions.*

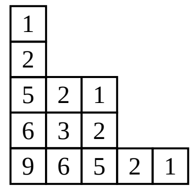


Figure 4: The diagram of the 4-core $(5, 3, 3, 1, 1)$ with its hook lengths.

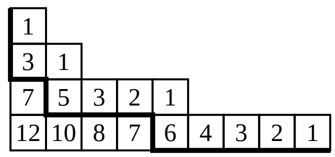


Figure 5: The diagram of $\lambda = (9, 5, 2, 1)$ with its hook lengths. The bold line is the 6-rim. The cells below that line form the 6-interior of λ and those above it form the 6-boundary.

See Figure 6 for an example. Note that k-cores and $k+1$-cores are k-shapes [LLMS, Proposition 10].

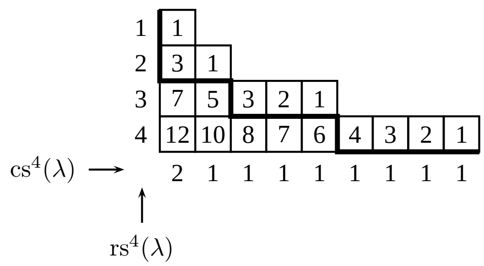


Figure 6: $\lambda = (9, 5, 2, 1)$ is a 4-shape.

For legibility reasons, we will sometimes omit the k-interior when representing k-shapes graphically, since a k-shape is uniquely determined by its k-boundary.

3 Irreducible k-shapes

3.1 Addition of rectangles

This section deals with an operation that builds a k-shape from another k-shape and a k- or $k-1$-rectangle. The definition of irreducible k-shapes and the proof of Theorem 1.1 rest on this operation. Note that it is a generalization of the construction presented in Theorem 10 of [LM04], which is equivalent, in our framework, to adding a k-rectangle to a $k+1$-core.

Definition 3.1 ([LLM03]) *A* k-rectangle *is a partition of the form $(\ell^{k+1-\ell})$, where ℓ satisfies $1 \le \ell \le k$.*

Evidently, k-rectangles are the partitions whose Ferrers diagram is a rectangle and whose largest hook length is k.

Lemma 3.2 *Let $r = (r_1, \ldots, r_m)$ and $c = (c_1, \ldots, c_p)$ be two compositions of the same integer n. Then there exists at most one skew partition λ/μ of weight n such that $\mathrm{rs}(\lambda/\mu) = r$ and $\mathrm{cs}(\lambda/\mu) = c$.*

Proof: We build the skew partition row by row, from top to bottom. At the i^{th} step ($1 \leq i \leq m$), we try to insert a row of length r_{m+1-i}. The first c_1 rows must be left-justified so that the first column contains c_1 cells, as desired. The subsequent $c_1 - c_2$ rows must be shifted by one so that the first column does not contain more than c_1 cells and the second column contains c_2 cells, and so on. Therefore, the resulting skew partition is unique. □

It must be noted that a solution does not always exist. For example, if $r = (3)$ and $c = (2, 1)$, the two conditions are incompatible, because the row condition implies that the skew partition has three columns whereas the column condition implies that it has two columns.

Definition 3.3 *Let λ be a k-shape, $r = \mathrm{rs}^k(\lambda)$ and $c = \mathrm{cs}^k(\lambda)$. For all $i \in \mathbb{N}^*$, let*

$$\mathbb{H}_i = [\max\{j \mid r_j > i\}, \min\{j \mid r_j < i\} - 1] \tag{2}$$
$$\mathbb{V}_i = [\max\{j \mid c_j > i\}, \min\{j \mid c_j < i\} - 1] \tag{3}$$

We use the convention that if π is a partition, we have $\pi_0 = \infty$ and $\pi_j = 0$ if $j > \ell(\pi)$.

The horizontal strip H_i is the set $\{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{H}_i\}$. Similarly, the vertical strip V_i is the set $\{(x, y) \in \mathbb{R}^2 \mid x \in \mathbb{V}_i\}$.

Note that the interval $\mathbb{H}_i$ (resp. $\mathbb{V}_i$) is reduced to a singleton if there is no integer j such that $r_j = i$ (resp. $c_j = i$). In this case, the corresponding strip is a single line. An example is the vertical strip V_2 in the k-shape shown on Figure 7.

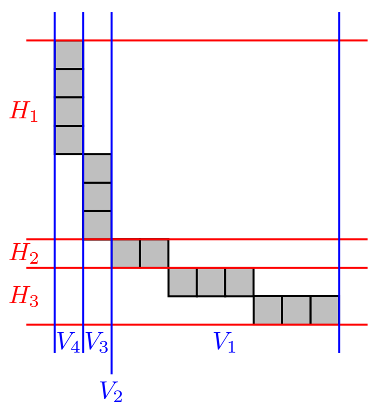


Figure 7: Horizontal and vertical strips in the diagram of the 4-shape $(10, 7, 4, 2, 2, 2, 1, 1, 1, 1)$. Note that V_2 is reduced to a single line.

Proposition 3.4 *Let λ be a k-shape and (u^v) be a k-rectangle or $k-1$-rectangle. Then there exists a point of the k-rim of λ that belongs to $H_u \cap V_v$.*

Proof (sketch): Using hook length considerations, we show that the upper right corner of the intersection $H_u \cap V_v$ belongs to the k-boundary while the lower left corner does not (see Figure 8). Consequently, a path between these two points must cross the k-rim. □

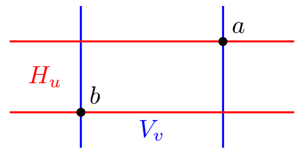


Figure 8: a is in the k-boundary and b in the k-interior; the k-rim therefore passes between these two points.

Proposition 3.5 *Let λ be a k-shape and (u^v) be a k-rectangle or $k-1$-rectangle. Then there exists a unique k-shape, which we denote by $\lambda + (u^v)$, such that*

$$\mathrm{rs}^k(\lambda + (u^v)) = \mathrm{rs}^k(\lambda) \cup (u^v) \tag{4}$$

$$\mathrm{cs}^k(\lambda + (u^v)) = \mathrm{cs}^k(\lambda) \cup (v^u) \tag{5}$$

Proof: We describe an algorithm that constructs $\partial^k(\lambda + (u^v))$ from $\partial^k(\lambda)$ by "inserting" the rectangle into the k-shape. Proposition 3.4 implies that we can decompose $\partial^k(\lambda)$ into three parts: the cells located to the right of p and above it, which we call the *foot* (region F in Figure 9), the cells to the left of the foot (region A) and the cells below the foot (region B).

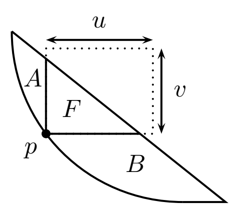


Figure 9: The decomposition of the k-boundary of a k-shape used for inserting a rectangle. Here, p is a point satisfying the conditions of Proposition 3.4.

The insertion process then consists in transforming this k-boundary into the skew partition shown on Figure 10, from which the partition $\lambda + (u^v)$ is easily reconstructed, as said before.

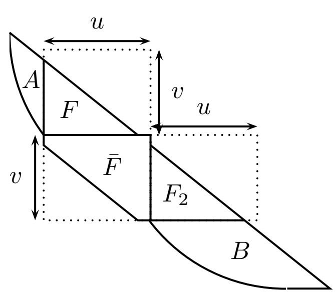


Figure 10: Result of the insertion process. F_2 is a copy of F and $\bar{F}$ is the complement of F in (u^v).

We call this skew partition $\tilde{\lambda}/\tilde{\mu}$. First, we must prove that it satisfies (4) and (5). If R_1 and R_2 are two regions of the diagram, let $R_1 \vee R_2$ denote their set-theoretic union. We have that

$$\begin{aligned}\mathrm{rs}(\tilde{\lambda}/\tilde{\mu}) &= \mathrm{rs}(B)\cdot\mathrm{rs}(\bar{F}\vee F_2)\cdot\mathrm{rs}(A\vee F)\\ &= \mathrm{rs}(B)\cdot(u^v)\cdot\mathrm{rs}(A\vee F)\\ &= \mathrm{rs}(\lambda/\mu)\cup(u^v)\end{aligned}$$

because $\mathrm{rs}(B)$ is a partition into parts $\geq u$ and $\mathrm{rs}(A\vee F)$ is a partition into parts $\leq u$, since λ is a k-shape. Therefore, $\tilde{\lambda}/\tilde{\mu}$ satisfies (4); the proof that it satisfies (5) is similar.

We now need to check that $\tilde{\mu}$ is indeed the k-interior of $\tilde{\lambda}$, *i.e.* that the hook lengths of $\tilde{\lambda}/\tilde{\mu}$ do not exceed k and those of $\tilde{\mu}$ are greater than k. To this end, we decompose $[\tilde{\lambda}]$ as shown on Figure 11.

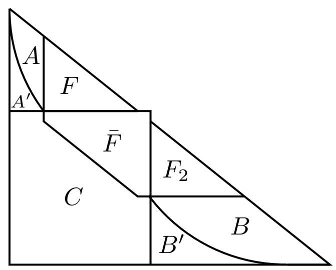


Figure 11: Decomposition of $[\tilde{\lambda}]$ to check the hook length conditions. The regions A', C and B' belong to the k-interior, the other regions belong to the k-boundary.

The hook lengths of the cells that were above p before the insertion (regions A, A', and F on Figure 11) have not been modified since no cells have been added or removed above those cells or to their right. The hook lengths of the cells that were to the right of p (regions B, B', and F_2) have not been modified either, for the same reason. A cell in the region $\bar{F}$ has at most $u-1$ cells to its right in the same row and at most $v-1$ cells above it in the same column: its hook length thus does not exceed $u+v-1\leq k$. Likewise, a cell in the region C has at least u cells to its right in the same row and v cells above it in he same column, and therefore has hook length at least $u+v+1\geq k+1$. All the hook lengths conditions are therefore satisfied.

The uniqueness is proved by Lemma 3.2. In particular, the result does not depend on the choice of p. □

See Figure 12 for an example of the insertion process.

Proposition 3.6 *Let λ be a k-shape and let $(u_1^{v_1})$ and $(u_2^{v_2})$ be two k- or $k-1$-rectangles. Then*

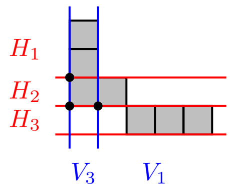
$$(\lambda+(u_1^{v_1}))+(u_2^{v_2}) = (\lambda+(u_2^{v_2}))+(u_1^{v_1}).$$

Proof: We have $\mathrm{rs}\left((\lambda+(u_1^{v_1}))+(u_2^{v_2})\right)=\mathrm{rs}(\lambda)\cup(u_1^{v_1})\cup(u_2^{v_2})=\mathrm{rs}\left((\lambda+(u_2^{v_2}))+(u_1^{v_1})\right)$ and likewise $\mathrm{cs}\left((\lambda+(u_1^{v_1}))+(u_2^{v_2})\right)=\mathrm{cs}\left((\lambda+(u_2^{v_2}))+(u_1^{v_1})\right)$. The proof is completed by using Lemma 3.2. □

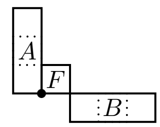
3.2 Main theorem

Definition 3.7 *We say that a k-shape is* irreducible *if it cannot be obtained from another k-shape by inserting a k- or $k-1$-rectangle as described in the proof of Proposition 3.5.*

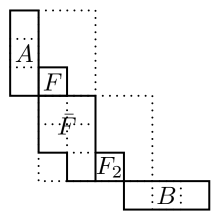
We first need to have a better characterization of the irreducible k-shapes.



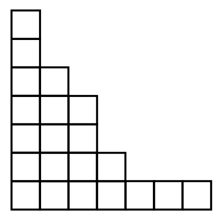
(a) The initial 4-shape $\lambda = (5,2,1,1)$ with the possible insertion points for the 4-rectangle (2^3).



(b) Decomposition of $\partial^4(\lambda)$ with respect to the insertion point $p = (1,1)$.



(c) The skew partition resulting from the insertion process.



(d) The final 4-shape $(7,4,3,3,2,1,1)$.

Figure 12: The insertion of the 4-rectangle (2^3) into the 4-shape $(5,2,1,1)$ produces the 4-shape $(7,4,3,3,2,1,1)$.

Proposition 3.8 *Let λ be a k-shape and (u^v) be a k- or $k-1$-rectangle. The following are equivalent:*

(i) there exists a k-shape μ such that $\lambda = \mu + (u^v)$,

(ii) there exist two points (x_1, y_1) and (x_2, y_2) of the k-rim of λ lying in $H_u \cap V_v$ such that $x_2 - x_1 \geq u$,

(iii) there exist two points (x_1, y_1) and (x_2, y_2) of the k-rim of λ lying in $H_u \cap V_v$ such that $y_2 - y_1 \geq v$.

By Proposition 3.8, in an irreducible k-shape the length of the k-rim is bounded, so that there are only finitely many of them. Therefore the generating function for irreducible k-shapes is a polynomial.

Thanks to the same proposition, we can also prove the following converse of Proposition 3.5:

Corollary 3.9 *Suppose that $\lambda, \lambda_1, \lambda_2$ are k-shapes and that there exist two different k- or $k-1$-rectangles $(u_1^{v_1})$ and $(u_2^{v_2})$ such that $\lambda = \lambda_1 + (u_1^{v_1}) = \lambda_2 + (u_2^{v_2})$. Then there exists a unique k-shape μ such that $\lambda = \mu + (u_1^{v_1}) + (u_2^{v_2})$.*

It has the following important consequence:

Corollary 3.10 *For each k-shape λ, there is a unique irreducible k-shape μ and a unique family of non-negative integers $(C_{(u^v)})$ indexed by k- and $k-1$-rectangles (u^v) such that*

$$\lambda = \mu + \sum_{(u^v)} C_{(u^v)} (u^v) \,. \tag{6}$$

Using this corollary, we can prove that the generating function for k-shapes is of the form shown in (1):

Corollary 3.11 *For $k \in \mathbb{N}$, the generating function for k-shapes is given by*

$$f_k(t) = \frac{P_k(t)}{\prod(1 - t^{uv})} \tag{7}$$

where the product is over the set of all k- or $k-1$-rectangles (u^v) and $P_k(t) \in \mathbb{N}[t]$ is the polynomial generating function for irreducible k-shapes.

4 Work in progress

4.1 Main conjecture

In this section we discuss some approaches we are currently exploring around the following conjecture:

Conjecture 4.1 *Irreducible k-shapes are counted by the Genocchi numbers.*

This conjecture has been extensively tested for all k up to 9.

We are considering two approaches: a direct bijection between irreducible k-shapes and some family of objects known to be counted by the Genocchi numbers (such as Dumont permutations of some kind, surjective pistols, certain tableaux…) and a recursive proof involving the Gandhi polynomials.

Indeed the following conjectural connection with Gandhi polynomials provides more evidence from the conjecture as well as what we believe is a good angle of attack to prove it. Let us recall the definition of those polynomials:

Definition 4.2 *For $k \geq 1$, the* Gandhi polynomials $P_{2k}(x)$ *are defined by the following recurrence:*

$$\begin{cases} P_2(x) = x^2 \\ P_{2k+2}(x) = x^2\left(P_{2k}(x+1) - P_{2k}(x)\right) \end{cases} \tag{8}$$

Gandhi [Gan70] conjectured that $P_{2k}(1) = G_{2k+2}$, which was later proved by Carlitz [Car71] and Riordan and Stein [RS73].

We need a small definition before stating a conjecture relating k-shapes and Gandhi polynomials.

Definition 4.3 *Let λ be a k-shape and ℓ be an integer such that $1 \leq \ell \leq k$. We say that $(\ell, k+1-\ell)$ is a* free k-site *in λ if there is no cell with $\ell - 1$ cells to its right in the same row and $k - \ell$ cells above it in the same column (note that such a cell would have hook length k).*

For example, the reader can check, looking at Figure 12, that the 4-shape $(5, 2, 1, 1)$ has 3 free 4-sites: $(1, 4)$, $(3, 2)$ and $(4, 1)$.

Conjecture 4.4 *Let $S(k, j)$ be the number of irreducible k-shapes with j free k-sites. Let $S_{2k}(x) = \sum_j S(k, j)x^j$. Then $S_{2k}(x) = P_{2k}(x)$.*

We aim to prove this conjecture by showing that the S_{2k} satisfy (8).

4.2 Algebraic perspectives

As we said in the introduction, the work presented here has a strong algebraic background. In particular, Lam et al [LLMS, Equation (11)] defined a symmetric function $\mathfrak{s}_\mu^{(k)}$ associated to each k-shape μ. By definition, $\mathfrak{s}_\mu^{(k)}$ reduces to the k-Schur function $s_\mu^{(k)}$ if μ is a k-core and to the $s_\mu^{(k+1)}$ if μ is a $k+1$-core. Therefore, those functions are not linearly independent. Nevertheless, the operation of adding a k-rectangle to a k-core reflects an algebraic relation on symmetric functions [LM07, Theorem 40]: for any k-core λ and k-rectangle (u^v), if $s_{(u^v)}$ is the usual Schur function associated to (u^v), then we have $s_{(u^v)} s_\lambda^{(k)} = s_{\lambda+(u^v)}^{(k)}$. It seems that the k-shape functions enjoy a similar property [Lap08]:

Conjecture 4.5 *Let λ be a k-shape and (u^v) be a k-rectangle or $k-1$-rectangle. Then*

$$s_{(u^v)}\mathfrak{s}_{\lambda}^{(k)} = \mathfrak{s}_{\lambda+(u^v)}^{(k)} \tag{9}$$

It seems moreover that this property passes to t-analogues replacing the Schur function by a vertex operator and the k-Schur functions by their graded version (see [LLMS, Equation (18)]).

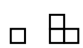
Acknowledgements

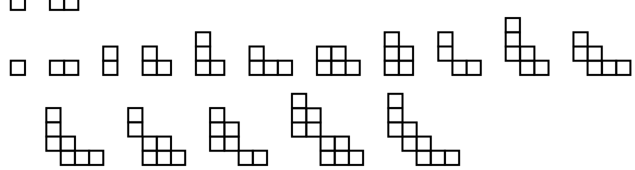
We wish to thank Luc Lapointe, who discussed the problem with us and provided us with useful references, and Matthieu Josuat-Vergès, who introduced us to various aspects of the combinatorics of the Genocchi numbers. We also point out that this research was driven by computer exploration using the open-source mathematical software `Sage` [S$^+$10] and its algebraic combinatorics features developed by the `Sage-Combinat` community [SCc10].

5 Table

Generating series for irreducible k-shapes:

$$\begin{aligned}
f_1(t) &= \frac{1}{1-t},\\
f_2(t) &= \frac{1}{(1-t)(1-t^2)^2},\\
f_3(t) &= \frac{1+t+t^3}{(1-t^2)^2(1-t^3)^2(1-t^4)},\\
f_4(t) &= \frac{1+t+2t^2+t^3+3t^4+2t^5+2t^6+3t^7+t^9+t^{10}}{(1-t^3)^2(1-t^4)^3(1-t^6)^2},\\
f_5(t) &= \frac{\begin{array}{c}1+t+2t^2+3t^3+3t^4+7t^5+6t^6+9t^7+9t^8+13t^9+13t^{10}+10t^{11}+13t^{12}+\\ 13t^{13}+13t^{14}+6t^{15}+10t^{16}+8t^{17}+5t^{18}+3t^{19}+3t^{20}+2t^{21}+t^{22}+t^{23}\end{array}}{(1-t^4)^2(1-t^5)^2(1-t^6)^2(1-t^8)^2(1-t^9)}.
\end{aligned}$$

Irreducible 3-shapes: $\varnothing$ 

Irreducible 4-shapes: $\varnothing$ 

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0-Hecke algebra actions on coinvariants and flags

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Abstract. By investigating the action of the 0-Hecke algebra on the coinvariant algebra and the complete flag variety, we interpret generating functions counting the permutations with fixed inverse descent set by their inversion number and major index.

Résumé. En étudiant l'action de l'algèbre de 0-Hecke sur l'algèbre coinvariante et la variété de drapeaux complète, nous interprétons les fonctions génératrices qui comptent les permutations avec un ensemble inverse de descentes fixé, selon leur nombre d'inversions et leur "major index".

Keywords: 0-Hecke algebra, Ribbon number, Descent monomial, Demazure operator.

1 Introduction

A composition I of an integer n gives rise to a descent class of permutations in the symmetric group $\mathfrak{S}_n$; the cardinality of this descent class is known as the *ribbon number* r_I and its inv-generating function is the *q-ribbon number* $r_I(q)$. Reiner and Stanton [15] defined a *(q,t)-ribbon number* $r_I(q,t)$, and gave an interpretation by representations of $\mathfrak{S}_n$ and $GL(n,\mathbb{F}_q)$.

Our main object here is to obtain similar interpretations of various ribbon numbers by representations of the 0-Hecke algebra $H_n(0)$ of type A. Norton [14] decomposed $H_n(0)$ into a direct sum of 2^{n-1} distinct indecomposable $H_n(0)$-submodules M_I indexed by compositions I of n. Consequently every indecomposable projective $H_n(0)$-module is isomorphic to M_I for some I, and every simple $H_n(0)$-module is isomorphic to $C_I = \mathrm{top}(M_I) = M_I/\mathrm{rad}\, M_I$ for some I.

1.1 Descent monomials and Demazure atoms

Our first result is related to the *descent monomials* within $\mathbb{Z}[\mathbf{x}] = \mathbb{Z}[x_1,\ldots,x_n]$

$$x_w = \prod_{i\in D(w)} x_{w(1)}\cdots x_{w(i)}, \quad w \in \mathfrak{S}_n,$$

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introduced by Garsia [8] as a $\mathbb{Z}$-basis for the coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$. The 0-Hecke algebra $H_n(0)$ acts on the coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ via the operators $\overline{\pi}_i = \pi_i - 1$, where π_i is the *Demazure operator* defined by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}.$$

Our first result shows that under this action:

- $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ is isomorphic to $H_n(0)$ as a (left) $H_n(0)$-module;
- $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n}) = \bigoplus_I N_I$, summed over all compositions of n, and $N_I \cong M_I$;
- $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ has a $\mathbb{Z}$-basis of certain *Demazure atoms* whose leading terms under some order are the descent monomials.

Since $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ has an extra grading by polynomial degree, it becomes a graded version of the left regular representation of $H_n(0)$.

1.2 A bigraded characteristic

Duchamp, Krob, Leclerc and Thibon [4] defined the *characteristic* of a finite dimensional $H_n(0)$-module M with simple composition factors $C_{I_1}, \dots, C_{I_k}$ to be

$$\mathcal{F}(M) = \sum_{i=1}^{k} F_{I_i}$$

where the F_I's are *quasi-ribbon functions* which form a basis for the algebra of quasi-symmetric functions. Krob and Thibon [10] showed that $\mathcal{F}(M_I)$ is the *ribbon schur function* s_I, and thus $\mathcal{F}(M)$ is symmetric whenever M is projective.

If $M = H_n(0)v$ is cyclic then the *length filtration*

$$H_n(0)^{(\ell)} = \bigoplus_{\ell(w)\geq \ell} \mathbb{Z}T_w$$

induces a filtration of $H_n(0)$-modules $M^{(\ell)} = H_n(0)^{(\ell)}v$, $k \geq 0$. This refines $\mathcal{F}(M)$ to a *graded characteristic*

$$\mathcal{F}_q(M) = \sum_{\ell\geq 0} q^\ell \mathcal{F}\left(M^{(\ell)}/M^{(\ell+1)}\right).$$

One has

$$\mathcal{F}_q(H_n(0)) = \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})} = \sum_I r_I(q) F_I$$

and taking a limit as $q \to 1$ gives

$$\mathcal{F}(H_n(0)) = \sum_I r_I F_I.$$

If M has another filtration by $H_n(0)$-modules M_d for $d \geq 0$, then one can look at the *bifiltration* by $H_n(0)$-modules $M^{(\ell,d)} = M^{(\ell)} \cap M_d$ for $\ell, d \geq 0$, and define the *bigraded characteristic* to be

$$\mathcal{F}_{q,t}(M) = \sum_{\ell,d\geq 0} q^\ell t^d \mathcal{F}\left(M^{(\ell,d)}/(M^{(\ell+1,d)} + M^{(\ell,d+1)})\right).$$

This happens to be the case for $M = \mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ with its length filtration (since $M \cong H_n(0)$) and its polynomial degree filtration by $M_d = \langle \overline{f} : \deg f \geq d\rangle_{\mathbb{Z}}$ for $d \geq 0$. Our next result is

- $\mathcal{F}_{q,t}\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right) = \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})}$

which completes the following picture.

$$\begin{tikzcd}
 & \boxed{\mathcal{F}(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=\!=} \sum_{w\in\mathfrak{S}_n} F_{D(w^{-1})}} & \\
\boxed{\begin{array}{c}\mathcal{F}_q(H_n(0)) \overset{\text{Krob-Thibon}}{=\!=\!=} \\ \sum_{w\in\mathfrak{S}_n} q^{\mathrm{inv}(w)} F_{D(w^{-1})}\end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q\leftrightarrow t", "\text{(Foata-Schützenberger)}"'] & & \boxed{\begin{array}{c}\mathcal{F}_t(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})) \overset{\text{New}}{=\!=} \\ \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{D(w^{-1})}\end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
 & \boxed{\begin{array}{c}\mathcal{F}_{q,t}\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right) \overset{\text{New}}{=\!=} \\ \sum_{w\in\mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})}\end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "q\to 1"'] &
\end{tikzcd}$$

Here the left inverse descent set $D(w^{-1})$ is identified with its descent composition, and the equality $\mathcal{F}_q(H_n(0)) \xlongequal{q\leftrightarrow t} \mathcal{F}_t(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ comes from the equidistribution of inv and maj over inverse descent classes, proved by Foata and Schützenberger [7]. We shall see in Section 3 that r_I and $r_I(q)$ appear as coefficients of F_I in $\mathcal{F}(H_n(0))$ and $\mathcal{F}_q(H_n(0))$.

1.3 Complete flag variety

Consider the general linear group $G = GL(n, \mathbb{F}_q)$ over a finite field $\mathbb{F}_q$ and its Borel subgroup B. The 0-Hecke algebra $H_n(0)$ acts on the *complete flag variety* $1_B^G = \mathbb{F}_q[G/B]$ by $T_w B = BwB$. This action induces an $H_n(0)$-module structure on

$$\mathrm{Hom}_{\mathbb{F}_q[G]}\left(1_B^G, \mathbb{F}_q[\mathbf{x}]\right) \cong \mathbb{F}_q[\mathbf{x}]^B$$

which is $\mathbb{F}_q[\mathbf{x}]^G$-linear, hence inducing an $H_n(0)$-module structure on $\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)$. Our next result is:

- $\mathcal{F}\left(1_B^G\right) = \sum_I r_I(q) F_I$;
- $\mathcal{F}_t\left(\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)\right) = \sum_I r_I(q,t) F_I$.

Therefore we have another picture as follows, which interprets all the ribbon numbers mentioned at the beginning.

$$
\begin{tikzcd}
 & \boxed{r_I \xlongequal{\text{Krob-Thibon}} \mathrm{mult}_{C_I}(H_n(0))} & \\
\boxed{\begin{array}{c} r_I(q) \xlongequal{\text{New}} \\ \mathrm{mult}_{C_I}(1_B^G) \end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q\leftrightarrow t"] & & \boxed{\begin{array}{c} r_I(t) \xlongequal{\text{New}} \\ \mathrm{grmult}_{C_I}\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right) \end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
 & \boxed{\begin{array}{c} r_I(q,t) \xlongequal{\text{New}} \\ \mathrm{grmult}_{C_I}\left(\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)\right) \end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "t\,\to\, t^{\frac{1}{q-1}}"', "q\,\to\,1"] &
\end{tikzcd}
$$

1.4 Generalizations to other types

The $H_n(0)$-actions on the coinvariants and complete flags can be generalized to the following setting. Let W be a Weyl group with weight lattice Λ. The 0-Hecke algebra $\mathscr{H}$ of type W acts on the group ring $\mathbb{Z}[\Lambda]$ via the operators $\overline{\pi}_i = \pi_i - 1$ where π_i was originally considered by Demazure [2] in this setting. Garsia and Stanton [9] constructed the descent monomials in $\mathbb{Z}[\Lambda]$, which form a free basis over $\mathbb{Z}[\Lambda]^W$. We prove that, similarly to $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$, taking a quotient of $\mathbb{Z}[\Lambda]$ by the ideal generated by "positive" W-invariants leads to an $\mathscr{H}$-module isomorphic to $\mathscr{H}$ (without an extra grading). We also have an $\mathscr{H}$-action on 1_B^G by $T_w B = BwB$, where G is a finite group with split BN-pair of characteristic $p > 0$, whose Weyl group is W. We determine the characteristic $\mathcal{F}(1_B^G)$ in the same manner as for type A.

In Section 2 we review the definitions for the various ribbon numbers and their interpretation by representations of $\mathfrak{S}_n$ and $GL(n,\mathbb{F}_q)$. In Section 3 we recall the representation theory of the 0-Hecke algebra. The result for the coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ is given in Section 4, and a similar result for $\mathbb{Z}[\Lambda]$ of the weight lattice Λ of a Weyl group is stated in Section 5. The $\mathscr{H}$-action on the complete flag variety 1_B^G and the coinvariant algebra of (G,B) is investigated in Section 6. Lastly some questions are asked in Section 7.

2 Ribbon numbers

We recall from Reiner and Stanton [15, §9, §10] the definitions and properties of the various ribbon numbers. Let $I = (i_1, \ldots, i_k)$ be a composition of n, let $\sigma_j = i_1 + \cdots + i_j$ for $j = 1, \ldots, k$, and let the *descent set* of I be $D(I) = \{\sigma_1, \ldots, \sigma_{k-1}\}$.

It is well known that compositions of n bijectively correspond to the subsets of $[n-1]$ via their descent set; they also bijectively correspond to the *ribbon diagrams*, *i.e.* connected skew Young diagrams without 2×2 boxes, whose row sizes from bottom to top are $i_1, \ldots, i_n$.

The *descent class* of I is the set of all permutations w in $\mathfrak{S}_n$ with $D(w) = D(I)$, and the *inverse descent class* is the set of all w in $\mathfrak{S}_n$ with $D(w^{-1}) = D(I)$. The *ribbon number* r_I is the cardinality of the descent class of I, and the q*-ribbon number* and t*-ribbon number* are

$$\begin{aligned} r_I(q) &= \sum_{w\in\mathfrak{S}_n : D(w)=D(I)} q^{\,\mathrm{inv}(w)} = [n]!_q \det\left(\frac{1}{[\sigma_j-\sigma_{i-1}]!_q}\right)_{i,j=1}^k, \\ r_I(t) &= \sum_{w\in\mathfrak{S}_n : D(w)=D(I)} t^{\,\mathrm{maj}(w^{-1})} = [n]!_t \det\left(\frac{1}{[\sigma_j-\sigma_{i-1}]!_t}\right)_{i,j=1}^k. \end{aligned}$$

where $[n]!_q = [n]_q[n-1]_q\cdots[1]_q$ and $[n]_q = 1+q+\cdots+q^{n-1}$. The notations make sense because $r_I(t) = r_I(q)|_{q=t}$, which is a consequence of the equidistribution of inv and maj on every inverse descent class, proved by Foata and Schützenberger [7]. A further generalization is the (q,t)*-ribbon number*

$$r_I(q,t) = n!_{q,t} \det\left(\varphi^{\sigma_{i-1}}\frac{1}{(\sigma_j-\sigma_{i-1})!_{q,t}}\right)_{i,j=1}^k$$

where $n!_{q,t} = (1-t^{q^n-1})(1-t^{q^n-q})\cdots(1-t^{q^n-q^{n-1}})$, and $\varphi : t\mapsto t^q$ is the *Frobenius operator*.

All these ribbon numbers can be calculated by similar determinantal formulae, and are interpreted by certain *homology representations* χ^I of $\mathfrak{S}_n$, and χ^I_q of $G = GL(n,\mathbb{F}_q)$, together with their intertwiners

$$M^I = \mathrm{Hom}_{\mathbb{Z}\mathfrak{S}_n}\left(\chi^I, \mathbb{Z}[\mathbf{x}]\right),\ \text{a } \mathbb{Z}[\mathbf{x}]^{\mathfrak{S}_n}\text{-module},$$

$$M^I_q = \mathrm{Hom}_{\mathbb{F}_qG}\left(\chi^I_q, \mathbb{F}_q[\mathbf{x}]\right),\ \text{an } \mathbb{F}_q[\mathbf{x}]^{G}\text{-module}.$$

Here χ^I (χ^I_q resp.) is the top homology of the *rank-selected Coxeter complex* $\Delta(\mathfrak{S}_n)_I$ (*Tits building* $\Delta(G)_I$ resp.). Precisely, one has the following picture.

$$\begin{tikzcd}
 & \boxed{r_I = \dim\chi^I} & \\
\boxed{\begin{array}{c} r_I(q) = \\ \dim\chi^I_q \end{array}} \arrow[ur, squiggly, "q\to 1"] \arrow[rr, leftrightarrow, "q\leftrightarrow t"] & & \boxed{\begin{array}{c} r_I(t) = \\ \mathrm{Hilb}(M^I/\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n}M^I, t) \end{array}} \arrow[ul, squiggly, "t\to 1"'] \\
 & \boxed{\begin{array}{c} r_I(q,t) = \\ \mathrm{Hilb}(M^I_q/\mathbb{F}_q[\mathbf{x}]_+^{G}M^I_q, t) \end{array}} \arrow[ul, squiggly, "t\to 1"] \arrow[ur, squiggly, "{t\to t^{\frac{1}{q-1}}},\ q\to 1"'] &
\end{tikzcd}$$

3 Representation theory of the 0-Hecke algebra

Recall from Norton [14] the representation theory of the 0-Hecke algebra. Let

$$W = \langle s_1,\ldots,s_\ell : s_i^2 = 1, (s_is_js_i\cdots)_{m_{ij}} = (s_js_is_j\cdots)_{m_{ij}}, 1\le i\ne j\le \ell\rangle$$

be a finite Coxeter group, where $(aba\cdots)_m$ denotes an alternating product of m terms. The 0-Hecke algebra $\mathscr{H}$ of type W is an associative $\mathbb{Z}$-algebra generated by $T_1,\dots,T_\ell$ with relations

$$\begin{cases} T_i^2 = -T_i, & 1\le i\le \ell,\\ (T_iT_jT_i\cdots)_{m_{ij}} = (T_jT_iT_j\cdots)_{m_{ij}}, & 1\le i\ne j\le \ell. \end{cases}$$

Norton [14] decomposed $\mathscr{H}$ into a direct sum of 2^ℓ distinct indecomposable submodules M_I indexed by compositions I of $\ell+1$, with $C_I = \mathrm{top}(M_I) = M_I/\mathrm{rad}\,M_I$ being the (one-dimensional) simple module given by

$$\rho_I(T_i) = \begin{cases} -1, & \text{if } i\in D(I),\\ 0, & \text{if } i\notin D(I). \end{cases}$$

This gives a complete list of indecomposable projective $\mathscr{H}$-modules and simple $\mathscr{H}$-modules.

To explicitly construct M_I inside $\mathscr{H}$, let $T_i' = T_i+1$, $1\le i\le\ell$. One can check that $(T_i')^2 = T_i'$, *i.e.* $T_iT_i' = 0$, and $(T_i'T_j'T_i'\cdots)_{m_{ij}} = (T_j'T_i'T_j'\cdots)_{m_{ij}}$, $1\le i\ne j\le\ell$; see [5, Lemma 3.1] or [14, Lemma 4.3]. Thus $T_w = T_{i_1}\cdots T_{i_k}$ and $T_w' = T_{i_1}'\cdots T_{i_k}'$ are both well-defined if $w = s_{i_1}\cdots s_{i_k}$ is reduced.

Given a composition $I = (i_1,\dots,i_k)$, let $\bar I = (i_k,\cdots,i_1)$ and let $I^\sim$ be the conjugate composition of I obtained by reflecting the ribbon diagram of I across the diagonal. For example, if $I=(2,1,3)$ then $\bar I = (3,1,2)$ and $I^\sim = (1,1,3,1)$. The descent class of I is an interval $[\alpha(I),\omega(I)]$ in the left weak order of W, where $\alpha(I)$ is the top element in the parabolic subgroup $W_{D(I)}$. One can write the module M_I in Norton's decomposition of $\mathscr{H}$ as $M_I = \mathscr{H}\cdot T_{\alpha(I)}T'_{\alpha(\bar I^\sim)}$, which has a $\mathbb{Z}$-basis given by

$$\left\{ T_wT'_{\alpha(\bar I^\sim)} : w\in[\alpha(I),\omega(I)] \right\}.$$

4 Coinvariant algebra of $\mathfrak{S}_n$

The symmetric group $\mathfrak{S}_n$ acts naturally on the polynomial ring $\mathbb{Z}[\mathbf{x}] = \mathbb{Z}[x_1,\dots,x_n]$ by permuting variables. The *Demazure operators* π_i are defined by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}$$

where s_i is the adjacent transposition $(i,i+1)$. One checks that the operators $\overline{\pi}_i = \pi_i - 1$ satisfy the same relations as T_i, preserve the polynomial grading of $\mathbb{Z}[\mathbf{x}]$, and are $\mathbb{Z}[\mathbf{x}]^{\mathfrak{S}_n}$-linear. Thus one has an $H_n(0)$-action on the coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ by sending T_i to $\overline{\pi}_i$, or sending T_w to $\overline{\pi}_w = \overline{\pi}_{i_1}\cdots\overline{\pi}_{i_k}$ if $w = s_{i_1}\cdots s_{i_k}$ is reduced.

One sees from the definition that $\overline{\pi}_i$ fixes all variables x_j except x_i and x_{i+1}, and

$$\overline{\pi}_i(x_i^a x_{i+1}^b) = \begin{cases} x_i^{a-1}x_{i+1}^{b+1} + x_i^{a-2}x_{i+1}^{b+2}\cdots + x_i^b x_{i+1}^a, & \text{if } a>b,\\ 0, & \text{if } a=b,\\ -x_i^a x_{i+1}^b - x_i^{a+1}x_{i+1}^{b-1} - \cdots - x_i^{b-1}x_{i+1}^{a+1}, & \text{if } a<b. \end{cases} \tag{1}$$

The module $\mathbb{Z}[\mathbf{x}]$ is free over $\mathbb{Z}[\mathbf{x}]^{\mathfrak{S}_n}$, with a basis consisting of the *descent monomials*

$$x_w = \prod_{i\in D(w)} x_{w(1)}\cdots x_{w(i)},\quad w\in\mathfrak{S}_n,$$

constructed by Garsia [8]. We shall obtain a $\mathbb{Z}$-basis for the $H_n(0)$-module $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ from certain *Demazure atoms*, *i.e.* $\overline{\pi}_w x_I$ for certain permutations w and compositions I, where

$$x_I = \prod_{i \in D(I)} x_1 \cdots x_i.$$

See Mason [13] for more information on the Demazure atoms. The leading terms of these Demazure atoms are exactly the descent monomials, under any linear extension of the following partial order. Given two monomials x^d and x^e, say $x^d \prec x^e$ if $\lambda(d) <_L \lambda(e)$ in the lexicographic order, where $\lambda(d)$ is the unique partition obtained from rearranging the exponent vector d, and similarly for $\lambda(e)$.

Lemma 4.1 *Suppose that I is a composition of n and w is a permutation in $\mathfrak{S}_n$ with $D(w) \subseteq D(I)$. Then*

$$\overline{\pi}_w x_I = w x_I + \sum_{d:\, x^d \prec x_I} c_d x^d$$

where $c_d \in \mathbb{Z}$; moreover, wx_I is a descent monomial if and only if $D(w) = D(I)$.

Theorem 4.2 *The coinvariant algebra $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ has a $\mathbb{Z}$-basis*

$$\{\overline{\pi}_w \overline{x}_I : w \in \mathfrak{S}_n, D(I) = D(w)\} \tag{2}$$

and decomposes into a direct sum of $H_n(0)$-modules

$$N_I = H_n(0) \cdot \overline{\pi}_{\alpha(I)} \overline{x}_I$$

as I runs through all compositions of n; moreover, N_I is isomorphic to $M_I \subseteq H_n(0)$ and has a basis $\{\overline{\pi}_w \overline{x}_I : w \in [\alpha(I), \omega(I)]\}$. Thus $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$ is isomorphic to $H_n(0)$ as a left $H_n(0)$-module.

Since each submodule N_I consists of homogeneous elements of degree $\mathrm{maj}(w)$ for any $w \in [\alpha(I), \omega(I)]$, one has

$$\mathcal{F}_{q,t}\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right) = \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} q^{\mathrm{inv}(w)} F_{D(w^{-1})} \tag{3}$$

which relates inv to the equidistributed maj (on every inverse descent class). In fact, one can get

$$\mathcal{F}_t\left(\mathbb{Z}[\mathbf{x}]\right) = \frac{1}{(1-t)(1-t^2)\cdots(1-t^n)} \sum_{w \in \mathfrak{S}_n} t^{\mathrm{maj}(w)} F_{D(w^{-1})} \tag{4}$$

directly from (1) using the filtration of $\mathbb{Z}[\mathbf{x}]$ induced from the following order (which appeared in Allen [1]): $x^d <_{ts} x^e$ if $\lambda(d) <_L \lambda(e)$, or if $\lambda(d) = \lambda(e)$ and $x^d <_L x^e$ in the lexicographic order. From (4) one can immediately deduce $\mathcal{F}_t\left(\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})\right)$, since $\overline{\pi}_i$ is $\mathbb{Z}[\mathbf{x}]^{\mathfrak{S}_n}$-linear.

5 Coinvariant algebra of Weyl group

Demazure's character formula [2] expresses the character of highest weight modules over a semisimple algebra using the *Demazure operators* π_i on the group ring $\mathbb{Z}[\Lambda]$ of the weight lattice Λ. If e^λ is the element in $\mathbb{Z}[\Lambda]$ corresponding to the weight $\lambda \in \Lambda$, and if $\lambda_1, \dots, \lambda_\ell$ denote the fundamental weights with $z_i = e^{\lambda_i}$, then

$$\mathbb{Z}[\Lambda] = \mathbb{Z}[z_1, \dots, z_\ell, z_1^{-1}, \dots, z_\ell^{-1}].$$

Suppose that $\alpha_1, \dots, \alpha_\ell$ are the simple roots which correspond to simple reflections $s_1, \dots, s_\ell$. Then the Demazure operators are defined by

$$\pi_i = \frac{1 - e^{-\alpha_i} s_i}{1 - e^{-\alpha_i}}.$$

It follows easily that

$$\pi_i(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda - \alpha_i} + \cdots + e^{s_i \lambda}, & \text{if } \langle \alpha_i^\vee, \lambda \rangle \geq 0, \\ 0, & \text{if } \langle \alpha_i^\vee, \lambda \rangle = -1, \\ -e^{\lambda + \alpha_i} - \cdots - e^{s_i \lambda - \alpha_i}, & \text{if } \langle \alpha_i^\vee, \lambda \rangle < -1. \end{cases} \tag{5}$$

See, for example, Kumar [12]. Demazure operators satisfy the braid relation [2, 5.5] and

$$s_i \pi_i = \pi_i, \ \ \pi_i^2 = \pi_i.$$

Hence the 0-Hecke algebra $\mathscr{H}$ of W acts on $\mathbb{Z}[\Lambda]$ by sending T_i to $\overline{\pi}_i = \pi_i - 1$.

Using the Stanley-Reisner ring of the Coxeter complex of W, Garsia and Stanton [9] showed that

$$\mathbb{Z}[\Lambda]^W = \mathbb{Z}[a_1, \dots, a_\ell]$$

where

$$a_i = \sum_{w \in W / W_{\{i\}^c}} e^{w \lambda_i}$$

and $\mathbb{Z}[\Lambda]$ has a free basis over $\mathbb{Z}[\Lambda]^W$, which consists of the *descent monomials*

$$z_w = \prod_{i \in D(w)} e^{w \lambda_i}, \ w \in W$$

(see also Steinberg [17]). This basis induces a $\mathbb{Z}$-basis for $\mathbb{Z}[\Lambda]/(a_1, \dots, a_\ell)$.

The $\mathscr{H}$-action on $\mathbb{Z}[\Lambda]$ is $\mathbb{Z}[\Lambda]^W$-linear, hence inducing an $\mathscr{H}$-action on $\mathbb{Z}[\Lambda]/(a_1, \dots, a_\ell)$.

For any weight λ in Λ, there exists a unique *dominant weight* (which is a nonnegative linear combination of $\lambda_1, \dots, \lambda_\ell$), denoted by $[\lambda]$, such that $\lambda = w[\lambda]$ for some w in W.

Lemma 5.1 *Given a composition I of $\ell + 1$, let $z_I = e^{\lambda_I}$ where*

$$\lambda_I = \sum_{i \in D(I)} \lambda_i.$$

Suppose that w lies in W with $D(w) \subseteq D(I)$. Then

$$\overline{\pi}_w z_I = e^{w \lambda_I} + \sum_{[\lambda] < \lambda_I} c_\lambda e^\lambda, \quad c_\lambda \in \mathbb{Z},$$

where $[\lambda] < \lambda_I$ means $\lambda_I - [\lambda]$ is a nonnegative linear combination of simple roots and $[\lambda] \neq \lambda_I$; moreover, $e^{w \lambda_I}$ is a descent monomial if and only if $D(w) = D(I)$.

Theorem 5.2 *The coinvariant algebra* $\mathbb{Z}[\Lambda]/(a_1,\dots,a_\ell)$ *has a* $\mathbb{Z}$*-basis*

$$\{\overline{\pi}_w \overline{z}_I : w \in W, D(I) = D(w)\}$$

and decomposes into a direct sum of $\mathscr{H}$*-modules*

$$N_I = \mathscr{H} \cdot \overline{\pi}_{\alpha(I)} \overline{z}_I$$

as I *runs through all compositions of* $\ell+1$*; moreover,* N_I *is isomorphic to* $M_I \subseteq \mathscr{H}$ *and has a basis* $\{\overline{\pi}_w \overline{z}_I : w \in [\alpha(I), \omega(I)]\}$ *. Thus* $\mathbb{Z}[\Lambda]/(a_1,\dots,a_\ell)$ *is isomorphic to* $\mathscr{H}$ *as an* $\mathscr{H}$*-module.*

Remark 5.3 Garsia and Stanton [9] pointed out a way to reduce the descent monomials in $\mathbb{Z}[\Lambda]$ to the descent monomials in $\mathbb{Z}[\mathbf{x}]$ for type A. However, it does not give Theorem 4.2 directly from Theorem 5.2; instead, one should consider the Demazure operators on $\mathbb{Z}[X(T)]$ where $X(T)$ is the character group of the maximal torus T of $GL(n,\mathbb{C})$.

6 Complete flag variety 1_B^G and coinvariants of (G,B)

Let G be a finite group with split BN-pair of characteristic $p > 0$, whose Weyl group W is generated by ℓ simple reflections. Let $1_B^G = \mathbb{Z}[B \setminus G]$ be the induction of the right trivial representation of B to G, *i.e.* the permutation representation on the right cosets $B \setminus G$.

Given a subset $S \subseteq G$, let $\overline{S} = \sum_{s\in S} s$ in $\mathbb{Z}[G]$. Then $1_B^G \cong \overline{B} \cdot \mathbb{Z}[G]$ and $\mathrm{End}_{\mathbb{Z}[G]}(1_B^G)$ has a basis $\{f_w : w \in \mathfrak{S}_n\}$, with f_w given by

$$f_w(\overline{B}) = \overline{BwB} = \overline{U}_w w \overline{B} \tag{6}$$

where U_w is the product of root subgroups U_α with $\alpha > 0, w^{-1}(\alpha) < 0$ [3, Proposition 1.7]. The endomorphism ring $\mathrm{End}_{\mathbb{Z}[G]}(1_B^G)$ is isomorphic to the Hecke algebra of W with parameters $q_i = |U_{s_i}|$, since the relations satisfied by $\{f_w\}$ are the same as those satisfied by the standard basis $\{T_w\}$. It follows that the 0-Hecke algebra $\mathscr{H}$ of type W acts on $1_B^G \otimes \mathbb{F}_p$ by (6). Dually, the left cosets of B give rise to a right $\mathscr{H}$-action. See Kuhn [11] for details.

Since we are mainly concerned with the 0-Hecke algebra, we shall write $1_B^G = 1_B^G \otimes \mathbb{F}_p$ for simplicity, and similar for $1_{P_I}^G$ where $P_I = BW_{D(I)^c}B$ is the *parabolic subgroup* of G indexed by the composition I of $\ell+1$. For type A, one has $G = GL(n,\mathbb{F}_q)$ and P_I is the group of all upper triangular block matrices with invertible diagonal blocks of sizes given by the parts of I.

To determine the simple factors of an $\mathscr{H}$-module, we develop the following lemma, where the (graded) characteristic is a natural extension of that of type A, with the F_I's simply being independent variables.

Lemma 6.1 *Given a finite dimensional graded* $\mathscr{H}$*-module* Q*, let* Q_I *be the submodule of elements that are annihilated by all* T_j *with* $j \notin D(I)$*, for any composition* I *of* $\ell+1$*. Then*

$$\mathcal{F}_t(Q) = \sum_I c_I(Q) F_I,$$

summed over all compositions I *of* $\ell+1$*, where*

$$c_I(Q) = \sum_{J : D(J) \subseteq D(I)} (-1)^{\ell(I,J)} \mathrm{Hilb}(Q_J, t).$$

Define the *q-ribbon number of type G* to be

$$r_I(G) = \sum_{w \in W : D(w) = D(I)} |U_w|.$$

The characteristic of 1_B^G can be obtained by using Lemma 6.1 and the observations $\left(1_B^G\right)_I = 1_{P_I}^G$,

$$|G/P_I| = \sum_{w \in W : D(w) \subseteq D(I)} |U_w|.$$

Theorem 6.2 $\mathcal{F}\left(1_B^G\right) = \sum_I r_I(G) F_I$.

If G is a finite group of Lie type over a finite field $\mathbb{F}_q$, then $|U_w| = q^{\ell(w)}$. In particular, for type A, *i.e.* when $G = GL(n, \mathbb{F}_q)$, one has $r_I(G) = r_I(q)$. Furthermore, G acts on $\mathbb{F}_q[\mathbf{x}]$ and the $H_n(0)$-action on 1_B^G induces an $H_n(0)$-module structure on

$$\mathrm{Hom}_{\mathbb{F}_q[G]}\left(1_B^G, \mathbb{F}_q[\mathbf{x}]\right) \cong \mathbb{F}_q[\mathbf{x}]^B$$

which is $\mathbb{F}_q[\mathbf{x}]^G$-linear, hence inducing an $H_n(0)$-module structure on $\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)$. Explicitly,

$$T_w(f) = \overline{U}_w w f, \ \text{ for all } f \text{ in } \mathbb{F}_q[\mathbf{x}]^B.$$

We observe that

$$\left(\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)\right)_I = \left\{\overline{f} : f \in \mathbb{F}_q[\mathbf{x}]^{P_I}\right\}.$$

It follows from Lemma 6.1 that

Theorem 6.3 *If $G = GL(n, \mathbb{F}_q)$ and B is the Borel subgroup of G, then*

$$\mathcal{F}_t\left(\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)\right) = \sum_I r_I(q,t) F_I.$$

We conclude this section with a description of the homology representation χ_q^I using the $(G, \mathscr{H})$-bimodule structure on 1_B^G, which is now spanned by left B-cosets. Smith [16] showed the following left G-module decomposition

$$1_B^G = \bigoplus_J \chi_q^J. \tag{7}$$

On the other hand, the decomposition of the 0-Hecke algebra

$$\mathscr{H} = \bigoplus_I \mathscr{H} \cdot T_{\alpha(I)} T'_{\alpha(\bar{I}^\sim)}$$

implies a unique way to write

$$1 = \sum_I h_I T_{\alpha(I)} T'_{\alpha(\bar{I}^\sim)}, \ h_I \in \mathscr{H}.$$

By the right action of $\mathscr{H}$ on 1_B^G, one has

$$1_B^G = \sum_I 1_B^G h_I T_{\alpha(I)} T'_{\alpha(\bar{I}^\sim)} \tag{8}$$

as a left G-module. We show that this is the same as the decomposition (7), that is,

Theorem 6.4

$$1_B^G h_I T_{\alpha(I)} T'_{\alpha(\bar{I}^\sim)} = \chi_q^I.$$

One sees from the above equality that the left G-module χ_q^I is in general not a right $\mathscr{H}$-module; however, the trivial G-representation $\chi_q^{(n)}$ and the Steinberg representation $\chi_q^{(1^n)}$ are right (isotypic) $\mathscr{H}$-modules.

7 Remaining Questions

The equidistribution of inv and maj was first proved on permutations of multisets by P.A. MacMahon in the 1910s; applying an inclusion-exclusion would give their equidistribution on inverse descent classes of $\mathfrak{S}_n$. However, the first proof for the latter result appearing in the literature was by Foata and Schützenberger [7] in 1970, using a bijection constructed earlier by Foata [6]. Is there an algebraic proof from the (q,t)-bigraded characteristic (3) of $\mathbb{Z}[\mathbf{x}]/(\mathbb{Z}[\mathbf{x}]_+^{\mathfrak{S}_n})$, which involves inv, maj, and inverse descents?

The next question is on the $\mathscr{H}$-module 1_B^G; its simple factors are given by Theorem 6.2, but its decomposition into indecomposable submodules is *not* obtained in general. Assume $G = GL(n, \mathbb{F}_q)$ below. Recall that

$$H_3(0) = M_3 \oplus M_{21} \oplus M_{12} \oplus M_{111}$$

where M_I is the indecomposable projective $H_3(0)$-module indexed by the composition I. For $n = 3$ we have candidates for a q-analogous $H_3(0)$-module decomposition

$$1_B^G \cong M_3 \oplus (M_{21} \oplus M_{12})^{\oplus \binom{q+1}{2}} \oplus (M_{111})^{\oplus q^3}$$

which has been checked correct for $q = 2, 3, 5, 7$, showing that 1_B^G is a projective $H_3(0)$-modules in these cases. Is this true for all *primes* q?

On the other hand, for $n = 3$, $q = 4, 8$, and $n = 4$, $q = 2, 3$, computations show that 1_B^G is *not* projective, although the characteristic of 1_B^G is always symmetric. In fact, using the RSK correspondence one can show that

$$\mathcal{F}(1_B^G) = \sum_{\lambda \vdash n} q^{b(\lambda)} \frac{[n]!_q}{\prod_{u \in \lambda} [h_u]_q} s_\lambda.$$

where h_u is the hook length of the box u in the Young diagram of λ and $b(\lambda) = \lambda_2 + 2\lambda_3 + 3\lambda_4 + \cdots$.

Computations also show that $\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)$ is *not* projective for $n = 3$, $q = 2, 3$. Besides being curious to know its decomposition, we wonder if there is any q-analogue of the Demazure operators, which might give another $H_n(0)$-action on $\mathbb{F}_q[\mathbf{x}]^B/(\mathbb{F}_q[\mathbf{x}]_+^G)$.

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The Incidence Hopf Algebra of Graphs

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Abstract. The *graph algebra* is a commutative, cocommutative, graded, connected incidence Hopf algebra, whose basis elements correspond to finite simple graphs and whose Hopf product and coproduct admit simple combinatorial descriptions. We give a new formula for the antipode in the graph algebra in terms of acyclic orientations; our formula contains many fewer terms than Schmitt's more general formula for the antipode in an incidence Hopf algebra. Applications include several formulas (some old and some new) for evaluations of the Tutte polynomial.

Résumé. *L'algèbre de graphes* est une algèbre d'incidence de Hopf commutative, cocommutative, graduée, et connexe, dont les éléments de base correspondent à des graphes finis simples et dont le produit et coproduit de Hopf admettent une description combinatoire simple. Nous présentons une nouvelle formule de l'antipode dans l'algèbre de graphes utilisant les orientations acycliques; notre formule contient beaucoup moins de termes que la formule générale de Schmitt pour l'antipode dans une algèbre d'incidence de Hopf. Les applications incluent plusieurs formules (connues et inconnues) pour les évaluations du polynôme de Tutte.

Keywords: combinatorial Hopf algebra, graph, chromatic polynomial, Tutte polynomial, acyclic orientation

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1 Introduction

The *graph algebra* $\mathcal{G}$ is a commutative, cocommutative, graded, connected Hopf algebra, whose basis elements correspond to finite simple graphs G, and whose Hopf product and coproduct admit simple combinatorial descriptions. The graph algebra was first considered by Schmitt in the context of incidence Hopf algebras [Sch94, §12] and furnishes an important example in the work of Aguiar, Bergeron and Sottile [ABS06, Example 4.5].

We derive a new formula (Theorem 3.1) for the Hopf antipode in $\mathcal{G}$. Our formula is specific to the graph algebra in that it involves acyclic orientations; therefore, it is not a consequence of Schmitt's general formula [Sch94, Thm. 4.1] for the antipode in an incidence Hopf algebra. Our formula turns out to be well suited for studying graph invariants, including the Tutte polynomial $T_G(x, y)$ and various specializations of it. The idea is to make $\mathcal{G}$ into a combinatorial Hopf algebra in the sense of Aguiar, Bergeron and Sottile [ABS06] by defining a character on it, then to define a graph invariant by means of a Hopf morphism to a polynomial ring. The antipode formula leads to combinatorial interpretations of the convolution inverses of several natural characters. When we view the Tutte polynomial itself as a character, its k-th convolution power itself is a Tutte evaluation at rational functions in x, y, k (Theorem 4.1). This implies several well-known formulas such as Stanley's formula for acyclic orientations in terms of the chromatic polynomial [Sta73]. Further enumerative consequences of Theorem 4.1 include interpretations of less familiar specializations of the Tutte polynomial (for example, $T_G(3,2)$), as well as an unusual-looking reciprocity relation between complete graphs of different sizes (Eqns. (15) and (16)).

This is an extended abstract of the full paper [HM10], containing background material and theorems but no proofs. Subsequently to writing this paper, we learned [Agu] that Aguiar and Ardila have independently discovered a more general antipode formula than ours, in the context of Hopf monoids (for which see [AM10]); their work will appear in a forthcoming paper.

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2 Hopf algebras

2.1 Basic definitions

We briefly review the basic facts about Hopf algebras, omitting the proofs. Good sources for the full details include Sweedler [Swe69] and (for combinatorial Hopf algebras) Aguiar, Bergeron and Sottile [ABS06].

Fix a field $\mathbb{F}$ (typically $\mathbb{C}$). A *bialgebra* $\mathcal{H}$ is a vector space over $\mathbb{F}$ equipped with linear maps

$$m : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}, \qquad u : \mathbb{F} \to \mathcal{H}, \qquad \Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}, \qquad \epsilon : \mathcal{H} \to \mathbb{F},$$

respectively the *multiplication*, *unit*, *comultiplication*, and *counit*, such that the following properties are satisfied: (1) $m\circ(m\otimes I) = m\circ(I\otimes m)$ (associativity); (2) $m\circ(u\otimes I) = m\circ(I\otimes u) = I$ (where I is the identity map on $\mathcal{H}$); (3) $(\Delta\otimes I)\circ\Delta = (I\otimes\Delta)\circ\Delta$ (coassociativity); (4) $(\epsilon\otimes I)\circ\Delta = (I\otimes\epsilon)\circ D = I$; and (5) Δ and ϵ are multiplicative (equivalently, m and u are comultiplicative). If there exists a bialgebra automorphism $S : \mathcal{H} \to \mathcal{H}$ such that $m\circ(S\otimes I)\circ\Delta = m\circ(I\otimes S)\circ\Delta = u\circ\epsilon$, we say that $\mathcal{H}$ is a *Hopf algebra*, and S is its *antipode*(i)

A Hopf algebra $\mathcal{H}$ is *graded* if $\mathcal{H} = \bigoplus_{n\geq 0}\mathcal{H}_n$ as vector spaces, and multiplication and comultiplication respect this decomposition, i.e.,

$$m(\mathcal{H}_i \otimes \mathcal{H}_j) \subseteq \mathcal{H}_{i+j} \quad \text{and} \quad \Delta(\mathcal{H}_k) \subseteq \sum_{i+j=k} \mathcal{H}_i \otimes \mathcal{H}_j.$$

Meanwhile, $\mathcal{H}$ is *connected* if $\dim(\mathcal{H}_0) = 1$. If $\mathcal{H}$ is a graded and connected bialgebra, then its antipode can be defined inductively as follows: $S(h) = h$ for $h \in \mathcal{H}_0$, and, then $(m\circ(S\otimes I)\circ\Delta)(h) = 0$ for $h \in \mathcal{H}_i$, $i > 0$. Most (if not all) of the Hopf algebras arising naturally in combinatorics are graded and connected, and every algebra we consider henceforth will be assumed to have these properties.

A *character* of a Hopf algebra $\mathcal{H}$ is a multiplicative linear map $\phi : \mathcal{H} \to \mathbb{F}$. The *convolution product* of two characters is $\phi * \psi = (\phi\otimes\psi)\circ\Delta$. That is, if $\Delta h = \sum_i h_1^{(i)} \otimes h_2^{(i)}$, then $(\phi*\psi)(h) = \sum_i \phi(h_1^{(i)})\psi(h_2^{(i)})$. (This formula can be writen more concisely in Sweedler notation: if $\Delta h = \sum h_1 \otimes h_2$, then $(\phi*\psi)(h) = \sum \phi(h_1)\psi(h_2)$.) Convolution makes the set of characters $\mathbb{X}(\mathcal{H})$ into a group, with identity ϵ and inverse given by $\phi^{-1} = \phi\circ S$. There is a natural involutive automorphism $\phi \mapsto \bar{\phi}$ of $\mathbb{X}(\mathcal{H})$ given by $\bar{\phi}(h) = (-1)^n\phi(h)$ for $h \in \mathcal{H}_n$. If $\mathcal{H}$ is a graded connected Hopf algebra and $\zeta \in \mathbb{X}(H)$, then the pair $(\mathcal{H},\zeta)$ is called a *combinatorial Hopf algebra*, or CHA for short. A *morphism* of CHAs $\Phi : (\mathcal{H},\zeta) \to (\mathcal{H}',\zeta')$ is a linear transformation $\mathcal{H} \to \mathcal{H}'$ that is a morphism of Hopf algebras (i.e., a linear transformation that preserves the operations of a bialgebra) such that $\zeta\circ\Phi = \zeta'$.

(i) It can be shown that S is the unique automorphism of $\mathcal{H}$ with this property.

2.2 The binomial and graph Hopf algebras

The *binomial Hopf algebra* is the ring of polynomials $\mathbb{F}[k]$ in one variable k, with the usual multiplicative structure; comultiplication $\Delta(f(k)) = f(k\otimes 1 + 1\otimes k)$; counit $\epsilon(f(k)) = \epsilon_0(f(k)) = f(0)$; and character $\epsilon_1(f(k)) = f(1)$. The following proposition is a consequence of work of Aguiar, Bergeron, and Sottile [ABS06, Thm. 4.1].

Proposition 2.1 (Polynomiality) *Every combinatorial Hopf algebra $(\mathcal{H}, \zeta)$ has a unique CHA morphism to $(\mathbb{F}[k], \epsilon_1)$.*

We regard this Hopf morphism as a way to associate a polynomial invariant $P_{\zeta,h}(k) = \zeta^k(h) \in \mathbb{F}[k]$ with each element $h \in \mathcal{H}$. In fact, Aguiar, Bergeron, and Sottile proved something much stronger: the algebra Q of quasisymmetric functions is a terminal object in the category of CHAs, i.e., every CHA has a unique morphism to Q. Composing this morphism with the principal specialization(ii) gives the morphism of Proposition 2.1. We will not use the full power of the Aguiar–Bergeron–Sottile theorem (which can be viewed as a way to associate a quasisymmetric-function invariant to each element of $\mathcal{H}$). Note that for $k \in \mathbb{Z}$, the identity $\zeta^k(h) = P_{\zeta,h}(k)$ follows from the definition of a CHA morphism; therefore, it is actually an identity of polynomials in k.

The *graph algebra*(iii) is the $\mathbb{F}$-vector space $\mathcal{G} = \bigoplus_{n\geq 0} \mathcal{G}_n$, where $\mathcal{G}_n$ is the linear span of isomorphism classes of simple graphs on n vertices. This is a graded connected Hopf algebra, with multiplication $m(G\otimes H) = G\cdot H = G\uplus H$ (where $\uplus$ denotes disjoint union); unit $u(1) = \emptyset$ (the graph with no vertices); comultiplication $\Delta(G) = \sum_{T\subseteq V(G)} G|_T \otimes G|_{\bar{T}}$ (where $G|_T$ denotes the induced subgraph on vertex set T, and $\bar{T} = V(G) \setminus T$); and counit

$$\epsilon(G) = \begin{cases} 1 & \text{if } G = \emptyset, \\ 0 & \text{if } G \neq \emptyset. \end{cases}$$

This Hopf algebra is commutative and cocommutative; in particular, its character group $\mathbb{X}(G)$ is abelian. As proved by Schmitt [Sch94, eq. (12.1)], the antipode in $\mathcal{G}$ is given by $S(G) = \sum_\pi (-1)^{|\pi|}|\pi|!G_\pi$, where the sum runs over all ordered partitions π of $V(G)$ into nonempty sets (or "blocks"), and G_π is the disjoint union of the induced subgraphs on the blocks. Here we have *two* canonical involutions on characters:

$$\bar{\phi}(G) = (-1)^{n(G)}\phi(G), \qquad \tilde{\phi}(G) = (-1)^{\mathrm{rk}(G))}\phi(G),$$

where $\mathrm{rk}(G)$ denotes the graph rank of G (that is, the number of edges in a spanning tree). (Note that $\phi \mapsto \tilde{\phi}$ is *not* an automorphism of $\mathbb{X}(G)$.) The graph algebra was studied by Schmitt [Sch94] and appears as the *chromatic algebra* in the work of Aguiar, Bergeron and Sottile [ABS06], where it is equipped with the character

$$\zeta(G) = \begin{cases} 1 & \text{if } G \text{ has no edges}, \\ 0 & \text{if } G \text{ has an edge}. \end{cases}$$

We will study several characters on $\mathcal{G}$ other than ζ.

(ii) If $F(x_1, x_2, \dots)$ is a formal power series, then its principal specialization is obtained by setting $x_i = 1$ and $x_i = 0$ for all $i > 1$.

(iii) The literature contains many other instances of "Hopf algebras of graphs"; for example, the algebra $\mathcal{G}$ is not the same as that studied by Novelli, Thibon and Thiéry [NTT04].

3 A new antipode formula

Our first result is a new formula for the Hopf antipode in $\mathcal{G}$. Unlike Schmitt's formula, our formula applies only to $\mathcal{G}$ and and does not generalize to other incidence algebras. On the other hand, our formula involves many fewer summands, which makes it useful for enumerative formulas involving characters.

Theorem 3.1 *Let G be a graph with vertex set $[n]$ and edge set E. Then*

$$S(G) = \sum_{\substack{F \subseteq E \\ F \text{ is a flat}}} (-1)^{n-\mathrm{rk}(F)} a(G/F) G_{V,F}$$

where $a(G)$ is the number of acyclic orientations of G, $\mathrm{rk}(F)$ is the rank of the flat F, and $G_{V,F}$ is the graph with vertices V and edges F.

For the proof, see [HM10]. An easy consequence is the following:

Proposition 3.2 *Let P be any family of graphs such that $G \uplus H \in P$ if and only if $G \in P$ and $H \in P$. That is, the function*

$$\chi_P(G) = \begin{cases} 1 & \text{if } G \in P, \\ 0 & \text{if } G \notin P \end{cases}$$

is a character. Then

$$\chi_P^{-1}(G) = \sum_{\text{flats } F \subseteq G:\ F \in P} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

Example 3.3 *Let P be the family of graphs with no edges. Then $\chi_P = \zeta$ and $\chi_P^{-1}(G)(-1)^n a(G)$, which is Stanley's well-known formula [Sta73].*

Example 3.4 *Let P be the family of acyclic graphs, and let $\alpha = \chi_P$. Then*

$$\alpha^{-1}(G) = \sum_{\text{acyclic flats } F} (-1)^{n-\mathrm{rk}(F)} a(G/F).$$

First, let $G = C_n$. The acyclic flats of G are just the sets of $n-2$ or fewer edges, so an elementary calculation (which we omit) gives $\alpha^{-1}(C_n) = (-1)^n + 1$, the Euler characteristic of an n-sphere. (For many other families P, the P-free flats of the n-cycle are just the flats, i.e., the edge sets of cardinality $\neq n-1$. In such cases, the same omitted calculation gives $\chi_P(C_n) = (-1)^n$.)

Second, let $G = K_n$. The acyclic flats of G are matchings; for $0 \leq k \leq \lfloor n/2 \rfloor$, the number of k-edge matchings is $n!/(2^k(n-2k)!k!)$, and contracting such a matching yields a graph whose underlying simple graph is K_{n-k}. Therefore, $\alpha^{-1}(K_n) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^{n-k} \frac{n!}{2^k(n-2k)!k!}(n-k)!$. These numbers (starting at $n=1$) are

$$-1, 1, 0, -6, 30, -90, 0, 2520, -22680, 113400, 0, -7484400, \ldots.$$

This is sequence A009775 in [Slo10]; the generating function is $-\tanh(\ln(1+x))$.

Example 3.5 *Fix any connected graph S. Say that G is S*-free *if it has no subgraph isomorphic to S. [Note: This is stronger than saying that G has no* induced *subgraph isomorphic to S.] Let η_S be the corresponding "avoidance character": $\eta_S(G) = 1$ if G is S-free, otherwise $\eta_S(G) = 0$. For example,*

$\eta_{K_1} = \epsilon$ and $\eta_{K_2} = \zeta$, and $\delta_m := \eta_{K_{m,1}}$ detects whether or not G has maximum degree $< m$. For an avoidance character, the sum in Proposition 3.2 is taken over all S-free flats F. For example, we have

$$\eta_{K_m}^{-1}(K_n) = \sum_{j=0}^{m-1} \binom{n}{j} (-1)^{n-j-1}(n-j)!.$$

Another consequence: if T is a tree with $r = n-1$ edges, then

$$\eta_S^{-1}(G) = \sum_{S\text{-free forests } F \subseteq T} (-1)^{r+1-|F|} 2^{r-|F|} = - \sum_{S\text{-free forests } F \subseteq T} (-2)^{r-|F|}.$$

Moreover, $P_{\eta_T}(T;k) = k^{n(T)} - k$.

Example 3.6 *Let S be a connected graph and η_S the corresponding avoidance character. Then $P_{\eta_S}(G;k)$ equals the number of k-colorings such that every color-induced subgraph is S-free. For instance, if S is the star $K_{m,1}$, then $P_{\eta_S}(G;k)$ is the number of k-colorings such that no vertex belongs to m or more monochromatic edges. This "degree-chromatic polynomial" counts colorings of G in which no color-induced subgraph has a vertex of degree $\geq m$; if $m = 1$, then we recover the usual chromatic polynomial. In general, two trees with the same number of vertices need not have the same degree-chromatic polynomials. For example, if G is the three-edge path on four vertices, H is the three-edge star, and S is the two-edge path, then $P_{\eta_S}(G;k) = k^4 - 2k^2 + k$ and $P_{\eta_S}(H;k) = k^4 - 3k^2 + 2k$. Based on experimental evidence, we conjecture that if T is any tree on n vertices, $m < n$, and $\eta = \delta_m$ (see Example 3.5), then*

$$P_\eta(T;k) = k^n - \sum_{v \in V(T)} \binom{d_T(v)}{m} k^{n-m} + (\textit{lower order terms}).$$

4 Tutte characters

The *Tutte polynomial* $T_G(x,y)$ is a powerful graph invariant. It can be viewed as a universal deletion-contraction invariant of graphs (in the sense that every graph invariant satisfying a deletion-contraction recurrence can be obtained from $T_G(x,y)$ via a standard "recipe" [Bol98, p. 340]. It is defined in closed form by the formula

$$T_G(x,y) \;=\; \sum_{A \subseteq E(G)} (x-1)^{\mathrm{rk}(G)-\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)}$$

where $\mathrm{rk}(A)$ is the graph rank of A, and $\mathrm{null}(A) = |A| - \mathrm{rk}(A)$ (the *nullity* of A). For much more on the background and application of the Tutte polynomial, see [BO92]. We note that $T_G(x,y)$ is multiplicative on connected components, so we can regard it as a character on the graph algebra:

$$\tau_{x,y}(G) = T_G(x,y).$$

We may regard x, y either as indeterminates or as (typically integer-valued) parameters. It is often more convenient to work with the *rank-nullity polynomial*

$$R_G(x,y) = \sum_{A \subseteq E} (x-1)^{\mathrm{rk}(A)} (y-1)^{\mathrm{null}(A)} = (x-1)^{\mathrm{rk}(G)} T_G(x/(x-1), y) \tag{1}$$

which carries the same information as $T_G(x, y)$, and is also multiplicative on connected components, hence is a character on $\mathcal{G}$. Note that $R_G(1, y) = 1$, and that

$$T_G(x, y) = (x-1)^{\mathrm{rk}(G)} R_G(x/(x-1), y). \tag{2}$$

Let $\rho_{x,y}$ denote the function $G \mapsto R_G(x, y)$, viewed as a character of the graph algebra $\mathcal{G}$. Let $P_{x,y}(G; k) = \rho^k_{x,y}(G)$ be the image of G under the CHA morphism $(\mathcal{G}, \rho_{x,y}) \to \mathbb{F}(x, y)[k]$ (see Proposition 2.1); note that $P_{x,y}(G; k)$ is a polynomial function of k.

For later use, we record the relationship between ρ and τ:

$$\tau_{x,y} = (x-1)^{\mathrm{rk}(G)} \rho_{x/(x-1),y}, \qquad \rho_{x,y} = (x-1)^{\mathrm{rk}(G)} \tau_{x/(x-1),y}. \tag{3}$$

In particular,

$$\tau_{0,y} = \widetilde{\rho_{0,y}} \quad \text{and} \quad \rho_{2,y} = \tau_{2,y}. \tag{4}$$

Our main theorem on Tutte characters is that $P_{x,y}(G; k)$ is itself a Tutte polynomial evaluation, as follows:

Theorem 4.1 *We have*

$$\rho^k_{x,y}(G) = P_{x,y}(G; k) = k^{c(G)} (x-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+x-1}{x-1}, y\right).$$

As is typical for Tutte polynomial identities, the idea of the proof is to show that the left-hand side satisfies a deletion-contraction recurrence.

4.1 Applications to Tutte polynomial evaluations

Theorem 4.1 has many enumerative consequences, some familiar and some less so. Many of the formulas we obtain resemble those in the work of Ardila [Ard07]; the precise connections remain to be investigated.

First, observe that setting $x = y = t$ in Theorem 4.1 yields

$$\rho^k_{t,t}(G) = P_{t,t}(G; k) = k^{c(G)} (t-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+t-1}{t-1}, t\right) = k^{c(G)} \bar{\chi} C_G(k; t) \tag{5}$$

where $\bar{\chi}$ denotes Crapo's coboundary polynomial(iv); see [MR05, p. 236] and [BO92, §6.3.F]. A consequence is the following pair of identities:

Corollary 4.2 *For $k \in \mathbb{Z}$ and y arbitrary, the Tutte characters $\tau_{2,y}$ and $\tau_{0,y}$ satisfy the identities*

$$(\tau_{2,y})^k (G) = k^{c(G)} T_G(k+1, y), \tag{6}$$

$$(\widetilde{\tau_{0,y}})^k (G) = k^{c(G)} (-1)^{\mathrm{rk}(G)} T_G(1-k, y). \tag{7}$$

In particular, $(\widetilde{\tau_{0,y}})^{-1} = \overline{\tau_{2,y}}$.

(iv) The bar is standard notation and has no relation to the involution $\phi \mapsto \bar{\phi}$ on $\mathbb{X}(\mathcal{G})$.

In this vein, we can find combinatorial interpretations of convolution powers of the characters $\tau_{2,2}$, $\tau_{2,0}$, $\widetilde{\tau_{0,2}}$, and $\widetilde{\tau_{0,0}}$. In the last case, we recover the standard formula for the chromatic polynomial as a specialization of the Tutte polynomial. Note that $\widetilde{\tau_{0,0}} = \tau_{0,0}$, because these characters are both zero on any graph with one or more edges.

This setup leads to combinatorial interpretations of other Tutte evaluations. If G is connected, then substituting $y = 2$ and $k = 2$ into (6) yields

$$2T(G;3,2) = P_{2,2}(G;2) = (\tau_{2,2} * \tau_{2,2})(G) = \sum_{U \subseteq V(G)} 2^{e(G|_U)} 2^{e(G|_{\bar{U}})} = \sum_{U \subseteq V(G)} 2^{e(G|_U)+e(G|_{\bar{U}})}. \quad (8)$$

That is, $T(G;3,2)$ counts the pairs (f, A), where f is a 2-coloring of G and A is a set of monochromatic edges.

In order to interpret more general powers of Tutte characters, we use (3) to rewrite the left-hand side of Theorem 4.1 as

$$k^{c(G)}(x-1)^{\mathrm{rk}(G)} T_G\left(\frac{k+x-1}{x-1}, y\right) = \sum_{V_1 \uplus \cdots \uplus V_k = V(G)} \prod_{i=1}^{k} (x-1)^{\mathrm{rk}(G_i)} \tau_{x/(x-1),y}(G_i)$$

where $G_i = G|_{V_i}$. Note that in the special case $G = K_n$, we have $G_i \cong K_{|V_i|}$ and $\mathrm{rk}(G_i) = |V_i| - 1$ for all i, so the equation simplifies to

$$(x-1)^{n-1} T_{K_n}\left(\frac{k+x-1}{x-1}, y\right) = k^{-1}(\tau_{x/(x-1),y})^k(K_n). \quad (9)$$

This equation has further enumerative consequences: setting $x = 2$ gives

$$T_{K_n}(k+1, y) = \frac{1}{k} \sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1! a_2! \ldots a_k!} \tau_{2,y}(K_{a_1}) \ldots \tau_{2,y}(K_{a_k}). \quad (10)$$

Setting $y = 0$ in (10), and observing that $\tau_{2,0}(K_a) = a!$ gives $T_{K_n}(k+1, 0) = (n+k-1)!/k!$ (which is not a new formula—it follows from the standard specialization of the Tutte polynomial to the chromatic polynomial, and the well-known formula $k(k-1)\cdots(k-n+1)$ for the chromatic polynomial of K_n). On the other hand, setting $y = 2$ in (10), and recalling that $\tau_{2,2}(K_a) = 2^{|E(K_a)|} = 2^{\binom{a}{2}}$, gives

$$T_{K_n}(k+1, 2) = \frac{1}{k} \sum_{a_1+\cdots+a_k=n} \frac{n!}{a_1! a_2! \ldots a_k!} 2^{\binom{a_1}{2}+\cdots+\binom{a_k}{2}} \quad (11)$$

This formula may be obtainable from the generating function for the coboundary polynomials of complete graphs, as computed by Ardila [Ard07, Thm. 4.1]; see also sequence A143543 in [Slo10]. Notice that setting $k = 2$ in (11) recovers (8).

It is natural to ask what happens when we set $x = 1$, since this specialization of the Tutte polynomial has well-known combinatorial interpretations in terms of, e.g., the chip-firing game [ML97] and parking functions [GS96]. The equations (1) and (2) degenerate upon direct substitution, but we can instead take the limit of both sides of Theorem 4.1 as $x \to 1$, obtaining (after some calculation, which we omit)

$$\rho_{1,y}^k(G) = k^{n(G)}.$$

We now examine what can be said about Tutte characters in light of the polynomiality principle (Proposition 2.1). Replacing x with $(k+x-1)/(x-1)$ in Theorem 4.1, we get

$$P_{(k+x-1)/(x-1),y}(G;k) = k^{c(G)}(k/(x-1))^{\mathrm{rk}(G)}T(G;x,y) = k^{n(G)}(x-1)^{-\mathrm{rk}(G)}T(G;x,y). \quad (12)$$

One consequence is a formula for the Tutte polynomial in terms of P:

$$T(G;x,y) = k^{-n(G)}(x-1)^{\mathrm{rk}(G)}P_{(k+x-1)/(x-1),y}(G;k). \quad (13)$$

In addition, this implies that the left-hand-side of (12) — which is an element of $\mathbb{F}(x,y)[k]$ — is actually just $k^{n(G)}$ times a rational function in x and y. Setting $k = x-1$ or $k = 1-x$, we can write down simpler formulas for the Tutte polynomial in terms of P:

$$T(G;x,y) = (x-1)^{-c(G)}P_{2,y}(G;x-1),$$
$$T(G;x,y) = (-1)^{n(G)}(x-1)^{c(G)}P_{0,y}(G;1-x).$$

5 A reciprocity relation between K_n and K_m

For each scalar $c \in \mathbb{C}$, define a character on $\mathcal{G}$ by $\xi_c(G) = c^{n(G)}$. It is not hard to see that

$$(\xi_c * \zeta)(G) = \sum_{\text{cocliques } Q} c^{n-|Q|}.$$

In particular, $(\xi_1 * \zeta)(G)$ is the number of cocliques in G, and $-(\xi_{-1} * \zeta)(G)$ is the reduced Euler characteristic of its clique complex.

Define a *k-near-coloring* to be a function $f : V \to [0,k]$, not necessarily surjective, such that each of the color classes $V_1 = f^{-1}(1), \ldots, V_k = f^{-1}(k)$, but not necessarily $V_0 = f^{-1}(0)$, is a coclique. Then

$$(\xi_c * \zeta)^k(G) = \sum_f (ck)^{|V_0|} = \sum_{V_0 \subseteq V(G)} (ck)^{|V_0|}(\#\text{ of } k\text{-colorings of } G - V_0). \quad (14)$$

To see the first equality in (14), consider a partition of V into $2k$ subsets. The union of the first k blocks is V_0, and the last k blocks are $V_1, \ldots, V_k$. Since V_0 is arbitrarily divided into k blocks, each k-near-coloring is counted $k^{|V_0|}$ times. Equation (14) implies that

$$(\xi_1 * \zeta^n)(K_m) = \sum_{W \subseteq [m]} \zeta^n(K_W) = \sum_{j=0}^{m} \binom{m}{j}(\#\text{ of } n\text{-colorings of } K_m) = \sum_{j=0}^{m} \frac{m!}{j!(m-j)!}\frac{n!}{(n-j)!}.$$

This expression is symmetric in n and m, which yields a surprising (to us, at least) reciprocity relation:

$$(\xi_1 * \zeta^n)(K_m) = (\xi_1 * \zeta^m)(K_n). \quad (15)$$

If we apply the bar involution to both sides of (15) (or, equivalently, redo the calculation with ξ_{-1} instead of ξ_1), we obtain

$$(\bar{\xi}_1 * \zeta^n)(K_m) = (-1)^{n+m}(\bar{\xi}_1 * \zeta^m)(K_n). \quad (16)$$

Experimental evidence indicates that

$$(\xi_1 * \zeta^{-1})(K_n) = (-1)^n D_n, \qquad (\xi_{-1} * \zeta^{-1})(K_n) = (-1)^n A_n,$$

where D_n is the number of derangements of $[n]$ [Slo10, sequence A000166] and B_n is the number of arrangements [Slo10, sequence A000522]. More generally, we conjecture that for all scalars k and c, the exponential generating function for $\xi_k * \zeta^c$ is

$$\sum_{n\geq 0}(\xi_k * \zeta^c)(K_n)\frac{x^n}{n!} \;=\; e^{-kx}(1-x)^{-c}$$

(see [Sta99, Example 5.1.2]).

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Bumping algorithm for set-valued shifted tableaux

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Abstract. We present an insertion algorithm of Robinson–Schensted type that applies to set-valued shifted Young tableaux. Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch and non set-valued shifted tableaux by Worley and Sagan. As an application, we obtain a Pieri rule for a K-theoretic analogue of the Schur Q-functions.

Résumé Nous présentons un algorithme d'insertion de Robinson–Schensted qui s'applique aux tableaux décalés à valeurs sur des ensembles. Notre algorithme est une généralisation de l'algorithme de Buch pour les tableaux à valeurs sur des ensembles et de l'algorithme de Worley et Sagan pour les tableaux décalés. Comme application, nous obtenons une formule de Pieri pour un analogue en K-théorie des Q-functions de Schur.

Keywords: set-valued shifted tableaux, insertion, Robinson–Schensted, Pieri rule, K-theory, Schur Q-functions

1 Introduction

This article is an extended abstract of the paper [INN] of the same title. Most details of the proofs are omitted.

In [IN], we introduced a non-homogeneous (K-theoretic) analogue of Schur Q-functions. These functions are labeled by strict partitions (or shifted Young diagrams), as are the original Q-functions. For a strict partition λ, the corresponding K-theoretic Schur Q-function $GQ_\lambda(x)$ can be expressed as a weighted generating function of *shifted set-valued semistandard tableaux* of shape λ, which are the central concern of this article.

The main result of the paper is a Robinson–Schensted type insertion algorithm for the shifted set-valued tableaux (Thm 3.4). Our algorithm is a generalization of both set-valued non-shifted tableaux by Buch [Bu] and non set-valued shifted tableaux by Worley [Wo] and Sagan [Sa]. As an immediate consequence of our algorithm, we have a Pieri rule for $GQ_\lambda(x)$ (Cor. 3.5).

The original purpose for introducing functions $GQ_\lambda(x)$ was to apply them to Schubert calculus. In [IN] we introduced function $GQ_\lambda(x|b)$ (resp. $GP_\lambda(x|b)$) with the *equivariant* parameter $b = (b_1, b_2, \ldots)$,

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which represents the structure sheaf of the Schubert variety indexed by λ in the K-ring of T-equivariant coherent sheaves on Langangian (resp. orthogonal) Grassmannian, where T is the maximal torus acting on the Grassmannians. Thus our Pieri rule gives an explicit description of K-theoretic Schubert structure constant for an arbitrary Schubert class times a special (one row type) Schubert class in the K-ring of Lagrangian Grassmannian.

Recently, a K-theoretic Littlewood-Richardson rule in terms of the *jeu de taquin* for odd orthogonal Grassmannians of maximal isotropic subspaces has been obtained by Clifford, Thomas and Yong [CTY]. Their method starts from a Pieri rule for the K-theory by Buch and Ravikumar [BR], which applies to cominuscule Grassmannians. Our approach differs from them substantially. We proceeded independently a different approach of tableaux insertion to result in the same formula as [BR], i.e. the counting of KLG-tableaux. But our method is only applicable to the case of Lagrangian Grassmannians, although there is a set valued tableaux description for $GP_\lambda(x)$.

Organization of the paper is as follows. In Section 2, we give the definition of shifted set-valued tableaux, and K-theoretic Schur Q-functions $GQ_\lambda(x)$. In Section 3, we present our main result, an existence of a Robinson–Schensted type bijection for set-valued shifted tableaux. As a corollary, we have a Pieri rule for $GQ_\lambda(x)$. Precise description of the bijection is given by a bumping algorithm which is given in Section 4. In Section 5, we discuss a variant of the bijection, which is analogous to the results by Sagan and Worley. In Section 6, we give an outline of the proof of the main theorem.

2 Shifted Young diagrams, set-valued tableaux

2.1 Shifted Young diagrams

Let Δ denote the set $\{(i,j) \in \mathbb{Z}^2 \mid 1 \le i \le j\}$. Any element $\alpha = (i,j)$ is called a *box*. If $i = j$, then (i,j) is called a *diagonal box*. A *shifted Young diagram* is any finite subset λ of Δ such that for each $\alpha = (i,j) \in \lambda$, any box $\beta = (i',j') \in \Delta$ satisfying $i' \le i$ and $j' \le j$ belongs to λ.

We define $\mathbb{S}$ to be the set of shifted Young diagrams. For $\lambda \in \mathbb{S}$, we define $|\lambda|$ to be the number of boxes in λ. For $\lambda, \mu \in \mathbb{S}$ such that $\lambda \subset \mu$, we define the skew shifted Young diagram μ/λ to be the set-theoretic difference $\mu - \lambda$.

Let $\alpha = (i,j), \beta = (i',j') \in \Delta$. We say that α is *weakly below* (resp. *weakly right of*) β if $i \ge i'$ (resp. $j \ge j'$). We say that α is *strictly below* (resp. *strictly right of*) β if $i > i'$ (resp. $j > j'$). We say that α is *directly below* (resp. *directly right of*) β if $i = i' + 1$ and $j = j'$ (resp. $i = i'$ and $j = j' + 1$).

We call a skew shifted diagram θ a *horizontal strip* (resp. *vertical strip*) if θ has no pair of boxes in the same column (resp. row). We call θ a *broken border strip* if θ contains no 2×2 square block.

2.2 Tableaux

Define a totally ordered set $\mathcal{B}$ to be disjoint union of sets $\mathcal{A} = \{1, 2, \ldots\}$ and $\mathcal{A}' = \{1', 2', \ldots\}$ with the following order:

$$1' < 1 < 2' < 2 < \cdots .$$

We define binary relations $\le_r$ and $\le_c$ on $\mathcal{B}$ by

$$x \le_r y \iff x = y \in \mathcal{A} \text{ or } x < y, \qquad x \le_c y \iff x = y \in \mathcal{A}' \text{or } x < y.$$

Note that $x \not\le_r y$ (resp. $x \not\le_c y$) is equivalent to $y \le_c x$ (resp. $y \le_r x$) for any $x, y \in \mathcal{B}$.

Let $\mathcal{X}$ denote the set of non-empty finite subsets of $\mathcal{B}$. We extend the relations $\leq_r, \leq_c$ on $\mathcal{X}$ by $A \leq_r B \Longleftrightarrow \max A \leq_r \min B$ and $A \leq_c B \Longleftrightarrow \max A \leq_c \min B$ for $A, B \in \mathcal{X}$.

Definition 2.1 (Shifted set-valued semistandard tableaux) *Let λ be a shifted Young diagram. A* set-valued semistandard tableau *of shape λ is a map T from the set of boxes in λ to $\mathcal{X}$ satisfying the following "semistandaredness" :*

1. *$T(\alpha) \leq_r T(\beta)$ if $\beta \in \lambda$ is directly right of $\alpha \in \lambda$.*
2. *$T(\alpha) \leq_c T(\beta)$ if $\beta \in \lambda$ is directly below $\alpha \in \lambda$.*

Example 2.2 An example of a set-valued tableau is given by the following:

$$T = \begin{array}{|c|c|c|c|} \hline 1' & 12' & 23 & 34' \\ \hline \multicolumn{1}{c|}{} & 2' & 4' & 6 \\ \cline{2-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 6 & \multicolumn{1}{|c}{} \\ \cline{3-3} \end{array}.$$

We denote by $\mathcal{T}(\lambda)$ the set of all set-valued tableaux of shape λ.

2.3 K-theoretic Q-Schur functions

Let $x = (x_1, x_2, \ldots)$ be a sequence of variables. Let $\lambda \in \mathbb{S}$ and $T \in \mathcal{T}(\lambda)$. We define the corresponding monomial $x^T = \prod_{i=1}^{\infty} x_i^{e_i(T)}$ where $e_i(T)$ denotes the total number of i and i' appearing in T. The *weight* of $T \in \mathcal{T}(\lambda)$ is defined to be $\beta^{|T|-|\lambda|} x^T$, where β is a formal parameter and $|T|$ is the total number of letters in T. The K-theoretic Q-Schur function $GQ_\lambda(x)$ is defined as the following formal sum of the weights of the elements in $\mathcal{T}(\lambda)$:

$$GQ_\lambda(x) = \sum_{T \in \mathcal{T}(\lambda)} \beta^{|T|-|\lambda|} x^T.$$

When $\beta = 0$ this becomes the Schur Q-function $Q_\lambda(x)$, and when $\beta = -1$ this represents K-theory Schubert class corresponding to λ for Lagrangian Grassmannians. See [IN] for other expressions of $GQ_\lambda(x)$ and geometric background.

3 Statements of main results

3.1 Admissible strips

Let $\theta = \lambda/\mu$ be a broken border strip. We consider a decomposition $\theta = C \sqcup C'$, with C, C' skew diagrams, i.e. there is a diagram ν satisfying $\mu \subset \nu \subset \lambda$ and $C = \lambda/\nu$ and $C' = \nu/\mu$. Such a decomposition of θ is called *admissible* of if the following conditions are satisfied:

1. in each of the diagrams C and C', there is no pair of boxes in the same row or column.
2. there is no diagonal box in C'.

A non-empty broken border strip θ is called a 1-*admissible strip* if there exists an admissible decomposition of θ. For a 1-admissible strip θ, we denote by $\mathcal{C}(\theta)$ the set of all admissible decompositions of θ. Later we define the notion of m-admissible decomposition of a broken border strip.

Example 3.1 The following is an example of a 1-admissible strip and its 1-admissible decomposition,

$$
\begin{array}{cccccccc}
\square & \square & \square & \square & \square & \square & 1' & 1 \\
 & \square & \square & \square & 1' & 1 & & \\
 & & \square & \square & 1 & & & \\
 & & & 1 & & & &
\end{array},
$$

where the boxes with entry 1's form C and $1'$'s form C'.

The next result shows the role of 1-admissible strip. The detailed construction of the map is given in Section 4. We define the weight of a 1-admissible strip θ to be $\beta^{|\theta|-1}$.

Proposition 3.2 *There is a weight preserving bijection:*

$$\phi : \mathcal{T}(\lambda) \times \mathcal{X} \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$$

where $\mu \in \mathbb{S}$ runs for those μ such that μ/λ is a 1-admissible strip.

3.2 Composable admissible strips

Let $\lambda, \mu, \nu \in \mathbb{S}$ be such that $\mu \subset \nu \subset \lambda$. Suppose $\theta_1 = \nu/\mu, \theta_2 = \lambda/\nu$ are 1-admissible strips. Let $(C_i', C_i) \in \mathcal{C}(\theta_i)\,(i = 1, 2)$.We say that (C_1', C_1) *precedes* (C_2', C_2) and denote $(C_1', C_1) \lhd (C_2', C_2)$, if the following conditions are satisfied:

1. $C_1' \cup C_2'$ is a vertical strip.
2. $C_1 \cup C_2$ is a horizontal strip.
3. Each box in C_2' is strictly below any box in C_1'.
4. Each box in C_2 is strictly right of any box in C_1.
5. If $C_1 \neq \emptyset$, then $C_2' = \emptyset$.

3.3 Main results

Let $\theta = \mu/\lambda$ be a broken border strip, and m be a positive integer. Suppose there is a nested sequence of shifted diagrams

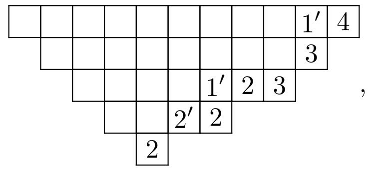
$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \cdots \subset \nu^{(m)} = \mu \tag{1}$$

such that $\theta^{(i)} := \nu^{(i)}/\nu^{(i-1)}$ $(1 \leq i \leq m)$ are 1-admissible strips. If, moreover, there is a sequence of 1-admissible decompositions $(C_i', C_i) \in \mathcal{C}(\theta^{(i)})$ $(1 \leq i \leq m)$ such that

$$(C_i', C_i) \lhd (C_{i+1}', C_{i+1}), \quad (1 \leq i \leq m-1). \tag{2}$$

then we say θ is an *m-admissible strip*. For an *m-admissible strip* θ, let $\mathcal{C}_m(\theta)$ denote the set of pairs $\big(\{\nu^{(i)}\}_{i=1}^m,\ \{(C_i', C_i)\}_{i=1}^m\big)$ satisfying the above conditions, which we call m-admissible decompositions of θ. Note $\mathcal{C}_1(\theta) = \mathcal{C}(\theta)$ since condition (2) is vacant for $m = 1$.

Example 3.3 The following is a 4-admissible strip



where the boxes with entry i are C_i, and i' are C_i'.

We denote by (m) the shifted diagram consisting of one row with m boxes. We simply denote $\mathcal{T}(m)$ for $\mathcal{T}((m))$. Recall that we define the weight of $T \in \mathcal{T}(\lambda)$ as $\beta^{|T|-|\lambda|}x^T$. Define the weight of $U \in \mathcal{C}_m(\theta)$ to be $\beta^{|\theta|-m}$.

Theorem 3.4 *By algorithm 4.4, we have a weight preserving bijection:*

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{T}(m) \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{C}_m(\mu/\lambda), \tag{3}$$

where μ runs for shifted diagrams μ such that $\mu \supset \lambda$ and μ/λ are m-admissible strips.

As an immediate consequence, we have the following.

Corollary 3.5 (Pieri rule) *We have*

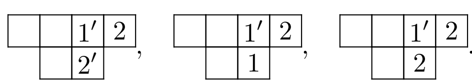
$$GQ_\lambda(x) \cdot GQ_m(x) = \sum_{\mu \supset \lambda} \beta^{|\mu|-|\lambda|-m} \# \mathcal{C}_m(\mu/\lambda) \times GQ_\mu(x),$$

where μ runs for shifted diagrams μ such that $\mu \supset \lambda$ and μ/λ are m-admissible strips.

For example we have

$$GQ_{2,1} \cdot GQ_2 = 2GQ_{4,1} + 2GQ_{3,2} + 3\beta GQ_{4,2} + \beta GQ_{5,1} + \beta GQ_{3,2,1} + \beta^2 GQ_{5,2} + \beta^2 GQ_{4,2,1}.$$

In order to give the coefficient of $GQ_{4,2}$, we count the elements in $\mathcal{C}_2(\mu/\lambda)$ with $\mu = (4,2), \lambda = (2,1)$:



N.B. The elements in $\mathcal{C}_m(\mu/\lambda)$ are exactly the KLG-tableaux of shape μ/λ with content $\{1, 2, ..., m\}$ in [BR].

4 Bumping algorithm

The aim of this section is to describe the bijection of Prop 3.2.

The input of our algorithm is a pair (T, w) with $T \in \mathcal{T}(\lambda)$ for some $\lambda \in \mathbb{S}$ and $w \in \mathcal{X}$. Basic output is a tableau T' of some shape $\mu \in \mathbb{S}$ such that $\mu \supset \lambda$. The skew diagram $\theta = \mu/\lambda$, the set of "new boxes", turns out to be a 1-admissible strip. We also have some "recording data" on θ which gives an element of $\mathcal{C}(\theta)$.

4.1 Parts of "L" shape of a tableau

Let $\lambda \in \mathbb{S}$. Let $\ell(\lambda)$ be the number of rows of λ. For $1 \leq t \leq \lambda_1$ we define a subset of λ by

$$L_t(\lambda) = \{(i,j) \in \lambda \mid i = t \text{ or } j = t\}.$$

For example, $L_1(\lambda)$ consists of the boxes in the first row. For $k \geq \ell(\lambda)$, $L_k(\lambda)$ is just the k-th column. In general, this is a subset of shape "L" including the diagonal box (t,t). Let $T \in \mathcal{T}(\lambda)$. By restriction we have a map $L_t(T) : L_t(\lambda) \to \mathcal{X}$, which we call the t-th *part* of T.

Our algorithm starts from inserting $w = w^{(0)} \in \mathcal{X}$ into $L_1 = L_1(T)$, the first row of T, resulting a row L_1' with possibly a new box at the right end, and a set $w^{(1)} \in \mathcal{X}$ "bumped out" from the procedure. Then we modify the original tableau $T = T^{(0)}$ by replacing L_1 with L_1' to obtain $T^{(1)}$. Next we insert $w^{(1)}$ into the second part of the modified tableau $T^{(2)}$. We repeat this procedure until no boxes are bumped out.

4.2 Insertion into a part of "L" shape (a rough idea)

We define a procedure to insert some sets $w \in \mathcal{X}$ into an L part X of a tableaux.

Here we present a rough idea of constructing the procedure. First, we look at the minimum letters of each boxes in order to decide the box into which a letter in w to be inserted, in the same manner as the classical bumping procedure (some letters go into empty box at the end). If we might simply insert these letters into X, some letters in w may violate the semistandardness, while some letters are not. So we eject some element in X before inserting w. Let $\hat{w}$ be the set of letters in w which do not conflict any original letters in X, and let $\check{w} := w - \hat{w}$ be the complement. If $\check{w} \neq \emptyset$, let $\check{u}$ be the set of elements in X that conflict some element in $\check{w}$. To ensure the semistandardness, we first eject the elements in $\check{u}$ from the tableau. Furthermore, if a letter in $\hat{w}$ is inserted into a non-empty box, we eject all the remaining (original) entries of the box. Thus any letter inserted into a non-empty box "does some work" (bumps out at least one letter). This feature is important for constructing the inverse algorithm.

There is a flaw in this idea. For example, we consider a tableau $T = \boxed{1'}$ and $w = w^{(1)} = \{1'\}$. According to the naive algorithm above, the resulting tableau is $T^{(1)} = \boxed{1'}$, and the ejected set is $w^{(2)} = \{1'\}$. Since the second part is empty, the final result is $\boxed{1'}\boxed{1'}$, which is not semistandard. This is a reason why we need the "unmark" process introduced in the next section. In fact, we should care for the case of inserting elements into the diagonal boxes.

4.3 Insertion into a diagonal box

Let $X \in \mathcal{X}$, and u be a subset of X. We insert $w \in \mathcal{X}$ into X, where we consider X to be a diagonal box.

Algorithm 4.1 (Bumping for a diagonal box)

input $X, w, u \in \mathcal{X}$ satisfying $u \subset X$ and $\max w \leq_c \min X$.

output Y, v.

procedure

1. If $X \neq u$, then let $Y = (X - u) \cup w$ and $v = u$; and return Y, v.
2. If $i' = \max(w) \in \mathcal{A}'$ and $i \in X,\ i' \notin X$, then let $Y = \{\ i\ \} \cup (w - \{\ i'\ \})$ and $v = X$; and return Y, v.

3. If $i' = \max(w) \in \mathcal{A}'$ and $i' \in X$, $i \notin X$, then let $Y = w$ and $v = \{ i \} \cup (X - \{ i' \})$; and return Y, v.
4. If $i' = \max(w) \in \mathcal{A}'$ and $i, i' \in X$, then let $Y = \{ i \} \cup w$ and $v = X - \{ i' \}$; and return Y, v.
5. Otherwise, let $Y = w$ and $v = X$; and return Y, v.

For example, if $u = X = \boxed{34}$ and $w = 13'$, then we apply (2) to obtain $Y = \boxed{13}$ rather than $\boxed{13'}$, and $u = 34$. Thus letter $3'$ is unprimed to be 3 in u. If $u = X = \boxed{3'4}$ and $w = 13'$, then we apply (3) to obtain $Y = \boxed{13'}$ and $u = 34$, rather than $u = 3'4$. In this case, two $3'$ are involved, and one may think of this process as umpriming "bigger" $3'$. Case (4) is a bit strange. If $u = X = \boxed{3'3}$ and $w = 3'$, then we have $Y = \boxed{3'3}$ and $u = 3$. This case we are unpriming "bigger" $3'$ also, and let it remain in the box.

4.4 Insertion into a part of "L" shape (definition)

Let T be a tableau of shape λ, and t be a positive integer such that $t \leq \lambda_1$. Let $X = L_t(T)$ be the t-th part of T. If $t = 1$, then X is a row: $X = (X_{(1,1)} \leq_r X_{(1,2)} \leq_r \cdots \leq_r X_{(1,\lambda_1)})$. If $t > \ell(\lambda)$ then X is a column: $X = (X_{(1,t)} \leq_c \cdots \leq_c X_{(k,t)})$ for some $k < t$. We say that X is a *pure column* in this case (note that X does not contain diagonal box). If $1 < t \leq \ell(\lambda)$ then $X = L_t(T)$ is a sequence of elements in $\mathcal{X}$:

$$X = (X_{(1,t)} \leq_c \cdots \leq_c X_{(t-1,t)} \leq_c X_{(t,t)} \leq_r X_{(t,t+1)} \leq_r \cdots \leq_r X_{(t,t+\lambda_t-1)}).$$

The following algorithm takes as an input a sequence of elements in $\mathcal{X}$ satisfying

$$X = (X_{-k} \leq_c \cdots \leq_c X_{-1} \leq_c X_0 \leq_r X_1 \leq_r \cdots \leq_r X_l),$$

for some $k, l \geq 0$, and $w \in \mathcal{X}$. If $k = 0$, we consider X as a row. Output is a triple (Y, Y_+, v), where Y is a sequence $Y = (Y_i)_{i=-k}^{l}$ satisfying the same condition as X, and $Y_+, v \in \mathcal{X} \cup \emptyset$. If $Y_+ \neq \emptyset$ we will make a new box with entry Y_+ at the right end of Y.

Algorithm 4.2 (Bumping rule for an L part)

input $X = (X_i)_{i=-k}^{l}$: tableau of L shape, i.e.

$$X = (X_{-k} \leq_c \cdots \leq_c X_{-1} \leq_c X_0 \leq_r X_1 \leq_r \cdots \leq_r X_l),$$

and $w \in \mathcal{X}$.

output Y tableau of L shape of the same length of X, and $Y_+, v \in \mathcal{X} \cup \emptyset$.

procedure

1. Define the subsets $w_{-k}, \ldots, w_{l+1}$ of w by

$$w_t = \begin{cases} \{x \in w \mid x \leq_r \min X_{-k}\} & (t = -k) \\ \{x \in w \mid \min X_{t-1} \leq_c x \leq_r \min X_i\} & (t = -k, \ldots, -1) \\ \{x \in w \mid \min X_{-1} \leq_c x \leq_c \min X_0\} & (t = 0) \\ \{x \in w \mid \min X_{t-1} \leq_r x \leq_c \min X_t\} & (t = 1, \ldots, l) \\ \{x \in w \mid \min X_l \leq_r x\} & (t = l + 1) \end{cases}$$

2. Decompose w_t into the subsets $\check{w}_t$ and $\hat{w}_t$ defined by

$$\hat{w}_t = \begin{cases} w_t & (t = -k) \\ \{\, x \in w_t \mid \max X_{t-1} \leq_c x \,\} & (t = -k+1, \ldots, 0) \\ \{\, x \in w_t \mid \max X_{t-1} \leq_r x \,\} & (t = 1, \ldots, l+1) \end{cases},$$

$\check{w}_t = w_t - \hat{w}_t$, for $t = -k, \ldots, l+1$,

3. Define $\check{u}_t$, $\hat{u}_k$, and u_k $(t = -k, \ldots, l)$ by:

$$\check{u}_t = \begin{cases} \emptyset & (\text{if } \check{w}_{t+1} = \emptyset) \\ \{\, y \in X_t \mid y \not\leq_c \min \check{w}_{t+1} \,\} & (\text{if } t = -k, \ldots, -1 \text{ and } \check{w}_{t+1} \neq \emptyset) \\ \{\, y \in X_t \mid y \not\leq_r \min \check{w}_{t+1} \,\} & (\text{if } t = 0, \ldots, l \text{ and } \check{w}_{t+1} \neq \emptyset) \end{cases},$$

$$\hat{u}_t = \begin{cases} \emptyset & (\text{if } \hat{w}_t = \emptyset) \\ X_t - \check{u}_t & (\text{if } \hat{w}_t \neq \emptyset) \end{cases}$$

$u_t = \hat{u}_t \cup \check{u}_t \subset X_t$.

4. Define $Y_t = (X_t - u_t) \cup w_t$ and $v_t = u_t$ for $t \neq 0$.
5. Let (Y_0, v_0) be the pair obtained from the triple (X_0, w_0, u_0) by Algorithm 4.1 if $l \geq 0$.
6. Let $Y = (Y_{-k}, \ldots, Y_l)$, $Y_+ = w_{l+1}$, and $v = \bigcup_{t=-k}^{l} v_t$; and return Y, Y_+, v.

Example 4.3 Let $X = (X_{-2}, X_{-1}; X_0; X_1, X_2, X_3)$ be

$$\begin{array}{|c|c|c|c|c|c|}\hline 13 & 4' & \boxed{5} & 56 & 8 & 9 \\ \hline \end{array}.$$

Let us insert $w = 25'6'79'9 \in \mathcal{X}$ into X. Since the minimums in X is

$$\begin{array}{|c|c|c|c|c|c|}\hline 1 & 4' & \boxed{5} & 5 & 8 & 9 \\ \hline \end{array},$$

we have $(w_{-2}, \ldots, w_4) = (\emptyset, 2, 5', \emptyset, 6'7, 9', 9)$. Since the maximums of X is

$$\begin{array}{|c|c|c|c|c|c|}\hline 3 & 4' & \boxed{5} & 6 & 8 & 9 \\ \hline \end{array},$$

we have

t	-2	-1	0	1	2	3	4
$\hat{w}_t$	$\emptyset$	$\emptyset$	$5'$	$\emptyset$	7	$9'$	9
$\check{w}_t$	$\emptyset$	2	$\emptyset$	$\emptyset$	$6'$	$\emptyset$	$\emptyset$
$\hat{u}_t$	$\emptyset$	$\emptyset$	5	$\emptyset$	8	9	$-$
$\check{u}_t$	3	$\emptyset$	$\emptyset$	6	$\emptyset$	$\emptyset$	$-$

Finally we get

$$Y = \begin{array}{|c|c|c|c|c|c|}\hline 1 & 24' & 5 & 5 & 6'7 & 9' \\ \hline \end{array},\ Y_+ = \{\, 9 \,\},\ u = \{\, 3, 5, 6, 8, 9 \,\}.$$

We need to define the bumping algorithm applicable also when $X = L_t(T)$ is a pure column case, i.e. $t > \ell(\lambda)$. However, extension of the algorithm to the column case is straightforward, so we omit detailed description here.

4.5 Insertion of w into arbitrary tableau

We define a procedure to insert an element $w \in \mathcal{X}$ into an arbitrary tableau T. In the procedure, we insert w into the first L part of the tableaux. When some letters are bumped out, we insert them into the second L part of the tableau. Then, while some letters are bumped out, we try to insert them into the next L part of the tableux until no letters are bumped out.

Algorithm 4.4

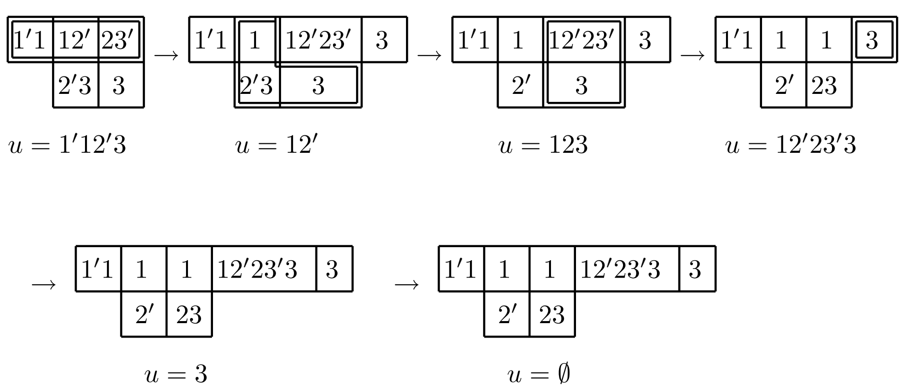
input $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$.

output U, S', S.

procedure

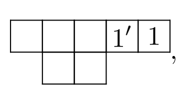
1. Let $u = w$, $U = T$, $S = \emptyset$ and $S' = \emptyset$.
2. While $u \neq \emptyset$, do the following:
   (a) Let X be the t-th L part of U,
   (b) Let (Y, Y_+, u) be the triple obtained from (X, u) by Algorithm 4.2.
   (c) Let U be the tableaux obtained from U by replacing the t-th L part by Y.
   (d) If $Y_+ \neq \emptyset$, then do the following:
      i. Add a new box to the end of t-th L part of U, and insert Y_+ into the box.
      ii. If X is a *pure column*, then add the new box to S, else add the new box to S'.
3. Return U, S' and S.

Example 4.5 Let T be the leftmost tableau below. We insert $w = \{1', 1, 2', 3\}$ into T as follows.



For each step, the relevant part of modification is enclosed.

Sets S' and S are as follows:



where the box with entry $1'$ (resp. 1) is S' (resp. S).

4.6 Definition of the map ϕ

In order to complete the description of the map ϕ, we need one more combinatorial idea. Let θ be a 1-admissible strip. We define an involution $\varrho : \mathcal{C}(\theta) \to \mathcal{C}(\theta)$. A box $\alpha \in \theta$ is said to be *isolated* if α is not a diagonal box and there is no other box than α in the row and column where α presents. For each isolated box, apply its entry the obvious involution $1 \mapsto 1'$, $1' \mapsto 1$, while the non-isolated boxes are untouched. The resulting decomposition of θ is obviously admissible. For example, we have

$$\begin{ytableau} {} & {} & {} & {} & {} & {} & {} & 1 \\ \none & {} & {} & {} & {} & 1' & 1 \\ \none & \none & {} & {} & 1' \\ \none & \none & \none & 1 \end{ytableau} \xrightarrow{\varrho} \begin{ytableau} {} & {} & {} & {} & {} & {} & {} & 1' \\ \none & {} & {} & {} & {} & 1' & 1 \\ \none & \none & {} & {} & 1 \\ \none & \none & \none & 1 \end{ytableau} .$$

It is obvious that ϱ is an involution.

Proposition 4.6 *Let $\lambda \in \mathbb{S}$, $T \in \mathcal{T}(\lambda)$, and $w \in \mathcal{X} = \mathcal{T}(1)$. We have by Algorithm 4.4 a tableau $U = (T \hookleftarrow w) \in \mathcal{T}(\mu)$ for some $\mu \in \mathbb{S}$ such that $\mu \supset \lambda$ and a decomposition (S', S) of $\theta = \mu/\lambda$. We have $(S', S) \in \mathcal{C}(\theta)$, and therefore θ is a 1-admissible strip.*

Let $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$ as in the above proposition. We define $\phi(T, w)$ to be $(U, \varrho(S', S)) \in \mathcal{T}(\mu) \times \mathcal{C}(\mu/\lambda)$.

4.7 Proof of Prop. 3.2

To show that ϕ is a bijection, we construct its inverse map. See [INN] for details.

5 Robinson–Schensted type correspondence

5.1 Quasi-standard tableaux

We will define a notion of "recording" tableaux in our setting. The resulting object is an analogue of a standard tableau, which we will call a *quasi-standard* tableau.

For $T \in \mathcal{T}(\lambda)$ and $w \in \mathcal{X}$ we denote by $T \hookleftarrow w$ the tableau given in Prop. 3.2. Let $T \in \mathcal{T}(\lambda)$ and $(w_1, \ldots, w_m) \in \mathcal{X}^m$. By the consecutive insertions

$$T^{(i)} = (\cdots((T \hookleftarrow w_1) \hookleftarrow w_2) \cdots \hookleftarrow w_i)$$

we have a tableaux $T^{(i)} \in \mathcal{T}(\nu^{(i)})$ for some shifted diagram $\nu^{(i)}$ and an element of $\mathcal{C}(\nu^{(i)}/\nu^{(i-1)})$ given by Proposition 3.2. Thus we have a nested sequence of shifted diagrams

$$\lambda = \nu^{(0)} \subset \nu^{(1)} \subset \nu^{(2)} \subset \cdots \subset \nu^{(m)} = \mu, \tag{4}$$

and also 1-admissible decompositions (C_i', C_i) of $\theta^{(i)} = \nu^{(i)}/\nu^{(i-1)}$. These objects are expressed as a tableau like

$$\begin{ytableau} {} & {} & {} & {} & {} & {} & {} & 1' & 1 & 2' & 4 \\ \none & {} & {} & {} & 1' & 2' & 2 & 3' & 4' & 4 \\ \none & \none & {} & 1' & 1 & 3' & 3 & 4' & 4 \\ \none & \none & \none & 1 & 2' & 3 & 4 \\ \none & \none & \none & \none & 2 \end{ytableau} ,$$

where the boxes filled with i (resp. i') are C_i (resp. C'_i).

We call such a tableau a *quasi-standard* tableau of degree m. The precise definition is the following.

Definition 5.1 *A map $U : \mu/\lambda \longrightarrow \mathcal{B}_m := \{1', 1, \dots, m', m\}$ is a quasi-standard tableau of degree m, if U is semistandard in the sense of Def. 2.1 and for any $1 \leq i \leq m$, $U^{-1}(\{i, i'\})$ is a 1-admissible strip with admissible decomposition given by $(U^{-1}(i'), U^{-1}(i))$.*

Let $\mathcal{S}_m(\mu/\lambda)$ denote the set of quasi-standard tableaux of degree m on μ/λ.

Remark. By the construction, $\mathcal{S}_1(\mu/\lambda)$ is non-empty if and only if $\theta = \mu/\lambda$ is an 1-admissible strip. Then we have $\mathcal{S}_1(\theta) = \mathcal{C}(\theta) = \mathcal{C}_1(\theta)$. For an m-admissible strip θ, the set $\mathcal{C}_m(\theta)$ is a subset of $\mathcal{S}_m(\theta)$.

5.2 Robinson–Schensted correspondence

The following result is an immediate consequence of Prop. 3.2.

Proposition 5.2 *Let $T \in \mathcal{T}(\lambda)$ and $(w_1, \dots, w_m) \in \mathcal{X}^m$. By consecutive insertions*

$$T' = (\cdots((T \hookleftarrow w_1) \hookleftarrow w_2) \cdots \hookleftarrow w_m)$$

we have a tableaux $T' \in \mathcal{T}(\mu)$ for some shifted diagram $\mu \supset \lambda$ and the recording tableau U. Then we have $U \in \mathcal{S}_m(\mu/\lambda)$. By this correspondence we have a weight preserving bijection

$$\phi_m : \mathcal{T}(\lambda) \times \mathcal{X}^m \longrightarrow \bigsqcup_{\mu} \mathcal{T}(\mu) \times \mathcal{S}_m(\mu/\lambda), \tag{5}$$

where the sum runs for shifted diagrams μ such that $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$.

Then we have immediately the following:

Corollary 5.3 *We have*

$$GQ_\lambda(x) \cdot GQ_1(x)^m = \sum_{\mu} \beta^{|\mu/\lambda| - m} \# \mathcal{S}_m(\mu/\lambda) \times GQ_\mu(x),$$

where the sum runs for shifted diagrams μ such that $\mathcal{S}_m(\mu/\lambda) \neq \emptyset$.

As a special case of $\lambda = \emptyset$, we have the following.

Corollary 5.4 (Robinson–Schensted correspondence) *There is a weight preserving bijection*

$$\mathcal{X}^m \longrightarrow \bigsqcup_{\lambda} \mathcal{T}(\lambda) \times \mathcal{S}_m(\lambda).$$

This bijection is a set-valued extension of the results in [Sa] and [Wo].

Example 5.5 Let $(w_1, w_2, w_3) = (2'3, 12'2, 134)$. By the correspondence in Cor. 5.3 we have pair of tableaux

$$\left(\begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline \multicolumn{1}{c|}{} & 23' & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array}, \quad \begin{array}{|c|c|c|c|c|} \hline 1 & 2' & 2 & 3' & 3 \\ \hline \multicolumn{1}{c|}{} & 2 & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array} \right),$$

as a result of bumping process:

$$\emptyset \overset{w_1}{\rightsquigarrow} \begin{array}{|c|} \hline 23' \\ \hline \end{array} \overset{w_2}{\rightsquigarrow} \begin{array}{|c|c|c|} \hline 12 & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & 3' & \multicolumn{1}{c}{} \\ \cline{2-2} \end{array} \overset{w_3}{\rightsquigarrow} \begin{array}{|c|c|c|c|c|} \hline 1 & 1 & 2 & 2 & 34 \\ \hline \multicolumn{1}{c|}{} & 23' & \multicolumn{3}{c}{} \\ \cline{2-2} \end{array}.$$

6 Outline of proof of Thm 3.4

Now we have the bijection ϕ_m in Prop. 5.2. Since a tableau in $\mathcal{T}(m)$ is a sequence in $\mathcal{X}$ such that

$$X_1 \leq_r \cdots \leq_r X_m,$$

we can think of $\mathcal{T}(m)$ as a subset of $\mathcal{X}^m$. Thus we only need to determine the image of $\mathcal{T}(\lambda) \times \mathcal{T}(m)$ under the map ϕ_m. The case $m = 1$ is obvious since $\mathcal{T}(1) = \mathcal{X}$. The case $m = 2$ is crucial.

Lemma 6.1 *Let $T \in \mathcal{T}(\lambda)$ and $w = (w_1, w_2) \in \mathcal{X}^2$, and*

$$\phi_2(T, w) = (T', (C_1', C_1), (C_2', C_2)).$$

Then the following are equivalent:

1. $w_1 \leq_r w_2$.
2. $(C_1', C_1) \lhd (C_2', C_2)$.

It is easy to see that the lemma leads to a proof of Thm 3.4. We show this lemma by an argument using "bumping routes". Details are given in [INN].

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Counting self-dual interval orders

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Abstract. In this paper, we first derive an explicit formula for the generating function that counts unlabeled interval orders (a.k.a. $(\mathbf{2}+\mathbf{2})$-free posets) with respect to several natural statistics, including their size, magnitude, and the number of minimal and maximal elements. In the second part of the paper, we derive a generating function for the number of self-dual unlabeled interval orders, with respect to the same statistics.

Our method is based on a bijective correspondence between interval orders and upper-triangular matrices in which each row and column has a positive entry.

Résumé. Dans cet article, on obtient une expression explicite pour la fonction génératrice du nombre des ensembles partiellement ordonnés (posets) qui évitent le motif $(\mathbf{2}+\mathbf{2})$. La fonction compte ces ensembles par rapport à plusieurs statistiques naturelles, incluant le nombre d'éléments, le nombre de niveaux, et le nombre d'éléments minimaux et maximaux. Dans la deuxième partie, on obtient une expression similaire pour la fonction génératrice des posets autoduaux évitant le motif $(\mathbf{2}+\mathbf{2})$.

On obtient ces résultats à l'aide d'une bijection entre les posets évitant $(\mathbf{2}+\mathbf{2})$ et les matrices triangulaires supérieures dont chaque ligne et chaque colonne contient un élément positif.

Keywords: interval orders, $(\mathbf{2}+\mathbf{2})$-free posets, self-dual posets

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1 Introduction

The aim of this paper is to enumerate interval orders (also known as $(\mathbf{2}+\mathbf{2})$-free posets) with respect to several natural poset statistics, including the size, the magnitude, and the number of minimal and maximal elements. We are mostly motivated by the generating function formulas recently obtained by Bousquet-Mélou et al. [3], Kitaev and Remmel [14], and Dukes et al. [5].

Although the formulas derived in this paper provide a common generalization of these previous results, the method we use is different. The previous results were derived using a recursive bijection between interval orders and ascent sequences, due to Bousquet-Mélou et al. [3]. In this paper, we instead use an encoding of interval orders by upper-triangular matrices without zero rows and zero columns (we call such matrices *Fishburn matrices*. Our approach is considerably simpler than the approach based on ascent sequences. More importantly, our approach allows to easily capture the notion of poset duality,

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which corresponds to transposition of Fishburn matrices. Consequently, we are easily able to adapt our method to the problem of enumerating self-dual interval orders, for which no explicit enumeration has been known so far.

Basic Notions

All the posets considered in this paper are assumed to be finite. We also assume that the posets are *unlabeled*, that is, isomorphic posets are taken to be identical. Let P be a poset with a strict order relation $\prec$. A *strict down-set* of an element $y \in P$ is the set $D(y)$ of all the elements of P that are smaller than y, i.e., $D(y) = \{x \in P;\ x \prec y\}$. Similarly, the *strict up-set* of y, denoted by $U(y)$, is the set $\{x \in P;\ x \succ y\}$. Note that y is a minimal element of P if and only if $D(y)$ is empty, and y is a maximal element if and only if $U(y)$ is empty.

For a poset P, the following conditions are known to be equivalent [2, 8]:

- P is $(\mathbf{2}+\mathbf{2})$-*free*, that is, P does not have an induced subposet isomorphic to the disjoint union of two chains of length two.
- P has an *interval representation*, that is, with each element $x \in P$ we may associate a real closed interval $[l_x, r_x]$, in such a way that $x \prec y$ if and only if $r_x < l_y$.
- For any two elements $x, y \in P$, the strict down-sets $D(x)$ and $D(y)$ are comparable by inclusion, i.e., $D(x) \subseteq D(y)$ or $D(y) \subseteq D(x)$.
- For any two elements $x, y \in P$, the strict up-sets $U(x)$ and $U(y)$ are comparable by inclusion.

The posets that satisfy these properties are known as *interval orders*. Let us briefly review some of their basic properties. For a more thorough exposition, see e.g. Fishburn's monograph [11].

Let P be an interval order. Two elements x and y of P are *indistinguishable* if $U(x) = U(y)$ and $D(x) = D(y)$. This is an equivalence relation on P. If no two distinct elements of P are indistinguishable, then P is said to be *primitive*. Every interval order P can be uniquely obtained from a primitive interval order P' by simultaneously replacing each element of P' by a positive number of 'duplicates'.

Since any two strict down-sets in P are comparable by inclusion, it is possible to arrange all the distinct strict down-sets into an increasing chain $D_1 \subsetneq D_2 \subsetneq \cdots \subsetneq D_m$, where m is the number of distinct strict down-sets determined by elements of P. An element $x \in P$ is said to have *level* i, if $D(x) = D_i$. Note that D_1 is always the empty set, and the elements of level 1 are exactly the minimal elements of P. Following Fishburn [9, 10], we call the number m of distinct strict down-sets the *magnitude* of P. It turns out that m is also equal to the number of distinct strict up-sets, and we can order the strict up-sets of P into a decreasing chain $U_1 \supsetneq U_2 \supsetneq \cdots \supsetneq U_m$, and we say that x has *up-level* i if $U(x) = U_i$. The maximal elements of P are precisely the elements of up-level m, and we have $U_m = \emptyset$. It can be shown [10] that an element of level i has an up-level greater than or equal to i. An interval representation of P can be obtained by mapping an element x with level i and up-level j to the (possibly degenerate) interval $[i, j]$. This is the unique representation of P by intervals with endpoints belonging to the set $[m] = \{1, 2, \ldots, m\}$, and in particular, there is no interval representation of P with fewer than m distinct endpoints.

The *dual* of a poset P is the poset $\overline{P}$ with the same elements as P and an order relation $\overline{\prec}$ defined by $x \overline{\prec} y \iff y \prec x$. A poset is *self-dual* if it is isomorphic to its dual. The dual of an interval order P of magnitude m is again an interval order of magnitude m, and an element of level i and up-level j in P has the level $m+1-j$ and up-level $m+1-i$ in $\overline{P}$.

Throughout this paper, an important part will be played by a bijective correspondence between interval orders and a certain kind of integer matrices, which we will call Fishburn matrices. We will state the key properties of the correspondence without proof. A slightly more technical, but essentially equivalent, version of this correspondence has been used by Fishburn [9, 11], where these matrices are called 'characteristic matrices'. The matrix representation has also been applied in several other papers related to interval orders [4, 6, 7].

A *Fishburn matrix* is an upper-triangular square matrix M of nonnegative integers with the property that every row and every column contains a nonzero entry. A Fishburn matrix is called *primitive* if all its entries are equal to 0 or 1. We always assume that a matrix has its rows numbered from top to bottom, and columns numbered left-to-right, starting with row and column number one. Let M_{ij} be the entry of M in row i and column j.

An interval order P of magnitude m corresponds to an $m\times m$ Fishburn matrix M with M_{ij} being equal to the number of elements of P that have level i and up-level j. Conversely, given an $m\times m$ Fishburn matrix M, we may recover the corresponding interval order P by taking the collection of intervals that contains precisely M_{ij} copies of the interval $[i,j]$, and taking this to be the interval representation of P.

This correspondence is a bijection between Fishburn matrices and interval orders. In fact, in this correspondence, each nonzero entry M_{ij} of M can be associated with a group of M_{ij} indistinguishable elements of P. Note that the sum of the i-th row of M is equal to the number of elements of level i in P, and similarly for column-sums and up-levels.

Primitive interval orders correspond to primitive Fishburn matrices. If the order P is mapped to a matrix M, then the dual order $\overline{P}$ is mapped to the matrix $\overline{M}$ obtained from M by transposition along the diagonal running from bottom-left to top-right. If a matrix M is equal to $\overline{M}$, we call it *self-dual*. Of course, self-dual matrices are representing precisely the self-dual interval orders.

Previous work and our results

Interval orders are equinumerous with several other combinatorial structures. Apart from the correspondence between interval orders and Fishburn matrices, there are also bijections mapping interval orders to other combinatorial objects, such as ascent sequences [3], Stoimenow matchings [17], or certain classes of pattern-avoiding permutations [3, 16]. Many of these combinatorial structures have been studied independently even before their relationship to interval orders was discovered.

The concept of interval order has been introduced by Fishburn [8] in 1970. In 1976, Andresen and Kjeldsen [1] have studied, under different terminology, an enumeration problem equivalent to counting the number of primitive Fishburn matrices with respect to their dimension and the number of elements in the first row, which in poset terminology corresponds to the number of primitive interval orders of a given magnitude and number of minimal elements. They obtained asymptotic bounds for the number of these matrices, as well as recurrence formulas that allowed them to compute several exact initial values.

In 1987, Haxell, McDonald and Thomason [12] have described the first efficient algorithm to compute the number of interval orders, using a recurrence derived using Fishburn matrices, which were already known to be equinumerous with interval orders, thanks to the work of Fishburn [11].

In 1998, Stoimenow [17] has introduced the concept of 'regular linearized chord diagram', later often referred to as a 'Stoimenow matching'. A Stoimenow matching of size n is a matching on the set $[2n]$ in which no two nested edges have adjacent endpoints. Stoimenow has introduced these matchings as a tool in the study of Vassiliev invariants of knots, and computed several asymptotic bounds on their number. Later, these bounds were improved by Zagier [19], who also showed that the generating function of Stoimenow matchings enumerated by their size admits a simple formula

$$F(x) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1 - (1-x)^i\right). \tag{1}$$

Recently, Bousquet-Mélou, Claesson, Dukes and Kitaev [3] have found a sequence of bijections, showing that interval orders are equinumerous with several other combinatorial objects, including Stoimenow matchings. They have also provided an alternative proof for (1), and derived a formula for the refined generating function that counts interval order with respect to their size and magnitude. This result has prompted renewed interest in this line of research. Dukes, Kitaev, Remmel and Steingrímsson [5] have found an expression for the generating function that enumerates primitive interval orders with respect to their size and magnitude, and deduced a formula that counts interval orders by their size, magnitude and the number of indistinguishable elements. Kitaev and Remmel [14] have obtained, among other results, the formula

$$F(x,y) = 1 + \sum_{n\geq 0} \frac{xy}{(1-xy)^{n+1}} \prod_{i=1}^{n} \left(1 - (1-x)^i\right), \tag{2}$$

where $F(x,y)$ is the generating function of interval orders in which x counts the size and y the number of minimal elements of the interval order. They conjectured that $F(x,y)$ can be also expressed in the following form:

$$F(x,y) = \sum_{n\geq 0} \prod_{i=1}^{n} \left(1 - (1-x)^{i-1}(1-xy)\right). \tag{3}$$

This conjecture has been subsequently confirmed by Yan [18] and independently by Levande [15]. Let us remark formula (3) also appears in Zagier's work [19, Theorem 1], but there it is interpreted in terms of Stoimenow matchings, not interval orders.

In Section 2 of this paper, we generalize the above-mentioned results of Bousquet-Mélou et al. [3], Dukes et al. [5], and Kitaev and Remmel [14], by obtaining a closed-form expression for the generation function of primitive interval orders, counted with respect to their magnitude, their size, and their number of minimal and maximal elements. From this expression, it is possible to directly derive the generating function of general interval orders, or of interval orders with bounded size of indistinguishability classes, counted with respect to the same statistics.

However, the main significance of our results is not in counting more statistics than previous papers, but rather in presenting a much simpler method to derive the generating functions. Previous results were largely based on a bijection, constructed by Bousquet-Mélou et al. [3], which maps interval orders to a certain kind of integer sequences, known as 'ascent sequences'. This bijection has a complicated recursive definition, which consequently leads to difficult recurrences for generating functions, which then require great ingenuity to be solved into closed forms. In contrast, the arguments in this paper are based on the encoding of interval orders as Fishburn matrices. We exploit a relationship between Fishburn matrices of dimension m and those of dimension $m+1$ to obtain a recurrence for the generating function that can be easily solved by elementary means.

To further illustrate the benefit of this approach, in Section 3 we enumerate self-dual interval orders, by a slightly more elaborate application of the same basic technique. The problem of counting self-dual interval orders with respect to their size, or counting primitive self-dual Fishburn matrices with respect to their dimension, does not seem to have been addressed by previous research, presumably because there is no known way to characterize self-duality in terms of ascent sequences. In view of this, it is remarkable that the expressions we obtain for the generating functions of self-dual objects are almost as simple as those of their non-self-dual counterparts.

2 Enumeration of Interval Orders

Recall that a primitive poset is a poset that does not contain any two indistinguishable elements. Our main concern will be to find an expression for the generating function of primitive interval orders, or equivalently, of 01-Fishburn matrices.

Let us call an element of a poset P *isolated* if it is not comparable to any other element of P. Note that an element is isolated if and only if it is both minimal and maximal. The number of isolated elements of an interval order P is equal to the value of the top-right cell of the corresponding Fishburn matrix. We will call the top-right cell of the matrix *the corner cell*.

Let us also say that an element of the poset P is *internal* if it is neither minimal nor maximal. We will consider these statistics of an interval order P:

- the magnitude of P, denoted by $\text{mag}(P)$,
- the number of isolated elements, denoted by $\text{iso}(P)$,
- the number of non-isolated minimal elements, denoted by $\min(P)$,
- the number of non-isolated maximal elements, denoted by $\max(P)$, and
- the number of internal elements, denoted by $\text{int}(P)$.

In particular, the size of P is equal to $\text{iso}(P) + \min(P) + \max(P) + \text{int}(P)$, and its number of minimal elements is equal to $\min(P) + \text{iso}(P)$. In Table 1, we summarize these statistics, with their interpretation both in terms of posets and in terms of matrices. As a matter of convenience, we restrict ourselves to non-empty interval orders and non-empty matrices in our arguments. If M is the Fishburn matrix representing an interval order P, we write $\text{iso}(M)$ as a synonym for $\text{iso}(P)$, and similarly for the other statistics. For an integer $n \geq 0$, we let $V_n(a, b)$ denote the polynomial $(a + 1)(b + 1)^n - 1$.

Let $\mathcal{P}$ be the set of all non-empty primitive interval orders, and let G be the generating function

$$G(t, v, w, x, y) = \sum_{P \in \mathcal{P}} t^{\text{mag}(P)} x^{\text{iso}(P)} y^{\min(P)} v^{\max(P)} w^{\text{int}(P)}.$$

We can now state the main result of this section.

Theorem 2.1 *The generating function $G(t, v, w, x, y)$ satisfies the identity*

$$G(t, v, w, x, y) = \sum_{n \geq 0} t^{n+1} \frac{V_n(x, y)}{1 + tV_n(v, w)} \prod_{i=0}^{n-1} \frac{V_i(v, w)}{1 + tV_i(v, w)}. \tag{4}$$

statistic	poset interpretation	matrix interpretation	variable
mag	magnitude	number of rows	t
iso	num. of isolated elements	value of corner cell	x
min	num. of non-isolated minimal elements	sum of the first row, except the corner cell	y
max	num. of non-isolated maximal elements	sum of the last column, except the corner cell	v
int	num. of internal elements	sum of cells outside first row and last column	w

Tab. 1: The statistics of interval orders

Remark 2.2 *Let $S_n \equiv S_n(t, v, w, x, y)$ denote the n-th summand of the sum on the right-hand side of* (4)*. Clearly, S_n is a multiple of t^{n+1}. Consequently, the sum on the right-hand side of* (4) *converges as a sum of power series in t. Moreover, S_n has total degree at least n in the variables v and w. Thus, the sum also converges as a sum of power series in v and w. Furthermore, for all k, the coefficient of t^k in S_n is a polynomial in v, w, x and y, and for all k, ℓ, the coefficient of $v^k w^\ell$ is a polynomial in t, x, and y.*

The properties stated in the above remark make the identity (4) amenable to many combinatorially meaningful substitutions. Before we state the proof of the theorem, we demonstrate several possible substitutions. Note that some of the formulas we derive have been previously obtained by different methods.

Corollary 2.3 *[5, Theorem 8] Let p_k be the number of primitive interval orders of size k. The generating function of p_k is equal to*

$$\sum_{k\geq 1} p_k x^k = G(1, x, x, x, x) = \sum_{n\geq 0} \prod_{i=0}^{n} \frac{V_i(x,x)}{1+V_i(x,x)} = \sum_{n\geq 0} \prod_{i=0}^{n} \left(1 - \frac{1}{(1+x)^{i+1}}\right).$$

Corollary 2.4 *Let m_k be the number of primitive $k \times k$ Fishburn matrices (or equivalently, the number of primitive interval orders of magnitude k). Then*

$$\sum_{m\geq 1} m_k t^k = G(t, 1, 1, 1, 1) = \sum_{n\geq 0} t^{n+1} \prod_{i=0}^{n} \frac{2^{i+1}-1}{1+t(2^{i+1}-1)}.$$

Let $\mathcal{G}$ be the set of non-empty interval orders, and let $G^*(t, v, w, x, y)$ be the corresponding generating function

$$G^*(t, v, w, x, y) = \sum_{P\in\mathcal{G}} t^{\mathrm{mag}(P)} x^{\mathrm{iso}(P)} y^{\mathrm{min}(P)} v^{\mathrm{max}(P)} w^{\mathrm{int}(P)}.$$

Each interval order can be uniquely obtained from a primitive interval order by replacing each element with a group of indistinguishable elements. In terms of generating functions, this means that

$$G^*(t, v, w, x, y) = G(t, \frac{v}{1-v}, \frac{w}{1-w}, \frac{x}{1-x}, \frac{y}{1-y}).$$

By substituting into (4), we get the next corollary.

Corollary 2.5 *The generating function $G^*(t, v, w, x, y)$ is equal to*

$$\sum_{n\geq 0} t^{n+1} \frac{1+V_n(-v,-w)}{1+V_n(-x,-y)} \cdot \frac{V_n(-x,-y)}{(t-1)V_n(-v,-w)-1} \cdot \prod_{i=0}^{n-1} \frac{V_i(-v,-w)}{(t-1)V_i(-v,-w)-1}.$$

From Corollary 2.5 we can obtain yet another proof of the formula (3) derived by Levande [15] and Yan [18] (and indirectly also by Zagier [19]) for the generating function of interval orders counted by their size and number of maximal elements (or equivalently size and number of minimal elements).

Corollary 2.6 *[15, 18, 19] Let $g_{k,\ell}$ be the number of interval orders with k elements and having exactly ℓ maximal elements (including isolated ones). We have*

$$\sum_{k,\ell\geq 1} g_{k,\ell}\, r^k s^\ell = G^*(1, rs, r, rs, r) = \sum_{n\geq 0}\prod_{i=0}^{n} -V_i(-rs,-r) = \sum_{n\geq 0}\prod_{i=0}^{n}(1-(1-rs)(1-r)^i).$$

Kitaev and Remmel have obtained a different expression for the generating function from the previous corollary. This alternative expression can also be derived from the general formula for G^*, by restricting the general formula to count minimal elements, rather than maximal ones.

Corollary 2.7 *[14] With $g_{k,\ell}$ as in Corollary 2.6, we have*

$$\sum_{k,\ell\geq 1} g_{k,\ell}\, r^k s^\ell = G^*(1, r, r, rs, rs) = \sum_{n\geq 0}\frac{rs}{(1-rs)^{n+1}}\prod_{i=0}^{n-1}\left(1-(1-r)^{i+1}\right). \qquad (5)$$

Let us now prove Theorem 2.1. Define $G_k(v,w,x,y) = [t^k]G(t,v,w,x,y)$, that is, G_k is the coefficient of t^k in G. We will state the proof in the language of Fishburn matrices rather than in the equivalent language of interval orders. It is thus convenient to view G_k as the generating function of the primitive $k\times k$ Fishburn matrices.

The next lemma provides the main idea in the proof of Theorem 2.1.

Lemma 2.8 *For any $k\geq 1$, we have*

$$G_{k+1}(v,w,x,y) = vG_k(v+w+vw, w, x+y+xy, y) - vG_k(v,w,x,y). \qquad (6)$$

Proof: Let $\mathcal{M}_k$ denote the set of primitive $k\times k$ Fishburn matrices. We will describe an operation which from a given matrix $M\in\mathcal{M}_k$ produces a (typically not unique) new matrix $M'\in\mathcal{M}_{k+1}$. The matrix M' is created by adding to M a new rightmost column and a new bottom row, and filling them according to these rules:

- $M'_{k+1,k+1}=1$, and all the other cells in row $k+1$ of M' have value 0.
- For every $j\leq k$, if $M_{j,k}=0$, then $M'_{j,k}=M'_{j,k+1}=0$.
- For every $j\leq k$, if $M_{j,k}=1$ we choose one of the three possibilities to fill $M'_{j,k}$ and $M'_{j,k+1}$: either $M'_{j,k}=0$ and $M'_{j,k+1}=1$, or $M'_{j,k}=M'_{j,k+1}=1$, or $M'_{j,k}=1$ and $M'_{j,k+1}=0$.
- Any other cell has the same value in M' as in M.

If M has p 1-cells in column k, then the above operation can produce 3^p matrices M'. All these 3^p matrices are upper-triangular, all of them have at least one 1-cell in each row, all of them have at least one 1-cell in each column different from column k, and all except for exactly one of them have at least one 1-cell in column k. In other words, for a given $M\in\mathcal{M}_k$ with p 1-cells in column k, the above operation produces $3^p - 1$ matrices M' from $\mathcal{M}_{k+1}$ (and one 'bad' matrix not belonging to $\mathcal{M}_{k+1}$). It is not difficult to see that each matrix $M' \in \mathcal{M}_{k+1}$ can be created in this way from exactly one matrix $M \in \mathcal{M}_k$.

More generally, suppose that $M \in \mathcal{M}_k$ is a matrix with $\mathrm{iso}(M) = a$, $\min(M) = b$, $\max(M) = c$ and $\mathrm{int}(M) = d$, that is, M contributes the monomial $x^a y^b v^c w^d$ into $G_k(v, w, x, y)$. Then all the Fishburn matrices produced from M have generating function

$$v\left((x + y + xy)^a y^b (v + w + vw)^c w^d - x^a y^b v^c w^d\right),$$

where the leftmost factor of v counts the 1-cell $(k + 1, k + 1)$ of M'. Summing this expression over all $M \in \mathcal{M}_k$ gives the recurrence from the statement of the lemma. □

An idea very similar to the previous proof has already been used by Haxell, McDonald, and Thomason [12] to obtain an efficient algorithm for the enumeration of interval orders. It is also closely related to the approach that Khamis [13] has used to derive a recurrence formula for the number of interval orders of a given size and height.

We now deduce Theorem 2.1 from Lemma 2.8 by a simple manipulation of series. For any power series $F(v, w, x, y)$, let $\sigma[F(v, w, x, y)]$ denote the power series $F(v + w + vw, w, x + y + xy, y)$, that is, σ represents the substitution of $v + w + vw$ for v and $x + y + xy$ for x. Let $\sigma^{(i)}[F(v, w, x, y)]$ denote the i-fold iteration of σ, i.e., $\sigma^{(0)}[F] = F$ and for $i \geq 1$, $\sigma^{(i)}[F] = \sigma\left[\sigma^{(i-1)}[F]\right]$. Note that

$$\sigma^{(i)}[F(v, w, x, y)] = F(V_i(v, w), w, V_i(x, y), y).$$

By writing G_k for $G_k(v, w, x, y)$ and G for $G(t, v, w, x, y)$, we can rewrite (6) as $G_{k+1} = v\sigma[G_k] - vG_k$, which implies $G - tG_1 = tv\sigma[G] - tvG$. Since $G_1 = x$, this simplifies into

$$G = \frac{tx}{1 + tv} + \frac{tv}{1 + tv}\sigma[G]. \tag{7}$$

Substituting repeatedly the right-hand side of (7) for the occurrence of G on the right-hand side, we obtain for any $m \geq 1$ the identity

$$G = \left(\sum_{n=0}^{m} \sigma^{(n)}\left[\frac{tx}{1 + tv}\right] \prod_{i=0}^{n-1} \sigma^{(i)}\left[\frac{tv}{1 + tv}\right]\right) + \left(\prod_{i=0}^{m} \sigma^{(i)}\left[\frac{tv}{1 + tv}\right]\right) \sigma^{(m+1)}[G]. \tag{8}$$

Since the rightmost summand of the right-hand side of (8) is $\mathcal{O}(t^{m+1})$, we can take the limit as $m \to \infty$, to obtain the identity in Theorem 2.1. This proves the theorem.

3 Self-dual posets

Recall that a $k \times k$ Fishburn matrix M represents a self-dual interval order if and only if M is invariant under transposition along the north-east diagonal, or in other words, if for each i, j the cell (i, j) has the same value as the cell $(k - j + 1, k - i + 1)$. We will say that the two cells (i, j) and $(k - j + 1, k - i + 1)$ form a *symmetric pair*.

As in the previous section, we will first concentrate on enumerating the primitive matrices. Unless otherwise noted, the generating functions are counting nonempty objects.

We distinguish three types of cells in a $k \times k$ matrix M: a cell (i,j) is a *diagonal cell* if $i+j=k+1$, i.e., (i,j) belongs to the north-east diagonal of the matrix. If $i+j<k+1$ (i.e., (i,j) is above and to the left of the diagonal) then (i,j) is a *North-West cell*, or *NW-cell*, while if $i+j>k+1$, then (i,j) is an *SE-cell*. The diagonal cells and SE-cells together uniquely determine a self-dual matrix.

Apart of the statistics introduced in Section 2 (see Table 1), we will also consider three new statistics of a matrix M.

- $\mathrm{diag}(M)$ is the sum of all the diagonal cells except for the corner cell.
- $\mathrm{se}(M)$ is the sum of all the SE-cells that do not belong to the last column.
- $\mathrm{nw}(M)$ is the sum of all the NW-cells that do not belong to the first row.

In particular, the sum of all cells in M is equal to $\mathrm{iso}(M)+\min(M)+\max(M)+\mathrm{se}(M)+\mathrm{nw}(M)+\mathrm{diag}(M)$. In a self-dual matrix we of course have $\min(M)=\max(M)$ and $\mathrm{se}(M)=\mathrm{nw}(M)$, so the above sum is equal to $\mathrm{iso}(M)+\mathrm{diag}(M)+2\max(M)+2\mathrm{se}(M)$.

If $k>1$, then among all the $k\times k$ primitive self-dual Fishburn matrices, exactly half have the corner cell filled with 1. This is because changing the corner cell from 1 to 0 cannot create an empty row or empty column, since the cells $(1,1)$ and (k,k) are always 1-cells and the value of the corner cell also does not affect the symmetry of the matrix. We will first enumerate the symmetric matrices whose corner cell has the fixed value 1, and then use the above observation to obtain the full count.

Let $\mathcal{S}^+$ be the set of primitive self-dual Fishburn matrices whose corner cell is equal to 1. Define the generating function $S^+(t,v,w,z)$ by

$$S^+(t,v,w,z)=\sum_{M\in\mathcal{S}^+} t^{\mathrm{mag}(M)}v^{\max(M)}w^{\mathrm{se}(M)}z^{\mathrm{diag}(M)}.$$

The next theorem is the key result of this section.

Theorem 3.1 *The generating function $S^+(t,v,w,z)$ is equal to*

$$\sum_{n\geq 0} t^{2n+1}\frac{1+tV_n(v,w)}{1+t^2V_n(v,w)}(1+z)^n(1+v)^n(1+w)^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{V_i(v,w)}{1+t^2V_i(v,w)}. \tag{9}$$

The comments in Remark 2.2 apply to the expression (9) as well.

Before we prove Theorem 3.1, we first state some of its consequences. Although Theorem 3.1 provides all the information we need for our enumerations, it is often more convenient to work with the closely related generating function that counts all Fishburn matrices, rather than just those that belong to $\mathcal{S}^+$. Let $\mathcal{S}$ be the set of primitive self-dual Fishburn matrices, and define

$$S(t,v,w,x,z)=\sum_{M\in\mathcal{S}} t^{\mathrm{mag}(M)}v^{\min(M)+\max(M)}w^{\mathrm{se}(M)+\mathrm{nw}(M)}x^{\mathrm{iso}(M)}z^{\mathrm{diag}(M)}.$$

Lemma 3.2 *The generating function S satisfies the identity*

$$S(t,v,w,x,z)=(1+x)S^+(t,v^2,w^2,z)-t. \tag{10}$$

Consequently, $S(t,v,w,x,z)$ is equal to

$$-t+(1+x)\sum_{n\geq 0} t^{2n+1}\frac{1+tV_n(v^2,w^2)}{1+t^2V_n(v^2,w^2)}(1+z)^n(1+v^2)^n(1+w^2)^{\binom{n}{2}}\prod_{i=0}^{n-1}\frac{V_i(v^2,w^2)}{1+t^2V_i(v^2,w^2)}.$$

Proof: The factor $(1+x)$ on the right-hand side of (10) corresponds to the fact that each primitive self-dual matrix either belongs to $\mathcal{S}^+$ or is obtained from a matrix in $\mathcal{S}^+$ by changing its corner cell from 1 to 0. The subtracted t accounts for the fact that the 1×1 matrix containing 0 is not a Fishburn matrix, even though it can be obtained from a matrix in $\mathcal{S}^+$ by changing the corner cell. The substitutions into S^+ are straightforward □

Corollary 3.3 *Let s_m be the number of self-dual primitive interval orders on m elements, with $s_0=1$. Then*

$$\sum_{m\geq 0} s_m x^m = 1+S(1,x,x,x) = \sum_{n\geq 0}(1+x)^{n+1}\prod_{i=0}^{n-1}\left((1+x^2)^{i+1}-1\right).$$

Corollary 3.4 *Let r_m be the number of primitive self-dual $m\times m$ Fishburn matrices. Then*

$$\sum_{m\geq 1} r_m t^m = S(t,1,1,1) = -t+\sum_{n\geq 0} 2^{\binom{2n+2}{2}} t^{2n+1}\frac{1+(2^{n+1}-1)t}{1+(2^{n+1}-1)t^2}\prod_{i=0}^{n-1}\frac{2^{i+1}-1}{1+(2^{i+1}-1)t^2}.$$

Let $S^*(t,v,w,x,z)$ be the generating function of (not necessarily primitive) self-dual interval orders, with variables having the same meaning as in $S(t,v,w,x,z)$. Clearly, a Fishburn matrix M representing a self-dual interval order may be obtained in a unique way from a matrix M' representing a primitive interval order, by changing each diagonal 1-cell of M' into an arbitrary non-zero cell, and by changing a symmetric pair of non-diagonal 1-cells of M' into a pair of nonzero cells having the same value. Thus,

$$\begin{aligned}S^*(t,v,w,x,z) &= \frac{1}{1-x}S^+\left(t,\frac{v^2}{1-v^2},\frac{w^2}{1-w^2},\frac{z}{1-z}\right)-t\\ &= -t+\sum_{n\geq 0}\frac{t^{2n+1}\left(1+(1-t)V_n(-v^2,-w^2)\right)\prod_{i=0}^{n-1}\frac{-V_i(-v^2,-w^2)}{1+(1-t^2)V_i(-v^2,-w^2)}}{(1-x)(1-z)^n(1-v^2)^n(1-w^2)^{\binom{n}{2}}\left(1+(1-t^2)V_n(-v^2,-w^2)\right)}.\end{aligned}$$

Corollary 3.5 *Let g_m be the number of self-dual interval orders on m elements, with $g_0=1$. Then*

$$\begin{aligned}\sum_{m\geq 0} g_m x^m = 1+S^*(1,x,x,x,x) &= \sum_{n\geq 0}\frac{1}{(1-x^2)^{\binom{n+1}{2}}(1-x)^{n+1}}\prod_{i=0}^{n-1}\left(1-(1-x^2)^{i+1}\right)\\ &= \sum_{n\geq 0}\frac{1}{(1-x)^{n+1}}\prod_{i=0}^{n-1}\left(\frac{1}{(1-x^2)^{i+1}}-1\right).\end{aligned}$$

The proof of Theorem 3.1 is based on the same general idea as the proof of Theorem 2.1. Let us define $S_k^+(v,w,z)=[t^k]S^+(t,v,w,z)$. The next lemma is the self-dual analogue of Lemma 2.8.

Lemma 3.6 *For any $k \geq 1$, we have*

$$S_{k+2}^{+}(v,w,z) = v(1+v)(1+z)S_k^{+}(v+w+vw,w,z) - vS_k^{+}(v,w,z). \tag{11}$$

Proof: Let $\mathcal{S}_k^+$ be the set of matrices of $\mathcal{S}^+$ of size $k \times k$. We will show how a given matrix $M \in \mathcal{S}_k^+$ can be extended into a matrix $M' \in \mathcal{S}_{k+2}^+$. Assume, just for the sake of this proof, that matrices in $\mathcal{S}_k^+$ have rows and columns indexed by $1, 2, \dots, k$, while matrices in $\mathcal{S}_{k+2}^+$ have rows and columns indexed by $0, 1, \dots, k+1$. Thus, if a cell (i,j) is a diagonal cell in $M \in \mathcal{S}_k^+$, then (i,j) is also a diagonal cell in $M' \in \mathcal{S}_{k+2}^+$, and similarly for SE-cells and NW-cells. The matrix M' is created from M by these rules:

- $M'_{k+1,k+1} = 1$, and any other cell in row $k+1$ of M' has value 0.
- $M'_{0,k+1} = 1$. Note that the cell $(0, k+1)$ is the corner cell of M'.
- $M'_{1,k}$ is filled arbitrarily by 0 or 1, and $M'_{1,k+1}$ is filled arbitrarily by 0 or 1 as well. (Recall that $M_{1,k} = 1$ by the definition of $\mathcal{S}^+$.)
- For any $j \in \{2, \dots, k\}$, if $M_{j,k} = 0$, then $M'_{j,k} = M'_{j,k+1} = 0$, while if $M_{j,k} = 1$, choose one of three possibilities: either $M'_{j,k} = 0$ and $M'_{j,k+1} = 1$, or $M'_{j,k} = M'_{j,k+1} = 1$, or $M'_{j,k} = 1$ and $M'_{j,k+1} = 0$.
- Any other SE-cell or diagonal cell of M' has the same value in M' as in M, and the NE-cells of M' are filled in order to form a self-dual matrix.

From a given matrix $M \in \mathcal{S}_k^+$ with $\max(M) = p$, this procedure creates $4 \cdot 3^p$ distinct self-dual matrices of size $(k+2) \times (k+2)$, which all belong to $\mathcal{S}_{k+2}^+$ except for one matrix, which has column k filled with zeros. If M is a matrix that contributes a monomial $v^p w^q z^r$ into the generating function $S_k^+(v,w,z)$, then the matrices in $\mathcal{S}_{k+2}^+$ created from M have the generating function

$$\sum_{\substack{M' \in \mathcal{S}_{k+2}^+ \\ M' \text{ obtained from } M}} v^{\max(M')} w^{\mathrm{se}(M')} z^{\mathrm{diag}(M')} = v\big((1+v)(1+z)(v+w+vw)^p w^q z^r - v^p w^q z^r\big). \tag{12}$$

It is easy to see that each matrix $M' \in \mathcal{S}_{k+2}^+$ can be generated by the above rules from a unique matrix $M \in \mathcal{S}_k^+$. By summing the identity (12) over all $M \in \mathcal{S}_k^+$, we obtain the identity (11). □

To prove Theorem 3.1 from Lemma 3.6, we can just imitate the proof of Theorem 2.1 from Lemma 2.8. We omit the straightforward argument.

4 Final remarks

The formulas of the form we derived in this paper are useful in providing an efficient way to explicitly compute the coefficients of the corresponding generating functions. It is not clear, however, whether one can use such formulas to extract information about the asymptotic growth of the coefficients. Zagier [19] has used formula (1), together with some highly non-trivial power series identities, to find a very precise asymptotic estimate of the number of interval orders on n elements. The corresponding formula for self-dual interval orders from Corollary 3.5 has a similar form, and perhaps it might be also used as a basis for an asymptotic estimate.

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A reciprocity approach to computing generating functions for permutations with no pattern matches

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Abstract. In this paper, we develop a new method to compute generating functions of the form

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}$$

where τ is a permutation that starts with 1, $\mathcal{NM}_n(\tau)$ is the set of permutations in the symmetric group S_n with no τ-matches, and for any permutation $\sigma \in S_n$, $\mathrm{LRMin}(\sigma)$ is the number of left-to-right minima of σ and $\mathrm{des}(\sigma)$ is the number of descents of σ. Our method does not compute $NM_\tau(t,x,y)$ directly, but assumes that

$$NM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where $U_\tau(t,y) = \sum_{n\geq 0} U_{\tau,n}(y)\frac{t^n}{n!}$ so that $U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)}$. We then use the so-called homomorphism method and the combinatorial interpretation of $NM_\tau(t,1,y)$ to develop recursions for the coefficient of $U_\tau(t,y)$.

Résumé. Dans cet article, nous développons une nouvelle méthode pour calculer les fonctions génératrices de la forme

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}$$

où τ est une permutation, $\mathcal{NM}_n(\tau)$ est l'ensemble des permutations dans le groupe symétrique S_n sans τ-matches, et pour toute permutation $\sigma \in S_n$, $\mathrm{LRMin}(\sigma)$ est le nombre de minima de gauche à droite de σ et $\mathrm{des}(\sigma)$ est le nombre de descentes de σ. Notre méthode ne calcule pas $NM_\tau(t,x,y)$ directement, mais suppose que

$$NM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

où $U_\tau(t,y) = \sum_{n\geq 0} U_{\tau,n}(y)\frac{t^n}{n!}$ de sorte que $U_\tau(t,y) = \frac{1}{NM_\tau(t,1,y)}$. Nous utilisons ensuite la méthode dite "de l'homomorphisme" et l'interprétation combinatoire de $NM_\tau(t,1,y)$ pour développer des récursions sur le coefficient de $U_\tau(t,y)$.

Keywords: permutation, pattern match, descent, left to right minimum, symmetric polynomial, exponential generating function

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1 Introduction

Given a sequence $\sigma = \sigma_1 \ldots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau = \tau_1 \ldots \tau_j$ in the symmetric group S_j, we say a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ has a τ-*match starting at position* i provided $\mathrm{red}(\sigma_i \ldots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. Given a permutation $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, we let $\mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. We say that σ_j is a *left-to-right minimum* of σ if $\sigma_j < \sigma_i$ for all $i < j$. We let $\mathrm{LRMin}(\sigma)$ denote the number of left-to-right minima of σ.

The main goal of this paper is to give a new method to compute generating functions of the form

$$NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} NM_\tau(x,y) \tag{1}$$

where

$$NM_\tau(x,y) = \sum_{\sigma\in\mathcal{NM}_n(\tau)} x^{\mathrm{LRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}, \tag{2}$$

$\tau \in S_j$, and $\mathcal{NM}_n(\tau)$ is the set of permutations in S_n with no τ-matches.

Our results where motivated by results of the authors in [6] where they introduced to the study of patterns in the cycle structure of permutations. That is, suppose that $\tau = \tau_1 \ldots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles $C_1 \ldots C_k$. We shall always write cycles in the form $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of C_i and we arrange the cycles by increasing smallest elements. That is, we arrange the cycles of σ so that $c_{0,1} < \cdots < c_{0,k}$. Then we say that σ has a *cycle-τ-match* (c-τ-match) if there is an i such that $C_i = (c_{0,i}, \ldots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\mathrm{red}(c_{r,i}c_{r+1,i} \ldots c_{r+j-1,i}) = \tau$ where we take indices of the form $r+s$ modulo p_i. Let $c\text{-}\tau\text{-mch}(\sigma)$ be the number of cycle-τ-matches in the permutation σ. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then $9\ 1\ 10$ is a cycle-τ-match in the first cycle and $7\ 5\ 8$ and $6\ 4\ 7$ are cycle-τ-matches in the third cycle so that $c\text{-}\tau\text{-mch}(\sigma) = 3$.

Given a cycle $C = (c_0, \ldots, c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\mathrm{cdes}(C) = 1 + \mathrm{des}(c_0 \ldots c_{p-1})$. Thus $\mathrm{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra 1 counts the descent pair $c_{p-1} > c_0$. For example if $C = (1,5,3,7,2)$, then $\mathrm{cdes}(C) = 3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C. By convention, if $C = (c_0)$ is a one-cycle, we let $\mathrm{cdes}(C) = 1$. If σ is a permutation in S_n with k cycles $C_1 \ldots C_k$, then we define $\mathrm{cdes}(\sigma) = \sum_{i=1}^k \mathrm{cdes}(C_i)$. We let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ.

In [6], Jones and Remmel studied generating functions of the form

$$NCM_\tau(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NCM}_n(\tau)} x^{\mathrm{cyc}(\sigma)} y^{\mathrm{cdes}(\sigma)}$$

where $\mathcal{NCM}_n(\tau)$ denote the set of permutations $\sigma \in S_n$ which have no cycle-τ-matches. The basic approach used in that paper was to use theory of exponential structures to reduce the problem of computing $NCM_\tau(t,x,y)$ to the problem of computing similar generating functions for n-cycles. That is, let $\mathcal{NCM}_{n,k}(\tau)$ denote the set of permutations σ of S_n with k cycles such that σ has no cycle-τ-matches and $\mathcal{L}_m^{ncm}(\tau)$ denote the set of m-cycles γ in S_m such γ has no cycle-τ-matches. The following theorem follows easily from the theory of exponential structures as is described in [13], for example.

Theorem 1

$$NCM_\tau(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{NCM}_{n,k}(\tau)} y^{\mathrm{cdes}(\sigma)} = e^{x \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(\tau)} y^{\mathrm{cdes}(C)}}. \quad (3)$$

It turns out that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $NCM_\tau(t,x,y)$ to the usual problem of finding the generating function of permutations that have no τ-matches. That is, suppose we are given a permutation $\sigma \in S_n$ with k cycles $C_1 \cdots C_k$. Assume we have arranged the cycles so that the smallest element in each cycle is on the left and we arrange the cycles by decreasing smallest elements. Then we let $\bar{\sigma}$ be the permutation that arises from $C_1 \cdots C_k$ by erasing all the parenthesis and commas. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$, then $\bar{\sigma} =$ 7 10 9 11 4 8 6 1 5 3 2. It is easy to see that the minimal elements of the cycles correspond to left-to-right minima in $\bar{\sigma}$. It is also easy to see that under our bijection $\sigma \to \bar{\sigma}$, that $\mathrm{cdes}(\sigma) = \mathrm{des}(\bar{\sigma}) + 1$ since every left-to-right minima is part of a descent pair in $\bar{\sigma}$. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$ so that $\bar{\sigma} =$ 7 10 9 11 4 8 6 1 5 3 2, $\mathrm{cdes}((7,10,9,11)) = 2$, $\mathrm{cdes}((4,8,6)) = 2$, and $\mathrm{cdes}((1,5,3,2)) = 3$ so that $\mathrm{cdes}(\sigma) = 2+2+3 = 7$ while $\mathrm{des}(\bar{\sigma}) = 6$. This given, Jones and Remmel [6] proved that if $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$, (1) σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima, (2) $\mathrm{cdes}(\sigma) = 1 + \mathrm{des}(\bar{\sigma})$, and (3) σ has no cycle-τ-matches if and only if $\bar{\sigma}$ has no τ-matches. It follows that if $\tau \in S_j$ and τ starts with 1, then $NCM_\tau(t,x,y) = NM_\tau(t,x,y)$. Hence, by Theorem 1, we should expect that $NM_\tau(t,x,y)$ is of the form $F(t,y)^x$ for some function $F(t,y)$. We should note that if a permutation τ does not start with 1, then it may be that case that $|\mathcal{NM}_n(\tau)| \neq |\mathcal{NCM}_n(\tau)|$. For example, Jones and Remmel [6] computed that $|\mathcal{NCM}_7(3142)| = 4236$ and $|\mathcal{NM}_7(3142)| = 4237$.

Jones and Remmel [6] were able compute functions of the form $NCM_\tau(t,x,y)$ when τ starts with 1 by combinatorially proving certain recursions for the $NCM_{\tau,n}(x,y)$ which led to certain sets of differential equations satisfied by $NCM_\tau(t,x,y)$. For example, using such methods, they were able to prove the following two theorems.

Theorem 2 *Let $\tau = \tau_1 \ldots \tau_j \in S_j$ where $j \geq 3$ and $\tau_1 = 1$ and $\tau_j = 2$. Then*

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \frac{1}{(1 - \int_0^t e^{(y-1)s - \frac{y^{\mathrm{des}(\tau)} s^{j-1}}{(j-1)!}} ds)^x} \quad (4)$$

Theorem 3 *Suppose that $\tau = 1\ 2\ \ldots j-1\ \gamma\ j$ where γ is a permutation of $j+1, \ldots, j+p$ and $j \geq 3$. Then*

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where $U_\tau(t,y) = \sum_{n\geq 0} U_{\tau,n}(y) \frac{t^n}{n!}$ *and for all* $n \geq 0$*,*

$$U_{\tau,n+j}(y) = (1-y)U_{\tau,n+j-1}(y) - y^{\mathrm{des}(\tau)} \binom{n}{p} U_{\tau,n-p+1}(y). \quad (5)$$

The main goal of this paper is develop a new method to obtain results similar to Theorem 3 for different classes of permutations. The basic idea of our method is not to try to compute $NCM_\tau(t,x,y)$ directly.

Instead, we assume that τ starts with 1 and

$$NM_\tau(t,x,y) = \left(\frac{1}{U_\tau(t,y)}\right)^x \text{ where } U_\tau(t,y) = 1 + \sum_{n\geq 1} U_{\tau,n}(y)\frac{t^n}{n!}. \tag{6}$$

Then clearly,

$$U_\tau(t,y) = \frac{1}{1+\sum_{n\geq 1} NM_{\tau,n}(1,y)\frac{t^n}{n!}}. \tag{7}$$

Remmel and various coauthors [1, 7, 8, 9, 10, 11, 12] developed a method called the homomorphism method to show that many generating functions involving permutation statistics can be derived by applying a homomorphism defined on the ring of symmetric functions Λ to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{8}$$

where

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1}\frac{1}{1-x_i t} \text{ and } E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t \tag{9}$$

are the generating functions of the homogeneous symmetric functions h_n and the elementary symmetric functions e_n in infinitely many variables $x_1, x_2, \ldots$. Now suppose that we define a homomorphism θ on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ by setting

$$\theta(e_n) = \frac{(-1)^n}{n!} NM_{\tau,n}(1,y).$$

Then

$$\theta(E(-t)) = \frac{1}{\sum_{n\geq 0} NM_{\tau,n}(1,y)\frac{t^n}{n!}} = U_\tau(t,y).$$

Thus $\theta(H(t))$ should equal $U_\tau(t,y)$. We shall then show how to use the combinatorial methods associated with the homomorphism method to develop recursions for the coefficient of $U_\tau(t,y)$ similar to those in Theorem 3. For example, we can show that

$$U_{1423,n}(y) = (1-y)U_{1423,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} (-y)^k \binom{n-k-2}{k} U_{1423,n-2k-1}(y) \text{ and} \tag{10}$$

$$U_{1324,n}(y) = (1-y)U_{1324,n-1}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{1324,n-2k+1}(y) \tag{11}$$

where C_k is the k-th Catalan number.

The outline of this paper is as follows. In Section 2, we shall briefly recall the background in the theory of symmetric functions that we will need for our proofs. Then in Section 3, we shall illustrate our method by proving (10) and we shall state some general results that follow from our methods.

2 Symmetric functions.

In this section, we give the necessary background on symmetric functions that will be needed for our proofs.

Given a partition $\lambda = (\lambda_1, \dots \lambda_\ell)$ where $0 < \lambda_1 \leq \cdots \leq \lambda_\ell$, we let $\ell(\lambda)$ be the number of nonzero integers in λ. If the sum of these integers is equal to n, then we say λ is a partition of n and write $\lambda \vdash n$.

Let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, \dots$. The n^{th} elementary symmetric function $e_n = e_n(x_1, x_2, \dots)$ and n^{th} homogeneous symmetric function $h_n = h_n(x_1, x_2, \dots)$ are defined are defined by the generating functions given in (9). For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$ and $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$. It is well known that that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ. In particular, $e_0, e_1, \dots$ is an algebraically independent set of generators for Λ and, hence, a ring homomorphism θ on Λ can be defined by simply specifying $\theta(e_n)$ for all n.

A key element of our proofs is the combinatorial description of the coefficients of the expansion of h_n in terms of the elementary symmetric functions e_λ given by Eğecioğlu and Remmel in [5]. They defined a λ-brick tabloid of shape (n) to be a rectangle of height 1 and length n chopped into "bricks" of lengths found in the partition λ. For example, Figure 1 shows one brick $(2, 3, 7)$-tabloid of shape (12).

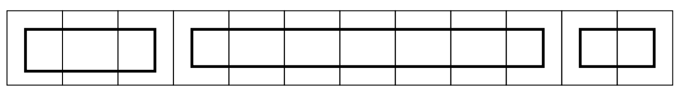

Fig. 1: A $(2, 3, 7)$-brick tabloid of shape (12).

Let $\mathcal{B}_{\lambda,n}$ denote the set of λ-brick tabloids of shape (n) and let $B_{\lambda,n}$ be the number of λ-brick tabloids of shape (n). If $B \in \mathcal{B}_{\lambda,n}$ we will write $B = (b_1, \dots, b_{\ell(\lambda)})$ if the lengths of the bricks in B, reading from left to right, are $b_1, \dots, b_{\ell(\lambda)}$. Through simple recursions, Eğecioğlu and Remmel [5] proved that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{12}$$

3 Computing $U_{1423,n}(y)$.

The main goal of this section is to prove the following theorem.

Theorem 4 *Let $\tau = 1423$. Then*

$$NM_\tau(t, x, y) = \left(\frac{1}{U_\tau(t, y)}\right)^x \text{ where } U_\tau(t, y) = 1 + \sum_{n \geq 1} U_{\tau,n}(y) \frac{t^n}{n!} \tag{13}$$

and $U_{\tau,1}(y) = -y$ and for $n > 1$,

$$U_{\tau,n}(y) = (1-y)U_{\tau,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} (-y)^k \binom{n-(k+2)}{k} U_{\tau,n-2k-1}(y). \tag{14}$$

Proof: We define a ring homomorphism on the ring of symmetric functions Λ by setting $\theta(e_0) = 1$ and

$$\theta(e_n) = \frac{(-1)^n}{n!} NM_{1423,n}(1,y) \text{ for } n \geq 1. \tag{15}$$

It follows that

$$\begin{aligned}
\theta(H(t)) &= \sum_{n\geq 0} \theta(h_n)t^n = \frac{1}{\theta(E(-t))} = \frac{1}{1+\sum_{n\geq 1}(-t)^n\theta(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1}\frac{t^n}{n!}NM_{1423,n}(1,y)} = 1 + \sum_{n\geq 1} U_{1423,n}(y)\frac{t^n}{n!}.
\end{aligned}$$

Thus $\theta(h_n) = \frac{U_{1423,n}(y)}{n!}$. Hence, to compute $U_{1423,n}(y)$, we must compute $n!\theta(h_n)$. It follows from (12) that

$$\begin{aligned}
n!\theta(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,n}\theta(e_\mu) \\
&= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \frac{(-1)^{b_i}}{b_i!} NM_{1423,b_i}(1,y) \\
&= \sum_{\mu\vdash n}(-1)^{\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}} \binom{n}{b_1,\ldots,b_{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} NM_{1423,b_i}(1,y).
\end{aligned} \tag{16}$$

Our next goal is to give a combinatorial interpretation to the right-hand side of (16). If we are given a brick tabloid $B = (b_1,\ldots,b_{\ell(\mu)})$, then we can interpret the multinomial coefficient $\binom{n}{b_1,\ldots,b_\mu}$ as all ways to assign sets $S_1,\ldots,S_{\ell(\mu)}$ to the bricks of B in such a way that $|S_i| = b_i$ for $i = 1,\ldots,\ell(\mu)$ and the sets $S_1,\ldots,S_{\ell(\mu)}$ form a set partition of $\{1,\ldots,n\}$. Next for each brick b_i, we use the factor

$$NM_{1423,b_i}(1,y) = \sum_{\sigma\in S_{b_i},\text{1423-mch}(\sigma)=0} y^{\text{des}(\sigma)+1}$$

to pick a rearrangement $\sigma^{(i)}$ of S_i which has no 1423-matches to put in cells of b_i and then we place a label of y on each cell that starts a descent in $\sigma^{(i)}$ plus a label of y on the last cell of b_i. Finally, we use the term $(-1)^{\ell(\mu)}$ to turn each label y at the end of brick to a $-y$. We let $\mathcal{O}_n$ denote the set of all objects created in this way. For each element $O \in \mathcal{O}_n$, we define the weight of O, $W(O)$, to be the product of y labels and the sign of O, $sgn(O)$, to be $(-1)^{\ell(\mu)}$. For example, such an object O constructed from the brick tabloid $B = (3,7,3)$ is pictured in Figure 2 where $W(O) = y^7$ and $sgn(O) = (-1)^3$. It follows that

$$U_{1423,n}(y) = \sum_{O\in\mathcal{O}_n} sgn(O)W(O). \tag{17}$$

Next we define a weight-preserving sign-reversing involution I on $\mathcal{O}_n$. Given an element $O \in \mathcal{O}_n$, scan the cells of O from left to right looking for the first cell c such that either (i) c is labeled with a y or (ii) c is a cell at the end of a brick b_i, the element in cell c is bigger than the element in the first cell of the

		−y		y		y		y	−y	y		−y
4	7	11	8	10	5	12	3	9	6	2	1	13

Fig. 2: An element of $\mathcal{O}_{13}$.

next brick b_{i+1}, and there is no 1423-match in elements of the cells in bricks b_i and b_{i+1}. In case (i), if c is a cell in brick b_j, then we split b_j in to two bricks b'_j and b''_j where b'_j contains all the cells of b_j up to an including cell c and b''_j consists of the remaining cells of b_j and we change to label on cell c from y to $-y$. In case (ii), we combine the two bricks b_i and b_{i+1} into a single brick b and change the label on cell c from $-y$ to y. For example, consider the element $O \in \mathcal{O}_{13}$ pictured in Figure 2. Even though the last element of brick 1 is bigger than the first element of brick 2, we can not combine these two bricks because the elements 7 11 8 10 would be a 1423-match. Thus the first place that we can apply the involution is on cell 5 which is labeled with a y so that $I(O)$ is the object pictured in Figure 3.

		−y		−y		y		y	−y	y		−y
4	7	11	8	10	5	12	3	9	6	2	1	13

Fig. 3: $I(O)$ for O in Figure 2.

We claim that I is an involution so that I^2 is the identity. To see this, consider case (i) where we split a brick b_j at cell c which is labeled with a y. In that case, we let a be the element in cell c and a' be the element in cell $c+1$ which must also be brick b_j. It must be the case that there is no cell labeled y before cell c since otherwise we would not use cell c to define the involution. However, we have to consider the possibility that when we spilt b_j into b'_j and b''_j that we might then be able to combine the brick b_{j-1} with b'_j because the element in that last cell of b_{j-1} is bigger that the element in the first cell of b'_j and there is no 1423-match in cells of b_{j-1} and b'_j. However, the only reason that we could not combine b_{j-1} and b_j is that there must be a 1423-match in cells of b_{j-1} and b_j. Clearly, this match must involve the element a'. Now a' can not be the first element of a 1423-match since otherwise the 1423-match would have occurred in b_j. Thus the 1423-match in cells of b_{j-1} and b_j must have also involved a. Since cell c was labeled with y, we must have $a > a'$ which means that the only possibility is that a plays the role of 4 in the 1423-match. But then, it cannot be that element in cell $c-1$ was part of brick b_{j-1} because that element must play the role of 1 in the 1423-match and we are assuming that the the element in that last cell of b_{j-1} is bigger that the element in the first cell of b'_j. But this would mean that there must have been a 1423-match in b_j in the first place which contradicts the fact that there are no 1423-matches in the cells of any brick in an object $O \in \mathcal{O}_n$. Thus one we apply case (i), the first action that we can take is combine bricks b'_j and b''_j so that $I^2(O) = O$.

If we are in case (ii), then again we can assume that there are no cells labeled y that occur before cell c. When we combine brick b_i and b_{i+1}, then we will label cell c with a y. It is clear that combining the elements of b_i and b_{i+1} cannot help us combine the resulting brick b with an earlier brick since it will be harder to have no 1423-matches with the larger brick b. Thus the first place cell c where we can apply the involution will again be cell c which is now labeled with a y so that $I^2(O) = O$ if we are in case (ii).

It is clear from our definitions that if $I(O) \neq O$, then $sgn(O)W(O) = -sgn(I(O))W(I(O))$. Hence it follows from (17) that

$$U_{1423,n}(y) = \sum_{O \in \mathcal{O}} sgn(O)W(O) = \sum_{O \in \mathcal{O}, I(O)=O} sgn(O)W(O). \tag{18}$$

Thus we must examine the fixed points of I. So assume that O is a fixed point of I. First it is easy to see that there can be no cells which are labeled with y so that elements in each brick of O must be increasing. Second we cannot combine two consecutive bricks b_i and b_{i+1} in O which means either that there is an increase between the bricks b_i and b_{i+1} or there is a decrease between the bricks b_i and b_{i+1} but there is a 1423-match in the cells of the bricks b_i and b_{i+1}. We claim that, in addition, the elements in the first cells of the bricks must form an increasing sequence, reading from left to right. That is, suppose that b_i and b_{i+1} are two consecutive bricks in a fixed point O of I and that $a > a'$ where a is the element in the first cell of b_i and a' is element in the first cell of b_{i+1}. Then clearly the element in the last cell of b_i must be bigger than a' so that it must be the case that there is a 1423-match in the cells of b_i and b_{i+1}. However a' is smallest element that resides in cells of of b_i and b_{i+1} which means that the only way that a' could be part of a 1423-match that occurs in the cells of b_i and b_{i+1} is to have a' play the role of 1. But then the 1423-match would be entirely contained in b_{i+1} which is impossible. Thus a' cannot be part of any 1423-match that occurs in the cells of b_i and b_{i+1}. However, this would mean that the 1423-match that occurs in the cells of b_i and b_{i+1} must either be contained entirely in the cells of b_i or entirely in the cells of b_{i+1} which again is impossible. Thus it must be the case that $a < a'$.

Thus we know that in a fixed point of I, 1 must be in the first cell. We claim that 2 must be either in cell 2 or 3. That is, suppose that 2 in cell c where $c \geq 4$. Then clearly cell c must be the first element of its brick since otherwise the cell $c-1$ would be labeled with y which is impossible for a fixed point of I. Now suppose that cell c is part of brick b_{i+1}. Then the element e in cell $c-1$ must be part of brick b_i. We claim that we must be able to combine bricks b_i and b_{i+1} in that case. Since 1 is in cell 1, we know that $e > 2$. Thus since I is a fixed point, it must be that there is 1423-match in the cells of b_i and b_{i+1}. Since $c \geq 4$, the element in cells $c-2$ and $c-1$ must both be greater than 2. Thus the only way that 2 could be part of 1423-match in the cells of b_i and b_{i+1} if it plays the role of 1 in 1423-match. But in that case the 1423-match would occur within the cells of b_{i+1} which is impossible. Thus 2 cannot be part of any 1423-match in the cells of b_i and b_{i+1} which would imply that 1423-match must occur entirely in b_i or entirely in b_{i+1}. Thus it must be the case that 2 lies in cell 2 or 3.

To summarize, we have proved the following proposition.

Proposition 5 *Suppose that $O \in \mathcal{O}_n$ and $I(O) = O$. Then the following hold.*

1. *The elements within each brick of O are increasing.*
2. *The first elements of each brick increase from left to right.*
3. *1 in cell 1 and 2 in cell 2 or 3.*
4. *If b_i and b_{i+1} are two consecutive bricks in O, then either (a) there is increase between b_i and b_{i+1} or (b) there is a decrease between b_i and b_{i+1} but there is 1423-match in the cells of b_i and b_{i+1}.*

Next we shall classify the fixed points $O \in \mathcal{O}_n$ of I. Suppose O is a fixed point of I which consists of bricks $b_1, \ldots, b_k$ reading from right to left.

Case 1. 2 is cell 2 in O.
In this case, there a two possibilities. Namely either (a) 1 is a brick by itself and is labeled with $-y$ or (b) 1 and 2 occupy the first two cells of the first brick. In case (a), we can remove the first brick and then we will be left with a fixed point of I in $\mathcal{O}_{n-1}$ after renumbering. Thus these fixed points of type (a) will contribute $-yU_{1423,n-1}(y)$ to $U_{1423,n}(y)$. In case (b), we can collapse the first brick by removing cell 1 and again, we will be left with a fixed point of I in $\mathcal{O}_{n-1}$ after renumbering. Thus fixed points of type (b) will contribute $U_{1423,n-1}(y)$ to $U_{1423,n}(y)$. Hence, the fixed points in Case 1 contribute $(1-y)U_{1423,n-1}(y)$ to $U_{1423,n}(y)$.

Case 2. 2 in cell 3 in O.
In this case, 2 must be the first cell of its brick since the elements are increasing in each brick. So suppose x_1 is in cell 2. Since 1 is in cell 1, it must be the case that $x_1 > 2$. We claim that x_1 cannot be in a brick b_2 by itself since otherwise 2 starts brick b_3 and we can combine bricks b_2 and b_3. That is, in such a situation, 2 would the smallest element in the cells of bricks b_2 and b_3. Hence, the only way that 2 could be part of 1423-match in the cells of bricks b_2 and b_3 is if 2 plays the role of 1 in that match. But then the 1423-match would be entirely contained in b_3 which is impossible. Thus we would be able to combine bricks b_2 and b_3 in such a situation. It follows that x_1 must be in b_1 and there is a 1423-match in the cells of b_1 and b_2. This means that 2 can not be the only element in brick b_2. Thus it must be the case that b_1 consists of 1 and x_1 and b_2 consists of 2 followed by at least one other element.

By using the same argument that we used to show that 2 must be either in cell 2 or 3 in a fixed point of I, we can show that in Case 2, 3 must be either in cell 4 or 5. Then we have the following two subcases.

Subcase 2.1. 3 is in cell 4.
In this case, brick b_1 consists of 1 followed by x_1 and contributes a $-y$ to $sgn(O)W(O)$. Then either (i) b_2 consists of the elements 2 and 3 or (ii) b_2 consists of 2 and 3 plus at least one additional element. In case (i), we can form a fixed point of $\mathcal{O}_{n-3}$ by removing brick b_1 and collapsing brick b_2 by removing 2 and renumbering. In this case, the first brick of the resulting object in $\mathcal{O}_{n-3}$ will start is brick of size 1 that contains the element 1. In case (ii), we can also form a fixed point of $\mathcal{O}_{n-3}$ by removing brick b_1 and collapsing brick b_2 by removing 2 and renumbering. In case (ii), the first brick of the resulting object in $\mathcal{O}_{n-3}$ will start with a brick that contains 1 plus some other elements. In this way, we can get an arbitrary fixed point of I in $\mathcal{O}_{n-3}$. We then have $\binom{n-3}{1}$ ways to pick x_1 so that the objects in Subcase 2.1 contribute $-y\binom{n-3}{1}U_{1423,n-3}(y)$ to $U_{1423,n}(y)$.

Subcase 2.2. 3 is in cell 5.
Let x_2 be the element in cell 4 of O. Since 1 is in cell 1 and 2 in cell 3, it follows that $x_2 > 3$ and hence 3 must start brick b_3. Thus b_1 consists of 1 and x_1, b_2 consists of 2 and x_2, and 1 x_1 2 x_2 must be a 1423-match so that we must have $x_1 > x_2$. There must also be a 1423-match contained in the cells of b_2 and b_3 which means that there must be at least one additional element in brick b_3.

By using the same argument that we used to show that 2 must be either in cell 2 or 3 in a fixed point of I, we can show that in Subcase 2.2, 4 must be either in cell 6 or 7. Then we have the following two subcases.

Subcase 2.2.1. 4 is in cell 6.

In this case, brick b_1 consists of 1 followed by x_1 and b_2 consists of 2 and x_2. Thus b_1 and b_2 contribute a factor of $(-y)^2$ to $sgn(O)W(O)$. Then either (i) b_3 consists of the elements 3 and 4 or (ii) b_3 consists of 3 and 4 plus at least one additional element. In case (i), we can form a fixed point of $\mathcal{O}_{n-5}$ by removing bricks b_1, and b_2 and collapsing brick b_3 by removing 3 and renumbering. In this case, the first brick of the resulting object in $\mathcal{O}_{n-5}$ will start is brick of size 1 that contains the element 1. In case (ii), we can also form a fixed point of $\mathcal{O}_{n-5}$ by removing bricks b_1 and b_2 and collapsing brick b_3 by removing 3 and renumbering. In case (ii), the first brick of the resulting object in $\mathcal{O}_{n-5}$ will start with a brick that contains 1 plus some other elements. In this way, we can get an arbitrary fixed point of I in $\mathcal{O}_{n-5}$. We then have $\binom{n-4}{2}$ ways to pick x_1 and x_2 so that the objects in Subcase 2.2.1 contribute $(-y)^2\binom{n-4}{2}U_{1423,n-5}(y)$ to $U_{1423,n}(y)$.

Subcase 2.2.2. 4 is in cell 7.
Let x_2 be the element in cell 4 of O and x_3 be the element in cell 6 of O. Since 1 is in cell 1, 2 in cell 3, and 3 is in cell 5, it follows that $x_3 > 4$ and hence 4 must start brick b_4. Thus b_1 consists of 1 and x_1, b_2 consists of 2 and x_2, and b_3 consists of 3 and x_3. Moreover, 1 x_1 2 x_2 and 2 x_2 3 x_3 must be 1423-matches so that we must have $x_1 > x_2 > x_3$. There must also be a 1423-match contained in the cells of b_3 and b_4 which means that there must be at least on additional element in brick b_4.

By using the same argument that we used to show that 2 must be either in cell 2 or 3 in a fixed point of I, we can show that in Subcase 2.2, 5 must be either in cell 8 or 9.

Continuing this type of reasoning, one can see that the general case is where b_i consists of i and x_i for $i = 1, \ldots, k$, and brick b_k starts with $k + 1$ and has at least one additional element. Moreover, for $i = 1, \ldots, k-1$, $i\ x_i\ i+1\ x_{i+1}$ must be a 1423-match so that we can conclude that $x_1 > x_2 > \cdots > x_k$. Also, there a 1423-match must occur in the cells of b_k and b_{k+1} which means that there must be at least 2 elements in brick b_{k+1}. Then by using the same argument that we used to show that 2 must be either in cell 2 or 3 in a fixed point of I, we can show that in this situation, $k + 2$ must be either in cell $2k + 2$ or cell $2k + 3$. We then have to consider several cases.

Case A. $n = 2k + 2$.
In this case $k+2$ must be in cell $2k+2$ which implies that $(x_1, x_2, \ldots, x_k) = (2k+2, 2k+1, \ldots, k+3)$. In this case, if we remove bricks $b_1, \ldots, b_k$, remove $k+1$ from brick b_{k+1}, and renumber, then we obtain the configuration which consists of a single brick of size 1 which contains 1. This is fixed point of I in $\mathcal{O}_{2k+2-(2k+1)} = \mathcal{O}_1$. Note in the this case $\lfloor \frac{n-2}{2} \rfloor = k$, $\binom{n-k-2}{k} = \binom{k}{k} = 1$ so that the contribution to $U_{1423,n}(y)$ is

$$(-y)^k \binom{n-k-2}{k} U_{1423,n-2k-1}(y) = (-y)^k U_{1423,1}(y) = (-y)^k(-y) = (-y)^{k+1}$$

as it should be.

Case B. $n = 2k + 3$.
In this case, $k + 2$ cannot be in cell $2k + 3$ since otherwise brick b_{k+1} must consist of $k + 1$ plus another element x_{k+1} and $k+2$ must be the only element in brick b_{k+2}. However, in that situation, $x_{k+1} > k+2$ and, hence, we could combine bricks b_{k+1} and b_{k+2} which would violate the fact that O is a fixed point

of I. Hence $k+2$ must be in cell $2k+2$. Let x_{k+1} be the element in cell $2k+3$. It is easy to see that $x_k, x_{k+1} > k+2$ and that $k\ x_k\ k+1\ k+2$ must be 1423-match. Then there are two possibilities, namely, either (a) b_{k+1} contains $k+1$ and $k+2$ and b_{k+2} is brick of size 1 which contains x_{k+1} or (b) b_{k+1} contains $k+1, k+2$, and x_{k+1}. In both cases, we can obtain of fixed point of I in $\mathcal{O}_2$ by removing bricks $b_1, \ldots, b_k$, removing $k+1$ from brick b_{k+1}, and renumbering. In case (a), we would obtain the configuration with 2 bricks of size 1 where 1 is in the first brick and 2 is in the second brick and in case (b) we would obtain the configuration where 1 and 2 are in the same brick. These are the two configurations in $\mathcal{O}_2$ which are clearly fixed points of I. Note that we know that $x_1 > x_2 > \cdots x_k$ but we have no condition on how x_{k+1} relates to $x_1, \ldots, x_k$. Thus we have $k+1$ ways to pick $x_1, \ldots, x_k$. However, note that $\lfloor \frac{n-2}{2} \rfloor = k$ and $\binom{n-(k+2)}{k} = \binom{k+1}{k} = k+1$. Thus the contribution of the objects in Case B is equal to

$$(-y)^k \binom{n-(k+2)}{k} U_{1423,n-(2k+1)} = (-y)^k (k+1) U_{1423,2}(y)$$

as it should be.

Case C. $n > 2k+3$ and $k+2$ is in cell $2k+2$.
In this case, we can argue as above that if we remove bricks $b_1, \ldots, b_k$, remove $k+1$ from brick b_{k+1}, renumber, then we can obtain an arbitrary fixed point of I in $\mathcal{O}_{n-2k-1}$. Since we must have 1423-match in the cells of b_k and b_{k+1}, it must be the case that $k\ x_k\ k+1\ k+2$ is 1423-match so that $x_k > k+2$. Thus we have $\binom{n-(k+2)}{k}$ ways to pick $x_1, \ldots, x_k$. Hence, the objects in Case C will contribute $(-y)^k \binom{n-(k+2)}{k} U_{1423,n-2k-1}(y)$ to $U_{1423,n}(y)$.

Case D. $n > 2k+3$ and $k+2$ is in cell $2k+3$.
In this case, we let x_{k+1} be the element in cell $2k+2$ of O. Since the positions of $1, \ldots, k+2$ are accounted for in O, we know that $x_{k+1} > k+2$. Now $k+2$ must the be first element in brick b_{k+2}. Moreover, it must be the case that $k\ x_k\ k+1\ x_{k+1}$ is 1423-match and that there is must be a 1423-match in the cells of b_{k+1} and b_{k+2} so that b_{k+2} must consist of at least 2 elements. Thus in Case D, we are back in the general case that we are considering except that we are guaranteed to start with sequence of $k+1$ bricks of size 2 rather that k bricks of size 2.

Thus we have shown that $U_{1423,n}(y)$ satisfies the recursion

$$U_{1423,n}(y) = (1-y)U_{1423,n-1}(y) + \sum_{k=1}^{\lfloor \frac{n-2}{2} \rfloor} (-y)^k \binom{n-(k+2)}{k} U_{1423,n-2k-1}(y)$$

with initial conditions that $U_0(y) = 1$ and $U_{1423,1}(y) = -y$. $\square$

Similar arguments can be used to prove this type of result for equations of the form (6). For example, we can show that if $\alpha = 1\ q\ 2\ 3 \ldots (q-1)$, then $U_{\alpha,n}(y)$ satisfies the recursion

$$U_{\alpha,n}(y) = (1-y)U_{\alpha,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{q-2} \rfloor + 1} (-y)^k \binom{n-(k-1)(q-3)-2}{k-1} U_{\alpha,n-(q-2)(k-1)-1}(y)$$

and if $\beta = 1\,2\ldots(q-3)(q-1)(q-2)q$ where $q \geq 5$, then $U_{\beta,n}(y)$ satisfies the recursion

$$U_{\beta,n}(y) = (1-y)U_{\beta,n-1}(y) + \sum_{k=2}^{\lfloor \frac{n-2}{q-2} \rfloor + 1} (-y)^k U_{\beta,n-(q-2)(k-1)-1}(y).$$

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Touchard-Riordan formulas, T-fractions, and Jacobi's triple product identity

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Abstract. We give a combinatorial proof of a Touchard-Riordan-like formula discovered by the first author. As a consequence we find a connection between his formula and Jacobi's triple product identity. We then give a combinatorial analog of Jacobi's triple product identity by showing that a finite sum can be interpreted as a generating function of weighted Schröder paths, so that the triple product identity is recovered by taking the limit. This can be stated in terms of some continued fractions called T-fractions, whose important property is the fact that they satisfy some functional equation. We show that this result permits to explain and generalize some Touchard-Riordan-like formulas appearing in enumerative problems.

Résumé. Nous donnons une preuve combinatoire d'une formule à la Touchard-Riordan due au premier auteur. En conséquence, nous faisons apparaître un lien entre cette formule et l'identité du produit triple de Jacobi. Nous donnons un analogue combinatoire à l'identité du produit triple en montrant qu'une somme finie peut être interprétée comme fonction génératrice de chemins de Schröder pondérés, de sorte que l'identité du produit triple s'obtient en passant à la limite. Ceci peut être énoncé en termes de fractions continues appelées T-fractions, dont la propriété importante est le fait qu'elle satisfont certaines équations fonctionnelles. Nous montrons que ce résultat permet d'expliquer et généraliser certaines formules à la Touchard-Riordan apparaissant dans des problèmes d'énumération.

Keywords: Jacobi's triple product identity, continued fractions, enumeration

1 Introduction

1.1 Touchard-Riordan formulas

The original result of Touchard [Tou52], later given more explicitly by Riordan [Rio75], answers the combinatorial problem of counting chord diagrams according to the number of crossings. It has also been stated in terms of a continued fraction by Read [Rea79], so that the Touchard-Riordan formula is:

$$[z^n]\left(\frac{1}{1-}\frac{[1]_q z}{1}\frac{}{-}\frac{[2]_q z}{1}\frac{}{-}\cdots\right) = \frac{1}{(1-q)^n}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)(-1)^k q^{\binom{k+1}{2}}, \tag{1}$$

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where $[z^n]$ is the operator that extracts the coefficient of z^n in a series, $[n]_q$ denotes $(1-q^n)/(1-q)$, and we use the notation for continued fractions as in (10). Recently, several variants have been derived. In particular, using continued fractions and basic hypergeometric series, the first author [JV10] proved the following formula in a slightly different form related with enumeration of alternating permutations:

$$[z^n]\left(\frac{1}{1-}\frac{[1]_q^2 z}{1}\,{}_{-}\frac{[2]_q^2 z}{1}\,{}_{-}\cdots\right) = \frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)q^{k(k+1)}\sum_{i=-k}^{k}(-q)^{-i^2}. \quad (2)$$

In the first part of this paper, we prove (2) combinatorially. To do this we introduce a combinatorial model whose weight sum is equal to $\sum_{i=-k}^{k}(-q)^{-i^2}$. As a consequence of the combinatorial proof we can let $k\to\infty$ and obtain

$$\prod_{i\ge 1}\frac{1-q^i}{1+q^i} = \sum_{i=-\infty}^{\infty}(-q)^{i^2}, \quad (3)$$

which is known as the special case ($y=-1$) of Jacobi's triple product identity:

$$\prod_{n\ge 1}(1-q^{2n})(1+yq^{2n-1})(1+y^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} y^n q^{n^2}. \quad (4)$$

1.2 Jacobi's triple product identity

Jacobi's triple product identity (JTP) is ubiquitous in various areas of mathematics and especially in analytical number theory. Quite a lot of different proofs, generalizations and variants are known, see for example [AB04, Ber06, Sch05, War05] and lots of references therein. Some classical particular cases or consequences are :

$$\prod_{i\ge 1}(1-q^i) = \sum_{j=-\infty}^{\infty}(-1)^j q^{\frac{j(3j-1)}{2}}, \quad (5)$$

which is known as Euler's pentagonal number theorem, and also (3) and

$$\prod_{i\ge 1}(1-q^i)^3 = \sum_{j=0}^{\infty}(-1)^j(2j+1)q^{\frac{j(j+1)}{2}}.$$

See [Ber06, Chapter 1] for a general reference about these identities.

Now we state the main result.

Theorem 1.1. *We define* $[n]_{y,q}=(1+yq^n)/(1-q)$*. Then*

$$[z^n]\left(\frac{1}{1-}\frac{[1]_{y,q}[1]_{y^{-1},q}\,z}{1}\,{}_{-}\frac{[2]_q^2\,z}{1}\,{}_{-}\frac{[3]_{y,q}[3]_{y^{-1},q}\,z}{1}\,{}_{-}\frac{[4]_q^2\,z}{1}\,{}_{-}\cdots\right)$$
$$= \frac{1}{(1-q)^{2n}}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)q^{k(k+1)}\sum_{i=-k}^{k}y^i q^{-i^2}. \quad (6)$$

Note that if $y = -1$, we get (2). In the second part of this paper we prove Theorem 1.1 by finding a functional equation satisfied by both sides of another equation equivalent to (6), see Theorem 4.1. Finding a combinatorial proof of Theorem 1.1 is still open. Our combinatorial model for (2) is naturally extended to a model for (6), and letting $k \to \infty$ as in the proof of (3), we obtain JTP. For this reason, Theorem 1.1 is a "finite version" of JTP.

1.3 T-fractions

Notice that we have the factor $\left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right)$ in the sums of (1), (2) and (6). This will be explained by the link between S-fractions and T-fractions (see Lemma 2.5). The family of T-fractions is the natural form of two-points Padé approximants [CPV+08], they also appear occasionally in combinatorics [RV96] but much less than S-fractions or J-fractions. We use here a particular kind of T-fractions as in Definition 2 below. They are related with weighted Schröder paths, so that our main result can be stated as an exact formula for a certain weight sum of Schröder paths. The main property of T-fractions we use to prove the result is the fact that they satisfy certain functional equations.

2 Preliminaries

2.1 Penaud's decomposition

Definition 1. A *Schröder path* of length $2n$ is a path from $(0, 0)$ to $(2n, 0)$ in $\mathbb{N}^2$ with three kinds of steps: an up step $(1, 1)$, a down step $(1, -1)$, and a horizontal step $(2, 0)$. A *Dyck path* is a Schröder path without any horizontal step. A *marked Schröder path* is a Schröder path in which each up step and down step may be marked. Let $\mathcal{S}_n$ (resp. $\overline{\mathcal{S}}_n$, $\mathcal{D}_n$, $\overline{\mathcal{D}}_n$) denote the set of Schröder paths (resp. marked Schröder paths, Dyck paths, marked Dyck paths) of length $2n$. Let $\overline{\mathcal{D}}_n^*$ denote the subset of $\overline{\mathcal{D}}_n$ consisting of the marked Dyck paths without any peak, i.e. an up step immediately followed by a down step, consisting two marked steps.

Note that $\mathcal{D}_n \subset \mathcal{S}_n$ and $\mathcal{D}_n^* \subset \mathcal{S}_n^*$. We will also consider that $\mathcal{D}_n \subset \mathcal{D}_n^*$ and $\mathcal{S}_n \subset \mathcal{S}_n^*$ by identifying a Schröder path with a marked Schröder path without any marked step.

Given sequences $\mathcal{A} = (a_1, a_2, \dots)$, $\mathcal{B} = (b_1, b_2, \dots)$ and a marked Schröder path p, we define the *weight* $\mathrm{wt}(p; A, B)$ to be the product of a_h (resp. b_h) for each unmarked up step (resp. unmarked down step) between height h and $h-1$, and -1 for each horizontal step (hence each marked step has weight 1). Since $\mathcal{S}_n \subset \mathcal{S}_n^*$, the weight is also defined for a Schröder path.

By definition, it is easy to see that

$$\sum_{p \in \mathcal{D}_n} \mathrm{wt}(p; \mathcal{A}, \mathcal{B}) = \sum_{p \in \overline{\mathcal{D}}_n} \mathrm{wt}(p; \mathcal{A}-1, \mathcal{B}-1), \qquad \sum_{p \in \mathcal{S}_k} \mathrm{wt}(p; \mathcal{A}, \mathcal{B}) = \sum_{p \in \overline{\mathcal{S}}_k} \mathrm{wt}(p; \mathcal{A}-1, \mathcal{B}-1), \quad (7)$$

where $\mathcal{A} - 1$ means the sequence $(a_1 - 1, a_2 - 1, \dots)$.

Now a generalized version of Penaud's decomposition can be describes as follows. Here, a *Dyck prefix* is a path in $\mathbb{N}^2$ from the origin to any point consisting of up steps and down steps.

Proposition 2.1. *Each $p \in \overline{\mathcal{D}}_n$ can be uniquely decomposed into (L, p') where L is a Dyck prefix of length $2n$ ending at height $2k$ and $p' \in \overline{\mathcal{D}}_k^*$ for some k. Moreover, for any sequences $\mathcal{A}$ and $\mathcal{B}$, we have* $\mathrm{wt}(p; \mathcal{A}, \mathcal{B}) = \mathrm{wt}(p'; \mathcal{A}, \mathcal{B})$.

It is well known that the number of Dyck prefixes of length $2n$ ending at height $2k$ is equal to $\binom{2n}{n-k} - \binom{2n}{n-k-1}$. Thus, from Proposition 2.1, we obtain

$$\sum_{p\in\overline{\mathcal{D}}_n} \mathrm{wt}(p;\mathcal{A},\mathcal{B}) = \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{A},\mathcal{B}). \tag{8}$$

By (7) and (8), we get the following proposition.

Proposition 2.2. *For any sequences $\mathcal{A}$ and $\mathcal{B}$, we have*

$$\sum_{p\in\mathcal{D}_n} \mathrm{wt}(p;\mathcal{A},\mathcal{B}) = \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{A}-1,\mathcal{B}-1).$$

On the other hand, by canceling a horizontal step and a peak with two marked steps, we obtain

$$\sum_{p\in\overline{\mathcal{S}}_k} \mathrm{wt}(p;\mathcal{A},\mathcal{B}) = \sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{A},\mathcal{B}). \tag{9}$$

By Proposition 2.2, (9) and (7), we get the following proposition.

Proposition 2.3. *For any sequences A and B, we have*

$$\sum_{p\in\mathcal{D}_n} \mathrm{wt}(p;\mathcal{A},\mathcal{B}) = \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \sum_{p\in\mathcal{S}_k} \mathrm{wt}(p;\mathcal{A},\mathcal{B}).$$

2.2 T-fractions and S-fractions

We will use the space-saving notation for continued fractions:

$$\frac{a_0}{b_0} - \frac{a_1}{b_1} - \frac{a_2}{b_2} - \cdots = \cfrac{a_0}{b_0 - \cfrac{a_1}{b_1 - \cfrac{a_2}{b_2 - \ddots}}}. \tag{10}$$

Definition 2. To any sequence $\lambda = \{\lambda_n\}_{n\geq 1}$, we associate the S-fraction $S_\lambda(z)$ and the T-fraction $T_\lambda(z)$:

$$S_\lambda(z) = \frac{1}{1-}\,\frac{\lambda_1 z}{1}\,-\,\frac{\lambda_2 z}{1}\,-\,\frac{\lambda_3 z}{1}\,-\,\ldots, \qquad T_\lambda(z) = \frac{1}{1+z-}\,\frac{\lambda_1 z}{1+z-}\,\frac{\lambda_2 z}{1+z-}\,\frac{\lambda_3 z}{1+z-}\,\ldots. \tag{11}$$

The combinatorial interpretation of S-fractions in terms of weighted Dyck paths is widely known, but the analogous result for T-fractions is not as common.

Lemma 2.4. *Let $\mathcal{A} = (a_1, a_2, \ldots)$, $\mathcal{B} = (b_1, b_2, \ldots)$ be two sequences and $\lambda_h = a_h b_h$. Then*

$$S_\lambda(z) = \sum_{n=0}^{\infty} z^n \sum_{p\in\mathcal{D}_n} \mathrm{wt}(p;\mathcal{A},\mathcal{B}), \qquad T_\lambda(z) = \sum_{n=0}^{\infty} z^n \sum_{p\in\mathcal{S}_n} \mathrm{wt}(p;\mathcal{A},\mathcal{B}). \tag{12}$$

Proof. This can be proved by a classical method, see for example [GJ04]. ☐

By Lemma 2.4 and Proposition 2.3, we obtain the following lemma which gives a relation between the coefficients of an S-fraction and a T-fraction. This is a key step in our proofs of the Touchard-Riordan-like formulas.

Lemma 2.5. *Given a sequence $\lambda = \{\lambda_n\}_{n\geq 1}$, we define $\mu = \{\mu_n\}_{n\geq 0}$ and $\nu = \{\nu_n\}_{n\geq 0}$ such that:*

$$\sum_{n=0}^{\infty} \mu_n z^n = S_\lambda(z), \qquad \sum_{n=0}^{\infty} \nu_n z^n = T_\lambda(z). \tag{13}$$

Then for any $n \geq 0$ we have the relation $\mu_n = \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \nu_k$.

3 Combinatorics on weighted Dyck paths

3.1 A combinatorial proof of (2)

From now on, we will fix two sequences

$$\mathcal{U} = ([1]_q, [2]_q, \ldots), \qquad \mathcal{V} = (1-q, 1-q^2, \ldots).$$

By Lemma 2.4, Proposition 2.2, and the fact that $\mathrm{wt}(p;\mathcal{U},\mathcal{U}) = \mathrm{wt}(p;\mathcal{V},\mathcal{V})/(1-q)^{2n}$ for all $p \in \mathcal{D}_n$, the left hand side of (2) is equal to

$$\sum_{p\in\mathcal{D}_n} \mathrm{wt}(p;\mathcal{U},\mathcal{U}) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) \sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{V}-1,\mathcal{V}-1). \tag{14}$$

Thus in order to get (2) it is sufficient to show the following identity:

$$\sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{V}-1,\mathcal{V}-1) = q^{k(k+1)} \sum_{i=-k}^{k} (-q)^{-i^2}. \tag{15}$$

We introduce some terminologies. We denote by δ_k the staircase partition $(k, k-1, \ldots, 1)$.

Definition 3. A δ_k*-configuration* is a pair (λ, A) of a partition $\lambda \subset \delta_{k-1}$ and a set A of arrows occupying a whole row or a whole column of δ_k/λ such that no inner corner of δ_k/λ is occupied by two arrows. Here, by an inner corner we mean a cell $c \in \delta_k/\lambda$ such that $\lambda \cup c$ is a partition. The *length* of an arrow is the number of cells occupied by the arrow. Let $\mathfrak{G}_k$ denote the set of δ_k-configurations. We define the q*-weight* of a δ_k-configuration $\Delta = (\lambda, A)$ to be

$$\mathrm{wt}_q(\Delta) = (-1)^{|A|} q^{2|\lambda| + \|A\|},$$

where $\|A\|$ is the sum of arrow lengths.

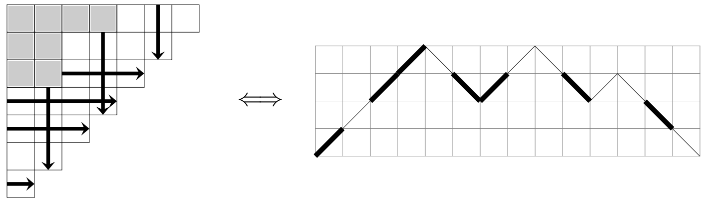

Figure 1: A δ_k-configuration and the corresponding marked Dyck path, where marked steps are the thicker steps.

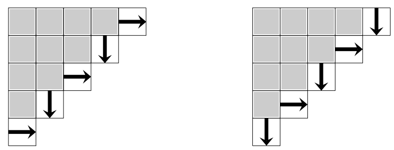

Figure 2: The two fixed points in $\mathfrak{G}_5$ which will not be sent to $\mathfrak{G}_4$.

For example, the q-weight of the δ_k-configuration in Figure 1 is $(-1)^7 q^{2\cdot 8+1+3+4+3+3+3+2}$.

There is a simple bijection between $\overline{\mathcal{D}}_k^*$ and $\mathfrak{G}_k$ as follows. For $\Delta = (\lambda, A) \in \mathfrak{G}_k$, the north-west border of δ_k/λ defines a marked Dyck path of length $2k$ where the marked steps correspond to the segments on the border with an arrow, see Figure 1. Moreover, if $p \in \overline{\mathcal{D}}_k^*$ corresponds to $\Delta \in \mathfrak{G}_k$, one can show that $\mathrm{wt}(p; \mathcal{V}-1, \mathcal{V}-1) = q^{k(k+1)} \mathrm{wt}_{q^{-1}}(\Delta)$. Thus we obtain the following lemma.

Lemma 3.1. *For any nonnegative integer k, we have*

$$\sum_{p \in \overline{\mathcal{D}}_k^*} \mathrm{wt}(p; \mathcal{V}-1, \mathcal{V}-1) = q^{k(k+1)} \sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_{q^{-1}}(\Delta).$$

We prove the proposition below by constructing a weight-preserving-sign-reversing involution on $\mathfrak{G}_k$. More precisely, we construct such an involution on $\mathfrak{G}_k$ such that the fixed point set is in weight-preserving bijection with $\mathfrak{G}_{k-1}$ except the two fixed points $\Delta = (\lambda, A)$ with $\lambda = \delta_{k-1}$ and A consisting of n arrows alternating as shown in Figure 2. Thus our involution implies

$$\sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_q(\Delta) = \sum_{\Delta \in \mathfrak{G}_{k-1}} \mathrm{wt}_q(\Delta) + 2(-q)^{k^2}.$$

The key idea of the involution is "moving" the arrows upwards or to the left to decrease the arrow length by 1. We omit the details.

Proposition 3.2. *For a nonnegative integer k, we have*

$$\sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_q(\Delta) = \sum_{i=-k}^{k} (-q)^{i^2}.$$

By Lemma 3.1 and Proposition 3.2 we obtain (15), thus completing the combinatorial proof of (2).

3.2 Limiting case $k \to \infty$ and a connection with JTP

For $\Delta = (\lambda, A) \in \mathfrak{G}_k$, if there is an arrow coming from a row or a column of λ, one can easily see that q^k divides $\mathrm{wt}_q(\Delta)$. In other words, if $\mathrm{wt}_q(\Delta)$ is not divisible by q^k, then the partition, the horizontal arrows and the vertical arrows are completely separated. Thus we can freely choose a partition, vertical arrows of distinct length and horizontal arrows of distinct length. This argument gives us the following.

Proposition 3.3. *For any nonnegative integer k, we have*

$$\sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_q(\Delta) \equiv \prod_{i \geq 1} \frac{1}{1-q^{2i}} \prod_{i\geq 1}(1-q^i) \prod_{i \geq 1}(1-q^i) = \prod_{i\geq 1} \frac{1-q^i}{1+q^i} \mod q^k.$$

Letting $k \to \infty$ in Propositions 3.2 and 3.3, we get (3).

Now we define the (y, q)-*weight* of a δ_k-configuration $\Delta = (\lambda, A)$ to be

$$\mathrm{wt}_{y,q}(\Delta) = (-1)^{|A|} q^{2|\lambda| + \|A\|} (-y)^{oh(A) - ov(A)},$$

where $oh(A)$ (resp. $ov(A)$) is the number of odd-length horizontal (resp. vertical) arrows in A. For example, the (y, q)-weight of the δ_k-configuration in Figure 1 is $(-1)^7 q^{2\cdot 8+1+3+4+3+3+3+2}(-y)^{3-2}$.

The proof of the lemma below is similar to that of Lemma 3.1. Here we define

$$\mathcal{J} = (1+yq, 1-q^2, 1+yq^3, 1-q^4, \dots), \qquad \mathcal{J}' = (1+y^{-1}q, 1-q^2, 1+y^{-1}q^3, 1-q^4, \dots).$$

Lemma 3.4. *For any nonnegative integer k, we have*

$$\sum_{p \in \overline{\mathcal{D}}_k^*} \mathrm{wt}(p; \mathcal{J}-1, \mathcal{J}'-1) = q^{k(k+1)} \sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_{y,q^{-1}}(\Delta).$$

By the same argument as in the proof of Proposition 3.3 together with

$$\begin{aligned}\prod_{i\geq 1} \frac{1}{1-q^{2i}} \prod_{i \geq 1}(1-q^{2i})(1+yq^{2i-1}) \prod_{i\geq 1}(1-q^{2i})(1+y^{-1}q^{2i-1}) \\ = \prod_{i \geq 1}(1-q^{2i})(1+yq^{2i-1})(1+y^{-1}q^{2i-1}),\end{aligned}$$

we obtain the following.

Proposition 3.5. *For any nonnegative integer k, we have*

$$\sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_{y,q}(\Delta) \equiv \prod_{i \geq 1}(1-q^{2i})(1+yq^{2i-1})(1+y^{-1}q^{2i-1}) \mod q^k. \tag{16}$$

Since the right hand side of (16) is one side of JTP, it is natural to guess

$$\sum_{\Delta \in \mathfrak{G}_k} \mathrm{wt}_{y,q}(\Delta) = \sum_{i=-k}^{k} y^i q^{i^2}. \tag{17}$$

Using Lemma 3.4 and the same argument as in (14), one can see that (17) is equivalent to Theorem 1.1. This is in fact the way the authors first discovered Theorem 1.1. Notice also that by Lemma 3.4, (9) and (7) in this order, we have

$$q^{k(k+1)} \sum_{\Delta\in\mathfrak{G}_k} \mathrm{wt}_{y,q^{-1}}(\Delta) = \sum_{p\in\overline{\mathcal{D}}_k^*} \mathrm{wt}(p;\mathcal{J}-1,\mathcal{J}'-1) = \sum_{p\in\overline{\mathcal{S}}_k} \mathrm{wt}(p;\mathcal{J}-1,\mathcal{J}'-1) = \sum_{p\in\mathcal{S}_k} \mathrm{wt}(p;\mathcal{J},\mathcal{J}').$$

Thus (17) is also equivalent to

$$\sum_{p\in\mathcal{S}_k} \mathrm{wt}(p;\mathcal{J},\mathcal{J}') = \sum_{j=-k}^{k} y^j q^{k(k+1)-j^2}. \tag{18}$$

We will prove this in the next section.

3.3 Generalized q-secant numbers

For two nonnegative integers a and b, we define

$$E_n^{a,b}(q) = [z^n]\left(\frac{1}{1-}\,\frac{[a+1]_q\,[b+1]_q\,z}{1}\,\frac{[a+2]_q\,[b+2]_q\,z}{1}\,\cdots\right).$$

Then $E_n^{0,0}(q)$ is a q-secant number and $E_n^{0,1}(q)$ is a q-tangent number, see [JV10].

By (2), we have

$$E_n^{0,0}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{k(k+1)} T_k(q^{-1}),$$

where $T_k(q) = \sum_{i=-k}^{k} (-q)^{i^2}$. Note that by (3) we have

$$\lim_{k\to\infty} T_k(q) = \frac{(q;q)_\infty}{(-q;q)_\infty},$$

where we use the usual notation $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $(a;q)_\infty = (1-a)(1-aq)\cdots$. We generalize this as follows.

Theorem 3.6. *For nonnegative integers a and b, there is a family $\{T_k^{a,b}(q)\}_{k\geq 0}$ of polynomials in q such that*

$$E_n^{a,b}(q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\binom{2n}{n-k} - \binom{2n}{n-k-1}\right) q^{k(k+1+a+b)} T_k^{a,b}(q^{-1}).$$

Moreover, for all $k \geq 0$, we have

$$T_k^{a,b}(q) \equiv \frac{1}{(q;q)_a (q;q)_b} \cdot \frac{(q;q)_\infty}{(-q;q)_\infty} \mod q^k,$$

which implies

$$\lim_{k\to\infty} T_k^{a,b}(q) = \frac{1}{(q;q)_a (q;q)_b} \cdot \frac{(q;q)_\infty}{(-q;q)_\infty}.$$

4 The finite version of Jacobi's triple product and consequences

4.1 Main result

By Lemma 2.5, Theorem 1.1 is equivalent to the following theorem. This is also equivalent to (17) and (18), and as we have seen, these identities give JTP when k tends to infinity.

Theorem 4.1. *There holds*

$$\sum_{k=0}^{\infty} z^k \sum_{j=-k}^{k} y^j q^{k(k+1)-j^2} = \frac{1}{1+z-}\,\frac{(1+qy)(1+qy^{-1})z}{1+z}\,\frac{}{-}\,\frac{(1-q^2)^2 z}{1+z}\,\frac{}{-}\cdots, \tag{19}$$

where the continued fraction being $T_\lambda(z)$ with $\lambda_n = (1+q^n y)(1+q^n y^{-1})$ for odd n and $\lambda_n = (1-q^n)^2$ for even n.

Proof. We show that both sides satisfy a common functional equation. Indeed, if $H(z)$ is the left-hand side of (19) and $c_{j,k}(z) = z^k y^j q^{k(k+1)-j^2}$, we have

$$H(z) = \sum_{j,k\in\mathbb{Z},\ k\geq|j|} c_{j,k}(z), \qquad c_{j,k+1}(z) = zq^2 c_{j,k}(zq^2), \tag{20}$$

and it follows that

$$H(z) - zq^2 H(zq^2) = \sum_{j\in\mathbb{Z}} c_{j,|j|}(z) = \frac{1}{1-yqz} + \frac{1}{1-y^{-1}qz} - 1. \tag{21}$$

To show the last equality, note that when $k=|j|$ the term k^2 cancels with $-j^2$ in $c_{j,k}(z)$ and splitting the j-sum according to the sign of j gives two geometric series. So $H(z)$ is the unique formal power series satisfying the functional equation:

$$H(z) = \frac{1}{1-yqz} + \frac{1}{1-y^{-1}qz} - 1 + zq^2 H(zq^2). \tag{22}$$

Uniqueness comes from the fact that by iterating (22), we can get more and more terms in the expansion of $H(z)$. It remains only to show that the continued fraction in the right-hand side of (19) satisfies the same functional equation, which is done in a separate lemma below. ☐

Lemma 4.2. *Let λ as in Theorem 4.1, then we have*

$$T_\lambda(z) = \frac{1}{1-yqz} + \frac{1}{1-y^{-1}qz} - 1 + zq^2 T_\lambda(zq^2). \tag{23}$$

Proof. We will identify 2×2-matrices and Möbius transformation in the usual way:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}[X] = \frac{aX+b}{cX+d}, \tag{24}$$

i.e. we use a bracket notation for the evaluation of a Möbius transformation. A continued fraction can be obtained by iterating such transformations. In the present case, we have

$$\frac{1}{1+z-\frac{(1+wqy)(1+wqy^{-1})z}{1+z-(1-wq^2)^2zX}} = \frac{z(1-wq^2)^2X-(1+z)}{z(1+z)(1-wq^2)^2X+(1+wqy)(1+wqy^{-1})z-(1+z)^2}, \tag{25}$$

so we can introduce the matrix

$$M(w,z) = \begin{pmatrix} z(1-wq^2)^2 & -1-z \\ z(1+z)(1-wq^2)^2 & (1+wqy)(1+wqy^{-1})z-(1+z)^2 \end{pmatrix}, \tag{26}$$

and we have

$$T_\lambda(z) = \prod_{n=0}^{\infty} M(q^{2n}, z). \tag{27}$$

The partial products are just the convergents of the continued fraction. More precisely, the infinite product is convergent in the following sense: the partial products are Möbius transformations, and these converge pointwise to the formal power series in the left-hand side of (27).

Let S be the matrix

$$S = \begin{pmatrix} zq^2 & \frac{1}{1-zqy}+\frac{1}{1-zqy^{-1}}-1 \\ 0 & 1 \end{pmatrix}. \tag{28}$$

The functional equation that we want to prove can be written:

$$\prod_{n=0}^{\infty} M(q^{2n}, z) = S\prod_{n=0}^{\infty} M(q^{2n}, zq^2). \tag{29}$$

By examining the previous equation, it is natural to introduce a matrix Ω_n by

$$\Omega_n = M(q^{2n}, z)^{-1}\cdots M(1,z)^{-1}SM(1,zq^2)\cdots M(q^{2n},zq^2), \tag{30}$$

where we understand that only even powers of q appear within the dots. It can be calculated explicitly, as given in Lemma 4.3 below, so that we obtain:

$$\Omega_n[0] = \frac{1-z^2q^2}{zq^{2n+3}(2qz-y-y^{-1})+1-z^2q^2}. \tag{31}$$

The important point is that from this closed form, we can check that $\Omega_n[0]$ is well-defined at $z=0$ (i.e. it has no pole at $z=0$). Let $w_n = \Omega_n[0]$, by definition of Ω_n we have:

$$M(1,z)\cdots M(q^{2n},z)[w_n] = SM(1,zq^2)\cdots M(q^{2n},zq^2)[0], \tag{32}$$

and at this point it remains only to let n tend to infinity in (32) to prove (29), which was a rewriting of (23). The only subtlety is in the left-hand side, where we need the fact that w_n is indeed a formal power series in z (as opposed to a formal Laurent series) to take the limit. More precisely, one can show by a straightforward calculation that the left-hand side in (32) does not depend on w_n up to a $O(z^{n+1})$, and this is why we can take the limit $n \to \infty$. ☐

Lemma 4.3. *The matrix Ω_n defined in* (30) *has the explicit form:*

$$\Omega_n = \frac{\begin{pmatrix} q^2z(2q^{2n+2}-zq^{2n+3}(y+y^{-1})+z^2q^2-1) & 1-z^2q^2 \\ (1-q^{2n+2})^2(z^2q^2-1)zq^2 & zq^{2n+3}(2qz-y-y^{-1})+1-z^2q^2 \end{pmatrix}}{(1-yzq)(1-y^{-1}zq)}. \tag{33}$$

Proof. Although calculations are quite cumbersome, there is a straightforward recursive verification of the given expression, using the relation, for $n \geq 1$,

$$\Omega_n = M(q^{2n}, z)^{-1}\Omega_{n-1}M(q^{2n}, zq^2), \tag{34}$$

where we define $\Omega_0 = S$. There are 4 coefficients in Ω_n, each appears as a sum of 4 terms when we expand the previous equation, and each of this term is a product of 3 coefficients of the matrices in (26) and (33). So there is a small "explosion" of the size of computations to perform. However, this is a verification that can be done with no particular cleverness, since expanding everything in (34) will clearly makes possible a term-by-term identification of both sides. We omit details and invite the unconvinced reader to use some computer algebra system for checking that the lemma is true. ☐

4.2 New Touchard-Riordan-like formulas

From Theorem 4.1, we can derive a whole family of S-fractions having associated Touchard-Riordan formulas, as given in the theorem below. A very interesting property of these is that there are exponential generating functions linked with trigonometric functions. The theorem below is also a wide generalization of the result in (2), which is related with a q-analog of secant numbers having exponential generating function $\sec(z)$. Note that in the definition of $[n]_q = (1-q^n)/(1-q)$, n can be any number, not necessarily an integer.

Theorem 4.4. *For any numbers a and b, we define $\mu_n(a,b,q)$ by*

$$\sum_{n=0}^{\infty} \mu_n(a,b,q) z^n = S_\lambda(z), \quad \textit{where} \quad \lambda_n = \begin{cases} [nb+a]_q\,[nb-a]_q & \textit{if } n \textit{ is odd}, \\ [nb]_q^2 & \textit{if } n \textit{ is even}. \end{cases} \tag{35}$$

Then we have

$$\mu_n(a,b,q) = \frac{1}{(1-q)^{2n}} \sum_{k=0}^{n} \left(\tbinom{2n}{n-k} - \tbinom{2n}{n-k-1} \right) \sum_{j=-k}^{k} (-1)^j q^{aj+b(k(k+1)-j^2)}, \tag{36}$$

and

$$\sum_{n=0}^{\infty} \mu_n(a,b,1) \frac{z^{2n}}{(2n)!} = \frac{\cos(az)}{\cos(bz)}. \tag{37}$$

Proof. Consider the identity obtained by substituting (y,q) with $(-q^a, q^b)$ in (19), and apply Lemma 2.5 to this T-fraction. This gives the desired formula for $(1-q)^{2n}\mu_n(a,b,q)$.

It remains only to obtain the exponential generating function of $\mu_n(a,b,1)$ as a ratio of cosines. Actually, this was essentially known by Stieltjes [Sti90] via analytical methods. It is also possible to prove this going through an addition formula satisfied by $\cos(az)/\cos(bz)$, and using a theorem of Stieltjes and Rogers, see for example in [GJ04, Chapter 5] (this method is generally well-suited for trigonometric functions). ☐

Note that $\mu(a, b, q)$ is a polynomial in q with nonnegative coefficients in the following situation: a and b are integers such that $0 \leq a < b$, and also (this is less obvious): a and b are half-integers satisfying the same inequalities. This implies that a function such as $\cos(z/2)/\cos(3z/2)$ is the exponential generating function for a sequence of nonnegative integers. Thus Theorem 4.4 opens some problems, for a better understanding of these quantities $\mu_n(a, b, q)$. We can ask if there is a combinatorial interpretation from which both the ordinary generating function and the exponential one (for $q = 1$) can be obtained. It would be quite remarkable to thus obtain the continued fraction on one side and the trigonometric function on the other side.

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Double homotopy Cohen-Macaulayness for the poset of injective words and the classical NC-partition lattice

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Abstract. In this paper we study topological properties of the poset of injective words and the lattice of classical non-crossing partitions. Specifically, it is shown that after the removal of the bottom and top elements (if existent) these posets are doubly Cohen-Macaulay. This extends the well-known result that those posets are shellable. Both results rely on a new poset fiber theorem, for doubly homotopy Cohen-Macaulay posets, which can be considered as an extension of the classical poset fiber theorem for homotopy Cohen-Macaulay posets.

Résumé. Dans cet article, nous étudions certaines propriétés topologiques du poset des mots injectifs et du treillis des partitions non-croisées classiques. Plus précisément, nous montrons qu'après suppression des plus petit et plus grand élément (s'ils existent), ces posets sont doublement Cohen-Macaulay. C'est une extension du fait bien connu que ces deux posets sont épluchables ("shellable"). Ces deux résultats reposent sur un nouveau théorème poset-fibre pour les posets doublement homotopiquement Cohen-Macaulay, que l'on peut voir comme extension du théorème poset-fibre classique pour les posets homotopiquement Cohen-Macaulay.

Keywords: injective words, non-crossing partitions, strongly constructible, doubly homotopy Cohen-Macaulay, poset fiber theorem

1 Introduction and results

This paper focuses on the study of the topology of two different posets – the poset of injective words on n letters (denoted by I_n) and the lattice of non-crossing partitions for the symmetric group (denoted by $\mathrm{NC}(S_n)$).

The results we obtain for those two posets rely on a new poset fiber theorem for doubly homotopy Cohen-Macaulay posets and intervals. This theorem can be seen as an extension of the classical poset fiber theorem for homotopy Cohen-Macaulay posets by Quillen [19].

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Theorem 1.1 *Let P be a graded poset, $I = (u, v)$ be an open interval in P and $x \in I$. Assume that $I - \{x\}$ is graded and that Q is a homotopy Cohen-Macaulay poset. Let further $f : P \to Q$ be a surjective rank-preserving poset map which satisfies the following conditions:*

(i) For every $q \in Q$ the fiber $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay.

(ii) There exists $q_0 \in Q$ such that

- *$f^{-1}(q_0) = \{x\}$ and $f(I) - \{q_0\}$ is homotopy Cohen-Macaulay, and*
- *for every $q > q_0$ and $p \in f^{-1}(q) \cap I$ the poset $[u, p] - \{x\}$ is homotopy Cohen-Macaulay.*

Then $I - \{x\}$ is homotopy Cohen-Macaulay as well. If for all $x \in I$ there exists a map satisfying the above conditions and if $\mathrm{rank}\,(I - \{x\}) = \mathrm{rank}(I)$, *then I is doubly homotopy Cohen-Macaulay.*

In the past, the poset of injective words as well as the one of classical non-crossing partitions have attracted the attention of a lot of different researchers and are well-studied objects.

It was shown by Farmer [13] in 1978 that the regular CW-complex K_n, whose face poset is I_n, is homotopy equivalent to a wedge of spheres of top dimension. Björner and Wachs [9] could strengthen this result by demonstrating that this complex is even CL-shellable. More recently, Reiner and Webb [20] computed the homology of K_n as an S_n-module and in [14] Hanlon and Hersh provided a refinement of this result by giving a Hodge type decomposition for the homology of K_n. Further generalizations of the complex of injective words K_n are considered in [16].

In this paper we are interested in the topological properties of the poset $I_n - \{\emptyset, x\}$, where $\emptyset$ denotes the empty word of I_n and $x \in I_n$ can be any word different from $\emptyset$. Using Theorem 1.1 we show that this poset, *i.e.*, its order complex, is homotopy Cohen-Macaulay. In particular, this yields the following result.

Theorem 1.2 *Let $n \geq 2$ and let $\emptyset \in I_n$ denote the empty word. Then $I_n - \{\emptyset\}$ is doubly homotopy Cohen-Macaulay.*

Our second main object of study is the poset of classical non-crossing partitions $\mathrm{NC}(S_n)$. This poset has been investigated by several people and it has been shown to be a graded, self-dual lattice [5]. Moreover, Björner and Edelman [6] established EL-shellability of $\mathrm{NC}(S_n)$. In personal communication, Athanasiadis proposed to study the problem of whether the proper part of $\mathrm{NC}(S_n)$, *i.e.*, the poset obtained from $\mathrm{NC}(S_n)$ after the removal of the maximum and the minimum element, is doubly homotopy Cohen-Macaulay. Using Theorem 1.1 we can give an affirmative answer to this question.

Theorem 1.3 *The proper part of the lattice of non-crossing partitions $\mathrm{NC}(S_n)$ is doubly homotopy Cohen-Macaulay for $n \geq 3$.*

The paper is structured as follows. Section 2.1 reviews background on posets and simplicial complexes. Undefined notions and concepts which were used in the introduction are defined and explained in this or one of the following sections. In Sections 2.2 and 2.3 we recall the definitions and some properties of the poset of injective words and the classical non-crossing partition lattice, respectively. In Section 3 we provide the proof of the poset fiber theorem for doubly homotopy Cohen-Macaulay intervals (Theorem 1.1) and derive another poset fiber theorem for doubly homotopy Cohen-Macaulay posets as a corollary (Corollary 3.3). Those two results are employed in Section 4 in order to show that $I_n - \{\emptyset\}$ and the proper part of $\mathrm{NC}(S_n)$ are both doubly homotopy Cohen-Macaulay posets (Theorems 1.2 and 1.3).

2 Preliminaries

2.1 Partial orders and simplicial complexes

Let $(P, \leq)$ be a finite partially ordered set (poset for short) and $x, y \in P$. We say that y *covers* x if $x < y$ and there is no $z \in P$ such that $x < z < y$. The poset P is called *bounded* if there exist elements $\hat{0}$ and $\hat{1}$ such that $\hat{0} \leq x \leq \hat{1}$ for every $x \in P$. The *proper part* $\bar{P}$ of a bounded poset P is the subposet $\bar{P} = P - \{\hat{0}, \hat{1}\}$ obtained after the removal of $\hat{0}$ and $\hat{1}$. A subset C of a poset P is called a *chain* if any two elements of C are comparable in P. Throughout this paper, we denote by $\{\hat{0}, \hat{1}\}$ the 2-element chain with $\hat{0} < \hat{1}$. The *length* of a (finite) chain C is equal to $|C| - 1$. We say that P is *graded* if all maximal chains of P have the same length and call this length the *rank* of P. Moreover, there exists a unique function $\mathrm{rank} : P \to \mathbb{N}$, called the *rank function* of P, such that $\mathrm{rank}(x) = 0$ if x is a minimal element of P, and $\mathrm{rank}(y) = \mathrm{rank}(x) + 1$ if y covers x. We say that x has *rank* i if $\mathrm{rank}(x) = i$. For $S \subseteq P$ the *order ideal* of P generated by S is the subposet $\langle S \rangle_P = \{x \in P \ : \ x \leq y \text{ for some } y \in S\}$. We omit the subscript P when it is clear from the context in which poset we are considering the order ideal. The same convention is used for intervals. For $x \in P$ we set $P_{<x} = \{p \in P \ : \ p < x\}$. Given two posets $(P, \leq_P)$ and $(Q, \leq_Q)$, a map $f : P \to Q$ is called a *poset map* if it is order-preserving, *i.e.*, $x \leq_P y$ implies $f(x) \leq_Q f(y)$ for all $x, y \in P$. If, in addition, f is a bijection with order-preserving inverse, then f is said to be a *poset isomorphism*. In this case, the posets P and Q are said to be *isomorphic*, and we write $P \cong Q$. Assuming that P and Q are graded, a map $f : P \to Q$ is called *rank-preserving* if for every $x \in P$ the rank of $f(x)$ in Q is equal to the rank of x in P. The *dual* of a poset P is the poset P^* on the same ground set as P with reversed ordering relations, *i.e.*, $x \leq_{P^*} y$ if and only if $y \leq_P x$. A poset P is called *self-dual* if and only if $P \cong P^*$ and it is *locally self-dual* if every closed interval of P is self-dual. The *direct product* of two posets P and Q is the poset $P \times Q$ on the set $\{(x, y) \ : \ x \in P, \ y \in Q\}$ for which $(x, y) \leq (x', y')$ holds in $P \times Q$ if $x \leq_P x'$ and $y \leq_Q y'$. The *ordinal sum* $P \oplus Q$ of P and Q is the poset defined on the disjoint union of P and Q with the order relation $x \leq y$ if (i) $x, y \in P$ and $x \leq_P y$, or (ii) $x, y \in Q$ and $x \leq_Q y$, or (iii) $x \in P$ and $y \in Q$. For more information on partially ordered sets we refer the reader to [21, Chapter 3].

An *abstract simplicial complex* Δ on a finite vertex set V is a collection of subsets of V such that $G \in \Delta$ and $F \subseteq G$ imply $F \in \Delta$. The elements of Δ are called *faces*. Inclusionwise maximal and 1-element faces are called *facets* and *vertices*, respectively. The dimension of a face $F \in \Delta$ is equal to $|F| - 1$ and is denoted by $\dim F$. The *dimension* of Δ is the maximum dimension of a face of Δ and is denoted by $\dim \Delta$. If all facets of Δ have the same dimension, then Δ is called *pure*. The *link* of a face F of Δ is defined as $\mathrm{link}_\Delta(\mathrm{F}) = \{\mathrm{G} \ : \ \mathrm{F} \cup \mathrm{G} \in \Delta, \mathrm{F} \cap \mathrm{G} = \emptyset\}$. All topological properties of an abstract simplicial complex Δ, we mention, refer to those of its geometric realization $\|\Delta\|$. The complex Δ is said to be *homotopy Cohen-Macaulay* if for all $F \in \Delta$ the link of F is topologically $(\dim(\mathrm{link}_\Delta(\mathrm{F})) - 1)$-connected. For a d-dimensional simplicial complex we have the following implications: shellable $\Rightarrow$ constructible $\Rightarrow$ homotopy Cohen-Macaulay $\Rightarrow$ homotopy equivalent to a wedge of d-dimensional spheres. For background concerning the topology of simplicial complexes we refer to [7] and [22].

To every poset P we associate its *order complex* $\Delta(P)$. The i-dimensional faces of $\Delta(P)$ are the chains of P of length i. If P is graded of rank n, then $\Delta(P)$ is pure of dimension n. All topological properties of P refer to those of $\|\Delta(P)\|$. We call a poset P *homotopy Cohen-Macaulay* or *shellable* if $\Delta(P)$ has this property.

2.2 The poset of injective words

A word ω over a finite alphabet A is called *injective* if no letter appears more than once. We denote by I_n the set of injective words on $\{1,\ldots,n\}$. The order relation on I_n is given by the containment of subwords, *i.e.*, $\omega_1\cdots\omega_s \le \sigma_1\cdots\sigma_r$ if and only if there exist $1 \le i_1 < i_2 < \cdots < i_s \le r$ such that $\omega_j = \sigma_{i_j}$ for $1 \le j \le s$. *E.g.*, we have $124 < 12345$ in I_5 and 12 and 23 are incomparable in each I_n for $n \ge 3$. Figure 1 illustrates the Hasse diagrams of the posets I_2 and I_3. We note that every closed interval of I_n is isomorphic to a Boolean algebra and as such shellable.

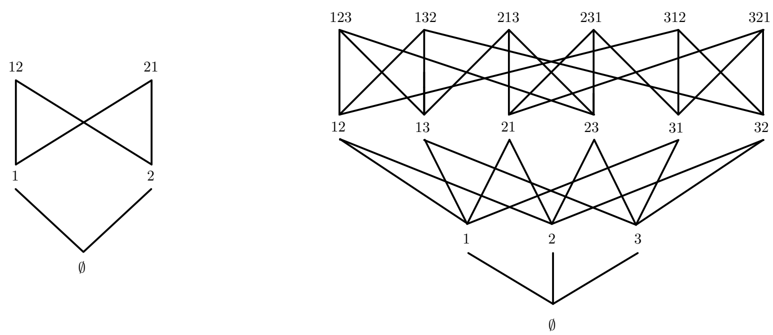

Fig. 1: The posets I_2 and I_3.

2.3 Non-crossing partitions

Let W be a finite Coxeter group and let T be its set of reflections. *E.g.*, if W equals the symmetric group S_n on $\{1,\ldots,n\}$, then the set T consists of all transpositions (ij) for $1 \le i < j \le n$. The *absolute length* $\ell_T(w)$ of an element $w \in W$ is the minimal integer r such that w can be written as a product of r reflections. The absolute length of an element of S_n equals n minus the number of cycles in its cycle decomposition. The *absolute order* on W is the poset $\mathrm{Abs}(W)$ defined by setting

$$\pi \le_T \mu \qquad \Longleftrightarrow \qquad \ell_T(\mu) = \ell_T(\pi) + \ell_T(\pi^{-1}\mu)$$

for all $\mu, \pi \in W$. This poset is graded with rank function ℓ_T and minimum element $e \in W$. It was shown in [12, Section 2] that for all $u, v \in S_n$ we have $u \le_T v$ if and only if

- every cycle in the cycle decomposition of u can be obtained from some cycle in the cycle decomposition of v by deleting elements, and
- any two cycles a and b of u which are obtained from the same cycle c of v are non-crossing with respect to c.

Here, disjoint cycles a and b are called *non-crossing* with respect to c if there does not exist a cycle $(ijkl)$ which is obtained from c by deleting elements such that i, k are elements of a and j, l are elements of b.

Let $c \in W$ be a Coxeter element. The interval

$$\mathrm{NC}(W,c) = [e,c] = \{w \in W \;:\; e \le_T w \le_T c\}$$

is called the poset of *non-crossing partitions*. It is well-known that for Coxeter elements $c, c' \in W$ it holds that $\mathrm{NC}(W, c) \cong \mathrm{NC}(W, c')$. We therefore often suppress c from the notation and just write $\mathrm{NC}(W)$.

The Coxeter elements of S_n are exactly the n-cycles. Figure 2 illustrates the Hasse diagrams of the posets $\mathrm{NC}(S_3)$ and $\mathrm{NC}(S_4)$.

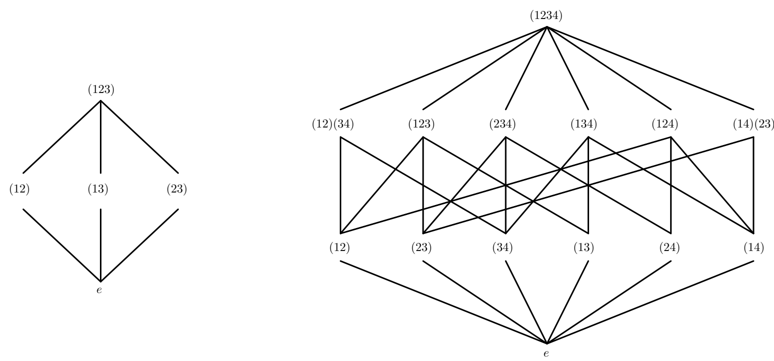

Fig. 2: The posets $\mathrm{NC}(S_3)$ and $\mathrm{NC}(S_4)$.

It follows from [1, Lemma 2.5.4] that $\mathrm{Abs}(W)$ is locally self-dual for every Coxeter group W. In particular, this implies the following corollary.

Corollary 2.1 *Let W be a finite Coxeter group with set of reflections T. Then, for all $u \in \mathrm{NC}(W)$ the principal lower order ideal $\langle u \rangle$ is self-dual. In particular, $\mathrm{NC}(W)$ is self-dual.*

For more information about Coxeter groups and non-crossing partitions we refer the reader to [1].

3 A poset fiber theorem for doubly homotopy Cohen-Macaulay posets

This section focuses on the proof of Theorem 1.1. We first recall the classical poset fiber theorem for homotopy Cohen-Macaulay posets by Quillen.

Theorem 3.1 [19, Corollary 9.7], [10, Theorem 5.1] *Let P and Q be graded posets. Let further $f : P \to Q$ be a surjective rank-preserving poset map. Assume that for every $q \in Q$ the fiber $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay. If Q is homotopy Cohen-Macaulay, then so is P.*

We will need the following result which follows from Remark 2.6 and Corollary 3.2 in [10].

Corollary 3.2 *Let P and Q be graded posets of rank n. Let $f : P \to Q$ be a surjective rank-preserving poset map such that for all $q \in Q$ the order complex $\Delta(Q_{>q})$ is $(n - \mathrm{rank}(q) - 2)$-connected and for all non-minimal $q \in Q$ the inclusion map*

$$\Delta\left(f^{-1}(Q_{<q})\right) \hookrightarrow \Delta\left(f^{-1}\left(\langle q \rangle\right)\right)$$

is homotopic to a constant map which sends $\Delta\left(f^{-1}(Q_{<q})\right)$ to c_q for some $c_q \in \Delta\left(f^{-1}\left(\langle q \rangle\right)\right)$. Then $\Delta(P)$ is $(n-1)$-connected if and only if Q is $(n-1)$-connected.

Proof of Theorem 1.1: It directly follows from Theorem 3.1 that the poset P is homotopy Cohen-Macaulay and hence so is the interval $I = (u, v)$. Let $\widetilde{I}$ denote the poset $I - \{x\}$ and let k be its rank. We need to verify that all links of faces $F \in \Delta(\widetilde{I})$ are $(\dim(\mathrm{link}_{\Delta(\widetilde{\mathrm{I}})}(\mathrm{F})) - 1)$-connected. The arguments we use are similar to those employed in the proof of [10, Theorem 5.1 (i)].

We first show that $\Delta(\widetilde{I}) = \mathrm{link}_{\Delta(\widetilde{\mathrm{I}})}(\emptyset)$ is $(k-1)$-connected. For this aim, we use Corollary 3.2.

Let $\widetilde{f} : \widetilde{I} \to f(I) - \{q_0\}$ denote the restriction of f to $\widetilde{I}$. This map is well-defined, since $f^{-1}(q_0) = \{x\}$, and it is a surjective poset map, because f is. Since f is rank-preserving and since $\widetilde{I}$ is graded by hypothesis, we deduce that $\widetilde{f}$ is rank-preserving. We set $\widetilde{J} = f(I) - \{q_0\}$ and by assumption we know that $\widetilde{J}$ is homotopy Cohen-Macaulay. In the following, consider $q \in \widetilde{J}$. Since $\Delta(\widetilde{J}_{>q})$ is the link of a face of $\Delta(\widetilde{J})$, we infer from the above that $\Delta(\widetilde{J}_{>q})$ is $(\mathrm{rank}(\widetilde{J}_{>q}) - 1) = (\mathrm{rank}(f(v)) - \mathrm{rank}(q) - 3)$-connected. This shows one of the conditions of Corollary 3.2 we need to verify. By assumption on f, the fiber $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay and therefore it is $(\mathrm{rank}(q) - 1)$-connected. As in the proof of Theorem 1.1 in [10], it follows that there exists a homotopy from the inclusion map $\Delta(f^{-1}(Q_{<q})) \hookrightarrow \Delta(f^{-1}(\langle q \rangle))$ to the constant map which sends $\Delta(f^{-1}(Q_{<q}))$ to $c_q \in \Delta(f^{-1}(\langle q \rangle))$. We can choose $c_q \in \Delta(\widetilde{f}^{-1}(\langle q \rangle)) \subseteq \widetilde{I}$. Then the above homotopy restricts to a homotopy from $\Delta(\widetilde{f}^{-1}(\widetilde{J}_{<q})) \hookrightarrow \Delta(\widetilde{f}^{-1}(\langle q \rangle))$ to the constant map which sends $\Delta(\widetilde{f}^{-1}(\widetilde{J}_{<q}))$ to c_q. Thus, $\Delta(\widetilde{f}^{-1}(\widetilde{J}_{<q})) \hookrightarrow \Delta(\widetilde{f}^{-1}(\langle q \rangle))$ is homotopic to a constant map. Finally, we can apply the Corollary aforementioned. Since, by homotopy Cohen-Macaulayness, $\widetilde{J}$ is $(k-1)$-connected, it follows that $\widetilde{I}$ is $(k-1)$-connected as well.

It remains to show that all links of proper faces $F \neq \emptyset$ of $\Delta(\widetilde{I})$ are $(\dim(\mathrm{link}_{\Delta(\widetilde{\mathrm{I}})}(\mathrm{F})) - 1)$-connected. Since the join of an s-connected and an r-connected complex is $(r+s-2)$-connected, it suffices to check open intervals and principal upper and lower order ideals (see *e.g.*, [11]).

Let (a, b) be an open interval in $\widetilde{I}$. Note that $(a, b)_P = (a, b)_I$. If $x \notin (a, b)_P$, then $(a, b)_I$ and $(a, b)_{\widetilde{I}}$ coincide. Since I is homotopy Cohen-Macaulay, it follows that $(a, b)_{\widetilde{I}}$ is $(\mathrm{rank}(b) - \mathrm{rank}(a) - 3)$-connected. Now let $a < x < b$ and let $c = f(b)$, *i.e.*, $b \in f^{-1}(c)$. From $b \neq v$, we infer that $b \in I$ and thus $b \in f^{-1}(c) \cap I$. Moreover, we have $c > q_0$ and by condition (ii) of the theorem, it follows that $[u, b]_P - \{x\}$ is homotopy Cohen-Macaulay. Since $(a, b)_{\widetilde{I}} = (a, b)_P - \{x\}$ is the link of a face of $[u, b]_P - \{x\}$, we conclude that $(a, b)_{\widetilde{I}}$ is $(\mathrm{rank}(b) - \mathrm{rank}(a) - 3)$-connected. The same reasoning shows that open principal lower order ideals $\widetilde{I}_{<p}$ of $\widetilde{I}$ are $(\mathrm{rank}(p) - \mathrm{rank}(u) - 3)$-connected.

Next, we show that for all $p \in \widetilde{I}$ the open principal upper order ideal $\widetilde{I}_{>p} = (p, v)_P - \{x\}$ is $(\mathrm{rank}(v) - \mathrm{rank}(p) - 3)$-connected. If $p \not< x$, then $(p, v)_P - \{x\} = (p, v)_P$, and the claim follows, because P is homotopy Cohen-Macaulay. Let now $p < x$. We consider the restriction of f to $P_{\geq p}$. To avoid confusion, let $\bar{f} : P_{\geq p} \to Q_{\geq f(p)}$ denote this restriction. We show that the map $\bar{f}$ is a surjective rank-preserving poset map, satisfying all assumptions of the theorem for the interval (p, v) and the element $x \in (p, v)$. Since, due to $u < p$, we have $\mathrm{rank}([p, v]_P - \{x\}) < \mathrm{rank}([u, v]_P - \{x\})$, we can then deduce by induction on the rank of the considered interval that $(p, v)_P - \{x\} = \widetilde{I}_{>p}$ is homotopy Cohen-Macaulay. In particular, we obtain that $\widetilde{I}_{>p}$ is $(\mathrm{rank}(v) - \mathrm{rank}(p) - 3)$-connected. For the verification of the assumptions, first note that $Q_{\geq f(p)}$ is homotopy Cohen-Macaulay because Q is. Clearly, $x \in (p, v)_P \subsetneq P_{\geq p}$. Since $\tilde{I}$ is graded by assumption, the same is true for $(p, v)_P - \{x\}$. Furthermore, f is a rank-preserving poset map, thus so is $\bar{f}$. To see that $\bar{f}$ is surjective, let $q \in Q_{\geq f(p)}$. Since f is rank-preserving and surjective and $f^{-1}(\langle q \rangle)$ is graded, all maximal elements of $f^{-1}(\langle q \rangle)$ are mapped to q and one of these has to be greater than p. Hence, $\bar{f}$ is surjective. For condition (i), note that for $q \in Q_{\geq f(p)}$ the fiber $\bar{f}^{-1}(\langle q\rangle)$ equals $f^{-1}(\langle q\rangle) \cap P_{\geq p}$. Thus, it is a closed principal upper order ideal of the homotopy Cohen-Macaulay poset $f^{-1}(\langle q\rangle)$ and as such homotopy Cohen-Macaulay.

It remains to verify condition (ii). Since $x > p$, we have $f(x) = q_0 \in Q_{\geq f(p)}$ and we obtain that $\bar{f}^{-1}(q_0) = \{x\}$. In addition, it holds that $\bar{f}((p,v)_P) - \{q_0\} = (f(I) - \{q_0\}) \cap (f(p), f(v))_Q$. Thus, $\bar{f}((p,v)_P) - \{q_0\}$ is an open principal upper order ideal of the homotopy Cohen-Macaulay poset $f(I) - \{q_0\}$ and as such homotopy Cohen-Macaulay.

Now let $q > q_0$ and let $\bar{p} \in \bar{f}^{-1}(q) \cap (p,v)_P$. The poset $[p,\bar{p}]_P - \{x\}$ is a closed interval of $[u,\bar{p}]_P - \{x\}$. Since by hypothesis the latter one is homotopy Cohen-Macaulay, so is $[p,\bar{p}]_P - \{x\}$. Finally, it follows by induction that $\widetilde{I}_{>p} = (p,v)_P - \{x\}$ is homotopy Cohen-Macaulay. This finishes the first part of the proof. The statement concerning double homotopy Cohen-Macaulayness follows directly from the definition of this property and the first part of the theorem. □

As a corollary of Theorem 1.1 one obtains the following poset fiber theorem for doubly homotopy Cohen-Macaulay posets, see [18] for the exact proof.

Corollary 3.3 *Let P be a graded poset without a minimum and a maximum element and let $x \in P$. Assume that $P - \{x\}$ is graded and that Q is a homotopy Cohen-Macaulay poset. Let further $f : P \to Q$ be a surjective rank-preserving poset map which satisfies the following conditions:*

(i) For every $q \in Q$ the fiber $f^{-1}(\langle q\rangle)$ is homotopy Cohen-Macaulay.

(ii) There exists $q_0 \in Q$ such that

- *$f^{-1}(q_0) = \{x\}$ and $Q - \{q_0\}$ is homotopy Cohen-Macaulay, and*
- *for every $q > q_0$ and $p \in f^{-1}(q)$ the poset $\langle p\rangle - \{x\}$ is homotopy Cohen-Macaulay.*

Then $P - \{x\}$ is homotopy Cohen-Macaulay as well. If for all $x \in P$ there exists a map satisfying the above conditions and if $\operatorname{rank}(P - \{x\}) = \operatorname{rank}(P)$*, then P is doubly homotopy Cohen-Macaulay.*

It is rather straightforward to give a generalization of Corollary 3.3 to k-Cohen-Macaulay posets where $k \geq 2$ (see [18, Proposition 3.4] for the exact result).

4 Applications of Corollary 3.3 and Theorem 1.1

This section provides applications of Corollary 3.3 and Theorem 1.1 to I_n and $\mathrm{NC}(S_n)$, respectively. We recall the notion of doubly homotopy Cohen-Macaulay posets.

Definition 4.1 *Let P be a homotopy Cohen-Macaulay poset. Then, P is called* doubly homotopy Cohen-Macaulay *if for every $x \in P$ the poset $P - \{x\}$ is homotopy Cohen-Macaulay of the same rank as P.*

For the proofs of Theorems 1.2 and 1.3 we will need the following technical result.

Theorem 4.2 *Let P be a poset of rank n with a minimum element $\hat{0}_P$. Let $\widetilde{P} = \bar{P}$ if P is bounded, and let $\widetilde{P} = P - \{\hat{0}_P\}$ if P does not have a maximum. Assume that $\widetilde{P}$ is doubly homotopy Cohen-Macaulay. Then, for every $x \in \widetilde{P}$ the poset $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ is homotopy Cohen-Macaulay of rank $n+1$.*

Sketch of the proof: Let $x \in \widetilde{P}$. We write $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ in the following way:

$$(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\} = \big((P - \{x\}) \times \{\hat{0}, \hat{1}\}\big) \cup \big((P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus \{(x, \hat{1})\} \oplus (P_{>x} \times \{\hat{1}\})\big). \quad (1)$$

The first part of the right-hand side of the above equation accounts for all chains in $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ not containing $(x, \hat{1})$. All chains in $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ passing through $(x, \hat{1})$, are captured by the second part of the right-hand side of Equation (1). Since homotopy Cohen-Macaulayness is preserved under taking direct products [11, Corollary 3.8] and ordinal sums [11, Corollary 3.4], it can be proved that the posets $(P - \{x\}) \times \{\hat{0}, \hat{1}\}$ and $(P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus \{(x, \hat{1})\} \oplus (P_{>x} \times \{\hat{1}\})$ are homotopy Cohen-Macaulay of rank $n + 1$. We now consider the intersection of those two posets. We have

$$\big((P - \{x\}) \times \{\hat{0}, \hat{1}\}\big) \cap \big((P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus \{(x, \hat{1})\} \oplus (P_{>x} \times \{\hat{1}\})\big) = (P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus (P_{>x} \times \{\hat{1}\}).$$

We obtain $(P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus (P_{>x} \times \{\hat{1}\})$ from $(P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus \{(x, \hat{1})\} \oplus (P_{>x} \times \{\hat{1}\})$ by deleting the element $(x, \hat{1})$. From the fact that rank-selection preserves homotopy Cohen-Macaulayness (see ***e.g.***, [6]) it follows that $(P_{<x} \times \{\hat{0}, \hat{1}\}) \oplus (P_{>x} \times \{\hat{1}\})$ is homotopy Cohen-Macaulay of rank n. If one applies Lemma 4.9 from [24] to the order complex of $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ as well as to its links, one obtains that $(P \times \{\hat{0}, \hat{1}\}) - \{(x, \hat{0})\}$ is homotopy Cohen-Macaulay of rank $n + 1$. $\square$

4.1 Proof of Theorem 1.2

Doubly shellable posets are defined in a similar way as doubly homotopy Cohen-Macaulay posets and it was shown by Baclawski [4, Corollary 4.3] that geometric lattices have this property. Combining this with the facts that the Boolean algebra is such a lattice and that homotopy Cohen-Macaulayness is implied by shellability, immediately yields the following.

Corollary 4.3 *The proper part of the Boolean algebra $\mathcal{B}_n$ is doubly homotopy Cohen-Macaulay.*

In various places of the proof of Theorem 1.2 we will employ the notion of strongly constructible posets. We therefore recall the definition of this notion which was introduced and studied in [3].

Definition 4.4 *A graded poset P of rank n with a minimum element is* strongly constructible *if either*

(i) P is bounded and pure shellable, or

(ii) P can be written as a union of two strongly constructible proper ideals J_1, J_2 of rank n such that the intersection $J_1 \cap J_2$ is a strongly constructible poset of rank at least $n - 1$.

Lemma 4.5 [3, Corollary 3.3, Proposition 3.6] *Let P be a strongly constructible poset. Then, P is homotopy Cohen-Macaulay.*

So as to show that the poset of injective words is doubly homotopy Cohen-Macaulay we will need the following simple lemma.

Lemma 4.6 *Let P be a strongly constructible poset of rank n and let $x \in P$ be a maximal element such that $P - \{x\}$ is graded of rank n. Then, the poset $P - \{x\}$ is strongly constructible of rank n.*

Proof: Let x be a maximal element of P such that $P-\{x\}$ is graded of rank n. Then, x cannot be the only maximal element of P. Since P is strongly constructible and – by the last argument – not bounded, there are proper ideals of P, say J_1 and J_2, which are strongly constructible of rank n and their intersection $J_1 \cap J_2$ is a strongly constructible ideal of rank at least $n-1$. Let $x \in J_1$ and $x \notin J_2$; the case $x \in J_2$ can be treated similarly. Using induction, we may assume that $J_1 = \langle x \rangle$. Since $P - \{x\}$ is graded of rank n, it follows that every element which is covered by x is also covered by at least one maximal element of J_2. Thus, $J_1 - \{x\} \subseteq J_2$ and therefore $P - \{x\} = (J_1 - \{x\}) \cup J_2 = J_2$, which by assumption is strongly constructible of rank n. □

Proof of Theorem 1.2:. Throughout this proof, we use $\tilde{I}_n$ to denote $I_n - \{\emptyset\}$. In order to show that $\tilde{I}_n$ is doubly homotopy Cohen-Macaulay we proceed by induction on n. If $n = 2$, then $\tilde{I}_2$ has two maximal elements (the words 12 and 21) and two elements (1 and 2) of rank 1. No matter which one of the elements 12, 21, 1 or 2 is removed from $\tilde{I}_2$ the poset obtained is homotopy Cohen-Macaulay of rank 1. Thus, $\tilde{I}_2$ is doubly homotopy Cohen-Macaulay. Now, assume that $n \geq 3$. Let $x \in \tilde{I}_n$ be a maximal element. Since I_n is shellable, it is in particular strongly constructible. Lemma 4.6 implies that $I_n - \{x\}$ is strongly constructible and using Lemma 4.5 we deduce that $\tilde{I}_n - \{x\}$ is homotopy Cohen-Macaulay.

Consider $x \in \tilde{I}_n$ that is not a maximal element. For every $w \in \tilde{I}_n$ let $\pi(w)$ denote the word obtained from w by deleting n. Note that $\pi(\tilde{I}_n) = I_{n-1}$, since $\pi(n) = \emptyset$. We define the map $f : \tilde{I}_n \to I_{n-1} \times \{\hat{0}, \hat{1}\} - \{(\emptyset, \hat{0})\}$ by letting

$$f(w) = \begin{cases} (\pi(w),\, \hat{0}), & \text{if } n \not\leq w, \\ (\pi(w),\, \hat{1}), & \text{if } n \leq w \end{cases}$$

for $w \in \tilde{I}_n$. Our aim is to apply Corollary 3.3 to this map. By definition, f is a rank-preserving map. We show that f is also a poset map and surjective. Let $u, v \in \tilde{I}_n$ with $u \leq v$. Suppose first that $n \not\leq v$. Then we also have $n \not\leq u$, thus $f(u) = (\pi(u), \hat{0}) = (u, \hat{0})$ and $f(v) = (\pi(v), \hat{0}) = (v, \hat{0})$. It follows that $f(u) \leq f(v)$. Suppose now that $n \leq v$. Then, $f(v) = (\pi(v), \hat{1})$ and $f(u)$ is either equal to $(\pi(u), \hat{0})$ or to $(\pi(u), \hat{1})$. Since $\pi(u) \leq \pi(v)$ and $\hat{0} < \hat{1}$, in both cases it holds that $f(u) \leq f(v)$. Altogether, this proves that f is a poset map. Let $w \in \tilde{I}_{n-1}$. Then $f^{-1}\left((w, \hat{0})\right) = \{w\}$ and every word obtained from w by inserting the letter n into some position of w lies in $f^{-1}\left((w, \hat{1})\right)$. Since $f^{-1}\left((\emptyset, \hat{1})\right) = n$, we obtain that f is surjective.

Next, we show that for $q \in I_{n-1} \times \{\hat{0}, \hat{1}\} - \{(\emptyset, \hat{0})\}$ the fiber $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay. By a straightforward but rather tedious computation one can verify that for those q it holds that $f^{-1}(\langle q \rangle) = \langle f^{-1}(q) \rangle_{\tilde{I}_n}$. So as to show that $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay, it thus suffices to show that $\langle f^{-1}(q) \rangle_{I_{n-1}} = \langle f^{-1}(q) \rangle_{I_n} - \{\emptyset\}$ has this property. If $q = (w, \hat{0})$ for some $w \in \tilde{I}_{n-1}$, then $\langle f^{-1}(q) \rangle_{\tilde{I}_n} = \langle w \rangle_{I_n} - \{\emptyset\}$. Since every interval in I_n is shellable (see Section 2.2), we infer that $f^{-1}(\langle q \rangle)$ is homotopy Cohen-Macaulay in this case. Now suppose that $q = (w, \hat{1})$ for $q \in I_{n-1}$. Without loss of generality, we may assume that $w = 123 \cdots k$, for some $k \leq n-1$. Then, $\langle f^{-1}(q) \rangle = \bigcup_{i=0}^{k} \langle 12 \cdots i\, n\, i+1 \cdots k \rangle_{\tilde{I}_n} = \bigcup_{i=0}^{k} \langle 12 \cdots i\, n\, i+1 \cdots k \rangle_{I_n} - \{\emptyset\}$. For every $i \in \{0, 1, \ldots, k\}$, the ideal $S_i = \langle 12 \cdots i\, n\, i+1 \cdots k \rangle_{I_n}$ is shellable and therefore strongly constructible and we have $\operatorname{rank}(S_i) = k+1$. We show by induction on j that the union $\bigcup_{i=0}^{j} S_i$ is strongly constructible as well. For this, it suffices to show that $S_j \cap \left(\bigcup_{i=0}^{j-1} S_i\right)$ is strongly constructible of rank k. We have

$$S_j \cap \left(\bigcup_{i=0}^{j-1} S_i\right) = \langle 12 \cdots k,\, 12 \cdots j-1\, n\, j+1 \cdots k \rangle_{I_n} = \langle 12 \cdots k \rangle_{I_n} \cup \langle 12 \cdots j-1\, n\, j+1 \cdots k \rangle_{I_n}.$$

Both ideals, $\langle 12\cdots k\rangle_{I_n}$ and $\langle 12\cdots j-1\, n\, j+1\cdots k\rangle_{I_n}$, are strongly constructible of rank k and their intersection is equal to $\langle 12\cdots j{-}1\, j{+}1\cdots k\rangle_{I_n}$, which is a strongly constructible ideal of rank $k{-}1$. This completes the induction and in particular implies that $\bigcup_{i=0}^{k} S_i$ is strongly constructible and by Lemma 4.5 homotopy Cohen-Macaulay. Since $\emptyset$ is the minimum of $\langle f^{-1}(q)\rangle_{I_n} = \bigcup_{i=0}^{k} S_i$, we deduce that $f^{-1}(\langle q\rangle)$ is homotopy Cohen-Macaulay.

Let $x \in \tilde{I}_n$. It is easy to see that $\tilde{I}_n - \{x\}$ is graded of rank $n-1$. It remains to verify condition (ii) of Corollary 1.1. Without loss of generality, we may assume that $x = 12\cdots k$ for some $1 \leq k \leq n-1$. Let $q_0 = (x, \hat{0})$. Clearly, $f(x) = q_0$ and $f^{-1}(q_0) = \{x\}$ by definition of f. By induction we may assume that $\tilde{I}_{n-1}$ is doubly homotopy Cohen-Macaulay and it now follows from Theorem 4.2 that $\big(I_{n-1} \times \{\hat{0}, \hat{1}\}\big) - \{q_0\}$ is homotopy Cohen-Macaulay. Since $(\emptyset, \hat{0})$ is the minimum of this poset, we conclude that $\big(I_{n-1} \times \{\hat{0}, \hat{1}\}\big) - \{(\emptyset, \hat{0}), q_0\}$ is homotopy Cohen-Macaulay. Let $q \in I_{n-1} \times \{\hat{0}, \hat{1}\} - \{(\emptyset, \hat{0})\}$ such that $q > q_0$ and let $p \in f^{-1}(q)$. We need to show that the ideal $\langle p\rangle_{\tilde{I}_n} - \{x\}$ is homotopy Cohen-Macaulay. We know from Section 2.2 that $\langle p\rangle_{I_n}$ is isomorphic to a Boolean algebra. Since $x < p$ (*i.e.*, x is not the maximal element of $\langle p\rangle_{I_n}$), we deduce from Corollary 4.3 that $\langle p\rangle_{I_n} - \{x\}$ is homotopy Cohen-Macaulay. Hence, so is $\langle p\rangle_{\tilde{I}_n} - \{x\}$. We can finally apply Corollary 3.3 which yields that $\tilde{I}_n - \{x\}$ is homotopy Cohen-Macaulay. □

4.2 Proof of Theorem 1.3

It was shown by Athanasiadis, Brady and Watt [2, Theorem 1.1] that for each finite Coxeter group the lattice of non-crossing partitions is shellable and in particular homotopy Cohen-Macaulay. In this section we show that the proper part of the lattice of classical non-crossing partitions is indeed doubly homotopy Cohen-Macaulay.

Proof of Theorem 1.3: Let $u \in \mathrm{NC}(S_n)$ for some n be a permutation of rank s. We show by induction on s that open intervals (e, u) of any non-crossing partition lattice are doubly homotopy Cohen-Macaulay. Without loss of generality, we can assume that u does not have a fixed point. For $s = 2$ the result is easy to check. It follows from [2, Theorem 1.1] and [1, Proposition 2.6.11] that (e, u) is shellable, hence homotopy Cohen-Macaulay. We need to show that for every $x \in (e, u)$ the poset $(e, u) - \{x\}$ is homotopy Cohen-Macaulay of rank $s - 2$. Since, by Lemma 2.1, $\langle u\rangle$ is self-dual, it is enough to consider those x of rank at most $\lfloor \frac{s}{2} \rfloor$. Note that such an x has to have a fixed point. Without loss of generality, we can assume that $x(n) = n$. We consider the following map from [17]. For every $w \in \mathrm{Abs}(S_n)$ let $\pi(w)$ be the permutation obtained from w by deleting n from its cycle decomposition. We define $g : \mathrm{Abs}(S_n) \to \mathrm{Abs}(S_{n-1}) \times \{\hat{0}, \hat{1}\}$ by letting

$$g(w) = \begin{cases} (\pi(w),\, \hat{0}), & \text{if } w(n) = n, \\ (\pi(w),\, \hat{1}), & \text{if } w(n) \neq n \end{cases}$$

for $w \in \mathrm{Abs}(S_n)$. Our goal is to apply Theorem 1.1 to the map g and the interval (e, u). In [17] it is shown that g is a surjective rank-preserving poset map whose fibers are homotopy Cohen-Macaulay. We note first that $(e, u) - \{x\}$ is graded. We consider the element $q_0 = (x, \hat{0}) \in g((e, u))$. By definition, $g^{-1}(q_0) = \{x\}$. Moreover, by $u(n) \neq n$, we know that the permutation $\pi(u)$ is of rank $s - 1$ and by induction, the interval $(e, \pi(u))$ is doubly homotopy Cohen-Macaulay. It follows from Theorem 4.2 that the poset $[e, \pi(u)) \times \{\hat{0}, \hat{1}\} - \{q_0\}$ is homotopy Cohen-Macaulay. Since $(e, \hat{0})$ is the minimum of this poset, we conclude that $g((e, u)) - \{q_0\} = [e, \pi(u)) \times \{\hat{0}, \hat{1}\} - \{(e, \hat{0}), q_0\}$ is homotopy Cohen-Macaulay.

It remains to verify the second part of condition (ii) of Theorem 1.1. Let $q \in g((e,u))$ such that $q > q_0$ and let $p \in g^{-1}(q) \cap (e,u)$. We need to show that $[e,p] - \{x\}$ is homotopy Cohen-Macaulay. Since the rank of p is at most $s-1$, the induction hypothesis implies that (e,p) is doubly homotopy Cohen-Macaulay. In particular, $[e,p] - \{x\}$ is homotopy Cohen-Macaulay. Finally, we can apply Theorem 1.1, which yields that $(e,u) - \{x\}$ is homotopy Cohen-Macaulay. From $\mathrm{rank}((e,u) - \{x\}) = \mathrm{rank}((e,u))$ we deduce that (e,u) is doubly homotopy Cohen-Macaulay. This finishes the proof since the proper part of any $\mathrm{NC}(S_n)$ is isomorphic to an interval in $\mathrm{NC}(S_{n+1})$ of the form (e,u). □

We want to remark that double homotopy Cohen-Macaulayness of the non-crossing partition lattice $\mathrm{NC}(S_n)$ can also be concluded by combining Theorem 6.3 and Theorem 3.1 in [15] and [23], respectively. It seems natural to ask whether also the proper parts of non-crossing partition lattices of other type are doubly homotopy Cohen-Macaulay. This turns out to be true for type B (see [18] for the proof of this result). (Note that in this case double homotopy Cohen-Macaulayness does not follow from [15] and [23].) After the reduction to the removal of elements having a fixed point, the proof for type B is literally the same as for type A. However, the question remains open for non-crossing partition lattices of other types.

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Skew quantum Murnaghan-Nakayama rule

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Abstract. In this extended abstract, we extend recent results of Assaf and McNamara, the skew Pieri rule and the skew Murnaghan-Nakayama rule, to a more general identity, which gives an elegant expansion of the product of a skew Schur function with a quantum power sum function in terms of skew Schur functions. We give two proofs, one completely bijective in the spirit of Assaf-McNamara's original proof, and one via Lam-Lauve-Sotille's skew Littlewood-Richardson rule.

Résumé. Dans cet article nous élargissons le cadre de résultats récents de Assaf et McNamara, la règle dissymétrique de Pieri et la règle dissymétrique de Murnaghan-Nakayama, pour obtenir une identité plus générale donnant un développement élégant du produit de la fonction de Schur dissymétrique par une somme de puissances quantiques, en termes de fonctions de Schur dissymétriques. Nous donnons deux démonstrations, la première suivant l'approche de Assaf-McNamara et la deuxième par le biais de la règle dissymétrique de Littlewood-Richardson obtenue par Lam-Lauve-Sotille.

Keywords: Murnaghan-Nakayama rule, Pieri rule, skew tableaux, Schur functions, q-analogue

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1 Introduction

Let us start with some basic definitions. A *partition* λ of n is an integer sequence $(\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ satisfying $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell > 0$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_\ell = n$; we use the notation $\lambda \vdash n$, $\ell = \ell(\lambda)$ (*length* of λ), $n = |\lambda|$ (*size* of λ), $\lambda_i = 0$ if $i > \ell(\lambda)$. We sometimes write $(\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots)$ if λ_1 is repeated k_1 times, $\lambda_2 < \lambda_1$ is repeated k_2 times etc. The *conjugate* partition of λ, denoted λ^c, is the partition $\mu = (\mu_1, \mu_2, \ldots, \mu_{\lambda_1})$ defined by $\mu_i = \max\{j \colon \lambda_j \geq i\}$. The *Young diagram* $[\lambda]$ of a partition λ is the set $\{(i,j) \colon 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$. For partitions λ, μ we say that $\mu \subseteq \lambda$ if $\mu_i \leq \lambda_i$ for all i. If $\mu \subseteq \lambda$, the *skew Young diagram* $[\lambda/\mu]$ of λ/μ is the set $\{(i,j) \colon 1 \leq i \leq \ell(\lambda), \mu_i < j \leq \lambda_i\}$. We denote $|\lambda| - |\mu|$ by $|\lambda/\mu|$. The elements of $[\lambda/\mu]$ are called *cells*. We treat λ and $\lambda/\emptyset$ as identical.

We say that λ/μ is a *horizontal strip* (respectively *vertical strip*) if $[\lambda/\mu]$ contains no 2×1 (respectively 1×2) block, equivalently, if $\lambda_i^c \leq \mu_i^c + 1$ (respectively $\lambda_i \leq \mu_i + 1$) for all i. We say that λ/μ is a *ribbon* if $[\lambda/\mu]$ is connected and if it contains no 2×2 block, and that λ/μ is a *broken ribbon* if $[\lambda/\mu]$ contains no 2×2 block, equivalently, if $\lambda_i \leq \mu_{i-1} + 1$ for $i \geq 2$. The Young diagram of a broken ribbon is a disjoint union of $\mathrm{rib}(\lambda/\mu)$ number of ribbons. The *height* $\mathrm{ht}(\lambda/\mu)$ (respectively *width* $\mathrm{wt}(\lambda/\mu)$) of a ribbon is the number of non-empty rows (respectively columns) of $[\lambda/\mu]$, minus 1. The height (respectively width) of a broken ribbon is the sum of heights (respectively widths) of the components. Clearly, λ/μ is a horizontal (respectively vertical) strip if and only if it is a broken ribbon of height (respectively width) 0.

A map $T\colon [\lambda/\mu] \to \mathbb{N}$ is called a *semistandard Young tableau of shape* λ/μ if $T_{i,j_1} \leq T_{i,j_2}$ for $j_1 < j_2$, and $T_{i_1,j} < T_{i_2,j}$ for $i_1 < i_2$. If T is bijective and maps to $\{1,\ldots,|\lambda|-|\mu|\}$, we say that it is a *standard Young tableau*. If T is a semistandard Young tableau, we denote by $t_i(T)$ the number of cells that map to i. Define the *skew Schur function* $s_{\lambda/\mu} = \sum_T x_1^{t_1(T)} x_2^{t_2(T)} \cdots$, where the sum is over all semistandard Young tableaux of shape λ/μ. A skew Schur function is a formal power series in $x_1, x_2, \ldots$, and it is easy to see that it is a symmetric function. Moreover, the set of *Schur functions* $\{s_\lambda\colon \lambda \text{ partition}\}$ is a basis of the space of symmetric functions. For more details, and for some of the amazing properties of Schur functions, see (7, §7).

There are several other bases of the space of symmetric functions. For the purposes of this paper, the most important one is the *power sum basis* $\{p_\lambda\colon \lambda \text{ partition}\}$, defined by $p_r = x_1^r + x_2^r + \ldots$, $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_\ell}$. Let us also mention the *monomial basis* $\{m_\lambda\colon \lambda \text{ partition}\}$, defined by $m_\lambda = \sum x_{\pi(1)}^{\lambda_1} \cdots x_{\pi(\ell)}^{\lambda_\ell}$, where the sum is over all injective maps $\pi\colon \{1,\ldots,\ell\} \to \mathbb{N}$.

The product of Schur functions can be uniquely expressed as a linear combination of Schur functions: $s_\lambda s_\mu = \sum c_{\lambda,\mu}^\nu s_\nu$. The coefficients $c_{\lambda,\mu}^\nu$ are called *Littlewood-Richardson coefficients* and can be computed using the celebrated *Littlewood-Richardson rule*, see (7, Appendix A1.3). This rule is quite complicated, but it is very simple if μ has only one row or column. Namely, we have the *Pieri rule – PR*:

$$s_\lambda \cdot s_r = \sum s_{\lambda^+}, \tag{1}$$

where the sum on the right is over all λ^+ such that λ^+/λ is a horizontal strip of size r. Similarly, the *conjugate Pieri rule – CPR* says that

$$s_\lambda \cdot s_{1^r} = \sum s_{\lambda^+}, \tag{2}$$

where the sum on the right is over all λ^+ such that λ^+/λ is a vertical strip of size r. See (7, §7.15).

The *Murnaghan-Nakayama rule – MNR* gives the product of a Schur function with a power sum symmetric function:

$$s_\lambda \cdot p_r = \sum (-1)^{\mathrm{ht}(\lambda^+/\lambda)} s_{\lambda^+}, \tag{3}$$

where the sum on the right is over all λ^+ such that λ^+/λ is a ribbon of size r. See (7, §7.17).

In (1) and (2), Assaf and McNamara found a beautiful extension of both PR and MNR.

Theorem 1 (Skew Pieri rule – SPR) *For partitions λ, μ, $\mu \subseteq \lambda$, we have*

$$s_{\lambda/\mu} \cdot s_r = \sum_j (-1)^j \sum s_{\lambda^+/\mu^-},$$

where the inner sum on the right is over all λ^+, μ^- such that λ^+/λ is a horizontal strip of size $r-j$, and μ/μ^- is a vertical strip of size j.

Corollary 2 (Conjugate skew Pieri rule – CSPR) *For partitions λ, μ, $\mu \subseteq \lambda$, we have*

$$s_{\lambda/\mu} \cdot s_{1^r} = \sum_j (-1)^j \sum s_{\lambda^+/\mu^-},$$

where the inner sum on the right is over all λ^+, μ^- such that λ^+/λ is a vertical strip of size $r-j$, and μ/μ^- is a horizontal strip of size j

CSPR can be proved from SPR via the involution ω on the algebra of symmetric functions which maps $s_{\lambda/\mu}$ to s_{λ^c/μ^c} and preserves the product. See (7, §7.6 and §7.14) for details.

Theorem 3 (Skew Murnaghan-Nakayama Rule – SMNR) *For partitions λ, μ, $\mu \subseteq \lambda$, we have*

$$s_{\lambda/\mu} \cdot p_r = \sum (-1)^{\mathrm{ht}(\lambda^+/\lambda)} s_{\lambda^+/\mu} - \sum (-1)^{\mathrm{ht}(\mu/\mu^-)} s_{\lambda/\mu^-},$$

where the first (respectively second) sum on the right is over all λ^+ (respectively μ^-) such that λ^+/λ (respectively μ/μ^-) is a ribbon of size r.

Note that while the Pieri rule and the Murnaghan-Nakayama rule give the expansion in terms of a basis, their skew versions give only one possible (but obviously special) expansion in terms of skew Schur functions, which are not a basis of the space of symmetric functions. Assaf and McNamara provide an elegant bijective proof of their skew Pieri rule (but not of the skew Murnaghan-Nakayama rule).

Define *quantum power sum symmetric functions* by

$$\widetilde{p}_r = \sum_{\tau \vdash r} (-1)^{\ell(\tau)-1} (q-1)^{\ell(\tau)-1} m_\tau, \qquad \widetilde{p}_\lambda = \widetilde{p}_{\lambda_1} \widetilde{p}_{\lambda_2} \cdots .$$

The functions $\widetilde{p}_\lambda$ have connections with representation theory (more precisely, characters of the Hecke algebra of type A; see for example (3, Theorem 6.5.3)). We have $\widetilde{p}_r|_{q=1} = m_r = p_r$, $\widetilde{p}_r|_{q=0} = \sum_{\tau \vdash r} m_\tau = s_r$, $\lim_{q\to\infty} \frac{\widetilde{p}_r}{q^{r-1}} = (-1)^{r-1} m_{1^r} = (-1)^{r-1} s_{1^r}$.

There exists a very natural generalization of the Murnaghan-Nakayama rule, the *quantum Murnaghan-Nakayama rule – QMNR*:

$$s_\lambda \cdot \widetilde{p}_r = (-1)^{r+1} \sum (-1)^{\mathrm{wt}(\lambda^+/\lambda)} q^{\mathrm{ht}(\lambda^+/\lambda)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)-1} s_{\lambda^+} \tag{4}$$

where the internal sum on the right is over λ^+ such that λ^+/λ is a broken ribbon of size r. See for example (3, Theorem 6.5.2) for a slightly different version and a (complicated) bijective proof via the classical definition of Schur functions.

The following is our main result, the skew quantum Murnaghan-Nakayama rule.

Theorem 4 (SQMNR) *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum_{j=0}^{r} (-1)^{r+1-j} \sum (-1)^{\mathrm{wt}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)} q^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (q-1)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} s_{\lambda^+/\mu^-},$$

where the internal sum on the right is over λ^+, μ^- such that λ^+/λ is a broken ribbon of size $r-j$, and μ/μ^- is a broken ribbon of size j.

There is an equivalent version of the statement that will be slightly more useful for our purposes.

Theorem 5 (SQMNR') *For partitions λ, μ, $\mu \subseteq \lambda$, and $r \geq 0$, we have*

$$s_{\lambda/\mu} \cdot \widetilde{p}_r = \sum (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1} s_{\lambda^+/\mu^-},$$

where the sum is over λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$.

The main theorem is a generalization of several statements. The following is a sample:

- $q = 0$: a term on the right-hand side of SQMNR' is non-zero if and only if $\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-) = 0$. In this case, λ^+/λ has height 0 (and is a horizontal strip) and μ/μ^- has width 0 (and is a vertical strip). As noted above, $\widetilde{p}_r|_{q=0} = s_r$. SQMNR' specializes to SPR due to Assaf-McNamara (1).

- $q = 1$: a term on the right of SQMNR' is non-zero if and only if $\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1 = 0$. In this case, one of λ^+/λ and μ/μ^- is empty, and the other one is a ribbon. We know that $\widetilde{p}_r|_{q=1} = p_r$. It is easy to see that SQMNR' becomes SMNR due to Assaf-McNamara (2).

- $q \to \infty$: divide SQMNR by q^{r-1} and let $q \to \infty$. It is easy to prove that this simplifies to CSPR.

- $\mu = \emptyset$: SQMNR is QMNR; we get the classical PR for $q = 0$, the classical MNR for $q = 1$, and the classical CPR if we let $q \to \infty$.

- $\lambda = \mu = \emptyset$: this gives the expansion of quantum power sum functions in the basis of Schur functions, $\widetilde{p}_r = \sum_{k=1}^r (-q)^{r-k} s_{k,1^{r-k}}$. We will verify this independently in Lemma 8.

This extended abstract is structured as follows. In Section 2, we describe the sign-reversing involution of Assaf and McNamara that was used to prove their skew Pieri rule. Furthermore, we show a variant of this involution that proves the conjugate skew Pieri rule. Note that this involution is actually simpler than the one in (1) (but, of course, does not provide a bijective proof of the skew Pieri rule itself). In Section 3, we present an extension of these involutions that proves the skew quantum Murnaghan-Nakayama rule. There is quite some work involved to interpret the right-hand side of SQMNR in an appropriate way, but once this is done the involution is just a natural combination of the two involutions in Section 2. In Section 4, we sketch another proof of SQMNR, via the skew Littlewood-Richardson rule of Lam-Lauve-Sotille (5); since this result (at the moment) only has an algebraic proof, this proof of SQMNR is not completely combinatorial. We finish with some concluding remarks in Section 5. A reader interested in a brief history of the subject and all the necessary references should consult (1). The full version of this paper is (4).

2 Proofs of the skew Pieri rule and its dual

One of the most important algorithms on semistandard Young tableaux is the Robinson-Schensted *row insertion*. Given a semistandard Young tableau T of shape λ and an integer k, we can *insert k into T* as follows. Define $k_1 = k$. Find the smallest j so that $T_{1j} > k_1$, replace T_{1j} by k_1, and define k_2 to be the previous value of T_{1j}. Then find the smallest j so that $T_{2j} > k_2$, replace T_{2j} by k_2, and define k_3 to be the previous value of T_{2j}. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda'_i+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion of k into T *exits in row i'*. See (7, §7.11) for details.

Now assume we have a semistandard Young tableau T of some *skew* shape λ/μ. We can *insert k into T* for some integer k in almost exactly the same way. Define $k_1 = k$. Find the smallest j, $\mu_1 < j \leq \lambda_1$, so that $T_{1j} > k_1$, replace T_{1j} by k_1, and define k_2 to be the previous value of T_{1j}. Then find the smallest j, $\mu_2 < j \leq \lambda_2$, so that $T_{2j} > k_2$, replace T_{2j} by k_2, and define k_3 to be the previous value of T_{2j}. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda_{i'}+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion of k into T *exits in row i'*.

There is another natural kind of insertion. Take i_0 so that $\mu_{i_0} < \lambda_{i_0}$ and either $i_0 = 1$ or $\mu_{i_0-1} > \mu_{i_0}$, and take $k_{i_0+1} = T_{i_0,\mu_{i_0}+1}$. We can *insert from row i_0 in T* as follows. Erase the entry $T_{i_0,\mu_{i_0}+1}$. Find the smallest j, $\mu_{i_0+1} < j \leq \lambda_{i_0+1}$, so that $T_{i_0+1,j} > k_{i_0+1}$, replace $T_{i_0+1,j}$ by k_{i_0+1}, and define k_{i_0+2} to be the previous value of $T_{i_0+1,j}$. Then find the smallest j, $\mu_{i_0+2} < j \leq \lambda_{i_0+2}$, so that $T_{i_0+2,j} > k_{i_0+2}$, replace $T_{i_0+2,j}$ by k_{i_0+2}, and define k_{i_0+3} to be the previous value of $T_{i_0+2,j}$. Continue until, for some i', all elements of row i' are $\leq k_{i'}$. Then define $T_{i',\lambda_{i'}+1} = k_{i'}$, and finish the algorithm. The result is again a semistandard Young tableau. We say that the insertion from row i_0 in T *exits in row i'*.

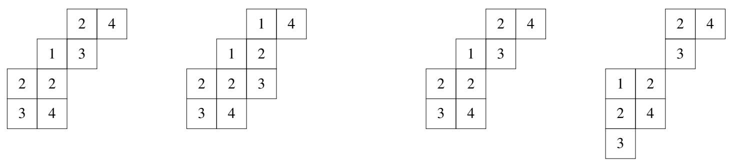

Fig. 1: Insertion of 1 into a tableau, and insertion from row 2 in a tableau.

Insertion has an inverse operation, *reverse insertion*. Say we are given a semistandard Young tableau T of shape λ/μ. Take i' so that $\lambda_{i'+1} < \lambda_{i'}$. We *reverse insert from row i' in T* as follows. Define $k_{i'-1} = T_{i',\lambda_{i'}}$. Erase the entry $T_{i',\lambda_{i'}}$. Find the largest j, $\mu_{i'-1} < j \leq \lambda_{i'-1}$, so that $T_{i'-1,j} < k_{i'-1}$, replace $T_{i'-1,j}$ by $k_{i'-1}$, and define $k_{i'-2}$ to be the previous value of $T_{i'-1,j}$. Then find the largest j, $\mu_{i'-2} < j \leq \lambda_{i'-2}$, so that $T_{i'-2,j} < k_{i'-2}$, replace $T_{i'-2,j}$ by $k_{i'-2}$, and define $k_{i'-3}$ to be the previous value of $T_{i'-2,j}$. Continue until we have k_{i_0}, where either $i_0 = 0$ or all elements of row i_0 are $\geq k_{i_0}$. If $i_0 = 0$, the result is a pair (S, k), where S is a semistandard Young tableau and $k = k_0$. We call k the *exiting integer*. If $i_0 \geq 1$ and all elements of row i_0 are $\geq k_{i_0}$, define $T_{\mu_{i_0}} = k_{i_0}$. The result is a semistandard Young tableau S. We say that the reverse insertion from row i' in T *exits in row i_0*.

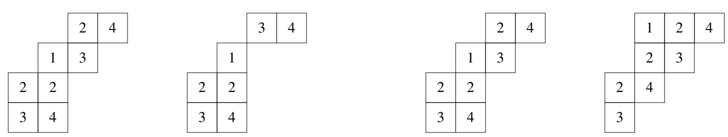

Fig. 2: Reverse insertion from rows 2 (which exits in row 0 with exiting integer 2) and 4 (which exits in row 1).

In (1), the operations of insertion and reverse insertion are proved to be inverses of one another in the following sense. If the insertion of an integer k into a semistandard Young tableau T exits in row i' and the resulting tableau is S, then the reverse insertion from row i' in S exits in row 0 and the result is (T, k). If the insertion from row i_0 into T exits in row i' and the resulting tableau is S, then the reverse insertion from row i' in S exits in row i_0 and the result is T. Similarly, if the reverse insertion from row i' in T exits in row 0 and the result is (S, k), then the insertion of k into S exits in row i' and the result is T. And if the reverse insertion from row i' in T exits in row $i_0 \geq 1$ and the result is S, then the insertion from row i_0 into S exits in row i' and the result is T.

The involution by Assaf and McNamara which proves the skew Pieri rule works as follows. Say we are given a skew shape λ/μ and a semistandard Young tableau T of shape λ^+/μ^-, where λ^+/λ is a horizontal strip and μ/μ^- is a vertical strip. Let v be the empty word. Let $i = \infty$ if $\mu = \mu^-$, and let i

be the top row of μ/μ^- otherwise. While $\lambda^+ \neq \lambda$ and the reverse insertion from row i', the top row of λ^+/λ, in T exits in row 0 and results in (S, k), attach k to the beginning of v, let $T = S$, and let λ^+/μ^- be the shape of the new T (note that $\lambda^+_{i'}$ is decreased by 1 and μ^- remains the same). If the while loop stops when $\lambda^+ \neq \lambda$ and the reverse insertion from row i' in T exits in row i_0, $0 < i_0 < i$, and results in S, let $T = S$. If the while loop stops when $\lambda^+ = \lambda$, $\mu \neq \mu^-$, or when $\lambda^+ \neq \lambda$ and the reverse insertion from row i' in T exits in row i_0, $i_0 \geq i$, insert from row i in T and call the resulting tableau T. Finish the algorithm by inserting the entries of v from left to right into T. The final result is a semistandard Young tableau of some shape λ^{++}/μ^{--}, we denote it $\Phi_{\lambda,\mu,\lambda^+,\mu^-}(T)$.

It turns out that Φ is an involution, and T is a fixed point if and only if $\mu = \mu^-$ and the while loop stops when $\lambda^+ = \lambda$. Such fixed points are in one-to-one correpondence with pairs (S, v), where S is a semistandard Young tableau of shape λ/μ and v is a weakly increasing word. Indeed, if we stop the algorithm after the while loop, we have exactly such a pair, and given a pair (S, v), we can insert the entries of v from left to right into S to get the corresponding T. Furthermore, if T is not a fixed point, then $|\mu^{--}| = |\mu^-| \pm 1$. It is easy to see that this shows the skew Pieri rule. See (1) for details and a precise proof, and (1), (2) or (4) for examples.

As mentioned in the introduction, the conjugate skew Pieri rule follows from SPR by applying the involution ω on the algebra of symmetric functions. There is, however, an involution in the spirit of Assaf-McNamara that proves CSPR.

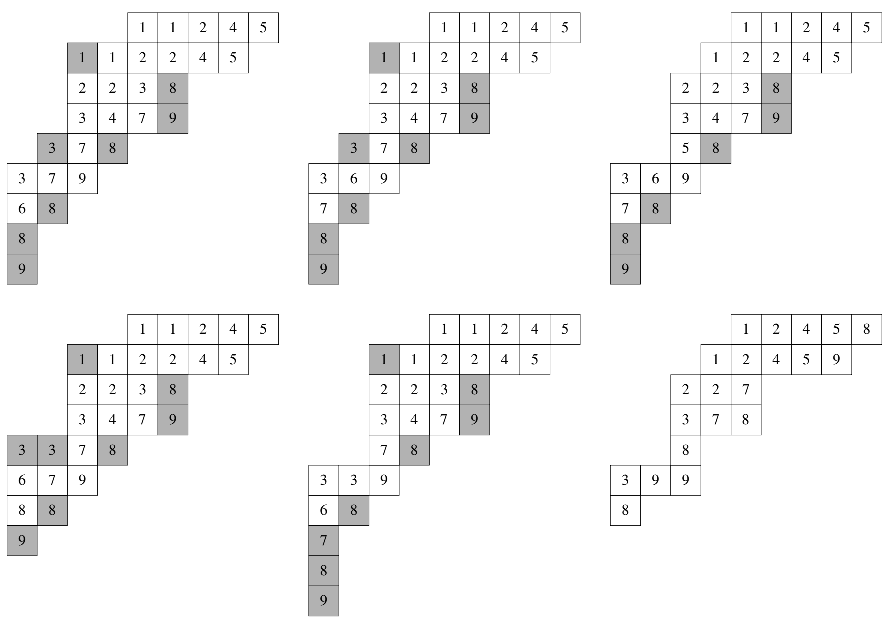

Fig. 3: Tableaux T_1, T_2, T_3 are on top; $\Psi(T_1)$ and $\Psi(T_2)$ are the bottom left and bottom middle tableaux, while T_3 is a fixed point of Ψ. The bottom right figure shows the tableau we get if we repeatedly reverse insert from the bottom row of T_3; the exiting integers are $1, 2, 3, 4, 5, 6$.

Fix λ, μ, r. A term on the right-hand side is represented by a semistandard Young tableau of shape λ^+/μ^-, where λ^+/λ is a vertical strip, μ/μ^- is a horizontal strip, and $|\lambda^+/\lambda| + |\mu/\mu^-| = r$. Such a tableau T is weighted by $(-1)^{|\mu/\mu^-|}$. Let i denote the bottom row of μ/μ^- (unless $\mu = \mu^-$, in which case take $i = 0$). Now reverse insert from row i', the bottom row of λ^+/λ, in T (unless $\lambda^+ = \lambda$). If the reverse insertion exits the diagram in row $\geq i$ (*except* in the case when $\mu = \mu^-$ and the reverse insertion exits in row 0), call this new diagram $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T)$. See Figure 3, left. If this reverse insertion exits the diagram in row $< i$, or if $\lambda^+ = \lambda$, insert from row i in T and call the result $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T)$. See Figure 3, middle. When $\mu = \mu^-$ and the reverse insertion exits in row 0, take $\Psi(T) = \Psi_{\lambda,\mu,\lambda^+,\mu^-}(T) = T$. See Figure 3, right.

Proposition 6 *The map $\Psi_{\lambda,\mu,\lambda^+,\mu^-}$ is an involution that is sign-reversing except on fixed points. Furthermore, the fixed points are in a bijective correspondence with elements on the left-hand side of CSPR.*

3 A bijective proof of the main theorem

The first step of our proof is to interpret the right-hand side of SQMNR' as a weighted sum over some combinatorial objects. The appropriate objects turn out to be semistandard Young tableaux with some cells colored gray. To motivate these colorings, observe the following. If we "glue" together a vertical strip and a horizontal strip in such a way that the result is a skew diagram, then this skew diagram cannot have any 2×2 squares. In other words, it is a broken ribbon. This also holds the other way around: if we are given a broken ribbon, we can break it up into a vertical strip and a horizontal strip.

Let us multiply both sides of SQMNR' by $1 - q$ and call this statement SQMNR":

$$s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r} (1-q)^{\ell(\tau)} m_\tau \right) = \sum_{\lambda^+,\mu^-} (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda) + \mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)} s_{\lambda^+/\mu^-}.$$

We have fixed λ, μ, r. Say that we are given λ^+, μ^- such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, and a semistandard Young tableau T of shape λ^+/μ^-. Our first goal is to break up each of the broken ribbons λ^+/λ and μ/μ^- into a vertical strip and a horizontal strip. More precisely, we wish to choose partitions λ', μ' such that λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips. We weight such a selection with

$$(-1)^{|\mu/\mu^-|} (-q)^{|\lambda^+/\lambda'| + |\mu'/\mu^-|}.$$

We color the cells of λ^+/λ' and μ'/μ^- gray and leave the other cells white. So our requirements are saying that the gray cells of λ^+/λ and the white cells of μ/μ^- form a vertical strip, and the white cells of λ^+/λ and the gray cells of μ/μ^- form a horizontal strip; also, the white cells form a diagram of some shape λ'/μ' for $\lambda \subseteq \lambda' \subseteq \lambda^+$, $\mu^- \subseteq \mu' \subseteq \mu$. Furthermore, the weight of such an object is $(-1)^{|\mu/\mu^-|}(-q)^j$, where j is the number of gray cells.

We claim that these objects indeed enumerate the right-hand side of SQMNR'.

Lemma 7 *For fixed $\lambda, \mu, \lambda^+, \mu^-$, we have*

$$\sum (-1)^{|\mu/\mu^-|} (-q)^{|\lambda^+/\lambda'| + |\mu'/\mu^-|} = (-1)^{|\mu/\mu^-|} (-q)^{\mathrm{ht}(\lambda^+/\lambda) + \mathrm{wt}(\mu/\mu^-)} (1-q)^{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)},$$

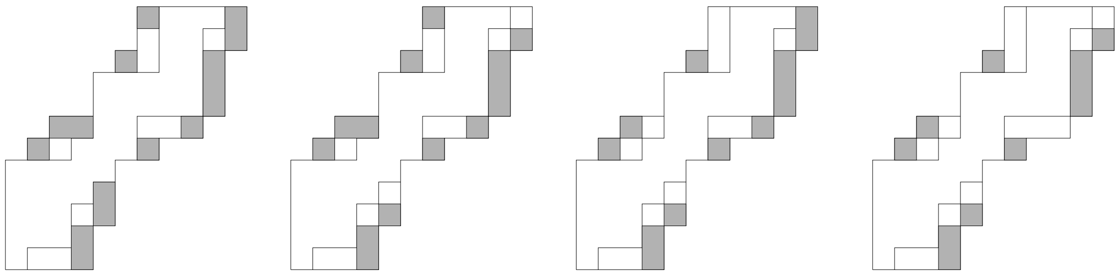

Fig. 4: Four examples with weights q^{16}, q^{14}, q^{13} and q^{11}.

where the sum on the left runs over all λ', μ' such that λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips.

Proof: For each cell of λ^+/λ, we have to decide whether or not to put it in λ'/λ or in λ^+/λ' (i.e. whether to make it white or gray). If a cell in λ^+/λ has a right neighbor in λ^+/λ, it cannot be in λ^+/λ', since its right neighbor would also have to be in λ^+/λ', and this would contradict the requirement that λ^+/λ' is a vertical strip. Similarly, if a cell in λ^+/λ has an upper neighbor in λ^+/λ, it cannot be in λ'/λ, since its upper neighbor would also have to be in λ'/λ, and this would contradict the requirement that λ'/λ is a horizontal strip.

This means that the colors of all the cells in λ^+/λ are determined, except for the top right cell of each ribbon of λ^+/λ, which can be either white or gray.

If a cell in μ/μ^- has a right neighbor in μ/μ^-, it cannot be in μ/μ', since its right neighbor would also have to be in μ/μ', and this would contradict the requirement that μ/μ' is a vertical strip. Similarly, if a cell in μ/μ^- has an upper neighbor in μ/μ^-, it cannot be in μ'/μ^-, since its upper neighbor would also have to be in μ'/μ^-, and this would contradict the requirement that μ'/μ^- is a horizontal strip.

This means that the colors of all the cells in μ/μ^- are determined, except for the top right cell of each ribbon of μ/μ^-, which can be either white or gray.

In other words, we have two choices for each top right cell of each ribbon of $(\lambda^+/\lambda) \cup (\mu/\mu^-)$. We have at least $\mathrm{ht}(\lambda^+/\lambda)$ gray cells in λ^+/λ, and at least $\mathrm{wt}(\mu/\mu^-)$ gray cells in μ/μ^-. So the weight of a term on the left-hand side is $(-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(-q)^j$, where j is the number of cells that are gray by choice, and these choices are made independently. Of course,

$$\sum_j \binom{\mathrm{rib}(\lambda^+/\lambda) + \mathrm{rib}(\mu/\mu^-)}{j} (-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(-q)^j =$$

$$= (-1)^{|\mu/\mu^-|}(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)},$$

which finishes the proof of the lemma. □

We have managed to rewrite SQMNR" as follows:

$$s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r} (1-q)^{\ell(\tau)} m_\tau \right) = \sum (-1)^{|\mu/\mu^-|}(-q)^{|\lambda^+/\lambda'|+|\mu'/\mu^-|} s_{\lambda^+/\mu^-},$$

where the sum is over all partitions $\lambda^+, \lambda', \mu^-, \mu'$ such that λ^+/λ and μ/μ^- are broken ribbons with $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, λ'/λ and μ'/μ^- are horizontal strips, and λ^+/λ' and μ/μ' are vertical strips.

For fixed λ, μ, r, a term on the right-hand side of SQMNR" therefore corresponds to a semistandard Young tableau T with some cells colored white and some cells colored gray, such that the following properties are satisfied:

- the shape of T is λ^+/μ^- for some $\lambda^+ \supseteq \lambda$ and $\mu^- \subseteq \mu$, $|\lambda^+/\lambda| + |\mu/\mu^-| = r$, and λ^+/λ and μ/μ^- are broken ribbons;
- the white cells form a skew diagram λ'/μ' for some partitions λ', μ', $\lambda' \supseteq \lambda$, $\mu' \subseteq \mu$;
- the white cells in λ^+/λ form a horizontal strip, and the white cells in μ/μ^- form a vertical strip;
- the gray cells in λ^+/λ form a vertical strip, and the gray cells in μ/μ^- a horizontal strip;

We call such an object a *colored tableau* of shape $(\lambda, \mu, \lambda', \mu', \lambda^+, \mu^-)$. We weight a colored tableau by

$$(-1)^{|\mu/\mu^-|}(-q)^{|\lambda^+/\lambda'|+|\mu'/\mu^-|}.$$

Perform the involution Ψ on the gray cells of a colored tableau. More specifically, find $\Psi_{\lambda',\mu',\lambda^+,\mu^-}(T)$. Since λ^+/λ' is a vertical strip and μ'/μ^- is a horizontal strip, the map is well defined. One gray cell is removed, and one gray cell is added in the process. The result is a colored tableau T' of shape $(\lambda, \mu, \lambda', \mu', \lambda^{++}, \mu^{--})$ for some λ^{++}, μ^{--}; it has the same white cells as T, the same number of gray cells as T, and with the property that $|\mu/\mu^{--}| = |\mu/\mu^-| \pm 1$ unless $T = T'$ is a fixed point.

This already cancels a large number of terms. The ones that remain correspond to fixed points of $\Psi_{\lambda',\mu',\lambda^+,\mu^-}$. Each such fixed point consists of a a semistandard Young tableau S of shape λ'/μ', where λ'/λ is a horizontal strip and μ/μ' is a vertical strip, and of a strictly decreasing word w. Such an object is weighted by $(-1)^{|\mu/\mu'|}(-q)^{|w|}$.

Now apply Assaf-McNamara involution Φ to the tableau. More specifically, find $\Phi_{\lambda,\mu,\lambda',\mu'}(S)$. This is well defined because λ'/λ is a horizontal strip and μ/μ' is a vertical strip. The result is a semistandard Young tableaux S' of some shape λ''/μ'' with the property that $|\mu/\mu''| = |\mu/\mu'| \pm 1$ unless $S = S'$ is a fixed point. This cancels more terms. The ones that remain correspond to fixed points of $\Phi_{\lambda,\mu,\lambda',\mu'}$, together with a strictly decreasing word w and the weight $(-q)^{|w|}$. Each such fixed point consists of a semistandard Young tableau R of shape λ/μ, together with a weakly increasing word v and a strictly decreasing word w. Such an object is weighted by $(-q)^{|w|}$. Furthermore, every such triple (R, v, w) appears as a non-canceling term on the right. Indeed, insert the elements of v into R to get a semistandard Young tableau S of shape λ'/μ for some partition λ' so that λ'/λ is a horizontal strip; then insert the elements of w into S and color the new cells gray to get a colored tableau T of shape λ^+/μ for some partition λ^+ so that λ^+/λ' is a vertical strip. Then applying Ψ and Φ to T yields (R, v, w).

It remains to enumerate all triples (R, v, w). If we want (v, w) to contain, say, τ_i copies of i, $1 \le i \le \ell$, we can choose any j-subset of $\{1, \ldots, \ell\}$ and put the elements in decreasing order to form w, and then put the remaining elements of the multiset $\{1^{\tau_1}, 2^{\tau_2}, \ldots, \ell^{\tau_\ell}\}$ in weakly increasing order to form v. Furthermore, the weight of (R, v, w) for these v and w is $(-q)^j$. That means that the right-hand side of SQMNR" becomes, after cancellations, $s_{\lambda/\mu} \cdot \left(\sum_{\tau \vdash r} (1-q)^{\ell(\tau)} m_\tau\right)$, which is the left-hand side of SQMNR".

4 A proof via skew Littlewood-Richardson rule

It is informative to use Lam-Lauve-Sotille's (5) skew Littlewood-Richardson rule to find another proof of SQMNR. We need three lemmas. The first lemma is a simple computation that allows us to replace the quantum power sum functions with "hook" Schur functions. The second lemma is technical and states that a certain property is preserved in jeu de taquin slides. And the third lemma sheds some light on connections between jeu de taquin, hooks, and decompositions of broken ribbons into vertical and horizontal strips.

Lemma 8 *For all r, we have*

$$\widetilde{p}_r = \sum_{k=1}^{r} (-q)^{r-k} s_{k,1^{r-k}}.$$

For the second lemma, we have to recall the celebrated *backward (respectively, forward) jeu de taquin slide* due to Schützenberger. Say we are given a standard Young tableau of shape λ/μ. Let $c = c_0$ be a cell that is not in λ/μ, shares the right or lower edge (respectively, the left or upper edge) with λ/μ, and such that $\lambda/\mu \cup c$ is a valid skew diagram. Let c_1 be the cell of λ/μ that shares an edge with c_0; if there are two such cells, take the one with the smaller entry (respectively, larger entry). Then move the entry occupying c_1 to c_0, look at the tableau entries below or to the right of c_1 (respectively, above or to the left of c_1), and repeat the same procedure. We continue until we reach the boundary of λ/μ, say in m moves. The new tableau is a standard Young tableau and is called $\mathrm{jdt}_c(T)$. We say that $c_0, c_1, \ldots, c_m$ is the *path* of the slide.

If T is a standard Young tableau of skew shape, we can repeatedly perform backward jeu de taquin slides. The final result S is a standard Young tableau of straight shape, and it is independent of the choices during the execution of the algorithm. We say that T *rectifies* to S. See (7, Appendix A1.2) for details and examples.

We say that a standard Young tableau T of shape λ/μ has the *k-NE property* if the following hold:

NE1 the entry in the last cell of the first non-empty row (i.e. the northeast cell) of λ/μ is k;

NE2 if $i < j < k$, then i appears strictly to the left of j in T;

NE3 if $j > i > k$, then i appears strictly above j in T.

Lemma 9 *If a tableau T has the k-NE property, its shape is a broken ribbon. Furthermore, the k-NE property is preserved in a jeu de taquin slide.*

Lemma 10 *Take r, k, $1 \leq k \leq r$, and let S be the standard Young tableau of shape $(k, 1^{r-k})$ with $1, 2, \ldots, k$ in the first row, and $k+1, k+2, \ldots, r$ in rows $2, 3, \ldots, r-k+1$. Choose a skew shape λ/μ. Then the number of standard Young tableaux of shape λ/μ that rectify to S is $\binom{\mathrm{rib}(\lambda/\mu)-1}{k-1-\mathrm{wt}(\lambda/\mu)}$ if λ/μ is a broken ribbon of size r, and 0 otherwise.*

Sketch of proof: Let us assume that λ/μ is a broken ribbon of size r and count the number of standard Young tableaux of shape λ/μ that have the k-NE property. Place k in the northeast cell. If a cell in λ/μ has a right neighbor in λ/μ, then the entry has to be less than k (otherwise both this entry and the entry to the right would be greater than k, and this would contradict NE3). Similarly, if a cell in λ/μ has an upper neighbor in λ/μ, then the entry has to be greater than k (otherwise both this entry and the entry above it

would be less than k, and this would contradict NE2).
This means that there are at least $\mathrm{wt}(\lambda/\mu)$ elements that are $< k$. We can choose the northeast element of any ribbon except the northeast ribbon and make it $< k$. Since there are $k-1$ elements total that are less than k, we have $\binom{\mathrm{rib}(\lambda/\mu)-1}{k-1-\mathrm{wt}(\lambda/\mu)}$ choices. The rest of the proof is easy and is left as an exercise for the reader. $\square$

Finally, recall the following result from (5). For standard Young tableaux T and S, we let $T * S$ be the tableau we get by placing T below and to the left of S.

Theorem 11 (Skew Littlewood-Richardson rule – SLRR) *Let λ, μ, σ, τ be partitions and fix a tableau S of shape σ. Then*

$$s_{\lambda/\mu}s_{\sigma/\tau} = \sum(-1)^{|R^-|}s_{\lambda^+/\mu^-},$$

where the sum is over triples (R^-, R^+, R) of standard Young tableaux of respective shapes $(\mu/\mu^-)^c$, λ^+/λ and τ such that $R^- * R^+ * R$ rectifies to S.

The lemmas indeed prove SQMNR as follows. By Lemma 8 and SLRR,

$$s_{\lambda/\mu}\cdot\widetilde{p}_r = \sum_{k=1}^{r}(-q)^{r-k}s_{\lambda/\mu}\cdot s_{k,1^{r-k}}. \qquad s_{\lambda/\mu}\cdot s_{k,1^{r-k}} = \sum_{R^-,R^+}(-1)^{|R^-|}s_{\lambda^+/\mu^-},$$

where the last sum is over $R^- \in \mathrm{SYT}((\mu/\mu^-)^c)$, $R^+ \in \mathrm{SYT}(\lambda^+/\lambda)$ such that $R^- * R^+$ rectifies to S, where S is the standard Young tableau of shape $(k,1^{r-k})$ with $1,2,\ldots,k$ in the first row, and $k+1,k+2,\ldots,r$ in rows $2,3,\ldots,r-k+1$. By Lemma 10, the sum on the right is over λ^+,μ^- such that λ^+/λ and μ/μ^- are broken ribbons, and for such λ^+,μ^-, the coefficient of s_{λ^+/μ^-} is $(-1)^{|\mu/\mu^-|}\binom{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}{k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-)}$. This means that the coefficient of s_{λ^+/μ^-} in $s_{\lambda/\mu}\cdot\widetilde{p}_r$ is

$$(-1)^{|\mu/\mu^-|}\sum_k(-q)^{r-k}\binom{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}{k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-)}.$$

Since $r = \mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)+\mathrm{wt}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)+\mathrm{ht}(\lambda^+/\lambda)+\mathrm{ht}(\mu/\mu^-)$, the sum equals

$$(-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}\sum_k(-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1-(k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-))}\binom{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}{k-1-\mathrm{wt}(\lambda^+/\lambda)-\mathrm{ht}(\mu/\mu^-)}$$

$$= (-q)^{\mathrm{ht}(\lambda^+/\lambda)+\mathrm{wt}(\mu/\mu^-)}(1-q)^{\mathrm{rib}(\lambda^+/\lambda)+\mathrm{rib}(\mu/\mu^-)-1}$$

by the binomial theorem. This is SQMNR'.

5 Final remarks

The motivation for this work was the open problem posed by Assaf and McNamara in (2): to find a combinatorial proof of the skew Murnaghan-Nakayama rule. Even though this paper provides a completely bijective proof of the skew *quantum* Murnaghan-Nakayama rule, which obviously specializes to the non-quantum rule, Assaf-McNamara's problem remains open. Indeed, plugging $q = 1$ into SQMNR", which is the identity we proved bijectively, gives 0 on both sides. To get SMNR, we have to divide SQMNR" by $1-q$ and then set $q=1$.

A more algebraic (and shorter) proof of SQMNR would run as follows. There is a short algebraic proof of SPR due to Lam in the appendix of (1). We have already mentioned that CSPR follows by applying the involution ω to SPR. It is easy to prove that $\sum_{\tau\vdash r}(1-q)^{\ell(\tau)}m_\tau = \sum_{k=0}^{r}(-q)^{r-k}s_k s_{1^{r-k}}$. So by SPR and CSPR, we have $s_{\lambda/\mu}\cdot\left(\sum_{\tau\vdash r}(1-q)^{\ell(\tau)}m_\tau\right) = \sum(-1)^{|\mu/\mu^-|}(-q)^{|\lambda^+/\lambda'|+|\mu'/\mu^-|}s_{\lambda^+/\mu^-}$, where the sum is over partitions $\lambda^+,\lambda',\mu^-,\mu'$ such that λ'/λ and μ'/μ^- are horizontal strips, λ^+/λ' and μ/μ' are vertical strips, $|\lambda^+/\lambda'|+|\mu'/\mu^-|=r-k$ and $|\lambda'/\lambda|+|\mu/\mu'|=k$. This implies SQMNR" and hence SQMNR via Lemma 7, which was proved independently of everything else in Section 3.

Lam-Lauve-Sotille's skew Littlewood-Richardson rule is very general, but the computation of actual coefficients in the expansion, i.e. counting all standard Young tableaux of a given shape that rectify to a given tableau, is complicated in practice. In light of Section 4, our work can be seen as one possible answer to the following question. For what special shapes of λ,μ,σ,τ can we actually compute the coefficients? SQMNR can be interpreted as saying that if $\tau=\emptyset$ and σ is a hook, the coefficients are certain binomial coefficients, while SPR says that the coefficient is ± 1 if $\tau=\emptyset$ and $\sigma=(r)$. It would be interesting to find other examples when the coefficients can be computed and yield elegant answers, both for Schur functions and for other Hopf algebras.

The quantum power sum functions $\widetilde{p}_r$ are special cases of the Hall-Littlewood polynomials. See (4, §5) for some conjectures involving Pieri rule-type products of (skew) Hall-Littlewood polynomials and Schur functions.

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Peter McNamara's excellent talk (2) at FPSAC'10 was the main motivation for this work. Sage (6) was of great help in guessing the right form for SQMNR.

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Closed paths whose steps are roots of unity

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Abstract. We give explicit formulas for the number $U_n(N)$ of closed polygonal paths of length N (starting from the origin) whose steps are n^{th} roots of unity, as well as asymptotic expressions for these numbers when $N \to \infty$. We also prove that the sequences $(U_n(N))_{N\geq 0}$ are P-recursive for each fixed $n \geq 1$ and leave open the problem of determining the values of N for which the *dual* sequences $(U_n(N))_{n\geq 1}$ are P-recursive.

Résumé. Nous donnons des formules explicites pour le nombre $U_n(N)$ de chemins polygonaux fermés de longueur N (débutant à l'origine) dont les pas sont des racines n-ièmes de l'unité, ainsi que des expressions asymptotiques pour ces nombres lorsque $N \to \infty$. Nous démontrons aussi que les suites $(U_n(N))_{N\geq 0}$ sont P-récursives pour chaque $n \geq 1$ fixé et laissons ouvert le problème de déterminer les valeurs de N pour lesquelles les suites *duales* $(U_n(N))_{n\geq 1}$ sont P-récursives.

Keywords: closed polygonal paths, roots of unity, P-recursive, asymptotics

1 Introduction

The subject of random walks is classical and appears in many areas of mathematics, physics and computer science (see, for example, $http://en.wikipedia.org/wiki/Random\_walks$). In this paper we combinatorially analyse a new type of closed random walks in the complex plane — a kind of restricted Brownian motion — whose steps are given by n^{th}-roots of unity. For $n \geq 1$, let $\Omega_n = \{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}$ be the set of all n-th roots of unity, where $\omega_n = \exp(2\pi i/n) \in \mathbb{C}$. A polygonal path of length N, starting at the origin in the complex plane, whose steps are n-th roots of unity can be encoded by the sequence $w = [\omega_n^{k_1}, \ldots, \omega_n^{k_N}]$ of its successive steps, $\omega_n^{k_j} \in \Omega_n, j = 1, \ldots, N$. For $\nu = 0, \ldots, n-1$, let m_ν be the number of times that ω_n^ν appears in w. We call the sequence $\vec{m} = [m_0, \ldots, m_{n-1}]$ the *type* of w, and write $\vec{m} = \text{type}(w)$. Of course, the path w is closed if and only if $\omega_n^{k_1} + \cdots + \omega_n^{k_N} = 0$ if and only if

$$m_0 + m_1\omega_n + m_2\omega_n^2 + \cdots + m_{n-1}\omega_n^{n-1} = 0. \tag{1.1}$$

We call a sequence $\vec{m} = [m_0, m_1, \ldots, m_{n-1}] \in \mathbb{N}^n$ *admissible* if (1.1) is satisfied. Figure 1 shows a closed pentagon made of 18-th roots of unity encoded by $[\omega_{18}^3, \omega_{18}^{11}, \omega_{18}^5, \omega_{18}^{12}, \omega_{18}^{17}]$ and a closed 11-gon made of 14-th roots of unity encoded by $[\omega_{14}^{12}, \omega_{14}, \omega_{14}^4, \omega_{14}^5, \omega_{14}^7, \omega_{14}^5\omega_{14}^{11}, \omega_{14}^{11}, \omega_{14}^9, \omega_{14}^3, \omega_{14}^{13}]$.

Clearly, the number of closed paths, of length N, with admissible type $\vec{m}$ is given by the multinomial coefficient $N!/m_0!m_1!\ldots m_{n-1}!$. This implies that the number $U_n(N)$ of closed polygonal paths of

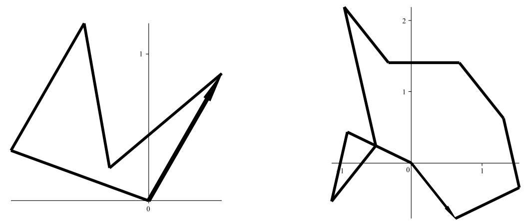

Fig. 1: Pentagon and 11-gon made of 18-th and 14-th roots of unity.

length N whose steps are n-th roots of unity is given by the formula

$$U_n(N) = \sum_{\substack{\vec{m}\,:\,\mathrm{admissible}\\ m_0+\cdots+m_{n-1}=N}} \frac{N!}{m_0!m_1!\cdots m_{n-1}!}. \tag{1.2}$$

In Section 2, we characterize admissibility and express the numbers $U_n(N)$ as *constant term* extractions in suitable rational expressions. We also give a formula from which the computation of the numbers $U_n(N)$ can be reduced to the computation of the numbers $U_q(N')$, where $N' \leq N$ and q is a suitable divisor of n. Section 3 is devoted to an analysis of recursive and asymptotic properties of the numbers $U_n(N)$. Finally, some tables are given.

2 Constant term and reduction formulas

To take advantage of formula (1.2) for $U_n(N)$ on a symbolic algebra system, we state first a simple characterization of admissibility for a sequence $\vec{m} \in \mathbb{N}^n$. This is done using the classical cyclotomic polynomials $\Phi_n(z) = \prod(z-\omega)$, where ω runs through the primitive n-th roots of unity. Equivalently, this means that $\omega = \exp(2k\pi i/n)$, where $1 \leq k \leq n$ and $\mathrm{GCD}(n,k) = 1$. Since $z^n - 1 = \prod_{d|n} \Phi_d(z)$, Moebius inversion implies that $\Phi_n(z) = \prod_{d|n}(x^d - 1)^{\mu(n/d)}$, where μ denotes the Moebius function. This shows that $\Phi_n(z)$ is a monic polynomial in $\mathbb{Z}[z]$ of degree $\varphi(n)$, the Euler function of n. The following very easy, but basic lemma characterizes admissibility.

Lemma 2.1 (criteria for admissibility). *For $n \geq 1$, the sequence $\vec{m} = [m_0, \ldots, m_{n-1}] \in \mathbb{N}^n$ is admissible if and only if the cyclotomic polynomial $\Phi_n(z)$ divides the polynomial*

$$P_{\vec{m}}(z) = m_0 + m_1 z + \cdots + m_{n-1}z^{n-1}.$$

Proof: Consider the euclidean division of $P_{\vec{m}}(z)$ by $\Phi_n(z)$ in the ring $\mathbb{Z}[z]$:

$$P_{\vec{m}}(z) = \Phi_n(z)Q_{\vec{m}}(z) + R_{\vec{m}}(z), \tag{2.1}$$

where $\deg R_{\vec{m}}(z) < \deg \Phi_n(z) = \varphi(n)$. Since $\Phi_n(\omega_n) = 0$ this shows that $\vec{m}$ is admissible if and only if $P_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(\omega_n) = 0$. But $R_{\vec{m}}(\omega_n) = 0$ if and only if $R_{\vec{m}}(z) = 0$ identically since $\Phi_n(z)$ is known to be the minimal polynomial of any of its roots and $\deg R_{\vec{m}} < \deg \Phi_n$. $\square$

Euclidean division shows that the coefficients of $R_{\vec{m}}(z)$ are $\mathbb{Z}$-linear combinations $l_k(m_0, \ldots, m_{n-1})$ of the m_i's. Hence, $\vec{m}$ is admissible if and only if $l_k(m_0, \ldots, m_{n-1}) = 0$ for $k = 0, \ldots, \varphi(n) - 1$. Table 1, made using the *rem* command in Maple gives the values of the l_k's for $n = 1, \ldots, 20$. For example, for $n = 6$, $\varphi(n) = 2$ and using Table 1, formula (1.2) takes the form

$$U_6(N) = \sum_{\substack{m_0+\cdots+m_5=N \\ m_0+m_5=m_2+m_3 \\ m_4+m_5=m_1+m_2}} \frac{N!}{m_0! \cdots m_5!}.$$

Note that, by the multinomial formula, this is equivalent to the following *constant term* formula

$$U_6(N) = \mathrm{CT}((t_1 + t_2 + \frac{t_1}{t_2} + \frac{t_2}{t_1} + t_1^{-1} + t_2^{-1})^N),$$

where $\mathrm{CT}(L(t_1, t_2, \ldots))$ denotes the constant term of the full expansion of L as a Laurent series in $t_1, t_2, \ldots$. This is generalized as follows.

Theorem 2.2 *There is a Laurent polynomial, $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$, such that $U_n(N) = \mathrm{CT}(\Lambda_n(t_1, \ldots, t_{\varphi(n)})^N)$. Moreover, $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$ is computed as follows. Let $m_0 + \cdots + m_{n-1}z^{n-1} = \Phi_n(z)Q(z) + R(z)$, where the remainder is $R(z) = \sum_{k=0}^{\varphi(n)-1} l_k(m_0, \ldots, m_{n-1})z^k$, with $l_k(m_0, \ldots, m_{n-1}) = \sum_{i=0}^{n-1} c_{k,i}m_i$, $c_{k,i} \in \mathbb{Z}$, $k = 0, \ldots, \varphi(n) - 1$. Then,*

$$\Lambda_n(t_1, \ldots, t_{\varphi(n)}) = \sum_{j=0}^{n-1} t_1^{c_{0,j}} t_2^{c_{1,j}} t_3^{c_{2,j}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}. \tag{2.2}$$

Proof: By the multinomial theorem,

$$\begin{aligned}
&\left(\sum_{j=0}^{n-1} t_1^{c_{0,j}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}\right)^N \\
&= \sum_{m_0+\cdots+m_{n-1}=N} \frac{N!}{m_0! \cdots m_{n-1}!} \left(t_1^{c_{0,0}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,0}}\right)^{m_0} \cdots \left(t_1^{c_{0,n-1}} \ldots t_{\varphi(n)}^{c_{\varphi(n)-1,n-1}}\right)^{m_{n-1}} \\
&= \sum_{m_0+\cdots+m_{n-1}=N} \frac{N!}{m_0! \cdots m_{n-1}!} t_1^{l_0(m_0,\ldots,m_{n-1})} \ldots t_{\varphi(n)}^{l_{\varphi(n)-1}(m_0,\ldots,m_{n-1})}.
\end{aligned}$$

The result follows since the constant term is given by taking the sum of the terms corresponding to the exponents $l_k = 0$ for $k = 0, \ldots, \varphi(n) - 1$. $\square$

Table 2 gives the rational functions $\Lambda_n(t_1, \ldots, t_{\varphi(n)})$ for $n = 1, \ldots, 20$. Let $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$ be the canonical decomposition of the integer n. By definition, the *radical* of n is the square-free integer $q = \mathrm{rad}(n) = p_1 \cdots p_s$ consisting of the product of the p_i's. The computation of the cyclotomic polynomial $\Phi_n(z)$ is greatly simplified by making use of the well-known reduction formula

$$\Phi_n(z) = \Phi_q(z^{n/q}), \quad q = \mathrm{rad}(n). \tag{2.3}$$

This implies that the computation of the exponential generating function of the sequence $(U_n(N))_{N\geq 0}$ is reduced to that of $(U_q(N))_{N\geq 0}$ as follows.

Proposition 2.3 (reduction formula for $U_n(N)$). *Let $n \geq 1$ and $q = \mathrm{rad}(n)$. Then,*

$$\sum_{N\geq 0} U_n(N)\frac{X^N}{N!} = \left(\sum_{N\geq 0} U_q(N)\frac{X^N}{N!}\right)^{n/q}. \tag{2.4}$$

Proof: Using the remainder function, we have by linearity,

$$R_{\vec{m}}(z) = \mathrm{rem}(P_{\vec{m}}(z), \Phi_n(z)) = \sum_{k=0}^{n-1} m_k \mathrm{rem}(z^k, \Phi_n(z)). \tag{2.5}$$

Now, for $0 \leq \nu \leq q-1$, consider the euclidean division

$$z^\nu = \Phi_q(z)Q_\nu(z) + \rho_\nu(z), \tag{2.6}$$

where $\rho_\nu(z) = \mathrm{rem}(z^\nu, \Phi_q(z))$. The substitution $z \to z^{n/q}$ in (2.6) followed by a multiplication by z^r gives, using (2.3), $z^{\nu n/q+r} = \Phi_q(z^{n/q})z^rQ_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q}) = \Phi_n(z)z^rQ_\nu(z^{n/q}) + z^r\rho_\nu(z^{n/q})$. Let $k = \nu n/q + r$, where $0 \leq r < n/q$. Then,

$$\deg z^r\rho_\nu(z^{n/q}) = r + \frac{n}{q}\deg\rho_\nu(z) \leq r + \frac{n}{q}(\varphi(q)-1) = r + \varphi(n) - \frac{n}{q} < \varphi(n).$$

This implies that $\mathrm{rem}(z^k, \Phi_n(z)) = z^r\rho_\nu(z^{n/q})$. Substituting this into (2.5) and collecting terms, we find that the $\varphi(n)$ conditions for admissibility, $[l_k(m_0, m_1, \ldots, m_{n-1}) = 0]_{0\leq k\leq\varphi(n)-1}$, split into n/q blocks of $\varphi(q)$ conditions, $[l_i(m_j, m_{\frac{n}{q}+j}, m_{2\frac{n}{q}+j}, \ldots, m_{(q-1)\frac{n}{q}+j}) = 0]_{0\leq i\leq\varphi(q)-1}, \quad 0 \leq j \leq \frac{n}{q}-1$, from which (2.4) follows. □

Table 3 gives the numerical values of $U_n(N)$ for $1 \leq n \leq 20$ and $0 \leq N \leq 20$.

3 Analysis of the sequences

Let us say that a path is *normalized* if its first step is the complex number 1 (i.e. the path starts *horizontally* along the positive real axis). Each normalized path $[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}]$ generates, by rotation, n distinct paths $\omega_n^k[1, \omega_n^{\nu_2}, \ldots, \omega_n^{\nu_N}] = [\omega_n^k, \omega_n^{k+\nu_2}, \ldots, \omega_n^{k+\nu_N}]$, $k = 0, 1, \ldots, n-1$. This implies that n divides $U_n(N)$ for every $n \geq 1$ and $N \geq 1$. As Tables 1 and 2 indicate, the structure of the sequence $(U_n(N))_{N\geq 0}$ heavily depend on the arithmetical nature of n. For example, let $n = p$ be a prime number. Then for such values of n, admissibility for a vector $\vec{m} \in \mathbb{N}^p$ means that $m_0 = m_1 = \cdots = m_{p-1}$ since, in this case, $\Phi_p(z) = 1 + z + \cdots + z^{p-1}$ and $R_{\vec{m}}(z) = (m_0 - m_{p-1}) + (m_1 - m_{p-1})z + \cdots + (m_{p-2} - m_{p-1})z^{p-2}$, (see Table 1, for example). Formula (1.2) then takes the form

$$U_p(N) = \frac{N!}{(\frac{N}{p})!^p} \quad \textit{if } p|N, \ 0 \textit{ otherwise}. \tag{3.1}$$

Note that when $p = 2$, (3.1) corresponds to the classical central binomial coefficients enumerating one-dimensional closed lattice paths of length N. When $p = 3$, (3.1) corresponds to the De Bruijn numbers (sequence A006480 in Sloane-Plouffe encyclopedia [Sloane(2010)]). For prime powers $n = p^\alpha$, we have by Proposition 2.3,

$$\sum_{N \geq 0} U_{p^\alpha}(N) \frac{X^N}{N!} = \left(\sum_{k \geq 0} \frac{X^{kp}}{k!^p} \right)^{p^{\alpha-1}} \tag{3.2}$$

since, in this case $q = p$. Note that when $n = 8 = 2^3$, then $U_8(N)$ is the number of 4-dimensional closed lattice paths in $\mathbb{Z}^4$ of length N starting at the origin (see sequence A039699 in Sloane). The reader can check that, more generally, $U_{2^\alpha}(N)$ is the number of closed lattice paths in $\mathbb{Z}^{2^{\alpha-1}}$ of length N starting at the origin. Interestingly enough, for any other dimension $d \neq 2^{\alpha-1}$, such a connection betweens lattice paths in $\mathbb{Z}^d$ and plane paths whose steps are roots of unity does not exist.

When n is not a prime power, the situation is more delicate. For example, if $n = 6$, then, using the Maple package GFUN [Salvy and Zimmermann(1994)], it can be seen that $(U_n(N))_{N \geq 0}$ satisfies the following linear recurrence with polynomial coefficients,

$$(N+3)^2 U_6(N+3) = (N+2)(N+3)U_6(N+2) + 24(N+2)^2 U_6(N+1) + 36(N+1)(N+2)U_6(N) \tag{3.3}$$

with initial conditions $U_6(0) = 1, U_6(1) = 0, U_6(2) = 6$. Such sequences are called polynomially recursive (P-recursive for short) and are characterized by the fact that their (ordinary or exponential) generating series are D-finite (i.e. satisfy a linear differential equation with polynomial coefficients). As a consequence, P-recursive sequences are closed under many operations including linear combinations, pointwise and Cauchy products [Stanley(1980)]. Moreover their asymptotic estimates, as $N \to \infty$, are well behaved. In our context, the general situation is summarized by Theorem 3.2. below. We need first the following technical lemma.

Lemma 3.1 *Let $\vec{t} = (t_1, \ldots, t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)}$. Then the Laurent polynomial Λ_n satisfies*

$$\max_{\substack{|t_\nu|=1 \\ 1 \leq \nu \leq \varphi(n)}} |\Lambda_n(\vec{t})| = n. \tag{3.4}$$

Moreover, if $n = p^\alpha$, a prime power, then the maximum value (3.4) *is attained precisely at the p distinct points $(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p})$, $\nu = 0, \ldots, p-1$ and we have $\Lambda_n(e^{2\pi i\nu/p}, \ldots, e^{2\pi i\nu/p}) = ne^{2\pi i\nu/p}$. If n is not a prime power, then the maximum value* (3.4) *is attained only at the point $(1, \ldots, 1)$ and we have $\Lambda_n(1, \ldots, 1) = n$.*

Proof: By Theorem 2.2, Λ_n can be written as a sum of n terms,

$$\Lambda_n(\vec{t}) = t_1 + \cdots + t_{\varphi(n)} + \Gamma_n(\vec{t}), \tag{3.5}$$

where Γ_n is a sum of $n - \varphi(n)$ unitary Laurent monomials in $t_1, \ldots, t_{\varphi(n)}$. Each of the n terms in Λ_n has modulus 1 when $|t_\nu| = 1$, $\nu = 1, \ldots, \varphi(n)$. Hence (3.4) follows from the triangular inequality and the fact that $\Lambda_n(1, \ldots, 1) = n$. Note that the maximum value in (3.4) is attained only at points $\vec{t^*} = (t_1^*, \ldots, t_{\varphi(n)}^*)$ for which the n monomials take a common value, $e^{i\theta^*}$, say. In particular, from (3.5), we must have $t_1^* = t_2^* = \cdots = t_{\varphi(n)}^* = e^{i\theta^*}$. We consider two cases:

(i) if $n = p^\alpha$, then it can be checked that each term in Γ_n has total degree $-(p-1)$. This implies that $e^{i\theta^*} = e^{-i(p-1)\theta^*}$. That is, $e^{i\theta^*}$ is a p-th root of unity: $e^{2\pi i\nu/p}$, $\nu = 0, \ldots, p-1$;

(ii) if $n \neq p^\alpha$, the situation is more delicate. If we can show that at least one of the terms in Γ_n has total degree 0, then the maximal value in (3.4) will be attained only at the point $(1, \ldots, 1)$, since this would imply that $e^{i\theta^*} = (e^{i\theta^*})^0 = 1$. The existence of such a 0-degree term is proved as follows. By (2.2), the general term $t_1^{c_{0,j}} t_2^{c_{1,j}} \cdots t_{\varphi(n)}^{c_{\varphi(n)-1,j}}$ has total degree $\sum_{k=0}^{\varphi(n)-1} c_{k,j}$. When $j = \varphi(n)$, this total degree is 0. To see this, note that $\sum_{k=0}^{\varphi(n)-1} c_{k,j} z^k = \mathrm{rem}(z^j, \Phi_n(z))$. Taking $j = \varphi(n)$, $z = 1$, this gives $\sum_{k=0}^{\varphi(n)-1} c_{k,\varphi(n)} = \mathrm{rem}(z^{\varphi(n)}, \Phi_n(z))|_{z=1} = (z^{\varphi(n)} - \Phi_n(z))|_{z=1} = 0$, since $\Phi_n(1) = 1$ when $n \neq p^\alpha$.

□

Theorem 3.2 *For any $n > 1$, we have an asymptotic estimate of the form*

$$U_n(N) \sim a_n \frac{n^N}{N^{\frac{1}{2}\varphi(n)}} \left(1 + \frac{b_{1,n}}{N} + \frac{b_{2,n}}{N^2} + \ldots\right), \quad \textit{as } N \to \infty, \tag{3.6}$$

where a_n, $b_{j,n}$ are independent of N. When $n = p^\alpha$ is a prime power, then N must be a multiple of p as it goes to infinity in (3.6)*. More explicitly, the leading coefficient a_n is given by*

$$a_n = \begin{cases} (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \textit{if } n \textit{ is not a prime power,} \\ p \cdot (n/2\pi)^{\frac{1}{2}\varphi(n)} / \sqrt{\prod_{p|n} p^{\varphi(n)/(p-1)}} & \textit{if } n = p^\alpha \textit{ is a prime power.} \end{cases}$$

For each $n \geq 1$, the sequence $(U_n(N))_{N \geq 0}$ is P-recursive but is not algebraic when $n > 2$.

Proof: In order to establish the asymptotic estimate (3.6), first note that the constant term extraction $U_n(N) = \mathrm{CT}(\Lambda_n(t_1, \ldots, t_{\varphi(n)})^N)$ can be expressed as the multiple integral

$$U_n(N) = \frac{1}{(2\pi)^{\varphi(n)}} \int \cdots \int_{(-\pi,\pi]^{\varphi(n)}} \Lambda_n(e^{iu_1}, \ldots, e^{iu_{\varphi(n)}})^N du_1 \cdots du_{\varphi(n)} \tag{3.7}$$

which is the average value of Λ_n^N over the $\varphi(n)$-dimensional torus $\{(t_1, \ldots, t_{\varphi(n)}) \in \mathbb{C}^{\varphi(n)} \,|\, |t_\nu| = 1, \nu = 1, \ldots, \varphi(n)\}$. Now by Theorem 2.2,

$$L_n(\vec{u}) := \Lambda_n(e^{iu_1}, \ldots, e^{iu_{\varphi(n)}}) = \sum_{j=0}^{n-1} e^{i\lambda_j(\vec{u})}, \tag{3.8}$$

where $\lambda_j(\vec{u}) = \sum_{k=0}^{\varphi(n)-1} c_{k,j} u_{k+1}$ is a real-valued linear combination of $u_1, \ldots, u_k$, $0 \leq j \leq \varphi(n) - 1$. By the triangular inequality, $|L_n(\vec{u})| \leq n$ for every $\vec{u} \in (-\pi, \pi]^{\varphi(n)}$. To obtain the asymptotic estimate of (3.6) it suffices to approximate (3.7) by a gaussian distribution around each point $\vec{u^*} = (u_1^*, \ldots, u_{\varphi(n)}^*) \in (-\pi, \pi]^{\varphi(n)}$ for which the maximum value $|L_n(\vec{u^*})| = |ne^{i\theta^*}| = n$ is attained. This is Laplace's method [De Bruijn(1981)]. By Lemma 3.1,

(i) if $n \neq p^\alpha$, then $\vec{u^*} = \vec{0}$ is the only point in $(-\pi, \pi]^{\varphi(n)}$ for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 0$;

(ii) if $n = p^\alpha$, then there are exactly p possible values of u^* for which $|L_n(\vec{u^*})| = n$. In fact $\theta^* = 2\nu\pi/p \mod 2\pi \in (-\pi, \pi], \nu = 0, \ldots, p-1$.

We conclude by estimating (3.7) by a sum of moments of gaussian distributions in the following way:

$$U_n(N) \sim \frac{n^N}{(2\pi)^{\varphi(n)}} \sum_{L_n(\vec{u^*})=ne^{i\theta^*}} e^{iN\theta^*} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} e^{-\frac{N}{2n}Q^*(\vec{u}-\vec{u^*})} H^*(\vec{u}-\vec{u^*})^N du_1 \dots du_{\varphi(n)},$$

where, for each $\vec{u^*}$ such that $L_n(\vec{u^*}) = ne^{i\theta^*}$,

$$\frac{1}{n}L_n(\vec{u}) = e^{i\theta^*}\left(1 - \frac{1}{2n}Q^*(\vec{u}-\vec{u^*}) + O(\| \vec{u}-\vec{u^*} \|^3)\right) = e^{i\theta^*}e^{-\frac{1}{2n}Q^*(\vec{u}-\vec{u^*})}H^*(\vec{u}-\vec{u^*}), \quad (3.9)$$

where $Q^*(\vec{v})$ is the positive definite quadratic form associated to the symmetric $\varphi(n) \times \varphi(n)$ matrix $K = CC^T$ in which $C = [c_{k,j}]_{0\le k\le \varphi(n)-1, 0\le j\le n-1}$, where the $c_{k,j}$'s are defined by (2.2) and $H^*(\vec{v}) = 1 + O(\| \vec{v} \|^3)$. It turns out that $\det(K) = \prod_{p|n} p^{\varphi(n)/(p-1)}$, which is a consequence of the known fact that the absolute value of the discriminant of $\Phi_n(z)$ is equal to $n^{\varphi(n)} \prod_{p|n} p^{\varphi(n)/(p-1)}$, for $n > 2$.

The P-recursivity of $(U_n(N))_{N\ge 0}$ is established as follows. Fix $n \ge 1$ and let $k = \varphi(n)$. We shall show that the series

$$\sum_{N\ge 0} U_n(N)X^N = \mathrm{CT}_{t_1,\dots,t_k} \frac{1}{1 - X\Lambda_n(t_1,\dots,t_k)} \quad (3.10)$$

is D-finite in X where $\mathrm{CT}_{t_1,\dots,t_k}$ means constant term extraction relative to the variables $t_1,\dots,t_k$. First, fix integers $m_1 > 0,\dots,m_k > 0$ in such a way that $t_1^{m_1}\dots t_k^{m_k}\Lambda_n(t_1,\dots,t_k)$ is a polynomial in $t_1,\dots,t_k$. The rational function

$$f(t_1,\dots,t_k,X) = \frac{1}{1 - t_1^{m_1}\dots t_k^{m_k}X\Lambda_n(t_1,\dots,t_k)} = \sum_{n_1,\dots,n_k,N\ge 0} a(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N \quad (3.11)$$

is obviously D-finite in the variables $t_1,\dots,t_k,X$. By Theorem 2.2, the numbers $U_n(N)$ can be expressed as the following coefficient extraction in $f(t_1,\dots,t_k,X)$:

$$U_n(N) = [t_1^{m_1N}\dots t_k^{m_kN}X^N]f(t_1,\dots,t_k,X).$$

Hence, by (3.10),

$$\sum_{N\ge 0} U_n(N)X^N = \sum_{N\ge 0} a(m_1N,\dots,m_kN,N)X^N. \quad (3.12)$$

Consider now the algebraic, hence D-finite, series

$$g(t_1,\dots,t_k,X) = \sum_{n_1,\dots,n_k,N\ge 0} b(n_1,\dots,n_k,N)t_1^{n_1}\dots t_k^{n_k}X^N,$$

where $b(n_1,\dots,n_k,N) = a(m_1n_1,\dots,m_kn_k,N)$. Formula (3.12) shows that

$$\sum_{N\ge 0} U_n(N)X^N = \sum_{N\ge 0} b(N,\dots,N,N)X^N$$

which is a (full) diagonal of $g(t_1,\dots,t_k,X)$. We conclude using the fact that any diagonal of a D-finite series is also D-finite, a result due to Lipshitz [Lipshitz(1988)]. The non algebraicity of $(U_n(N))_{N\ge 0}$, for each $n > 2$, follows from the fact that $\varphi(n)$ is even and the dominant term of the asymptotic formula contains $N^{-\text{positive integer}}$. This is incoherent with Puiseux expansion around an algebraic singularity. □

A better control of the coefficients $b_{j,n}$ can be achieved by a smooth local change of variables, $\vec{u} = \vec{u^*} + \vec{g}(\vec{w})$, $\vec{g}(\vec{0}) = \vec{0}$ in (3.9) such that $\frac{1}{n} L_n(\vec{u}) = e^{i\theta^*} e^{-\frac{1}{2n} Q^*(\vec{w})}$. This is always possible by Morse Lemma [Morse(1925)]. The first terms of the asymptotic estimates of Theorem 3.2 are given in Table 4 for $n = 1, \ldots, 20$.

Corollary 3.3 *If n is not a prime power, then $\exists N_0 = N_0(n)$ such that $U_n(N) > 0$ for $N \geq N_0$.* □

The sequences $(U_n(N))_{N\geq 0}$, $n = 1, 2, \ldots$, can be considered in a *dual* way: for each fixed N, one can consider the sequence $(U_n(N))_{n\geq 1}$ by reading each column of Table 3. The first five of these dual sequences, $(U_n(0))_{n\geq 1}, (U_n(1))_{n\geq 1}, \ldots, (U_n(4))_{n\geq 1}$, are P-recursive. The fifth one, $(U_n(4))_{n\geq 1}$, can be described as follows: $U_n(4) = 3n(n-1)\chi(2|n)$, where $\chi(T(n)) = 1$ if $T(n)$ is true and 0 otherwise. This can be checked by noting that closed paths of length 4 whose steps are n^{th} roots of unity are (possibly degenerated and non-convex) rhombuses. Following extensive computations we conjecture that $(U_n(5))_{n\geq 1}$ is also P-recursive and is of the form $U_n(5) = 24n\chi(5|n) + 20n(n-3)\chi(6|n)$. We leave open the problem of determining the values of N for which $(U_n(N))_{n\geq 1}$ is P-recursive.

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n	*Linear combinations for admissibility*
1	$[m_0]$
2	$[m_0 - m_1]$
3	$[m_0 - m_2, m_1 - m_2]$
4	$[m_0 - m_2, m_1 - m_3]$
5	$[m_0 - m_4, m_1 - m_4, m_2 - m_4, m_3 - m_4]$
6	$[m_0 - m_2 - m_3 + m_5, m_1 + m_2 - m_4 - m_5]$
7	$[m_0 - m_6, m_1 - m_6, m_2 - m_6, m_3 - m_6, m_4 - m_6, m_5 - m_6]$
8	$[m_0 - m_4, m_1 - m_5, m_2 - m_6, m_3 - m_7]$
9	$[m_0 - m_6, m_1 - m_7, m_2 - m_8, m_3 - m_6, m_4 - m_7, m_5 - m_8]$
10	$[m_0 - m_4 - m_5 + m_9, m_1 + m_4 - m_6 - m_9, m_2 - m_4 - m_7 + m_9, m_3 + m_4 - m_8 - m_9]$
11	$[m_0 - m_{10}, m_1 - m_{10}, m_2 - m_{10}, m_3 - m_{10}, m_4 - m_{10}, m_5 - m_{10}, m_6 - m_{10}, m_7 - m_{10},$ $m_8 - m_{10}, m_9 - m_{10}]$
12	$[m_0 - m_4 - m_6 + m_{10}, m_1 - m_5 - m_7 + m_{11}, m_2 + m_4 - m_8 - m_{10}, m_3 + m_5 - m_9 - m_{11}]$
13	$[m_0 - m_{12}, m_1 - m_{12}, m_2 - m_{12}, m_3 - m_{12}, m_4 - m_{12}, m_5 - m_{12}, m_6 - m_{12}, m_7 - m_{12},$ $m_8 - m_{12}, m_9 - m_{12}, m_{10} - m_{12}, m_{11} - m_{12}]$
14	$[m_0 - m_6 - m_7 + m_{13}, m_1 + m_6 - m_8 - m_{13}, m_2 - m_6 - m_9 + m_{13}, m_3 + m_6 - m_{10} - m_{13},$ $m_4 - m_6 - m_{11} + m_{13}, m_5 + m_6 - m_{12} - m_{13}]$
15	$[m_0 - m_8 - m_9 - m_{10} + m_{13} + m_{14}, m_1 + m_8 - m_{11} - m_{13}, m_2 + m_9 - m_{12} - m_{14},$ $m_3 - m_8 - m_9 + m_{14}, m_4 + m_8 - m_{13} - m_{14}, m_5 - m_8 - m_{10} + m_{13},$ $m_6 - m_9 - m_{11} + m_{14}, m_7 + m_8 + m_9 - m_{12} - m_{13} - m_{14}]$
16	$[m_0 - m_8, m_1 - m_9, m_2 - m_{10}, m_3 - m_{11}, m_4 - m_{12}, m_5 - m_{13}, m_6 - m_{14}, m_7 - m_{15}]$
17	$[m_0 - m_{16}, m_1 - m_{16}, m_2 - m_{16}, m_3 - m_{16}, m_4 - m_{16}, m_5 - m_{16}, m_6 - m_{16}, m_7 - m_{16},$ $m_8 - m_{16}, m_9 - m_{16}, m_{10} - m_{16}, m_{11} - m_{16}, m_{12} - m_{16}, m_{13} - m_{16}, m_{14} - m_{16}, m_{15} - m_{16}]$
18	$[m_0 - m_6 - m_9 + m_{15}, m_1 - m_7 - m_{10} + m_{16}, m_2 - m_8 - m_{11} + m_{17}, m_3 + m_6 - m_{12} - m_{15},$ $m_4 + m_7 - m_{13} - m_{16}, m_5 + m_8 - m_{14} - m_{17}]$
19	$[m_0 - m_{18}, m_1 - m_{18}, m_2 - m_{18}, m_3 - m_{18}, m_4 - m_{18}, m_5 - m_{18}, m_6 - m_{18}, m_7 - m_{18}, m_8 - m_{18},$ $m_9 - m_{18}, m_{10} - m_{18}, m_{11} - m_{18}, m_{12} - m_{18}, m_{13} - m_{18}, m_{14} - m_{18}, m_{15} - m_{18}, m_{16} - m_{18},$ $m_{17} - m_{18}]$
20	$[m_0 - m_8 - m_{10} + m_{18}, m_1 - m_9 - m_{11} + m_{19}, m_2 + m_8 - m_{12} - m_{18}, m_3 + m_9 - m_{13} - m_{19},$ $m_4 - m_8 - m_{14} + m_{18}, m_5 - m_9 - m_{15} + m_{19}, m_6 + m_8 - m_{16} - m_{18}, m_7 + m_9 - m_{17} - m_{19}]$

Tab. 1: The linear combinations $(l_k(\vec{m}))_{0 \le k \le \varphi(n)-1}$ for admissibility, $n = 1, \ldots, 20$.

n	$\Lambda_n(t_1,\ldots,t_{\varphi(n)})$
1	t_1
2	$\left(t_1+t_1{}^{-1}\right)$
3	$\left(t_1+t_2+\frac{1}{t_1t_2}\right)$
4	$\left(t_1+t_2+t_1{}^{-1}+t_2{}^{-1}\right)$
5	$\left(t_1+t_2+t_3+t_4+\frac{1}{t_1t_2t_3t_4}\right)$
6	$\left(t_1+t_2+\frac{t_1}{t_2}+\frac{t_2}{t_1}+t_1{}^{-1}+t_2{}^{-1}\right)$
7	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_2t_3t_4t_5t_6}\right)$
8	$\left(t_1+t_2+t_3+t_4+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}\right)$
9	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{1}{t_1t_4}+\frac{1}{t_2t_5}+\frac{1}{t_3t_6}\right)$
10	$\left(t_1+t_2+t_3+t_4+\frac{t_1t_3}{t_2t_4}+\frac{t_2t_4}{t_1t_3}+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}\right)$
11	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}}\right)$
12	$\left(t_1+t_2+t_3+t_4+\frac{t_1}{t_3}+\frac{t_3}{t_1}+\frac{t_2}{t_4}+\frac{t_4}{t_2}+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}\right)$
13	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_9+t_{10}+t_{11}+t_{12}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}}\right)$
14	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1t_3t_5}{t_2t_4t_6}+\frac{t_2t_4t_6}{t_1t_3t_5}+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}+t_5{}^{-1}+t_6{}^{-1}\right)$
15	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+\frac{t_1t_4t_7}{t_3t_5t_8}+\frac{t_2t_5t_8}{t_1t_4t_6}+\frac{t_1t_6}{t_2t_5t_8}+\frac{t_3t_8}{t_1t_4t_7}+\frac{1}{t_1t_6}+\frac{1}{t_2t_7}+\frac{1}{t_3t_8}\right)$
16	$\left(t_1+t_2+t_3+t_4+t_5+t_6+t_7+t_8+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}+t_5{}^{-1}+t_6{}^{-1}+t_7{}^{-1}+t_8{}^{-1}\right)$
17	$\left(t_1+t_2+\cdots+t_{16}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}}\right)$
18	$\left(t_1+t_2+t_3+t_4+t_5+t_6+\frac{t_1}{t_4}+\frac{t_4}{t_1}+\frac{t_2}{t_5}+\frac{t_5}{t_2}+\frac{t_3}{t_6}+\frac{t_6}{t_3}+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}+t_5{}^{-1}+t_6{}^{-1}\right)$
19	$\left(t_1+t_2+\cdots+t_{18}+\frac{1}{t_1t_2t_3t_4t_5t_6t_7t_8t_9t_{10}t_{11}t_{12}t_{13}t_{14}t_{15}t_{16}t_{17}t_{18}}\right)$
20	$\left(t_1+t_2+\cdots+t_8+\frac{t_1t_5}{t_3t_7}+\frac{t_3t_7}{t_1t_5}+\frac{t_2t_6}{t_4t_8}+\frac{t_4t_8}{t_2t_6}+t_1{}^{-1}+t_2{}^{-1}+t_3{}^{-1}+t_4{}^{-1}+t_5{}^{-1}+t_6{}^{-1}+t_7{}^{-1}+t_8{}^{-1}\right)$

Tab. 2: The Laurent polynomials Λ_n for $n=1,\cdots,20$.

n	$U_n(N),\ N = 0, \ldots, 20$
1	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0$
2	$1, 0, 2, 0, 6, 0, 20, 0, 70, 0, 252, 0, 924, 0, 3432, 0, 12870, 0, 48620, 0, 184756$
3	$1, 0, 0, 6, 0, 0, 90, 0, 0, 1680, 0, 0, 34650, 0, 0, 756756, 0, 0, 17153136, 0, 0$
4	$1, 0, 4, 0, 36, 0, 400, 0, 4900, 0, 63504, 0, 853776, 0, 11778624, 0, 165636900, 0, 2363904400, 0, 34134779536$
5	$1, 0, 0, 0, 0, 120, 0, 0, 0, 0, 113400, 0, 0, 0, 0, 168168000, 0, 0, 0, 0, 305540235000$
6	$1, 0, 6, 12, 90, 360, 2040, 10080, 54810, 290640, 1588356, 8676360, 47977776, 266378112, 1488801600,$ $8355739392, 47104393050, 266482019232, 1512589408044, 8610448069080, 49144928795820$
7	$1, 0, 0, 0, 0, 0, 0, 5040, 0, 0, 0, 0, 0, 0, 681080400, 0, 0, 0, 0, 0, 0$
8	$1, 0, 8, 0, 168, 0, 5120, 0, 190120, 0, 7939008, 0, 357713664, 0, 16993726464, 0, 839358285480, 0,$ $42714450658880, 0, 2225741588095168$
9	$1, 0, 0, 18, 0, 0, 2430, 0, 0, 640080, 0, 0, 215488350, 0, 0, 84569753268, 0, 0, 36905812607664, 0, 0$
10	$1, 0, 10, 0, 270, 240, 10900, 25200, 551950, 2116800, 32458860, 169092000, 2120787900, 13427013600$ $149506414200, 1075081207200, 11143223412750, 87198375264000, 865743970019500, 7171730187336000,$ 69416724049550020
11	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 39916800, 0, 0, 0, 0, 0, 0, 0, 0, 0$
12	$1, 0, 12, 24, 396, 2160, 23160, 186480, 1845900, 17213280, 171575712, 1703560320, 17365421304,$ $178323713568, 1856554560432, 19487791106784, 206411964321420, 2201711191213248, 23642813637773616$ $255355132936441824, 2772650461148938656$
13	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 6227020800, 0, 0, 0, 0, 0, 0, 0$
14	$1, 0, 14, 0, 546, 0, 32900, 10080, 2570050, 2540160, 238935564, 465696000, 25142196156, 76886409600,$ $2900343069624, 12211317518400, 359067702643650, 1915829643087360, 47006105030584700,$ $300455419743198720, 6437718469449262996$
15	$1, 0, 0, 30, 0, 360, 7650, 0, 302400, 4544400, 11226600, 324324000, 4310633250, 24324300000, 437404968000,$ $5634178329780, 45972927000000, 697866761592000, 8962716395833200, 88725951057744000,$ 1258898645656852200
16	$1, 0, 16, 0, 720, 0, 50560, 0, 4649680, 0, 514031616, 0, 64941883776, 0, 9071319628800, 0, 1369263687414480,$ $0, 219705672931613440, 0, 37024402443528248320$
17	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 355687428096000, 0, 0, 0$
18	$1, 0, 18, 36, 918, 5400, 82800, 801360, 10907190, 132053040, 1802041668, 24199809480, 340640607384,$ $4834708246368, 70229958125184, 1032223723667136, 15391538570569590, 231935110984687968,$ $3531542904056225916, 54244559313713885688, 839979883121036697468$
19	$1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 121645100408832000, 0$
20	$1, 0, 20, 0, 1140, 480, 102800, 151200, 12310900, 38707200, 1812247920, 9574488000, 313983978000,$ $2391608419200, 62051403928800, 611744666332800, 13627749414064500, 160896284989440000,$ $3253345101771050000, 43527416858084016000, 829176006298475046640$

Tab. 3: The sequences $(U_n(N))_{0 \le N \le 20}$ for $n = 1, \ldots, 20$.

n	*Asymptotic estimate of* $U_n(N)$ *as* $N \to \infty$	*Extra condition*
1	0	*NIL*
2	$\frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{2^N}{\sqrt{N}} \left(1 - \frac{1}{4N} + \frac{1}{32N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
3	$\frac{3\sqrt{3}}{2\pi} \cdot \frac{3^N}{N} \left(1 - \frac{2}{3N} + \frac{2}{9N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
4	$\frac{2}{\pi} \cdot \frac{4^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{8N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
5	$\frac{25\sqrt{5}}{4\pi^2} \cdot \frac{5^N}{N^2} \left(1 - \frac{2}{N} + \frac{2}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 5$
6	$\frac{\sqrt{3}}{2\pi} \cdot \frac{6^N}{N} \left(1 - \frac{1}{2N} + \frac{1}{12N^2} + O(\frac{1}{N^3})\right)$	*NIL*
7	$\frac{343\sqrt{7}}{8\pi^3} \cdot \frac{7^N}{N^3} \left(1 - \frac{4}{N} + \frac{8}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 7$
8	$\frac{8}{\pi^2} \cdot \frac{8^N}{N^2} \left(1 - \frac{1}{N} + \frac{1}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
9	$\frac{243\sqrt{3}}{8\pi^3} \cdot \frac{9^N}{N^3} \left(1 - \frac{3}{N} + \frac{4}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 3$
10	$\frac{5\sqrt{5}}{4\pi^2} \cdot \frac{10^N}{N^2} \left(1 - \frac{1}{N} + \frac{3}{4N^2} + O(\frac{1}{N^3})\right)$	*NIL*
11	$\frac{161051\sqrt{11}}{32\pi^5} \cdot \frac{11^N}{N^5} \left(1 - \frac{10}{N} + \frac{50}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{11}$
12	$\frac{3}{\pi^2} \cdot \frac{12^N}{N^2} \left(1 - \frac{1}{N} + \frac{2}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
13	$\frac{4826809\sqrt{13}}{64\pi^6} \cdot \frac{13^N}{N^6} \left(1 - \frac{14}{N} + \frac{98}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{13}$
14	$\frac{49\sqrt{7}}{8\pi^3} \cdot \frac{14^N}{N^3} \left(1 - \frac{3}{2N} + \frac{3}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*
15	$\frac{1125}{16\pi^4} \cdot \frac{15^N}{N^4} \left(1 - \frac{4}{N} + \frac{25}{3N^2} + O(\frac{1}{N^3})\right)$	*NIL*
16	$\frac{512}{\pi^4} \cdot \frac{16^N}{N^4} \left(1 - \frac{2}{N} + \frac{9}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod 2$
17	$\frac{6975757441\sqrt{17}}{256\pi^8} \cdot \frac{17^N}{N^8} \left(1 - \frac{24}{N} + \frac{288}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{17}$
18	$\frac{81\sqrt{3}}{8\pi^3} \cdot \frac{18^N}{N^3} \left(1 - \frac{3}{2N} + \frac{5}{2N^2} + O(\frac{1}{N^3})\right)$	*NIL*
19	$\frac{322687697779\sqrt{19}}{512\pi^9} \cdot \frac{19^N}{N^9} \left(1 - \frac{30}{N} + \frac{450}{N^2} + O(\frac{1}{N^3})\right)$	$N \equiv 0 \pmod{19}$
20	$\frac{125}{\pi^4} \cdot \frac{20^N}{N^4} \left(1 - \frac{2}{N} + \frac{7}{N^2} + O(\frac{1}{N^3})\right)$	*NIL*

Tab. 4: Asymptotic estimates of $U_n(N)$ as $N \to \infty$, for $n = 1, \ldots, 20$.

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Minkowski decompositions of associahedra

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Abstract. Realisations of associahedra can be obtained from the classical permutahedron by removing some of its facets and the set of facets is determined by the diagonals of certain labeled convex planar n-gons as shown by Hohlweg and Lange (2007). Ardila, Benedetti, and Doker (2010) expressed polytopes of this type as Minkowski sums and differences of scaled faces of a standard simplex and computed the corresponding coefficients y_I by Möbius inversion from the z_I if tight right-hand sides z_I for all inequalities of the permutahedron are assumed. Given an associahedron of Hohlweg and Lange, we first characterise all tight values z_I in terms of non-crossing diagonals of the associated labeled n-gon, simplify the formula of Ardila et al., and characterise the remaining terms combinatorially.

Résumé. Dans un article paru en 2007, Hohlweg et Lange décrivent des associaèdres réalisés à partir du permutoédre en enlevant certaines de ses facettes. Ces facettes sont déterminées par les diagonales d'une famille de n-gones étiquetés. En 2010, Ardila, Benedetti et Doker ont montré que ces polytopes s'expriment par des sommes et différences de Minkowski de faces pondérées d'un simplexe. De plus, si les coefficients z_I des inégalités décrivant l'associaèdre à partir du permutoédre sont optimaux, alors les coefficients y_I correspondants sont calculés par une inversion de Möbius. Étant donné un tel associaèdre, nous décrivons d'abord les valeurs optimales z_I en termes de diagonales non croisées d'un certain n-gone étiqueté, ensuite nous simplifions la formule de Ardila et al. pour finalement décrire combinatoirement les termes restants.

Keywords: reflection and Coxeter groups, lattice polytopes, associahedra, Minkowski sums

1 Generalised permutahedra and associahedra

Generalised permutahedra and Minkowski decompositions. A generalised permutahedron according to A. Postnikov is a convex $(n-1)$-polytope that has the following description by inequalities

$$P_n(\{z_I\}) := \left\{ \boldsymbol{x} \in \mathbb{R}^n \;\middle|\; \textstyle\sum_{i\in[n]} x_i = z_{[n]} \text{ and } \sum_{i\in I} x_i \geq z_I \text{ for } \varnothing \subset I \subset [n] \right\},$$

where $[n]$ denotes the set $\{1, 2, \cdots, n\}$, [Pos09]. The classical permutahedron, as described for example by G. M. Ziegler, [Zie98], corresponds to the choice $z_I = \frac{|I|(|I|+1)}{2}$ for $\varnothing \subset I \subseteq [n]$. Obviously, some of the above inequalities may be redundant for $P_n(\{z_I\})$ and sufficiently small increases and decreases of z_I of redundant inequalities do not change the set $P_n(\{z_I\})$ unless the inequality $\sum_{i\in I} x_i \geq z_I$ is tight. As described next, all generalised permutahedra have a Minkowski decomposition into faces of a simplex and the coefficients y_I of this decomposition can be computed if *all* (tight) values z_I are known.

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For any two polytopes P and Q, the Minkowski sum $P+Q$ is defined as $\{p+q \mid p \in P,\ q \in Q\}$. In contrast, we define the Minkowski difference $P-Q$ of P and Q only if there is a polytope R such that $P = Q+R$. We are interested in decompositions of generalised permutahedra into Minkowski sums and differences of faces of the $(n-1)$-dimensional standard simplex

$$\Delta_n = \operatorname{conv}\{e_1, e_2, \cdots, e_n\},$$

where e_i is a standard basis vector of $\mathbb{R}^n$. The faces Δ_I of Δ_n are given by $\operatorname{conv}\{e_i\}_{i\in I}$ for $I \subseteq [n]$. If a polytope P is the Minkowski sum and difference of scaled faces of Δ_n, we say that P has a Minkowski decomposition into faces of the standard simplex. The following two results are key observations.

Lemma 1.1 ([ABD10, Lemma 2.1]) $P_n(\{z_I\}) + P_n(\{z'_I\}) = P_n(\{z_I + z'_I\})$.

If we consider the function $I \longmapsto z_I$ that assigns every subset of $[n]$ the corresponding tight value z_I of $P_n(\{z_I\})$, then the Möbius inverse of this function assigns to I the coefficient y_I of a Minkowski decomposition of $P_n(\{z_I\})$ into faces of the standard simplex:

Proposition 1.2 ([ABD10, Proposition 2.3])
Every generalised permutahedron $P_n(\{z_I\})$ can be written uniquely as a Minkowski sum and difference of faces of Δ_n:

$$P_n(\{z_I\}) = \sum_{I\subseteq[n]} y_I \Delta_I$$

where $y_I = \sum_{J\subseteq I}(-1)^{|I\setminus J|} z_J$ for each $I \subseteq [n]$.

Studying properties of generalised permutahedra, Postnikov proved a weaker version of Proposition 1.2 that requires $y_I \geq 0$ for all $I \subseteq [n]$, [Pos09]. Although we can compute the values y_I theoretically if all tight values z_I are known, the formula of Proposition 1.2 is computationally expensive.

We could stop here and be fascinated how the Möbius inversion relates the description by half spaces and Minkowski decompositions. Nevertheless, we go further and study this relationship for certain realisations of associahedra, a certain subclass of generalised permutahedra described in the following subsection. It turns out that the formula of Ardila, Benedetti & Doker can be simplified significantly and that there is a combinatorial interpretation of the terms that remain after this simplification. The simplified formula extracts the combinatorial core data for the Möbius inversion: which subsets J of I are essential to compute y_I. In general, the sets J obtained for the same I but different realisations of an n-dimensional associahedron are different. Nevertheless, the sets J have a simple combinatorial characterisation as shown in Section 3.

Associahedra as generalised permutahedra. Associahedra form a class of combinatorially equivalent simple polytopes and can be realised as generalised permutahedra. They are often defined by specifying their 1-skeleton or graph. A theorem of G. Kalai, [Kal88], implies that the face lattice of an $(n-1)$-dimensional associahedron As_{n-1} is in fact completely determined by this graph. Now, the graph of an associahedron is isomorphic to a graph with all triangulations (without new vertices) of a convex and plane $(n+2)$-gon Q as vertex set and all pairs of distinct triangulations that differ in precisely one proper diagonal[(i)] as edge set. Alternatively, the edges of As_{n-1} are in bijection with the set of triangulations with

[(i)] A proper diagonal is a line segment connecting a pair of vertices of Q whose relative interior is contained in the interior of Q. A non-proper diagonal is a diagonal that connects vertices adjacent in ∂Q and a degenerate diagonal is a diagonal where the end-points are equal.

one proper diagonal removed. Similarly, k-faces of As_{n-1} are in bijection to triangulations of Q with k proper diagonals deleted. In particular, the facets of As_{n-1} are in bijection with the proper diagonals of Q. J.-L. Loday published a beautiful construction of associahedra in 2004, [Lod04]. This construction was generalised by C. Hohlweg and C. Lange, [HL07], and explicitly describes realisations of As_{n-1} as generalised permutahedra that depend on combinatorics induced by the choice of a Coxeter element c of the symmetric group Σ_n on n elements. Figure 1 shows two realisations of As_3 for different choices of c. Before explaining how to obtain these realisations, we stop for some general remarks.

S. Fomin and A. Zelevinsky introduced generalised associahedra in the context of cluster algebras of finite type, [FZ03], and it is well-known that associahedra and generalised associahedra associated to cluster algebras of type A are combinatorially equivalent. The construction of [HL07] was generalised by C. Hohlweg, C. Lange, and H. Thomas, [HLT11] to generalised associahedra. Their construction depends also on choosing a Coxeter element c and the normal vectors of the facets are determined by combinatorial properties of c. Since the normal fans of these realisations turn out to be Cambrian fans as described by N. Reading and D. Speyer, [RS09], the obtained realisations are generalised associahedra associated to any given cluster algebra of finite type. N. Reading and D. Speyer conjectured a linear isomorphism between Cambrian fans and g-vector fans associated to cluster algebras of finite type with acyclic initial seed introduced by S. Fomin and A. Zelevinsky, [FZ07]. They proved their conjecture up to an assumption of another conjecture of [FZ07]. S.-W. Yang and A. Zelevinsky gave an alternative proof of the conjecture of Reading and Speyer in [YZ08]. We emphazise in this context that the results of Section 2 and 3 can be read along these lines: the computations of z_I and y_I for fixed I and varying c involve sums over different choices of z_{R_δ} where the choice of δ depends on c. Moreover, the values z_{R_δ} that occur in these sums can be choosen within a large class as described for example in [HLT11]. From this point of view, we suggest that combinatorial properties of the g-vector fan for cluster algebras of finite type A with respect to an acyclic initial seed are reflected by the Minkowski decompositions studied in this manuscript.

We now return to the construction of [HL07] and give a brief outline. The choice of a Coxeter element c corresponds to a partition of $[n]$ into a *down set* D_c and an *up set* U_c:

$$\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\} \qquad \text{and} \qquad \mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}.$$

This partition induces a labeling of Q with label set $[n+1]_0 := [n+1] \cup \{0\}$ as follows. Pick two vertices of Q which are the end-points of a path with $\ell + 2$ vertices on the boundary of Q, label the vertices of this path counter-clockwise increasing using the label set $\overline{\mathsf{D}}_c := \mathsf{D}_c \cup \{0, n+1\}$ and label the remaining path clockwise increasing using the label set U_c. Without loss of generality, we shall always assume that the label set D_c is on the right-hand side of the diagonal $\{0, n+1\}$ oriented form 0 to $n+1$, see Figure 1 for two examples. We derive the values z_I for some subsets $I \subset [n]$ obtained from this labeled $(n+2)$-gon Q using proper diagonals of Q as follows. Orient each proper diagonal δ from the smaller to the larger labeled end-point of δ, associate to δ the set R_δ that consists of all labels on the strict right-hand side of δ, and replace the elements 0 and $n+1$ by the smaller respectively larger label of the end-points contained in U_c if possible. For each proper diagonal δ we have $R_\delta \subseteq [n]$ but for $n > 2$, obviously not every subset of $[n]$ is of this type. We set

$$\tilde{z}_I^c := \begin{cases} \frac{|I|(|I|+1)}{2} & \text{if } I = R_\delta \text{ for some proper diagonal } \delta, \\ -\infty & \text{else,} \end{cases}$$

compare Table 1 for the examples of Figure 1. In [HL07] it is shown that $P_n(\{\tilde{z}_I^c\})$ is in fact an associa-

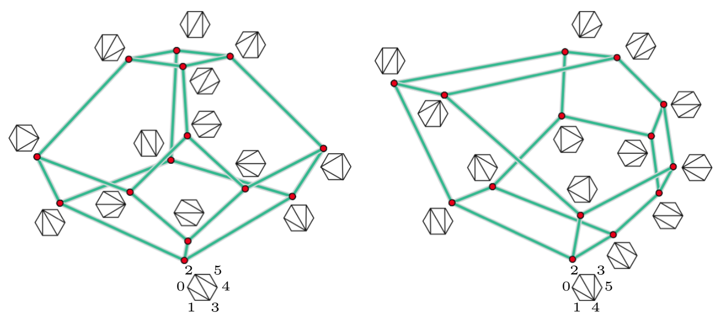

Fig. 1: Two 3-dimensional associahedra $\mathsf{As}_3^c = P_4(\{\tilde{z}_I^c\})$ with vertex coordinates computed for differently choosen Coxeter elements according to [HL07] after application of an orthogonal transformation. The different Coxeter elements are encoded by different labelings of hexagons as indicated.

$\mathsf{D}_c = \{1,3,4\}$ and $\mathsf{U}_c = \{2\}$:

δ	$\{0,3\}$	$\{0,4\}$	$\{0,5\}$	$\{1,2\}$	$\{1,4\}$	$\{1,5\}$	$\{2,3\}$	$\{2,4\}$	$\{3,5\}$
R_δ	$\{1\}$	$\{1,3\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{3\}$	$\{3,4\}$	$\{1,2\}$	$\{1,2,3\}$	$\{4\}$
$\tilde{z}_{R_\delta}$	1	3	6	6	1	3	3	6	1

$\mathsf{D}_c = \{1,4\}$ and $\mathsf{U}_c = \{2,3\}$:

δ	$\{0,4\}$	$\{2,4\}$	$\{3,4\}$	$\{0,5\}$	$\{0,3\}$	$\{1,2\}$	$\{2,5\}$	$\{1,3\}$	$\{1,5\}$
R_δ	$\{1\}$	$\{1,2\}$	$\{1,2,3\}$	$\{1,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,4\}$	$\{3,4\}$	$\{4\}$
$\tilde{z}_{R_\delta}$	1	3	6	3	6	6	6	3	1

Tab. 1: The tables list R_δ and z_I associated to the proper diagonal δ of a labeled hexagon. The upper table corresponds to the associahedron shown on the left of Figure 1, the bottom one to the one on the right.

hedron of dimension $n-1$ realised in $\mathbb{R}^n$ for every choice of c. We refer to these realisations as As_{n-1}^c. As shown by C. Hohlweg, C. Lange, and H. Thomas, one also obtains associahedra if the finite values of $\tilde{z}_I^c$ are replaced by the corresponding right-hand sides of a permutahedron obtained as convex hull of a (non-degenerate) Σ_n-orbit of a point of $\mathbb{R}^n$, where Σ_n acts by permutation of coordinates, [HLT11]. The results of Sections 2 and 3 remain true in this situation.

Some instances of As_{n-1}^c have been studied earlier. For example, the realisations of J.-L. Loday, [Lod04], and of G. Rote, F. Santos, and I. Streinu, [RSS03], related to one-dimensional point configurations, are affinely equivalent to As_{n-1}^c if $\mathsf{U}_c = \varnothing$ or $\mathsf{U}_c = \{2,3\}$. Moreover, G. Rote, F. Santos, and I. Streinu point out in Section 5.3 that their realisation is not affinely equivalent to the realisation of F. Chapoton, S. Fomin, and A. Zelevinsky, [CFZ02], which in turn is affinely equivalent to As_{n-1}^c if $\mathsf{U}_c = \{2\}$ or $\mathsf{U}_c = \{3\}$.

Outline. In this article, we study Minkowski decompositions of a family of realisations of associahedra. In the next section, we explain a combinatorial procedure to compute tight values z_I for the redundant inequalities from the irredundant ones for the realisations of [HL07], compare Theorem 2.8. A key step for the computation is the up and down interval decomposition of I defined in Definition 2.2. In Section 3, we simplify the formula for y_I of Proposition 1.2 and show that most terms in that alternating sum cancel. It turns out that at most four summands remain and they are combinatorially characterised by properties of the up and down interval decomposition of I, see Theorem 3.1 for the precise statement. A full version of this extended abstract that includes all proofs will be published later.

2 Tight values for all z_I^c for As_{n-1}^c

As already mentioned, the facet-defining inequalities for As_{n-1}^c correspond to proper diagonals of Q and these are precisely the irredundant inequalities for the generalised permutahedron $P_n(\{\tilde{z}_I^c\})$. As motivated in the previous section, we want to determine tight values for all z_I, $I \subseteq [n]$, in order to compute the coefficients y_I of the Minkowski decomposition of As_{n-1}^c described by F. Ardila, C. Benedetti, and J. Doker. Since we already know that $\tilde{z}_I^c$ is a tight value if and only if $I = R_\delta$ for some proper diagonal of Q, we now aim for a description of the missing $\tilde{z}_I^c$. The concept of an up and down interval decomposition induced by the partitioning $\mathsf{D}_c \cup \mathsf{U}_c$ (or equivalently induced by c) of a given interval $I \subset [n]$ is a key concept that we introduce first.

Definition 2.1 (up and down intervals)
Let $\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}$ be the partition of $[n]$ induced by a Coxeter element c.

(a) A set $S \subseteq [n]$ is a non-empty interval of $[n]$ if $S = \{r, r+1, \cdots, s\}$ for some $0 < r \leq s < n$. We write S as closed interval $[r,s]$ (end-points included) or as open interval $(r-1, s+1)$ (end-points not included). An empty interval is an open interval $(k, k+1)$ for some $1 \leq k < n$.

(b) A non-empty open down interval is a set $S \subseteq \mathsf{D}_c$ such that $S = \{d_r < d_{r+1} < \cdots < d_s\}$ for some $1 \leq r \leq s \leq \ell$. We write S as open down interval $(d_{r-1}, d_{s+1})_{\mathsf{D}_c}$ where we allow $d_{r-1} = 0$ and $d_{s+1} = n+1$, i.e. $d_{r-1}, d_{s+1} \in \overline{\mathsf{D}}_c$. For $1 \leq r \leq \ell - 1$, we have the empty down interval $(d_r, d_{r+1})_{\mathsf{D}_c}$.

(c) A closed up interval is a non-empty set $S \subseteq \mathsf{U}_c$ such that $S = \{u_r < u_{r+1} < \cdots < u_s\}$ for some $1 \leq r \leq s \leq \ell$. We write $[u_r, u_s]_{\mathsf{U}_c}$.

We often omit the words *open* and *closed* when we consider down and up intervals. There will not be any ambiguity, because we are not going to deal with closed down intervals and open up intervals. Up intervals are always non-empty, while down intervals may be empty. It turns out to be convenient to distinguish the empty down intervals $(d_r, d_{r+1})_{\mathsf{D}_c}$ and $(d_s, d_{s+1})_{\mathsf{D}_c}$ if $r \neq s$ although they are equal as sets.

Definition 2.2 (up and down interval decomposition)
Let $\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}$ be the partition of $[n]$ induced by a Coxeter element c and $I \subseteq [n]$ be non-empty.

(a) An up and down interval decomposition of type (v, w) of I is a partition of I into disjoint up and down intervals $I_1^{\mathsf{U}}, \cdots, I_w^{\mathsf{U}}$ and $I_1^{\mathsf{D}}, \cdots, I_v^{\mathsf{D}}$ obtained by the following procedure.

1. *Suppose there are $\tilde{v}$ non-empty inclusion maximal down intervals of I denoted by $\tilde{I}_k^{\mathsf{D}} = (\tilde{a}_k, \tilde{b}_k)_{\mathsf{D}_c}$, $1 \leq k \leq \tilde{v}$, with $\tilde{b}_k \leq \tilde{a}_{k+1}$ for $1 \leq k < \tilde{v}$. Let $E_i^{\mathsf{D}} = (d_{r_i}, d_{r_i+1})_{\mathsf{D}_c}$ denote all empty down intervals with $\tilde{b}_k \leq d_{r_i} < d_{r_i+1} \leq \tilde{a}_{k+1}$ for $0 \leq k \leq \tilde{v}$, $\tilde{b}_0 = 0$, and $\tilde{a}_{\tilde{v}+1} = n+1$. Denote the open intervals $(\tilde{a}_i, \tilde{b}_i)$ and (d_{r_i}, d_{r_i+1}) of $[n]$ by $\tilde{I}_i$ and E_i respectively.*
2. *Consider all up intervals of I which are contained in (and inclusion maximal within) some interval $\tilde{I}_i$ or E_i obtained in Step 1 and denote these up intervals by*
$$I_1^{\mathsf{U}} = [\alpha_1, \beta_1]_{\mathsf{U}_c}, \cdots, I_w^{\mathsf{U}} = [\alpha_w, \beta_w]_{\mathsf{U}_c}.$$
Without loss of generality, we assume $\alpha_i \leq \beta_i < \alpha_{i+1}$.
3. *A down interval $I_i^{\mathsf{D}} = (a_i, b_i)_{\mathsf{D}_c}$, $1 \leq i \leq w$, is a down interval obtained in Step 1 that is either a non-empty down interval $\tilde{I}_k^{\mathsf{D}}$ or an empty down interval E_k^{D} with the additional property that there is some up interval I_j^{U} obtained in Step 2 such that $I_j^{\mathsf{U}} \subseteq E_k$. Without loss of generality, we assume $b_i \leq a_{i+1}$ for $1 \leq i < w$.*

(b) An up and down interval decomposition of type $(1, w)$ is called nested. A nested component of I is an inclusion-maximal subset J of I such that the up and down decomposition of J is nested.

The following example illustrates Lemma 2.4 and Lemma 2.5.

Example 2.3
Let $\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $\mathsf{U}_c = \{u_1 < u_2 < \cdots < u_m\}$ be the partition of $[n]$ induced by a Coxeter element c. The proper diagonals $\delta = \{a, b\}$, $a < b$, of the labeled polygon Q are in bijection to certain non-empty proper subsets $I \subset [n]$ that have an up and down interval decomposition of type $(1, 0)$, $(1, 1)$, or $(1, 2)$. More precisely, we have

(a) $R_\delta = (a, b)_{\mathsf{D}_c}$ if and only if R_δ has an up and down decomposition of type $(1, 0)$.

(b) $R_\delta = (0, b)_{\mathsf{D}_c} \cup [u_1, a]_{\mathsf{U}_c}$ or $R_\delta = (a, n+1)_{\mathsf{D}_c} \cup [b, u_m]_{\mathsf{U}_c}$ if and only if R_δ has a decomposition of type $(1, 1)$.

(c) $R_\delta = (0, n+1)_{\mathsf{D}_c} \cup [u_1, a]_{\mathsf{U}_c} \cup [b, u_m]_{Up_c}$ if and only if R_δ has an up and down decomposition of type $(1, 2)$.

Lemma 2.4
Given a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c and a non-empty subset $I \subseteq [n]$. Let I have a nested up and down interval decomposition of type $(1, 0)$ with down interval $(a, b)_{\mathsf{D}_c}$ and no up interval $[\alpha_i, \beta_i]_{\mathsf{U}_c}$. Associate the diagonal $\delta_1 := \{a, b\}$ to I.

(a) The diagonal δ_1 is a proper diagonal if and only if $I \subset [n]$.

(b) If the diagonal δ_1 is not proper then $\delta_1 = \{0, n\}$, in particular $\mathsf{U}_c = \varnothing$.

(c) $I = R_{\delta_1}$.

Lemma 2.5
Given a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ *induced by the Coxeter element* c *and a non-empty subset* $I \subseteq [n]$*. Let* I *have a nested up and down interval decomposition of type* $(1, w)$ *with* $w \geq 1$*, down interval* $(a, b)_{\mathsf{D}_c}$*, and up intervals* $[\alpha_i, \beta_i]_{\mathsf{U}_c}$ *for* $1 \leq i \leq w$*. Associate* $w + 1$ *diagonals* δ_i *to* I*:*

$$\delta_1 := \{a, \alpha_1\}, \qquad \delta_i := \{\beta_{i-1}, \alpha_i\} \text{ for } 1 < i < w + 1, \text{ and } \qquad \delta_{w+1} := \{\beta_w, b\}.$$

Then
(a) The diagonals δ_i *are proper diagonals of* Q *for* $1 < i < w + 1$*.*
(b) The diagonal δ_1 *is either a proper diagonal or the edge* $\{0, u_1\}$ *of* Q*.*
(c) The diagonal δ_{w+1} *is either a proper diagonal or the edge* $\{u_m, n + 1\}$ *of* Q*.*
(d) The diagonals δ_i *are non-crossing.*
(e) Let $W \subseteq [w+1]$ *be the index set of the proper diagonals among* $\delta_1, \cdots, \delta_{w+1}$ *and* m *be the maximal element of* W*. Then*

$$I = R_{\delta_m} \setminus \left(\bigcup_{i \in W \setminus \{m\}} [n] \setminus R_{\delta_i} \right).$$

Proof: We only prove statement (e). The up and down interval decomposition of I guarantees $I \subseteq R_{\delta_i}$ for $i \in W$. Thus

$$R_{\delta_m} \cap R_{\delta_i} = R_{\delta_m} \setminus ([n] \setminus R_{\delta_i}), \qquad \text{for } i \in W \setminus \{m\}$$

and

$$I = \bigcap_{i \in W} R_{\delta_i} = \bigcap_{i \in W \setminus \{m\}} (R_{\delta_m} \cap R_{\delta_i}) = R_{\delta_m} \setminus \Big(\bigcup_{i \in W \setminus \{m\}} [n] \setminus R_{\delta_i} \Big).$$

□

Lemma 2.6
Given a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ *induced by a Coxeter element* c*. Let* I *be a non-empty proper subset of* $[n]$ *with up and down interval decomposition of type* (v, w)*. Then there are* v *nested components of type* $(1, w_1), \cdots, (1, w_v)$ *with* $w = \sum_{j=1}^{v} w_j$*. For* $1 \leq i \leq v$ *and* $1 \leq j \leq w_i$ *let* $[\alpha_{i,j}, \beta_{i,j}]_{\mathsf{U}_c}$ *denote the up intervals nested in the down interval* $(a_i, b_i)_{\mathsf{D}_c}$ *and associate* $w_i + 1$ *diagonals* $\delta_{i,j}$*,* $1 \leq j \leq w_i + 1$*, to each nested component in the same way as in Lemma 2.4 and Proposition 2.5. Then*
(a) The diagonals $\delta_{i,j}$ *are non-crossing.*
(b) As in Proposition 2.5 it can happen that we have to adjust to index sets W_1 *and* W_v *to avoid non-proper diagonals.*

$$I = \bigcup_{i=1}^{v} \left(R_{\delta_{i,m_i}} \setminus \Big(\bigcup_{j \in W_i \setminus \{m_i\}} [n] \setminus R_{\delta_{i,j}} \Big) \right).$$

Proof: There are precisely v nested components, one for every down interval. Moreover, for each up interval $[\alpha, \beta]_{\mathsf{U}_c}$ exists a unique down interval $(a, b)_{\mathsf{D}_c}$ with $[\alpha, \beta]_{\mathsf{U}_c} \subseteq (a, b)$. Depending on the type, apply Lemma 2.4 or Lemma 2.5 to each nested component. Combining the statements gives the claim. □

Definition 2.7 *Given a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c. Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v, w), nested components of type $(1, w_1), \cdots, (1, w_v)$ and associated diagonals $\delta_{i,j}$. The subset $\mathcal{D}_I$ of proper diagonals of $\{\delta_{i,j} \mid 1 \le i \le v \text{ and } 1 \le j\}$ is called set of proper diagonals associated to I. Similarly, we say that $\delta \in \mathcal{D}_I$ is a proper diagonal associated to I.*

Theorem 2.8
Given a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c. Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type (v, w). We use the notation of Lemma 2.6 for its up and down interval decomposition. For non-empty $I \subseteq [n]$ we set

$$z_I^c := \sum_{i=1}^{v} \left(\sum_{j \in W_i} \tilde{z}^c_{R_{\delta_{i,j}}} - (|W_i| - 1) z_{[n]} \right).$$

Then $P(\{z_I^c\}) = P(\{\tilde{z}_I^c\}) = \mathsf{As}^c_{n-1}$ and all z_I^c are tight.

Proof: The verification of the inequality is a straightforward calculation:

$$\begin{aligned}
\sum_{i \in I} x_i &= \sum_{k=1}^{v} \left(\sum_{i \in R_{\delta_{k,m_k}} \setminus (\bigcup_{\ell \in W_k \setminus \{m_k\}} [n] \setminus R_{\delta_{k,\ell}})} x_i \right) \\
&= \sum_{k=1}^{v} \left(\sum_{i \in R_{\delta_{k,m_k}}} x_i - \sum_{\ell \in W_k \setminus \{m_k\}} \sum_{i \in [n] \setminus R_{\delta_{k,\ell}}} x_i \right) \\
&= \sum_{k=1}^{v} \left(\sum_{i \in R_{\delta_{k,m_k}}} x_i + \sum_{\ell \in W_k \setminus \{m_k\}} \left(- \sum_{i \in [n] \setminus R_{\delta_{k,\ell}}} x_i \right) \right) \\
&\ge \sum_{k=1}^{v} \left(\tilde{z}^c_{R_{\delta_{k,w_k}}} + \sum_{\ell \in W_k \setminus \{m_k\}} \left(\tilde{z}^c_{R_{\delta_{k,\ell}}} - z_{[n]} \right) \right).
\end{aligned}$$

The second equality holds, since the sets $[n] \setminus R_{\delta_{k,\ell}}$ are for fixed k and varying ℓ pairwise disjoint. The last inequality holds, since $\sum_{i \in R_\delta} x_i \ge \tilde{z}^c_{R_\delta}$ is equivalent to $-\sum_{i \in [n] \setminus R_\delta} x_i \ge \tilde{z}^c_{R_\delta} - z_{[n]}$ for every proper diagonal δ. Since the irredundant facets of the associahedron As^c_{n-1} correspond to inequalities for proper diagonals δ and these inequalities are tight for As^c_{n-1}, the claim follows. □

As expected, we have $z^c_{R_\delta} = \tilde{z}^c_{R_\delta}$ for all proper diagonals δ, since $v = |W_1| = 1$.

3 Combinatorial description and efficient computation of y_I

Proposition 1.2 together with Theorem 2.8 provides a way to compute all Minkowski coefficients y_I since all tight values z_I^c for As^c_{n-1} can be computed. The main goal of this section is to provide the simplified formula for y_I of Theorem 3.1 which consists of at most four non-zero summand for each $I \subseteq [n]$ and has a combinatorial interpretation.

Throughout this section, we use the following notation and make some general assumptions unless explicitly mentioned otherwise. Let $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ be a partition of $[n]$ induced by a Coxeter element c with $\mathsf{D}_c = \{d_1 = 1 < d_2 < \cdots < d_\ell = n\}$ and $\mathsf{U}_c = \{u_1 < \cdots < u_m\}$. Consider a non-empty subset $I \subseteq [n]$ with up and down interval decomposition of type (v,k). If I has a nested up and down interval decomposition, then in particular $v = 1$ and $I = (a,b)_{\mathsf{D}_c} \cup \bigcup_{i=1}^{k} [\alpha_i, \beta_i]_{\mathsf{U}_c}$ with $\alpha_k < \beta_k \leq \alpha_{k+1}$ as before. In this situation, we denote the smallest (respectively largest) element of I by γ (respectively Γ) and consider the diagonals

$$\delta_1 = \{a,b\}, \qquad \delta_2 = \{a,\Gamma\}, \qquad \delta_3 = \{\gamma,b\}, \qquad \text{and} \qquad \delta_4 = \{\gamma,\Gamma\}.$$

The subset of proper diagonals of $\{\delta_1,\delta_2,\delta_3,\delta_4\}$ is denoted by $\mathscr{D}_I$. It is convenient to use the mnemonic δ for diagonals, but unfortunately the meaning of δ_i in this and the preceeding section (starting with Lemma 2.4) is not consistent. We now extend our definition of R_δ and z_{R_δ} to non-proper and degenerate diagonals δ. Is $\delta = \{x,y\}$ not a proper diagonal, we set

$$R_\delta := \begin{cases} \varnothing & \text{if } x,y \in \overline{\mathsf{D}}_c \\ [n] & \text{otherwise,} \end{cases} \qquad \text{and} \qquad z^c_{R_\delta} := \begin{cases} 0 & \text{if } R_\delta = \varnothing \\ \frac{n(n+1)}{2} & \text{if } R_\delta = [n]. \end{cases}$$

The main result of this section is

Theorem 3.1
Let I be a non-empty subset of $[n]$ with a nested up and down interval decompositionof type $(1,k)$. Then

$$y_I = \sum_{\delta \in \mathscr{D}_I} (-1)^{|I \setminus R_\delta|} z^c_{R_\delta}.$$

The ideas used to prove Theorem 3.1 also prove the following corollary.

Corollary 3.2
Let I be a non-empty subset of $[n]$ with a nested up and down interval decompositionof type (v,k) and $v > 1$. Then $y_I = 0$.

The rest of this section sketches the proof of Theorem 3.1. We start with some general remarks and then sketch the proof of Theorem 3.1 if $\mathscr{D}_I = \{\delta_1,\delta_2,\delta_3,\delta_4\}$. The same techniques combined with a more detailed statement of Lemma 3.3 settle the remaining cases of Theorem 3.1, see Corollary 3.4.

Combining Proposition 1.2 and Theorem 2.8, we obtain

$$y_I = \sum_{J \subseteq I} (-1)^{|I \setminus J|} z^c_J = \sum_{J \subseteq I} (-1)^{|I \setminus J|} \sum_{i=1}^{v_J} \left(\tilde{z}^c_{R_{\delta^J_{i,m^J_i}}} + \sum_{j \in W^J_i \setminus \{m^J_i\}} \left(\tilde{z}^c_{R_{\delta^J_{i,j}}} - z_{[n]} \right) \right).$$

This formula is rather awkward at first sight but it turns out that the right-hand side simplifies significantly for our instances As^c_{n-1}. We say that a proper diagonal δ (associated to J) is of type $\tilde{z}^c_{R_\delta}$ (in the expression for y_I), if there exists a set $J \subseteq I$ and an index $i \in [v_J]$ such that $\delta = \delta^J_{i,m^J_i}$. Similarly, we say that a proper diagonal δ (associated to J) is of type $\left(\tilde{z}^c_{R_\delta} - z_{[n]}\right)$ (in the expression for y_I), if there exists a set J and indices $i \in [v_J]$ and $j \in W^J_i \setminus \{m^J_i\}$ such that $\delta = \delta^J_{i,j}$.

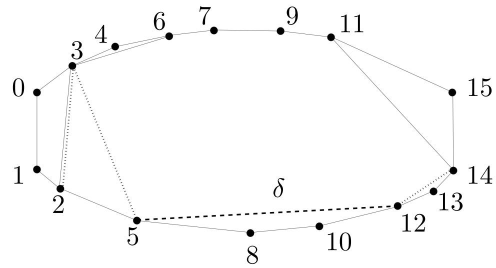

Fig. 2: Here $I = (2,14)_{\mathsf{D}_c} \cup [3,3]_{\mathsf{U}_c} \cup [6,11]_{\mathsf{U}_c}$ with $\gamma = 3$ and $\Gamma = 13$. For $\delta = \{5,12\}$, the up and down interval decomposition of R_δ is of type $(1,0)$. Moreover, δ is associated to $\{8,10\}$, $\{3,8,10\}$, $\{3,8,10,13\}$, and $\{8,10,13\}$ since $(2,5) \cap I = \{3\}$ and $(12,14) \cap I = \{13\}$. Thus the contribution of δ to y_I vanishes. The only diagonals associated to $J \subseteq I$ with up and down interval decomposition of type $(1,0)$ and non-vanishing contribution to y_I are diagonals associated to only one subset $J \subseteq I$, i.e. $\delta_1 = \{2,14\}$ and $\delta_2 = \{2,13\}$ in this example.

Lemma 3.3

Let I be a non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1,k)$.

(a) There is no partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c and no non-empty $I \subset [n]$ such that $\mathscr{D}_I$ is one of the following sets:

$$\varnothing, \quad \{\delta_2\}, \quad \{\delta_3\}, \quad \{\delta_4\}, \quad \{\delta_1,\delta_2\}, \quad \{\delta_1,\delta_3\}, \quad \{\delta_2,\delta_4\}, \quad \textit{or} \quad \{\delta_3,\delta_4\}.$$

(b) There is a partition $[n] = \mathsf{D}_c \sqcup \mathsf{U}_c$ induced by a Coxeter element c and a non-empty $I \subset [n]$ such that $\mathscr{D}_I$ is one of the following sets:

$$\{\delta_1\},\ \{\delta_1,\delta_4\},\ \{\delta_2,\delta_3\},\ \{\delta_1,\delta_2,\delta_3\},\ \{\delta_1,\delta_2,\delta_4\},\ \{\delta_1,\delta_3,\delta_4\},\ \{\delta_2,\delta_3,\delta_4\},\ \textit{or}\ \{\delta_1,\delta_2,\delta_3,\delta_4\}.$$

The proof of Part (a) is left to the reader, while a more detailed analysis for Part (b) is implicitly given (and needed) for Corollary 3.4. We now sketch the proof of Theorem 3.1 if $\mathscr{D}_I = \{\delta_1,\delta_2,\delta_3,\delta_4\}$.

Proof: We have to analyse the equation for y_I given above. Let δ be some diagonal $\delta_{i,j}^J$ that occurs on the equation's right-hand side. In particular, δ is a proper and non-degenerate diagonal, since δ is a diagonal of an up and down interval decomposition. By Example 2.3, the up and down interval decomposition of R_δ is either of type $(1,0)$, $(1,1)$ or $(1,2)$. A good understanding which sets $S \subseteq I$ satisfy $\delta \in \mathcal{D}_S$ and in which cases δ associated to S is of type $\tilde{z}^c_{R_\delta}$ or of type $(\tilde{z}^c_{R_\delta} - z_{[n]})$ in the expression for y_I is essential for the simplification. The complete picture is a case study of the mentioned three cases for R_δ, two of them split further into subcases. In order to illustrate some of the arguments used, we show the simplest case where R_δ has an up and down interval decomposition of type $(1,0)$.

1. R_δ has up and down decomposition of type $(1,0)$, see Figure 2.
   Then $R_\delta = (\tilde{a},\tilde{b})_{\mathsf{D}_c} \subseteq (a,b)_{\mathsf{D}_c}$ and $R_\delta \subseteq I$. Let $S \subseteq I$ be a set with $\delta \in \mathcal{D}_S$. Then $(\tilde{a},\tilde{b})_{\mathsf{D}_c}$ is a nested component of type $(1,0)$ of S and all other nested components are subsets of $(a,\tilde{a}) \cap I$ and $(\tilde{b},b) \cap I$. It follows that a set S satisfies $\delta \in \mathcal{D}_S$ if and only if

$$R_\delta \subseteq S \subseteq R_\delta \cup \big((a,\tilde{a}) \cap I\big) \cup \big((\tilde{b},b) \cap I\big).$$

   We now collect all terms for $\tilde{z}^c_{R_\delta}$ in the expression for y_I. Since δ is a proper diagonal, we have $\tilde{z}^c_{R_\delta} \neq 0$ and the resulting alternating sum vanishes if and only if there is more than one term of this type, that is, if and only if $\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)\neq\varnothing$. If $\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)=\varnothing$, we obtain $(-1)^{|I\setminus R_\delta|}\tilde{z}^c_{R_\delta}$ as contribution for y_I. Moreover, the condition $\big((a,\tilde{a})\cap I\big)\cup\big((\tilde{b},b)\cap I\big)=\varnothing$ guarantees $\delta\in\mathscr{D}_I$. The diagonal δ_1 is always of this type. Similarly, we have $\delta_2\in\mathscr{D}_I$ if $\Gamma\in\mathsf{D}_c$, $\delta_3\in\mathscr{D}_I$ if $\gamma\in\mathsf{D}_c$, and $\delta_4\in\mathscr{D}_I$ if $\gamma,\Gamma\in\mathsf{D}_c$.

Once the three cases are settled, we have a smaller number of diagonals that have a potential contribution to y_I. We now proceed by another case study and distinguish the four cases

1. $\gamma,\Gamma\in\mathsf{D}_c$,
2. $\gamma\in\mathsf{U}_c$ and $\Gamma\in\mathsf{D}_c$,
3. $\gamma,\Gamma\in\mathsf{U}_c$, and
4. $\gamma\in\mathsf{D}_c$, $\Gamma\in\mathsf{U}_c$.

Collecting the diagonals in each of these four cases from the previous analysis, additional terms cancel and the claim follows. □

We end this section listing the precise statements needed to settle the degenerate cases $\mathscr{D}_I\neq\{\delta_1,\delta_2,\delta_3,\delta_4\}$ of Theorem 3.1 in the following corollary.

Corollary 3.4
Let I be non-empty proper subset of $[n]$ with up and down interval decomposition of type $(1,k)$.

(a) Suppose that I satisfies one of the following conditions

(i) $\mathscr{D}_I=\{\delta_1\}$,
(ii) $\mathscr{D}_I=\{\delta_1,\delta_3,\delta_4\}$, $(a,b)_{\mathsf{D}}=\{\Gamma\}$, and $\gamma\in\mathsf{U}_c$
(iii) $\mathscr{D}_I=\{\delta_1,\delta_2,\delta_4\}$, $(a,b)_{\mathsf{D}}=\{\gamma\}$, and $\Gamma\in\mathsf{U}_c$
(iv) $\mathscr{D}_I=\{\delta_1,\delta_2,\delta_3\}$ and $(a,b)_{\mathsf{D}}=\{\gamma,\Gamma\}$, or
(v) $\mathscr{D}_I=\{\delta_2,\delta_3,\delta_4\}$ and $(a,b)_{\mathsf{D}}=\varnothing$.

Then the Minkowski coefficient y_I of As^c_{n-1} is

$$y_I=\sum_{\delta\in\mathscr{D}_I}(-1)^{|I\setminus R_\delta|}z_{R_{\delta_1}}.$$

(b) Suppose that I satisfies one of the following conditions

(i) $\mathscr{D}_I=\{\delta_1,\delta_4\}$,
(ii) $\mathscr{D}_I=\{\delta_2,\delta_3\}$,
(iii) $\mathscr{D}_I=\{\delta_1,\delta_3,\delta_4\}$ and $\bigcup_{i=1}^k[\alpha_i,\beta_i]_{\mathsf{U}_c}=\{\Gamma\}$,
(iv) $\mathscr{D}_I=\{\delta_1,\delta_2,\delta_4\}$ and $\bigcup_{i=1}^k[\alpha_i,\beta_i]_{\mathsf{U}_c}=\{\gamma\}$,
(v) $\mathscr{D}_I=\{\delta_1,\delta_2,\delta_3\}$ and $\bigcup_{i=1}^k[\alpha_i,\beta_i]_{\mathsf{U}_c}=\{\gamma,\Gamma\}$.

Then the Minkowski coefficient y_I of As^c_{n-1} is

$$y_I=(-1)^{|\{\gamma,\Gamma\}|}z_{[n]}+\sum_{\delta\in\mathscr{D}_I}(-1)^{|I\setminus R_\delta|}z_{R_{\delta_1}}.$$

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Hierarchical Zonotopal Power Ideals

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Abstract. Zonotopal algebra deals with ideals and vector spaces of polynomials that are related to several combinatorial and geometric structures defined by a finite sequence of vectors. Given such a sequence X, an integer $k \geq -1$ and an upper set in the lattice of flats of the matroid defined by X, we define and study the associated *hierarchical zonotopal power ideal*. This ideal is generated by powers of linear forms. Its Hilbert series depends only on the matroid structure of X. It is related to various other matroid invariants, *e. g.* the shelling polynomial and the characteristic polynomial.

This work unifies and generalizes results by Ardila-Postnikov on power ideals and by Holtz-Ron and Holtz-Ron-Xu on (hierarchical) zonotopal algebra. We also generalize a result on zonotopal Cox modules due to Sturmfels-Xu.

Résumé. La théorie de l'algèbre "zonotopique" s'occupe d'idéaux et d'espaces vectoriels de polynômes qui ont un rapport avec plusieurs structures combinatoires et géométriques définies par des suites finies de vecteurs. Étant donné une telle suite X, un nombre entier $k \geq -1$ et un ensemble supérieur dans le treillis des plans du matroïde défini par X, nous définissons et étudions l'*idéal hiérarchique zonotopique*, engendré par des puissances de formes linéaires. Sa série de Hilbert dépend seulement de la structure matroïdale de X. Il existe des relations avec d'autres invariants de matroïdes, tels que le polynôme d'épluchage et le polynôme caractéristique.

Ce travail unifie et généralise des résultats d'Ardila-Postnikov sur les idéaux de puissances et de Holtz-Ron et Holtz-Ron-Xu sur l'algèbre zonotopique (hiérarchique). Nous généralisons aussi un résultat sur les modules de Cox zonotopiques, dû à Sturmfels-Xu.

Keywords: matroids, Tutte polynomials, power ideals, zonotopal algebra, Hilbert series

1 Introduction

Let $X = (x_1, \ldots, x_N) \subseteq \mathbb{R}^r$ be a sequence of vectors that span $\mathbb{R}^r$. For a vector η, let $m(\eta)$ denote the number of vectors in X that are *not* perpendicular to η. A vector $v \in \mathbb{R}^r$ defines a linear polynomial $p_v := \sum_i v_i t_i \in \mathbb{R}[t_1, \ldots, t_r]$. For $Y \subseteq X$, let $p_Y := \prod_{x \in Y} p_x$. Then define

$$\mathcal{P}(X) := \operatorname{span}\{p_Y : X \setminus Y \text{ spans } \mathbb{R}^r\} \qquad \text{and} \qquad \mathcal{I}(X) := \operatorname{ideal}\{p_\eta^{m(\eta)} : \eta \neq 0\} \tag{1.1}$$

The following theorem and several generalizations are well known:

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Theorem 1.1 ([1, 5, 11])**.**

$$\mathcal{P}(X) = \ker \mathcal{I}(X) := \operatorname{span}\left\{ q \in \mathbb{R}[t_1, \ldots, t_r] : f\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_r}\right) q = 0 \textit{ for all } f \in \mathcal{I}(X) \right\} \quad (1.2)$$

In addition, $\mathcal{I}(X) = \mathcal{I}'(X) := \{p_\eta^{m(\eta)}$ *: the vectors in* X *that are perpendicular to* η *span a hyperplane*$\}$

We show that a statement as in Theorem 1.1 holds in a far more general setting: we study the kernel of the *hierarchical zonotopal power ideal*

$$\mathcal{I}(X, k, J) := \operatorname{ideal}\{p_\eta^{m(\eta)+k+\chi_J(\eta)} : \eta \neq 0\} \quad (1.3)$$

where $k \geq -1$ is an integer and χ_J is the indicator function of an upper set J in the lattice of flats of the matroid defined by X. We study those spaces in a slightly more abstract setting, *e. g.* $\mathcal{P}(X, k, J)$ is contained in the symmetric algebra over some $\mathbb{K}$-vector space, where $\mathbb{K}$ is a field of characteristic zero.

The choice of a sequence of vectors X defines a large number of objects in various mathematical fields which are all related to zonotopal algebra [11]. Examples include combinatorics (matroids, matroid and graph polynomials, generalized parking functions and chip firing games if X is graphic [15, 18]), discrete geometry (hyperplane arrangements, zonotopes and tilings of zonotopes), approximation theory (box splines [6], least map interpolation) and algebraic geometry (Cox rings, fat point ideals [1, 10, 19]).

Central $\mathcal{P}$-spaces (in our terminology the kernel of $\mathcal{I}'(X, 0, \{X\})$) were introduced in the literature on approximation theory around 1990 (*e. g.* [5]). A dual space called $\mathcal{D}(X)$ appeared almost 30 years ago. See [11, Section 1.2] for a historic survey.

Recently, Olga Holtz and Amos Ron coined the term *zonotopal algebra* [11]. They introduced internal ($k = -1$) and external ($k = +1$) $\mathcal{P}$-spaces and $\mathcal{D}$-spaces. Federico Ardila and Alexander Postnikov constructed $\mathcal{P}$-spaces for arbitrary integers $k \geq -1$ (*Combinatorics and geometry of power ideals* [1], presented at FPSAC 2009). Olga Holtz, Amos Ron, and Zhiqiang Xu [12] introduced hierarchical zonotopal spaces, *i. e.* structures that depend on the choice of an upper set J in addition to X and k. They studied semi-internal and semi-external spaces (*i. e.* $k = -1$ and $k = 0$ and some special upper sets J). The central case was treated in an algebraic setting in [8]. Other related results include [2, 20].

The *least map* [7] assigns to a finite set $S \subseteq \mathbb{R}^r$ of cardinality m an m-dimensional space of homogeneous polynomials in $\mathbb{R}[t_1, \ldots, t_r]$. Holtz and Ron [11] showed that in the internal, central and external case, $\mathcal{P}$-spaces can be obtained via the least map if $X \subseteq \mathbb{Z}^r$ is unimodular. In those cases, the $\mathcal{P}$-spaces are obtained by choosing the set S as a certain subset of the set of integral points in the zonotope

$$Z(X) := \left\{ \sum_{i=1}^{N} \lambda_i x_i : 0 \leq \lambda_i \leq 1 \right\} \quad (1.4)$$

There is also a discrete theory, where differential operators are replaced by difference operators. ([8] and *A Tutte polynomial for toric arrangements* (Luca Moci) [16], presented at FPSAC 2010). In this theory, the connection with lattice points in zonotope remains valid even if X is not unimodular.

As an example for the connections between zonotopal algebra and combinatorics, we now explain various relationships between zonotopal spaces and matroid/graph polynomials. They can be deduced from the fact that both, the matroid/graph polynomials [9] and the Hilbert series of the zonotopal spaces [1] are evaluations of the Tutte polynomial.

$\mathrm{Hilb}(\mathcal{P}(X, 0, \{X\}), t)$ equals the shelling polynomial [3] $h_{\Delta(X^*)}(t)$ of the matroid dual to X with the coefficients reversed. The t^{N-r-i} coefficient of $h_{\Delta(X^*)}(t+1)$ equals the number of independent sets of cardinality i in the matroid X. Let X_G denote the reduced oriented incidence matrix of a connected graph G, *i. e.* the matrix that is obtained from the oriented incidence matrix by removing one row so that it has full rank. The Hilbert series of $\mathcal{P}(X_G, -1, \{X_G\})$ is related to the flow polynomial ϕ_G. By duality it is also related to the chromatic polynomial of the graph resp. the characteristic polynomial of the matroid in the general case. The connection is as follows: if G is connected, then

$$\phi_G(t) = (t-1)^{N-r}\,\mathrm{Hilb}(\mathcal{P}(X_G, -1, \{X_G\}), 1/(1-t)) \tag{1.5}$$

The four color theorem is equivalent to the following statement: if G is a planar graph and G^* denotes its dual, then

$$\mathrm{Hilb}(\mathcal{P}(X_{G^*}, -1, \{X_{G^*}\}), -1/3) > 0 \tag{1.6}$$

Organization of the article: In Section 2, we introduce our notation and review the mathematical background. In Section 3, we define hierarchical zonotopal power ideals. A simple description of their kernels is given by our Main Theorem. In Section 4, we construct bases for the vector spaces $\mathcal{P}(X, k, J)$ and we deduce several formulas for their Hilbert series. In Section 5, we apply our results to prove a statement about zonotopal Cox modules that were defined by Bernd Sturmfels and Zhiqiang Xu [19]. In Section 6, we give some examples. The full paper is available on the arXiv [14].

2 Preliminaries

Notation: The following notation is used throughout this paper: $\mathbb{K}$ is a fixed field of characteristic zero. V denotes a finite-dimensional $\mathbb{K}$-vector space of dimension $r \geq 1$ and $U := V^*$ its dual. Our main object of study is a finite sequence $X = (x_1, \ldots, x_N) \subseteq U$ whose elements span U. We slightly abuse notation by using the symbol $\subseteq$ for subsequences. For $Y \subseteq X$, the deletion $X \setminus Y$ denotes the deletion of a subsequence and not the deletion of a subset, *i. e.* $(x_1, x_2) \setminus (x_1) = (x_2)$ even if $x_1 = x_2$. The order of the elements in X is irrelevant except in Section 4.

Matroids and posets: Let $X = (x_1, \ldots, x_N)$ be a finite sequence whose elements span U. Let $\mathfrak{M}(X) := \{I \subseteq \{1, \ldots, N\} : \{x_i : i \in I\} \text{ linearly independent}\}$. Then $\mathfrak{M}(X)$ is a matroid of rank r on N elements. X is called a $\mathbb{K}$-*representation* of the matroid $\mathfrak{M}(X)$. For more information about matroids, see Oxley's book [17].

We now introduce some additional matroid theoretic concepts. To facilitate notation, we always write X instead of $\mathfrak{M}(X)$. The *rank* of $Y \subseteq X$ is defined as the cardinality of a maximal independent set contained in Y. It is denoted $\mathrm{rk}(Y)$. The *closure* of Y in X is defined as $\mathrm{cl}_X(Y) := \{x \in X : \mathrm{rk}(Y \cup x) = \mathrm{rk}(Y)\}$. $C \subseteq X$ is called a *flat* if $C = \mathrm{cl}(C)$. A *hyperplane* is a flat of rank $r-1$. The set of all hyperplanes in X is denoted by $\mathcal{H} = \mathcal{H}(X)$.

Given a flat $C \subseteq X$, we call $\eta \in V$ a *defining normal* for C if $C = \{x \in X : \eta(x) = 0\}$. Note that for hyperplanes, there is a unique defining normal (up to scaling). The set of bases of the matroid X (*i. e.* the subsequences of X of cardinality r and rank r) is denoted $\mathbb{B}(X)$. If $x = 0$, then x is called a *loop*. If $\mathrm{rk}(X \setminus x) = r - 1$, then x is called a *coloop*.

The set of flats of a given matroid X ordered by inclusion forms a lattice (*i. e.* a poset with joins and meets) called the *lattice of flats* $\mathcal{L}(X)$. An *upper set* $J \subseteq \mathcal{L}(X)$ is an upward closed set, *i. e.* $C \subseteq C'$,

$C \in J$ implies $C' \in J$. We call $C \in \mathcal{L}(X)$ a *maximal missing flat* if $C \notin J$ and C is maximal with this property.

The *Tutte polynomial* [4] $T_X(x,y) := \sum_{A \subseteq X} (x-1)^{r-\mathrm{rk}(A)}(y-1)^{|A|-\mathrm{rk}(A)}$ captures a lot of information about the matroid $\mathfrak{M}(X)$.

Algebra: $\mathrm{Sym}(V)$ denotes the *symmetric algebra* over V. This is a base-free version of the ring of polynomials over V. The choice of a basis $B = \{b_1, \ldots, b_n\} \subseteq V$ yields an isomorphism $\mathrm{Sym}(V) \cong \mathbb{K}[b_1, \ldots, b_r]$.

A *derivation* on $\mathrm{Sym}(V)$ is a $\mathbb{K}$-linear map D satisfying Leibniz's law, *i. e.* $D(fg) = D(f)g + fD(g)$ for $f, g \in \mathrm{Sym}(V)$. For $v \in V$, we define the *directional derivative* in direction v, $D_v : \mathrm{Sym}(U) \to \mathrm{Sym}(U)$ as the unique derivation which satisfies $D_v(u) = v(u)$ for all $u \in U$. For $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, this definition agrees with the analytic definition of the directional derivative. It follows that $\mathrm{Sym}(V)$ can be identified with the ring of differential operators on U.

Now we define a pairing $\langle \cdot, \cdot \rangle : \mathrm{Sym}(U) \times \mathrm{Sym}(V) \to \mathbb{K}$ by $\langle f, q \rangle := (q(D)f)(0)$ where $q(D)f$ means that q acts as a differential operator on f.

A *graded vector space* is a vector space V that decomposes into a direct sum $V = \bigoplus_{i \geq 0} V_i$. For a graded vector space, we define its *Hilbert series* as the formal power series $\mathrm{Hilb}(V,t) := \sum_{i \geq 0} \dim(V_i) t^i$. A *graded algebra* V has the additional property $V_i V_j \subseteq V_{i+j}$. We use the symmetric algebra with its natural grading. This grading is characterized by the property that the degree 1 elements are exactly the ones that are contained in V. A $\mathbb{Z}^n$*-multigraded ring* R is defined similarly: R decomposes into a direct sum $R = \bigoplus_{\boldsymbol{a} \in \mathbb{Z}^n} R_{\boldsymbol{a}}$ and $R_{\boldsymbol{a}} R_{\boldsymbol{b}} \subseteq R_{\boldsymbol{a}+\boldsymbol{b}}$.

A linear map $f : V \to W$ induces an algebra homomorphism $\mathrm{Sym}(f) : \mathrm{Sym}(V) \to \mathrm{Sym}(W)$.

Homogeneous ideals and their kernels: An ideal $\mathcal{I} \subseteq \mathrm{Sym}(V)$ is called a *power ideal* [1] if $\mathcal{I} = \mathrm{ideal}\{D_\eta^{e_\eta} : \eta \in Z\}$ for some $Z \subseteq V \setminus \{0\}$ and $e \in \mathbb{Z}_{\geq 0}^Z$.(i) D_η denotes the image of η under the canonical injection $V \hookrightarrow \mathrm{Sym}(V)$. By definition, power ideals are homogeneous.

Definition 2.1. Let $\mathcal{I} \subseteq \mathrm{Sym}(V)$ be a homogeneous ideal. Its *kernel* (or inverse system) is defined as

$$\ker \mathcal{I} := \{f \in \mathrm{Sym}(U) : \langle f, q \rangle = 0 \text{ for all } q \in \mathcal{I}\} \tag{2.1}$$

It is known that for a homogeneous ideal $\mathcal{I} \subseteq \mathrm{Sym}(V)$ of finite codimension the Hilbert series of $\ker \mathcal{I}$ and $\mathrm{Sym}(V)/\mathcal{I}$ are equal. (ii)

A remark on the notation: As zonotopal spaces were studied by people from different fields, the notation and the level of abstraction used in the literature varies. Authors with a background in spline theory usually work over $\mathbb{R}^n$ while other authors prefer an abstract setting as we do.

Since the Euclidean setting captures all the important parts of the theory, a reader with no background in abstract algebra may safely make the following substitutions: $\mathbb{K} = \mathbb{R}$, $U \cong V \cong \mathbb{R}^n$. $f \in \mathbb{R}[t_1, \ldots, t_r] = \mathrm{Sym}(V)$ acts on $\mathbb{R}[t_1, \ldots, t_r] = \mathrm{Sym}(U)$ as the differential operator that is obtained from f by substituting $t_i \mapsto \frac{\partial}{\partial t_i}$.

(i) In the original definition in [1], 1 is added to every exponent.

(ii) $\ker \mathcal{I}$ is sometimes defined slightly differently in the literature: The pairing $\langle \bullet, \bullet \rangle$ is defined on $\mathrm{Sym}(V)^* \times \mathrm{Sym}(V)$ and $\ker \mathcal{I}$ is the subset of $\mathrm{Sym}(V)^*$ that is annihilated by $\mathcal{I}$. In our setting, both definitions agree.

Some authors work in the dual setting and consider a central hyperplane arrangement instead of a finite sequence of vectors X. While hierarchical zonotopal power ideals can be defined in both settings, it is natural for us to work with vectors as we are also interested in the zonotope $Z(X)$.

3 Hierarchical zonotopal power ideals and their kernels

In this section, we define hierarchical zonotopal power ideals. Our Main Theorem gives a simple description of their kernels.

Definitions and the Main Theorem: Recall that $U = V^*$ denotes an r-dimensional vector space over a field $\mathbb{K}$ of characteristic zero and $X = (x_1, \ldots, x_N)$ denotes a finite sequence whose elements span U.

A vector $\eta \in V$ defines a flat $C \subseteq X$. Define $m_X(C) = m_X(\eta) := |X \setminus C|$. Given an upper set $J \subseteq \mathcal{L}(X)$, $\chi_J : \mathcal{L}(X) \to \{0,1\}$ denotes its indicator function. The index of χ and m is omitted if it is clear which upper set is meant. We define χ for arbitrary sets $A \subseteq X$ as $\chi(A) := \chi(\mathrm{cl}(A))$.

For a given $x \in U$, we denote the image of X under the canonical injection $U \hookrightarrow \mathrm{Sym}(U)$ by p_x. For $Y \subseteq X$, we define $p_Y := \prod_{x \in Y} p_x$. For $\eta \in V$, we write D_η for the image of η under the canonical injection $V \hookrightarrow \mathrm{Sym}(V)$ in order to stress the fact that we mostly think of $\mathrm{Sym}(V)$ as the algebra generated by the directional derivatives on $\mathrm{Sym}(U)$.

Definition 3.1 (Hierarchical zonotopal power ideals and $\mathcal{P}$-spaces)**.** Let $\mathbb{K}$ be a field of characteristic zero, V be a finite-dimensional $\mathbb{K}$-vector space of dimension $r \geq 1$ and $U = V^*$. Let $X = (x_1, \ldots, x_N) \subseteq U$ be a finite sequence whose elements span U. Let $k \geq -1$ be an integer and let $J \subseteq \mathcal{L}(X)$ be a non-empty upper set, where $\mathcal{L}(X)$ denotes the lattice of flats of the matroid defined by X.

Let $\chi : \mathcal{L}(X) \to \{0,1\}$ denote the indicator function of J. Let $E : \mathcal{L}(X) \to V$ be a function that assigns a defining normal to every flat. Now define

$$\mathcal{I}'(X,k,J,E) := \mathrm{ideal}\big\{ D_{E(C)}^{m(C)+k+\chi(C)} : C \text{ hyperplane or maximal missing flat} \big\} \tag{3.1}$$

$$\mathcal{I}(X,k,J) := \mathrm{ideal}\big\{ D_\eta^{m(\eta)+k+\chi(\eta)} : \eta \in V \setminus \{0\} \big\} \subseteq \mathrm{Sym}(V) \tag{3.2}$$

$$\mathcal{P}(X,k,J) := \mathrm{span}\, S(X,k,J) \subseteq \mathrm{Sym}(U) \tag{3.3}$$

where

$$S(X,k,J) := \begin{cases} \{ f p_Y : Y \subseteq X, 0 \leq \deg f \leq \chi(X \setminus Y) + k - 1 \} & \text{for } k \geq 1 \\ \{ p_Y : Y \subseteq X,\ \mathrm{cl}(X \setminus Y) \in J \} & \text{for } k = 0 \\ \{ p_Y : |Y \setminus C| < m(C) - 1 + \chi(C) \text{ for all } C \in \mathcal{L}(X) \setminus \{X\} \} & \text{for } k = -1 \end{cases}$$

Note that the definition of $S(X,0,J)$ can be seen as a special case of the definition of $S(X,k,J)$ for $k \geq 1$. For examples, see Section 6, Remark 3.7, and Proposition 3.8.

Theorem 3.2 (Main Theorem)**.** *We use the same terminology as in Definition* 3.1*. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$, i. e. J contains all hyperplanes in X. Then,*

$$\mathcal{P}(X,k,J) = \ker \mathcal{I}(X,k,J) \subseteq \ker \mathcal{I}'(X,k,J,E) \tag{3.4}$$

Furthermore, for $k \in \{-1,0\}$, $\mathcal{I}'(X,k,J,E)$ is independent of the choice of E and $\mathcal{P}(X,k,J) = \ker \mathcal{I}(X,k,J) = \ker \mathcal{I}'(X,k,J,E)$.

Corollary 3.3. *In the setting of the Main Theorem,* $\mathcal{P}(X,k,\mathcal{L}(X)) = \mathcal{P}(X,k+1,\{X\})$.

Corollary 3.4. *The Hilbert series of* $\mathcal{P}(X,k,J)$ *depends only on the matroid* $\mathfrak{M}(X)$*, but not on the representation* X.

Remark 3.5. One might wonder if similar theorems can be proved for $k \le -2$. One would of course need to impose extra conditions on X to ensure that the exponents appearing in the definition of the ideals are non-negative. It is easy to prove that $\mathcal{I}$ and $\mathcal{I}'$ are equal in this case. However, we do not know how to construct a generating set for the kernel. A different approach would be required: in general, it is not spanned by a set of polynomials of type p_Y for some $Y \subseteq X$ [1].

Basic results: In this paragraph, we state an important lemma and we give an explicit formula for $\mathcal{P}(X,k,J)$ in two particularly simple cases.

Lemma 3.6. $\mathcal{P}(X,k,J) \subseteq \ker \mathcal{I}(X,k,J) \subseteq \ker \mathcal{I}'(X,k,J)$ *holds for all* $k \ge -1$ *and all* $J \subseteq \mathcal{L}(X)$.

Remark 3.7. Suppose that $\dim U = 1$ and that X contains N' non-zero entries. Let $x \in U$ and $y \in V$ be non-zero vectors. Note that $\emptyset$ is the only hyperplane in X. Hence, $\mathcal{I}'(X,k,J) = \mathcal{I}(X,k,J) = \mathrm{ideal}\{D_y^{N'+k+\chi(\emptyset)}\}$ and $\mathcal{P}(X,k,J) = \mathrm{span}\{p_x^i : i \in \{0,1,\dots,N'-1+k+\chi(\emptyset)\}\}$.

Proposition 3.8. *We use the same terminology as in Definition* 3.1. *Let* $X = (x_1,\dots,x_r)$ *be a basis for* U. *Then* $\mathcal{P}(X,k,J) = \ker \mathcal{I}(X,k,J) \subseteq \ker \mathcal{I}'(X,k,J,E)$. *Furthermore, for* $k \in \{-1,0\}$, $\ker \mathcal{I}(X,k,J) = \ker \mathcal{I}'(X,k,J,E)$ *for arbitrary* E.

More precisely, writing $p_i := p_{x_i}$ *as shorthand notation, we get*

$$\mathcal{P}(X,k,J) = \mathrm{span}\left\{ \prod_{i\in I} p_i^{a_i+1} : I \subseteq [r],\, a_i \in \mathbb{Z}_{\ge 0}, \sum_{i \in I} a_i \le k + \chi(X \setminus \{x_i : i \in I\}) - 1 \right\} \tag{3.5}$$

For $k = 0$*, this specializes to* $\mathcal{P}(X,0,J) = \mathrm{span}\{p_Y : X \setminus Y \in J\}$. *For* $k = -1$, $\mathcal{P}(X,-1,J) = \mathrm{span}\{1\}$ *if* $J \supseteq \mathcal{H}$ *and* $\mathcal{P}(X,-1,J) = \{0\}$ *otherwise.*

For a two-dimensional example, see Example 6.1 and Figure 1.

Deletion and Contraction: In this paragraph, we define deletion and contraction for matroids X and upper sets J.

Fix an element $x \in X$. The *deletion* of x is the matroid defined by the sequence $X \setminus x$. Let $\pi_x : U \to U/x$ denote the projection to the quotient space. The *contraction* of x is the matroid defined by the sequence X/x which contains the images of the elements of $X \setminus x$ under π_x.

Let $Y \subseteq X \setminus x$. We write $\bar{Y}$ to denote the subsequence of X/x with the same index set as Y and vice versa.

Let $J \subseteq \mathcal{L}(X)$ be an upper set. Then define

$$J \setminus x := \{C \setminus x : C \in J \text{ and } C = \mathrm{cl}(C \setminus x)\} \subseteq \mathcal{L}(X \setminus x) \tag{3.6}$$

$$J/x := \{(\overline{C \setminus x}) : x \in C \in J\} \subseteq \mathcal{L}(X/x) \tag{3.7}$$

On the proof of the Main Theorem: The Main Theorem can be proven by induction. Proposition 3.8 is used as the base case. Deletion-contraction gives rises to a short exact sequence from which the result follows. This short exact sequence is recorded in the following proposition:

Proposition 3.9. *We use the same terminology as in Definition* 3.1. *Suppose that $x \in X$ is neither a loop nor a coloop. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$ or $J = \{X\}$. Then, the following sequence is exact:*

$$0 \to \ker \mathcal{I}(X \setminus x, k, J \setminus x) \xrightarrow{\cdot p_x} \ker \mathcal{I}(X, k, J) \xrightarrow{\mathrm{Sym}(\pi_x)} \ker \mathcal{I}(X/x, k, J/x) \to 0 \tag{3.8}$$

Remark 3.10. We do not know if the Main Theorem holds for $k = -1$ and $J \not\supseteq \mathcal{H}$. In general, our proof does not work in this case and Proposition 3.9 is false. The difficulty of the case $k = -1$ was already observed by Holtz and Ron. They conjectured that the Main Theorem holds in the internal case *i. e.* for $k = -1$ and $J = \{X\}$, but they were unable to prove it [11, Conjecture 6.1]. An incorrect prove of this special case has appeared in the literature [1, Part 3 of Proposition 4.5].

4 Bases and Hilbert series

In this section, we show how to select a basis for $\mathcal{P}(X, k, J)$ from $S(X, k, J)$ for $k \geq 0$ and we give several formulas for the Hilbert series of $\mathcal{P}(X, k, J)$.

Bases: Our construction of a basis depends on the order on X. This order is used to define internal and external activity with respect to a given basis (see [4, Section 6.6.] for a reference). Recall that $\mathbb{B}(X)$ denotes the set of all bases $B \subseteq X$. Fix a basis $B \in \mathbb{B}(X)$. $b \in B$ is called *internally active* if $b = \max(X \setminus \mathrm{cl}(B \setminus b))$, *i. e.* b is the maximal element of the unique cocircuit contained in $(X \setminus B) \cup b$. The set of internally active elements in B is denoted $I(B)$. $x \in X \setminus B$ is called *externally active* if $x \in \mathrm{cl}\{b \in B : b \leq x\}$, *i. e.* x is the maximal element of the unique circuit contained in $B \cup x$. The set of externally active elements with respect to B is denoted $E(B)$.

We generalize [1, Proposition 4.21] to hierarchical spaces:

Definition 4.1. We use the same terminology as in Definition 3.1. In addition, let $k \geq 0$. Then define

$$\Gamma(X, k, J) := \big\{(B, I, \boldsymbol{a}_I) : B \in \mathbb{B}(X),\ I \subseteq I(B),\ \boldsymbol{a}_I \in \mathbb{Z}^I_{\geq 0}, \sum_{x \in I} a_x \leq k + \chi((B \cup E(B)) \setminus I) - 1\big\}$$

$$\mathcal{B}(X, k, J) := \Big\{p_{X \setminus (B \cup E(B))} \prod_{x \in I} p_x^{a_x + 1} : (B, I, \boldsymbol{a}_I) \in \Gamma(X, k, J)\Big\} \subseteq \mathrm{Sym}(U) \tag{4.1}$$

Theorem 4.2 (Basis Theorem)**.** *We use the same terminology as in Definition* 3.1. *Let $k \geq 0$. Then $\mathcal{B}(X, k, J)$ is a basis for $\mathcal{P}(X, k, J)$.*

Remark 4.3. We do not know if there is a simple method to construct bases for $\mathcal{P}(X, -1, J)$. This difficulty was already observed in the internal case by Holtz and Ron [11].

Recursive formulas for the Hilbert series: The following statement is a direct consequence of Proposition 3.9 and of the Main Theorem:

Corollary 4.4. *We use the same terminology as in Definition* 3.1. *Let $x \in X$ be an element that is not a coloop. For $k = -1$, we assume in addition that $J \supseteq \mathcal{H}$ or $J = \{X\}$, i. e. J contains either all or no hyperplanes. Then,*

$$\operatorname{Hilb}(\mathcal{P}(X,k,J)),t) = \begin{cases} \operatorname{Hilb}(\mathcal{P}(X\setminus x,k,J\setminus x),t) & \text{if } x \text{ is a loop} \\ t\operatorname{Hilb}(\mathcal{P}(X\setminus x,k,J\setminus x),t) + \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x),t) & \text{otherwise} \end{cases}$$

For coloops, the situation is more complicated and requires an additional definition. Fix a coloop $x \in X$. Then, $X \setminus x$ is a hyperplane and the following is an upper set:

$$\widehat{J/x} := \{\bar{C} : x \notin C \in J\} \cup \{\overline{X\setminus x}\} \subseteq \mathcal{L}(X/x) \tag{4.2}$$

$\widehat{J/x}$ forgets about the flats containing x, whereas J/x forgets about the flats not containing x. While the latter is always an upper set in $\mathcal{L}(X/x)$, some elements of $\widehat{J/x}$ are not closed unless $X \setminus x$ is a hyperplane.

Theorem 4.5. *We use the same terminology as in Definition* 3.1. *Let $x \in X$ be a coloop and $k \geq 0$. Then,*

$$\operatorname{Hilb}(\mathcal{P}(X,k,J)),t) = \begin{cases} \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x)),t) & \\ +\sum_{j=0}^{k} t^{j+1}\operatorname{Hilb}(\mathcal{P}(X/x,k-j,\widehat{J/x}),t) & \text{if } X\setminus x \in J \\ \operatorname{Hilb}(\mathcal{P}(X/x,k,J/x)),t) & \\ +\sum_{j=0}^{k-1} t^{j+1}\operatorname{Hilb}(\mathcal{P}(X/x,k-j,\widehat{J/x}),t) & \text{if } X\setminus x \notin J \end{cases} \tag{4.3}$$

For $k=-1$, $\operatorname{Hilb}(\ker\mathcal{I}(X,-1,J)),t) = \operatorname{Hilb}(\ker\mathcal{I}(X/x,k,J/x)),t)$ if $X\setminus x \in J$ and otherwise $\operatorname{Hilb}(\ker\mathcal{I}(X,-1,J)),t) = 0$. This holds for arbitrary non-empty upper sets $J \subseteq \mathcal{L}(X)$.

For an example, see Example 6.1. We actually prove a more general statement, namely decomposition formulas for the $\mathcal{P}$-spaces of type $\mathcal{P}(X,k,J) \cong \mathcal{P}(X/x,k,J/x) \oplus \bigoplus_j p_x^{j+1}\mathcal{P}(X/x,k-j,\widehat{J/x})$.

Combinatorial formulas for $k \geq 0$: Theorem 4.2 provides a method to compute the Hilbert series of a $\mathcal{P}$-space combinatorially:

Corollary 4.6. *We use the same terminology as in Definition* 3.1. *Let $k \geq 0$. Then,*

$$\operatorname{Hilb}(\mathcal{P}(X,k,J),t) = \sum_{B\in\mathbb{B}(X)} t^{N-r-|E(B)|}\left(1 + \sum_{\emptyset\neq I\subseteq I(B)} \sum_{j=0}^{\chi((B\cup E(B))\setminus I)+k-1} t^{|I|+j}\binom{j+|I|-1}{|I|-1}\right) \tag{4.4}$$

where $E(B)$ and $I(B)$ denote the sets of externally resp. internally active elements.

From this, we can deduce a result, which relates the dimension of $\mathcal{P}(X,0,J)$ and the number of independent sets satisfying a certain property. This was already proven with a different method by Holtz, Ron, and Xu [12].

Corollary 4.7. $\dim\mathcal{P}(X,0,J) = |\{Y \subseteq X : Y \text{ independent}, \operatorname{cl}(Y) \in J\}|$

Corollary 4.6 gives a formula in terms of the internal and external activity of the bases of X. For $k=0$, there is also a subset expansion formula similar to the one for the Tutte polynomial. In the internal, central and external case, the Hilbert series of the $\mathcal{P}$-spaces are evaluations of the Tutte polynomial [1]. In particular, $\operatorname{Hilb}(\mathcal{P}(X,0,\{X\}),t) = t^{N-r}T_X\left(1,\frac{1}{t}\right)$ and $\operatorname{Hilb}(\mathcal{P}(X,1,\{X\}),t) = t^{N-r}T_X\left(1+t,\frac{1}{t}\right)$. The following theorem provides a formula for the semi-external case which "interpolates" between those two formulas:

Theorem 4.8. *We use the same terminology as in Definition* 3.1.

$$\operatorname{Hilb}(\mathcal{P}(X,0,J),t) = t^{N-r} \sum_{\substack{A \subseteq X \\ \chi(A)=1}} t^{r-\operatorname{rk}(A)} \left(t^{-1}-1\right)^{|A|-\operatorname{rk}(A)} \tag{4.5}$$

The case $k=-1$**:** For $k=-1$, we do not know if there is such a nice formula as in Corollary 4.6 or Theorem 4.8. However, in a special case, a formula is known.

Fix $C_0 \in \mathcal{L}(X)$ and set $J_{C_0} := \{C \in \mathcal{L}(X) : C \supseteq C_0\}$. All maximal missing flats in J_{C_0} are hyperplanes. They have unique defining normals (up to scaling). Holtz, Ron, and Xu [12] showed that $\ker \mathcal{I}(X,-1,J_{C_0}) = \ker \mathcal{I}'(X,-1,J_{C_0}) = \bigcap_{x \in C_0} \mathcal{P}(X \setminus x, 0, \{X \setminus x\})$.

Fix an independent set $K \subseteq X$ that spans C_0. Choose an order on X s. t. K is maximal and define $\mathbb{B}_-(X, J_{C_0}) = \{B \in \mathbb{B}(X) : I(B) \cap K = \emptyset\}$. Then, the following theorem holds:

Theorem 4.9 ([12, p. 20])**.** *We use the same terminology as in Definition* 3.1. *In addition, let* $C_0 \in \mathcal{L}(X)$. *Then,*

$$\operatorname{Hilb}(\ker \mathcal{I}(X,-1,J_{C_0}),t) = \sum_{B \in \mathbb{B}_-(X,J_{C_0})} t^{N-r-|E(B)|} \tag{4.6}$$

5 Zonotopal Cox Rings

In this section, we briefly describe the zonotopal Cox rings defined by Sturmfels and Xu [19] and we show that our Main Theorem can be used to generalize a result on zonotopal Cox modules due to Ardila and Postnikov [1].

Fix m vectors $D_1, \ldots, D_m \in V$ and $\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^m_{\geq 0}$. Sturmfels and Xu [19] introduced the Cox-Nagata ring $R^G \subseteq \mathbb{K}[s_1, \ldots, s_m, t_1, \ldots, t_m]$. This is the ring of polynomials that are invariant under the action of a certain group G which depends on the vectors $D_1, \ldots, D_m$. It is multigraded with a $\mathbb{Z}^{m+1}$-grading. For $r \geq 3$, R^G is equal to the Cox ring of the variety X_G which is gotten from $\mathbb{P}^{r-1}$ by blowing up the points $D_1, \ldots, D_m$. Cox rings have received a considerable amount of attention in the recent literature in algebraic geometry. See [13] for a survey.

Cox-Nagata rings are closely related to power ideals: let $\mathcal{I}_{\boldsymbol{u}} := \operatorname{ideal}\{D_1^{u_1+1}, \ldots, D_m^{u_m+1}\}$ and let $\mathcal{I}^{-1}_{d,\boldsymbol{u}}$ denote the homogeneous component of grade d of $\ker \mathcal{I}_{\boldsymbol{u}}$. Then, $R^G_{(d,\boldsymbol{u})}$, the homogeneous component of R^G of grade $(d, \boldsymbol{u})$, is naturally isomorphic to $\mathcal{I}_{d,\boldsymbol{u}}$ ([19, Proposition 2.1]).

Cox-Nagata rings are an object of great interest but in general, it is quite difficult to understand their structure. However, for some choices of the vectors $D_1, \ldots, D_m$, we understand a natural subring of the Cox-Nagata ring very well.

Let $\mathcal{H} = \{H_1, \ldots, H_m\}$ denote the set of hyperplanes in $\mathcal{L}(X)$. Let $\mathfrak{H} \in \{0,1\}^{m \times N}$ denote the non-containment vector-hyperplane matrix, *i. e.* the 0-1 matrix whose (i,j) entry is 1 if and only if x_j is not contained in H_i.

Sturmfels and Xu defined the following structures: the *zonotopal Cox ring*

$$\mathcal{Z}(X) := \bigoplus_{(d,\boldsymbol{a}) \in \mathbb{Z}^{N+1}_{\geq 0}} R^G_{(d,\mathfrak{H}\boldsymbol{a})} \tag{5.1}$$

and $\mathcal{Z}(X,\boldsymbol{w}) := \bigoplus_{(d,\boldsymbol{a})\in\mathbb{Z}^{N+1}_{\geq 0}} R^G_{(d,\mathfrak{H}\boldsymbol{a}+\boldsymbol{w})}$, the *zonotopal Cox module* of shift $\boldsymbol{w}$ for $\boldsymbol{w}\in\mathbb{Z}^n$.

Let $X(\boldsymbol{a})$ denote the sequence of $\sum_i a_i$ vectors in U that is obtained from X by replacing each x_i by a_i copies of itself and let $\boldsymbol{e} := (1,\dots,1)\in\mathbb{Z}^m_{\geq 0}$. Ardila and Postnikov prove the following isomorphisms [1, Proposition 6.3]: $R^G_{(d,\mathfrak{H}\boldsymbol{a})} \cong \mathcal{P}(X(\boldsymbol{a}),1,\{X\})_d$, $R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e})} \cong \mathcal{P}(X(\boldsymbol{a}),0,\{X\})_d$, and $R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e})} \cong \mathcal{P}(X(\boldsymbol{a}),-1,\{X\})_d$

Using the Main Theorem, we can generalize those results. Let $J_{\boldsymbol{b}} := \{C\in\mathcal{L}(X) : b_H = 1 \text{ for all } H\subseteq C\}$. We have the following results on *hierarchical zonotopal Cox modules* and their Hilbert series:

Proposition 5.1. *We use the same terminology as in Definition* 3.1. *For the graded components of the* semi-external zonotopal Cox module $\mathcal{Z}(X,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e}+\boldsymbol{b})$*, the following holds:*

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e}+\boldsymbol{b})} \cong \mathcal{P}(X(\boldsymbol{a}),0,J_{\boldsymbol{b}})_d \qquad \text{for all } d \tag{5.2}$$

Proposition 5.2. *We use the same terminology as in Definition* 3.1. *Let $C_0\in\mathcal{L}(X)$ be a fixed flat and $J_{C_0} := \{C\in\mathcal{L}(X) : C\supseteq C_0\}$ (cf. Subsection* 4*). If $\boldsymbol{b}\in\{0,1\}^{\mathcal{H}}$ satisfies $b_H = 1$ if and only if $H\supseteq C_0$, then for the graded components of the* semi-internal zonotopal Cox module $\mathcal{Z}(X,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e}+\boldsymbol{b})$)*, the following holds:*

$$R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e}+\boldsymbol{b})} \cong \ker\mathcal{I}(X(\boldsymbol{a}),-1,J_{C_0})_d \qquad \text{for all } d \tag{5.3}$$

Corollary 5.3. *In the setting of Proposition* 5.1*, the dimension of $R^G_{(d,\mathfrak{H}\boldsymbol{a}-\boldsymbol{e}+\boldsymbol{b})}$ equals the coefficient of t^d in*

$$\operatorname{Hilb}(\mathcal{P}(X(\boldsymbol{a}),0,J_{\boldsymbol{b}}),t) = t^{\sum_i a_i - r}\sum_{\substack{A\subseteq X\\ \chi(A)=1}} t^{r-\operatorname{rk}(A)} \sum_{\substack{1\leq s_i\leq a_i\\ \boldsymbol{s}\in\mathbb{Z}^A_{\geq 0},\, x_i\in A}} \left(\prod_i \binom{a_i}{s_i}\right)\left(\frac{1}{t}-1\right)^{\sum_i s_i - \operatorname{rk}(A)}$$

Corollary 5.4. *In the setting of Proposition* 5.2*, the dimension of $R^G_{(d,\mathfrak{H}\boldsymbol{a}-2\boldsymbol{e}+\boldsymbol{b})}$ equals the coefficient of t^d in*

$$\operatorname{Hilb}(\ker\mathcal{I}(X(\boldsymbol{a}),-1,J_{C_0}),t) = \sum_{B\in\mathbb{B}_-(X,J_{C_0})} \sum_{\substack{0\leq s_i\leq a_i-1\\ \boldsymbol{s}\in\mathbb{Z}^B_{\geq 0}}} t^{\sum_{i\,:\,x_i\notin E(B)} a_i - r - \sum_{x_i\in B} s_i} \tag{5.4}$$

6 Examples

In this section, we give examples for the structures and constructions appearing in this extended abstract. Here, we do the following identifications: $\operatorname{Sym}(V) = \operatorname{Sym}(U) = \mathbb{K}[x,y]$ and $\mathbb{K}[x,y]/x = \mathbb{K}[y]$.

$$X_1 := \begin{bmatrix} 1 & 0 & 1\\ 0 & 1 & 1\end{bmatrix} = (x_1,x_2,x_3) \tag{6.1}$$

Let $J_1 := \{X,(x_1),(x_3)\}$. The set of bases is $\mathbb{B}(X) = \{(x_1x_2),(x_1x_3),(x_2x_3)\}$ and the ideal is $\mathcal{I}(X_1,0,J_1) = \operatorname{ideal}\{x^2,xy^2,y^3\}$.

$$S(X_1,0,J_1) = \{1,\, p_{x_1},\, p_{x_2},\, p_{x_3},\, p_{x_1x_2},\, p_{x_2x_3}\} \quad \rightsquigarrow \quad \mathcal{P}(X,0,J_1) = \operatorname{span}\{1,x,y,xy,y^2\}$$
$$\Gamma(X_1,0,J_1) = \{((x_1x_2),\emptyset,0),((x_1x_3),\emptyset,0),((x_1x_3),(x_3),0),((x_2x_3),\emptyset,0),((x_2x_3),(x_2),\emptyset,0)\}$$
$$\mathcal{B}(X_1,0,J_1) = \{p_\emptyset,\, p_{x_2},\, p_{x_2x_3},\, p_{x_1},\, p_{x_1x_2}\}$$

Now we consider the deletion and contraction of x_1. For the upper set J_1, we obtain $J_1 \setminus x_1 = \{(x_2,x_3),(x_3)\}$ and $J_1/x_1 = \{(\bar{x}_2,\bar{x}_3),\bar{\emptyset}\}$.

$$X_1 \setminus x_1 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = (x_2,x_3) \qquad X_1/x_1 = \begin{bmatrix} 1 & 1 \end{bmatrix} = (\bar{x}_2,\bar{x}_3)$$
$$\mathcal{I}(X_1 \setminus x_1, 0, J_1 \setminus x_1) = \operatorname{ideal}\{x,y^2\} \qquad \mathcal{I}(X_1/x_1,0,J_1/x_1) = \operatorname{ideal}\{y^3\}$$
$$\mathcal{P}(X_1 \setminus x_1, 0, J_1 \setminus x_1) = \operatorname{span}\{1,y\} \qquad \mathcal{P}(X_1/x_1,0,J_1/x_1) = \operatorname{span}\{1,y,y^2\}$$

Example 6.1. This is an example for the recursion in Theorem 4.5 and for Proposition 3.8. Let $X_2 :=$

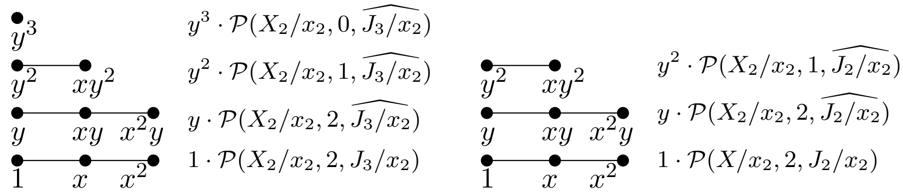

Fig. 1: On the left, we see $\mathcal{P}(X_2,2,J_3)$ and on the right we see $\mathcal{P}(X_2,2,J_2)$. For both spaces, the decompositions corresponding to Theorem 4.5 are shown.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (x_1,x_2)$, $J_2 := \{X_2\}$, and $J_3 := \{X_2,(x_1)\}$. This implies $\widehat{J_2/x_2} = \widehat{J_3/x_2} = \{(x_1)\}$. For a graphic description of the $\mathcal{P}$-spaces involved in the decomposition, see Figure 1.

$$\mathcal{I}(X_2,2,J_2) = \operatorname{ideal}\{x^3,y^3,x^2y^2\} \qquad \mathcal{P}(X_2,2,J_2) = \operatorname{span}\{1,x,y,\ x^2,xy,y^2,\ x^2y,xy^2\}$$
$$\mathcal{I}(X_2,2,J_3) = \operatorname{ideal}\{x^3,y^4,x^2y^2,xy^3\} \quad \mathcal{P}(X_2,2,J_3) = \operatorname{span}\{1,x,y,\ x^2,xy,y^2,\ x^2y,xy^2,y^3\}$$

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Special Cases of the Parking Functions Conjecture and Upper-Triangular Matrices

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Abstract. We examine the $q = 1$ and $t = 0$ special cases of the parking functions conjecture. The parking functions conjecture states that the Hilbert series for the space of diagonal harmonics is equal to the bivariate generating function of $area$ and $dinv$ over the set of parking functions. Haglund recently proved that the Hilbert series for the space of diagonal harmonics is equal to a bivariate generating function over the set of Tesler matrices–upper-triangular matrices with every hook sum equal to one. We give a combinatorial interpretation of the Haglund generating function at $q = 1$ and prove the corresponding case of the parking functions conjecture (first proven by Garsia and Haiman). We also discuss a possible proof of the $t = 0$ case consistent with this combinatorial interpretation. We conclude by briefy discusing possible refinements of the parking functions conjecture arising from this research and point of view. **Note added in proof:** We have since found such a proof of the $t = 0$ case and conjectured more detailed refinements. This research will most likely be presented in full in a forthcoming article.

Résumé. On examine les cas spéciaux $q = 1$ et $t = 0$ de la conjecture des fonctions de stationnement. Cette conjecture déclare que la série de Hilbert pour l'espace des harmoniques diagonaux est égale à la fonction génératrice bivariée (paramètres $area$ et $dinv$) sur l'ensemble des fonctions de stationnement. Haglund a prouvé récemment que la série de Hilbert pour l'espace des harmoniques diagonaux est égale à une fonction génératrice bivariée sur l'ensemble des matrices de Tesler triangulaires supérieures dont la somme de chaque équerre vaut un. On donne une interprétation combinatoire de la fonction génératrice de Haglund pour $q = 1$ et on prouve le cas correspondant de la conjecture dans le cas des fonctions de stationnement (prouvé d'abord par Garsia et Haiman). On discute aussi d'une preuve possible du cas $t = 0$, cohérente avec cette interprétation combinatoire. On conclut en discutant brièvement les raffinements possibles de la conjecture des fonctions de stationnement de ce point de vue. **Note ajoutée sur épreuve:** j'ai trouvé depuis cet article une preuve du cas $t = 0$ et conjecturé des raffinements possibles. Ces résultats seront probablement présentés dans un article ultérieur.

Keywords: parking function, Hilbert series, diagonal harmonics

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1 Introduction

A *parking function* of length n is a sequence $a_1 a_2 \cdots a_n$ such that, for all $1 \leq i \leq n$,

$$|f^{-1}(\{1, 2, \cdots, i\})| \geq i.$$

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Let P_n be the set of parking functions of length n. The parking functions conjecture (see [Hag08]) states that the Hilbert series for the space of diagonal harmonics is given by the following bivariate generating function over P_n.

$$\sum_{a_1 a_2 \cdots a_n \in P_n} q^{dinv(a_1 a_2 \cdots a_n)} t^{area(a_1 a_2 \cdots a_n)},$$

where $area$ and $dinv$ are statistics defined on parking functions.

Let the m-th *hook sum* of an upper-triangular matrix $A = [a_{i,j}]_{1 \leq i \leq j \leq n}$ be given by

$$\sum_{r=m}^{n} a_{m,r} - \sum_{k=1}^{m-1} a_{k,m}.$$

Let A be a *Tesler matrix* if and only if it has non-negative integer entries and every hook sum is equal to one. Let T_n be the set of $n \times n$ Tesler matrices. The following matrix is an example of a member of T_5:

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 1 & 0 & 1 \\ & & 1 & 1 & 0 \\ & & & 2 & 0 \\ & & & & 2. \end{matrix}$$

Let $[n]_{q,t}$ be the polynomial $\sum_{i=0}^{n-1} q^i t^{n-1-i}$. If $A \in T_n$, let $extra(A)$ be n less than the number of nonzero entries of A. (By construction, A must have at least one nonzero entry per row.)

Haglund [Hag] recently proved that the Hilbert series of diagonal harmonics is given by the following bivariate generating function over T_n:

$$\sum_{A=[a_{i,j}] \in T_n} ((q-1)(1-t))^{extra(A)} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}.$$

As a result of Haglund's proof, the parking functions conjecture can now be considered equivalent to the following identity of bivariate generating functions:

$$\sum_{a_1 a_2 \cdots a_n \in P_n} q^{dinv(a_1 a_2 \cdots a_n)} t^{area(a_1 a_2 \cdots a_n)} = \sum_{A=[a_{i,j}] \in T_n} ((q-1)(1-t))^{extra(A)} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}. \qquad (1)$$

In the following extended abstract, we will prove the $q = 1$ special case of Equation (1) and give an outline of a proof of the $t = 0$ special case. The $q = 1$ special case was first proven by Garsia and Haiman [GH96].

2 The $q = 1$ special case

Let $T_n^* = \{A \in T_n : extra(A) = 0\}$. Alternately, let T_n^* be the set of $n \times n$ Tesler matrices with precisely one nonzero entry per row. Let $[n]_t = [n]_{q,t}|_{q=1} = \sum_{i=0}^{n-1} t^i$. For $a_1 a_2 \cdots a_n \in P_n$, let $area(a_1 a_2 \cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i$.

When Equation (1) is evaluated at $q = 1$, the following equation results:

$$\sum_{a_1a_2\cdots a_n \in P_n} t^{area(a_1a_2\cdots a_n)} = \sum_{A=[a_{i,j}]\in T_n^*} \prod_{a_{i,j}>0} [a_{i,j}]_t. \tag{2}$$

To prove Equation (2), we will begin by constructing a surjective map ϕ from P_n to S_n. (**Note added in proof:** Haglund pointed out that this map sends each parking function to the inverse of the permutation it "parks" to. We will retain our original description.) We will then give a simple proof that $|T_n^*| = n!$. We will then construct a bijection between S_n and T_n^* by associating each permutation $\pi \in S_n$ to a unique Tesler matrix $C_\pi \in T_n^*$. Finally, we will prove that, for each $\pi \in S_n$,

$$\sum_{\phi(a_1a_2\cdots a_n)=\pi} t^{area(a_1a_2\cdots a_n)} = \prod_{C_\pi=[a_{i,j}],a_{i,j}>0} [a_{i,j}]_t.$$

This will prove Equation (2).

2.1 A surjective map from P_n to S_n.

Let the function ϕ on the set of n-sequences of positive integers be defined as follows: Given an n-sequence of positive integers $a_1a_2\cdots a_n$, let the sequence $b_1b_2\cdots b_n$ be defined recursively by

- $b_1 = a_1$, and
- For $i > 1$, let b_i be the smallest integer greater than or equal to a_i that is not a member of $\{b_1, b_2, \ldots, b_{i-1}\}$.

Let $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$.

Lemma 1. *1. If $a_1a_2\cdots a_n \in S_n$, then $\phi(a_1a_2\cdots a_n) = a_1a_2\cdots a_n$.*

2. $\phi(a_1a_2\cdots a_n) \in S_n$ if and only if $a_1a_2\cdots a_n \in P_n$.
Therefore, ϕ is a surjective map from P_n to S_n.

Proof. 1. If $a_1a_2\cdots a_n \in S_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$ as above, then $b_1 = a_1$. Assume inductively that $a_j = b_j$ for all $j \le i-1$. Then $a_i \notin \{b_1, b_2, \cdots, b_{i-1}\} = \{a_1, a_2, \cdots, a_{i-1}\}$, and a_i is obviously the smallest such integer greater than or equal to a_i. Therefore $a_i = b_i$ for all i, and $\phi(a_1a_2\cdots a_n) = a_1a_2\cdots a_n$.

2. If $a_1a_2\cdots a_n \notin P_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$ as above, then there must be an integer j such that $|\{i : a_i \le j\}| < j$. Since $b_i \ge a_i$ for all i, $b_i \le j$ only if $a_i \le j$. Therefore there are strictly fewer than k integers i such that $b_i \le j$. Therefore there are strictly fewer than j integers i such that $b_i \le j$. Therefore $b_1b_2\cdots b_n \notin S_n$. Therefore $b_1b_2\cdots b_n \in S_n$ only if $a_1a_2\cdots a_n \in P_n$.

   If $a_1a_2\cdots a_n \in P_n$ and $\phi(a_1a_2\cdots a_n) = b_1b_2\cdots b_n$, then $b_1b_2\cdots b_n \notin S_n$ if and only if, for some $R \in [n]$, there does not exist an integer i such that $b_i = R$. Assume R is minimal with this property. Therefore $[R-1] \subset \{b_1, b_2, \cdots, b_n\}$.

   Because $a_1a_2\cdots a_n$ is a parking function, there must be at least R integers i such that $a_i \le R$. Let $c_1 < c_2 < \cdots < c_P$ be the increasing rearrangement of these integers, so $P \ge R$. For each $1 \le j \le P$, $a_{c_j} \le R$ and $b_{c_j} \ne R$. If $b_{c_j} > R$, then R would be an integer less than b_{c_j} that is greater than a_{c_j} and not in $\{b_1, b_2, \cdots, b_{c_j-1}\}$. This contradicts the definition of b_{c_j}. Since $b_{c_j} \neq R$ by definition, $b_{c_j} < R$ for all j. Therefore $b_{c_1}, b_{c_2}, \ldots, b_{c_P}$ is a sequence of P distinct integers strictly less than R with $P \geq R$. This is impossible, and therefore $b_1 b_2 \cdots b_n \in S_n$.

Therefore ϕ is a surjective map from P_n to S_n. □

Let $\pi_1 \pi_2 \cdots \pi_n$ be a parking function of length n. Let the functions $g_\pi(i)$ be defined by $g_\pi(i) = min(\pi_i - \pi_j : i < j, \pi_i > \pi_j)$, with $g_\pi(i) = \pi_i$ if there is no such π_j. For example, if $\pi = 24153$, then $(g_\pi(1), g_\pi(2), g_\pi(3), g_\pi(4), g_\pi(5)) = (1, 1, 1, 2, 3)$. Alternately, $g_\pi(i)$ is the largest integer such that $\pi_i - 1, \pi_i - 2, \cdots, \pi_i - g_\pi(i) + 1$ all appear before π_i in π.

Theorem 2. *Let $\pi = \pi_1 \pi_2 \cdots \pi_n$ be a permutation in S_n.*

1. $\phi^{-1}(\pi_1 \pi_2 \cdots \pi_n) = \{a_1 a_2 \cdots a_n : \pi_i \geq a_i \geq \pi_i - g_\pi(i) + 1\}$.
2. $|\phi^{-1}(\pi_1 \pi_2 \cdots \pi_n)| = \prod_{i=1}^n g_\pi(i)$.
3. $$(n+1)^{n-1} = \sum_{\pi \in S_n} \prod_{i=1}^n g_\pi(i).$$
4. $$\sum_{\phi(a_1 a_2 \cdots a_n) = \pi} t^{area(a_1 a_2 \cdots a_n)} = \prod_{i=1}^n [g_\pi(i)]_t.$$

Proof. (sketch) Consider $\phi^{-1}(24153)$. The parking function $23153 \notin \phi^{-1}(24153)$, since ϕ will keep the second entry equal to 3, as another 3 has not yet appeared. In fact, $\phi(23153) = 23154$. However, $24143 \in \phi^{-1}(24153)$, since ϕ will change the fourth entry from 4 to 5, since another 4 will have appeared in the second entry. Similarly, $344215 \notin \phi^{-1}(346215)$, since ϕ will not change the third entry from a 4 to a 6. Instead, because the integer 5 between 4 and 6 has not yet appeared, ϕ will change the third entry from 4 to 5, and $\phi(344215) = 345216$. In general, π_i can be replaced by an integer k and still have ϕ change k back to π_i if and only if k, and all integers between k and π_i, appear before π_i. Alternately, π_i can be replaced by k if and only if $\pi_i \geq k \geq \pi_i - g_\pi(i) + 1$. The rest of the theorem follows directly or from the definition $area(a_1 a_2 \cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^n a_i$. □

2.2 A weight-preserving bijection between T_n^* and S_n.

Recall that T_n^* is the set of Tesler matrices with precisely one nonzero entry per row.

Lemma 3. *A Tesler matrix in T_n^* is uniquely determined by the locations of its nonzero entries. In particular, $|T_n^*| = n!$.*

Proof. (sketch) There are n choices about where to put the nonzero entry in the first row, which must equal 1. There are then $n-1$ choices about where to put the nonzero entry in the second row, which must equal 1 unless the nonzero entry in the first row was in the second column, in which case it must equal 2. Continuing in this fashion from top to bottom it is clear that the condition that every hook sum is equal to one uniquely determines the entries once their location is determined. □

(This lemma was realized by the author along with Mirkó Visontai).

We will now define a map from S_n to T_n^*. Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$, let $h_\pi(i)$ be the smallest integer such that $i < h_\pi(i)$, $\pi_i < \pi_{h_\pi(i)}$, and, if $\pi_i < k < \pi_{h_\pi(i)}$, then $k \in \{\pi_1, \pi_2, \ldots, \pi_{i-1}\}$. Alternately, $h_\pi(i)$ is the location of the first entry to the right of π_i that is larger than π_i but such that every integer in between it and π_i appears to the left of π_i. (If there is no such integer let $h_\pi(i) = i$.) For example, if $\pi = 24153$, then $(h_\pi(1), h_\pi(2), h_\pi(3), h_\pi(4), h_\pi(5)) = (5, 4, 5, 4, 5)$.

By construction, $i \le h_\pi(i) \le n$ for all i. Therefore, let $C_\pi \in T_n^*$ be the unique Tesler matrix with nonzero entries precisely at $(i, h_\pi(i))$ for all $1 \le i \le n$. For example, if $\pi = 24153$, then

$$C_\pi = \begin{matrix} 0 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 \\ & & 0 & 0 & 1 \\ & & & 2 & 0 \\ & & & & 3. \end{matrix}$$

We will refer to the following inductive construction of a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ as the *bumping algorithm*. Given a fixed integer $1 \le m_1 \le n$ and permutation $\pi^* = \pi_1^*\pi_2^*\cdots\pi_{n-1}^*$, let $\pi_1\pi_2\cdots\pi_n$ be defined as follows: If $m_1 = 1$, then $\pi_1\pi_2\cdots\pi_n = (n+1)\pi_1^*\pi_2^*\cdots\pi_{n-1}^*$. If $m_1 > 1$, then

- $\pi_1 = \pi_{m_1-1}^*$.
- $\pi_i = \pi_{i-1}^*$ if $\pi_{i-1}^* < \pi_{m_1-1}^*$.
- $\pi_i = \pi_{i-1}^* + 1$ if $\pi_{i-1}^* \ge \pi_{m_1-1}^*$.

For example, if $\pi^* = 412635$ and $m_1 = 4$, then the bumping algorithm results in $\pi = 2513746$.

Theorem 4. *1. If π is constructed by the bumping algorithm from m_1 and π^*, then $h_\pi(1) = m_1$ and $h_\pi(i) = h_{\pi^*}(i-1) + 1$ for $i > 2$.*

2. Therefore the map $\pi \to C_\pi$ is a bijection between S_n and T_n^*.

Proof. (sketch) The first part follows from the definition of h_π and inductively demonstrates that for every sequence $m_1m_2\cdots m_n$ such that $i \le m_i \le n$ for all i, there is at least one permutation $\pi = \pi_1\pi_2\cdots\pi_n$ with $h_\pi(i) = m_i$ for all i. This proves the second part. □

Given a permutation $\pi \in S_n$, let D_π be the increasing forest on $[n]$ constructed inductively as follows: Given a permutation $\pi^* = \pi_1^*\pi_2^*\cdots\pi_{n-1}^*$ and integer $1 \le m_1 \le n$, increase the label of every node of D_{π^*} by one. Add 1 as a leaf of the node now labelled m_1. If $m_1 = 1$, leave 1 as an isolated node. Let D_π be the resulting increasing forest, where π is constructed by the bumping algorithm from π^* and m_1. (Increasing forests are well-known to be equinumerous with permutations, see [Sta97].)

Theorem 5. *For each $i \in [n]$, the parent of the node labelled i in the increasing forest D_π is the node labelled $h_\pi(i)$.*

Proof. The parent of the node labelled 1 is the node labelled m_1. For $i > 1$, assume inductively that the parent of the node labelled $i-1$ in D_{π^*} is the node labelled $h_{\pi^*}(i-1)$. Then the parent of the node labelled i in D_π is the node labelled $h_{\pi^*}(i-1) + 1 = h_\pi(i)$. □

Theorem 6. *1. Given $\pi \in S_n$, the Tesler matrix $C_\pi \in T_n^*$ has the entry $g_\pi(i)$ at $(i, h_\pi(i))$. Equivalently, if an upper-triangular matrix has only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$, then it is a Tesler matrix.*

2.
$$\prod_{C_\pi=[a_{i,j}],a_{i,j}>0}[a_{i,j}]_t = \prod_{i=1}^{n}[g_\pi(i)]_t = \sum_{\phi(a_1a_2\cdots a_n)=\pi} t^{area(a_1a_2\cdots a_n)}.$$

Proof. (sketch) From the definitions of g_π and h_π and taking $h_\pi^0(j) = j$, it can be seen that

$$g_\pi(i) = |\{j : h_\pi^r(j) = i\}|.$$

Alternately, $g_\pi(i)$ is the number of integers eventually sent to i by h_π. Using Theorem (5), h_π can be seen as a function that sends an integer labelling a node in the increasing forest D_π to the integer labelling the parent of that node. Therefore $g_\pi(i)$ is the number of descendents in D_π of the node labelled i, including the node itself.

If an upper-triangular matrix has only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$, then the i-th hook sum is $g_\pi(i) - \sum_{j:h_\pi(j)=i} g_\pi(j)$. This is equal to the number of descendents in D_π of the node labelled i (including itself) minus the number of descendents of the nodes with i as a parent (including themselves) This must equal 1. Therefore the upper-triangular matrix with only the nonzero entries $g_\pi(i)$ at $(i, h_\pi(i))$ is the unique Tesler matrix $C_\pi \in T_n^*$. □

This proves Equation (2).

3 The $t = 0$ special case

Note added in proof: The version of this extended abstract originally submitted described our efforts to find a proof of Equation (3). We have since completed a proof of Equation (3), which we will give the broad outlines of below. A more complete description of both proofs may likely be in a forthcoming article.

Recall that, if $a_1a_2\cdots a_n$ is a parking function of length n, then $area(a_1a_2\cdots a_n) = \binom{n+1}{2} - \sum_{i=1}^{n} a_i$.

Lemma 7. *A parking function $a_1a_2\cdots a_n \in P_n$ has $area$ equal to zero if and only if it is a permutation in S_n.*

Therefore, when Equation (1) is evaluated at $t = 0$, the following equation results:

$$\sum_{\pi_1\pi_2\cdots\pi_n\in S_n} q^{dinv(\pi_1\pi_2\cdots\pi_n)} = \sum_{A=[a_{i,j}]\in T_n} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)}. \tag{3}$$

The $dinv$ statistic is complicated to define on general parking functions (see [Hag08]). However, it is easier to define on permutations: For $\pi_1\pi_2\cdots\pi_n \in S_n$, let

$$dinv(\pi_1\pi_2\cdots\pi_n) = |\{(i,j) : i < j, \pi_i < \pi_j\}|.$$

Alternately, $dinv$ gives the number of pairs of entries of $\pi_1\pi_2\cdots\pi_n$ that are not inversions. It is well known that $dinv$ is a *Mahonian* statistic when applied to permutations, so

$$\sum_{\pi_1\pi_2\cdots\pi_n\in S_n} q^{dinv(\pi_1\pi_2\cdots\pi_n)} = \prod_{i=1}^{n}[i]_q.$$

We will sketch a proof of Equation (3). In particular, we will partition the set of $n\times n$ Tesler matrices T_n into $n!$ disjoint subsets T_π indexed by the permutations $\pi\in S_n$. These subsets T_π are such that, for each $\pi\in S_n$,

$$\sum_{A=[a_{i,j}]\in T_\pi} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)} = q^{dinv(\pi)}.$$

This will prove Equaiton (3).

3.1 Decoding Tesler Matrices

Given a Tesler matrix $A=[a_{i,j}]\in T_n$, let A^* be the multiset of ordered pairs and integers such that $(i,j)\in A^*$ with multiplicity k if and only if $a_{i,j}=k$ for $i<j$ and $i\in A^*$ with multiplicity k if and only if $a_{i,i}=k$. For example, if

$$A = \begin{matrix} 0 & 1 & 0 & 0 & 0 \\ & 0 & 2 & 0 & 0 \\ & & 1 & 1 & 1, \\ & & & 1 & 1 \\ & & & & 3 \end{matrix}$$

then $A^*=\{(1,2),(2,3),(2,3),3,(3,4),(3,5),4,(4,5),5,5,5\}$. From each such A^*, we can construct a list of n vertically-arranged sets, each written in increasing order, such that:

- a is immediately before b in a set if and only if $(a,b)\in A^*$.
- The last entries in each set are the single integers in A^*, weakly decreasing from the top set to the bottom set.

In particular, we want to construct such a list of n vertically-arranged sets, each written in increasing order, such that:

- The last entry in each set must be greater than or equal to any entry in a set below.
- The first entry in each set does not appear in any set below, and is the only entry in that set where this is the case.
- If a is immediately before b in a set and c is immediately before b in a set above, then $c\geq b$.

For example, one list of sets resulting from $\{(1,2),(2,3),(2,3),3,(3,4),(3,5),4,(4,5),5,5,5\}$ satisfying the above rules is

$$\begin{array}{rrrrr} 1 & 2 & 3 & 4 & 5 \\ & & 2 & 3 & 5 \\ & & & & 5 \\ & & & & 4 \\ & & & & 3 \end{array}$$

We will omit the proof of the following theorem stating that this construction is bijective.

Theorem 8. *1. There is precisely one list of sets satisfying the above rules for each matrix $A \in T_n$.*

2. If a list of sets satisfies the above rules, then the first entry of each set, read from top to bottom, gives a permutation π.

For a fixed $\pi \in S_n$, let T_π be the set of Tesler matrices $A \in T_n$ such that the unique list of sets satisfying the above rules gives the permutation π. For example, T_{31254} consists of the following Tesler matrices (each paired with the corresponding list of sets)

$$\begin{array}{rrr} & 3 & 5 \\ 1 & 2 & 5 \\ & 2 & 5 \\ & & 5 \\ & & 4 \end{array} \quad \begin{array}{rrrrr} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 2 \\ & & 0 & 0 & 1 \\ & & & 1 & 0 \\ & & & & 4 \end{array} \qquad\qquad \begin{array}{rrr} 3 & 4 & 5 \\ 1 & 2 & 5 \\ & 2 & 5 \\ & & 5 \\ & & 4 \end{array} \quad \begin{array}{rrrrr} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 2 \\ & & 0 & 1 & 0 \\ & & & 1 & 1 \\ & & & & 4 \end{array}$$

$$\begin{array}{rrrr} & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 \\ & & 2 & 5 \\ & & & 5 \\ & & & 4 \end{array} \quad \begin{array}{rrrrr} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 1 & 1 \\ & & 0 & 1 & 0 \\ & & & 1 & 2 \\ & & & & 4 \end{array} \qquad\qquad \begin{array}{rrr} 3 & 4 & 5 \\ 1 & 4 & 5 \\ & 2 & 5 \\ & & 5 \\ & & 4 \end{array} \quad \begin{array}{rrrrr} 0 & 0 & 0 & 1 & 0 \\ & 0 & 0 & 0 & 1 \\ & & 0 & 1 & 0 \\ & & & 1 & 2 \\ & & & & 4 \end{array}$$

$$\begin{array}{rrrr} & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 \\ & 2 & 4 & 5 \\ & & & 5 \\ & & & 4 \end{array} \quad \begin{array}{rrrrr} 0 & 1 & 0 & 0 & 0 \\ & 0 & 0 & 2 & 0 \\ & & 0 & 1 & 0. \\ & & & 1 & 3 \\ & & & & 4 \end{array}$$

Note that the Tesler matrix C_{31254} is in T_{31254}, and that the sum of $(q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)}$ over T_{31254} is equal to $q^7 = q^{dinv(31254)}$. Finally, note that the term of q^7 comes from the term of the generating function corresponding to the last Tesler matrix. In general, for $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, T_π will always contain precisely one Tesler matrix such that the first set in the corresponding list of sets consists of π_1 followed by everything after and larger than π_1, the second set consists of π_2 followed by everything after and larger than π_2, and so on. This Tesler matrix will always give a term of $q^{dinv(\pi)}$. It is possible to define an involution that cancels out the other terms, but we will not describe the involution in this extended abstract.

Theorem 9. *The following equation holds due to an involution:*

$$\sum_{A=[a_{i,j}]\in T_\pi} (q-1)^{extra(A)} q^{\sum_{a_{i,j}>0}(a_{i,j}-1)} = q^{dinv(\pi)}.$$

3.2 A non-Mahonian statistic on S_n

Having sketched a combinatorial proof of Equation (3), we will now discuss how it relates to the combinatorial proof presented earlier of Equation (2).

We can define a statistic akin to $dinv$ by replacing the t-analogues in Equation (2) with q,t-analogues. For $a_1a_2\cdots a_n \in P_n$ with $\phi(a_1a_2\cdots a_n) = \pi_1\pi_2\cdots\pi_n$, let $pdinv(a_1a_2\cdots a_n)$ be defined by

$$pdinv(a_1a_2\cdots a_n) = \sum_{i=1}^{n}(a_i - \pi_i + g_\pi(i) + 1).$$

Alternately, $pdinv(a_1a_2\cdots a_n)$ is the sum of how much there is left that can be subtracted from each entry and still leave a parking function in $\phi^{-1}(\pi_1\pi_2\cdots\pi_n)$.

Lemma 10. *1. $pdinv$ satisfies the following generating function identity:*

$$\sum_{a_1a_2\cdots a_n \in P_n} q^{pdinv(a_1a_2\cdots a_n)} t^{area(a_1a_2\cdots a_n)} = \sum_{A=[a_{i,j}]\in T_n^*} \prod_{a_{i,j}>0} [a_{i,j}]_{q,t}.$$

2. $pdinv$ is equidistributed with $area$ over P_n and is therefore equidistributed with $dinv$ over P_n.

Proof. The first part of the lemma is clear from the alternate description of $pdinv$. The equidistribution of $area$ and $pdinv$ follows from the fact that each $[a_{i,j}]_{q,t}$ is symmetric in q and t. The equidistribution $dinv$ and $pdinv$ then follows from the equidistribution of $area$ and $dinv$. □

However, $(pdinv, area)$ is *not* joint-equidistributed with $(dinv, area)$ over P_n. If it were, then there would be no need for the terms of Equation (1) corresponding to Tesler matrices with more than one entry per row. For example, the sum of $q^{pdinv(a_1a_2a_3)}$ over S_3, the set of parking functions of length 3 with zero area, is equal to $1 + 3q + q^2 + q^3$, while the sum of $q^{dinv(a_1a_2a_3)}$ over S_3 is equal to $1 + 2q + 2q^2 + q^3$. ($dinv$ being Mahonian over S_3.)

We can therefore think of the Tesler matrices with more than one entry per row as somehow de-coupling and re-coupling $pdinv$ and $area$ in some parking functions to obtain (conjecturally) the $(dinv, area)$ joint distribution.

As with $dinv$, $pdinv$ is difficult to interpret combinatorially as a statistic on P_n but easier to interpret combinatorially as a statistic on S_n. In particular, $pdinv$ enumerates a subset of non-inversion pairs.

Theorem 11. *For $\pi_1\pi_2\cdots\pi_n \in S_n$, $pdinv(\pi_1\pi_2\cdots\pi_n)$ is the number of pairs (i,j) such that $i < j$, $\pi_i < \pi_j$ and, for all $i < j < k$, $\pi_k \notin [\pi_i, \pi_j]$. Alternately, $pdinv(\pi_1\pi_2\cdots\pi_n)$ is the number of pairs (i,j) such that π_j forms a non-inversion with π_i and there is no integer after it in between π_i and π_j in magnitude.*

Proof. (sketch) By definition, $pdinv(\pi_1\pi_2\cdots\pi_n) = -n + \sum_{i=1}^{n} g_\pi(i)$. Recall that $g_\pi(i)$ is the largest integer such that $\pi_i - 1, \pi_i - 2, \cdots, \pi_i - g_\pi(i) + 1$ all appear before π_i in π. π_i will form a non-inversion with each of these integers, and for each integer there can be no integer after π_i in between it and π_i in magnitude. Since there are $g_\pi(i) - 1$ such integers, there will be $-n + \sum_{i=1}^{n} g_\pi(i) = pdinv(\pi_1\pi_2\cdots\pi_n)$ such pairs in all. □

Looking only at the $t = 0$ special case, we can therefore think of the effect of the matrices in T_π as adding these "missing" anti-inversions back to $pdinv(\pi)$ to result in $dinv(\pi)$. Each of these missing anti-inversions comes from the $pdinv$ of another parking function, and so on.

4 Further Directions and Conjectures

Note added in proof: This section has been updated to account for our current work and conjectures.

The proof of the $t = 0$ special case, Equation (3), shows that the Tesler matrices can be divided into disjoint subsets T_π indexed by the permutations $\pi \in S_n$ such that the Haglund generating function is q-positive at $t = 0$ when summed over each subset. The full Haglund generating function is not q, t-positive when summed over each T_π. However, we have verified for $n \leq 7$ that the Haglund generating function is q, t-positive when summed over the union of T_π over naturally-defined subsets of S_n. This suggests that the parking functions conjecture might be refinable. Unfortunately, the resulting positive generating functions do not match the generating function of $dinv$ and $area$ over the corresponding parking functions. This suggests that some alternate statistic f, such that $(f, area)$ is joint-equidistributed with $(dinv, area)$, might better illuminate the combinatorics of the parking functions conjecture.

Also, we are unfamiliar with the work surrounding the Kreveras theorem, which states that parking functions with area k are equinumerous with spanning forests with k inversions, but it apparently lacks a "nice" bijective proof, and it is possible our proof of Equation (2) could yield one through our use of spanning increasing forests. See Pak [Pak] and Kreveras [Kre80].

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Matrices with restricted entries and q-analogues of permutations (extended abstract)

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Abstract. We study the functions that count matrices of given rank over a finite field with specified positions equal to zero. We show that these matrices are q-analogues of permutations with certain restricted values. We obtain a simple closed formula for the number of invertible matrices with zero diagonal, a q-analogue of derangements, and a curious relationship between invertible skew-symmetric matrices and invertible symmetric matrices with zero diagonal. In addition, we provide recursions to enumerate matrices and symmetric matrices with zero diagonal by rank. Finally, we provide a brief exposition of polynomiality results for enumeration questions related to those mentioned, and give several open questions.

Résumé. Nous étudions certaines fonctions qui comptent des matrices à coefficients dans un corps fini d'un rang donné ayant certaines entrées égales à zéro. Nous montrons que ces matrices sont des q-analogues des permutations avec certaines valeurs restreintes, et nous obtenons une formule simple et fermée pour calculer le nombre de matrices inversibles avec zéro sur toute la diagonale. De plus nous donnons des récursions pour énumérer par le rang les matrices et les matrices symétriques avec des zéros sur la diagonale. Pour finir, nous faisons un exposé concis des résultats sur la polynomialité des fonctions énumératives liées à celles qui sont mentionnées antérieurement, et nous incluons plusieurs questions ouvertes.

Resumen. Estudiamos ciertas funciones que cuentan matrices con un rango dado sobre un campo finito y con ciertas entradas iguales a cero. Mostramos que estas matrices son un q-análogo de permutaciones con ciertos valores restringidos. También obtenemos una recursión simple y cerrada para el número de matrices invertibles con ceros en toda la diagonal. Además, damos recursiones para enumerar matrices y matrices simétricas con ceros en la diagonal por rango. Finálmente, damos una exposición breve de resultados sobre la polinomialidad de funciones enumerativas relacionadas a las anteriormente mencionadas e incluimos varias preguntas abiertas.

Keywords: linear algebra over finite fields, q-analogues, derangements

1 Introduction

Fix a prime power q. Let $\mathbf{F}_q$ denote the field with q elements and let $\mathbf{GL}(n, q)$ denote the group of $n \times n$ invertible matrices over $\mathbf{F}_q$. The **support** of a matrix (A_{ij}) is the set of indices (i, j) such that $A_{ij} \neq 0$.

Our work was initially motivated by the following question of Richard Stanley: how many matrices in $\mathbf{GL}(n,q)$ have support avoiding the diagonal entries? The answer to this question is

$$q^{\binom{n-1}{2}-1}(q-1)^n\left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}[n-i]_q!\right),$$

which is proven in Proposition 2.1 as part of a more general result. This question has a natural combinatorial appeal and is reminiscent of the work of Buckheister (Buc) and Bender (Ben) enumerating invertible matrices over $\mathbf{F}_q$ with trace zero (see also (Sta1, Prop. 1.10.15)). It also falls naturally into two broader contexts, the study of q-analogues of permutations and the study of polynomiality results for certain counting problems related to algebraic varieties over $\mathbf{F}_q$.

In the former context, we consider the following situation: fix $m, n \geq 1$, $r \geq 0$, and $S \subset \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Let T_q be the set of $m \times n$ matrices A over $\mathbf{F}_q$ with rank r and support contained in the complement of S. Also, let T_1 be the set of 0-1 matrices with exactly r 1's, no two of which lie in the same row or column, and with support contained in the complement of S (i.e., the set of **rook placements** avoiding S). We have that T_1 is a q**-analogue** of T_q, in the following precise sense:

Proposition 5.1 *We have* $\#T_q \equiv \#T_1 \cdot (q-1)^r \pmod{(q-1)^{r+1}}$.

In particular, when $\#T_q$ is a polynomial function of q we have that $\#T_q$ is divisible by $(q-1)^r$ and $\#T_q/(q-1)^r|_{q=1} = \#T_1$. Thus, rank r matrices whose support avoids the set S can be seen as a q-analogue of rook placements that avoid S. Applying this to our situation where S is the set of diagonal entries, we get that the set of invertible matrices avoiding the diagonal is a q-analogue of the set of derangements, a fact that can also be seen directly from the explicit formula above. (There is also a more conceptual explanation for this using the Bruhat decomposition of $\mathbf{GL}(n,q)$: see (LLMPSZ, Sec. 2.2).)

Note that for an arbitrary set S of positions, the function $\#T_q$ need not be a polynomial in q. (Stembridge (Ste1) gives an example of non-polynomial $\#T_q$ with $n = m = 7$, $r = 7$, and a set S with $\#S = 28$.) The second context concerns the question of which sets S give a polynomial $\#T_q$ and is deeply related to a speculation of Kontsevich from 1997 (see Stanley (Sta2) and Stembridge (Ste1)) that was proven false by Belkale and Brosnan (BB). We provide further background on this topic in Section 5.

We close this introduction with a summary of the results of our paper.

Section 2 is concerned with Stanley's question on the enumeration of matrices in $\mathbf{GL}(n,q)$ with zero diagonal. We attack this problem by enumerating larger classes of matrices. We provide two recursions, one based on the size of the matrix and the other based on the rank of the matrix, and we provide a closed-form solution for the first recursion.

In Section 3, we enumerate symmetric matrices in $\mathbf{GL}(n,q)$ whose support avoids the diagonal in the case that n is even. These matrices may be viewed as a q-analogue of fixed point-free involutions. A curious byproduct of our formula (originally due to Jones (Jon)) is that it also counts the number of symmetric matrices in $\mathbf{GL}(n-1,q)$ and the number of skew-symmetric matrices in $\mathbf{GL}(n,q)$. In fact, the varieties associated to these three classes of matrices are pairwise non-isomorphic, and we have not found a satisfactory reason that their solution sets have the same size.

In Section 4, we attack the general problem of enumerating symmetric matrices with zeroes on the diagonal with given rank. We provide recursions for arbitrary rank and solve the full rank case to obtain the enumeration of symmetric matrices in $\mathbf{GL}(n,q)$ when n is odd. The situation in this case is significantly more complicated than in Sections 2 and 3.

Finally, in Section 5 we revisit Proposition 5.1, discuss the two broader contexts mentioned above, and give some open questions about families of sets S for which $\#T_q(m \times n, S, r)$ is a polynomial in q.

For a complete version of this extended abstract, see (LLMPSZ).

Notation

Given an integer n, we define the q-number $[n]_q = \frac{q^n-1}{q-1}$, the q-factorial $[n]_q! = [n]_q \cdot [n-1]_q \cdot [n-2]_q \cdots$ and the q-double factorial $[n]_q!! = [n]_q \cdot [n-2]_q \cdot [n-4]_q \cdots$. In addition, we use a number of invented notations; to avoid confusion and for easy reference, we include a table of these functions here. The last column indicates the sections in which the notation is used.

set	# set	description	section
$\mathrm{Mat}_0(n,k,r)$	$\mathrm{mat}_0(n,k,r)$	set of $n \times n$ matrices of rank r over $\mathbf{F}_q$ with first k diagonal entries equal to zero	2
$\mathrm{Sym}(n)$	$\mathrm{sym}(n)$	set of $n \times n$ symmetric invertible matrices over $\mathbf{F}_q$	3, 4
$\mathrm{Sym}(n,r)$	$\mathrm{sym}(n,r)$	set of $n \times n$ symmetric matrices over $\mathbf{F}_q$ of rank r	4
$\mathrm{Sym}_0(n,r)$	$\mathrm{sym}_0(n,r)$	set of $n \times n$ symmetric matrices with rank r over $\mathbf{F}_q$ with diagonal entries equal to zero	4
$\mathrm{S}(n,k)$	$\mathrm{s}(n,k)$	set of $n \times n$ symmetric invertible matrices with first k diagonal entries equal to zero	3, 4
$\mathrm{Sym}_0(n,k,r)$	$\mathrm{sym}_0(n,k,r)$	set of $n \times n$ symmetric matrices with rank r with first k diagonal entries equal to zero	4
$T_q(m \times n, S, r)$	$\#T_q(m \times n, S, r)$	set of $m \times n$ matrices over $\mathbf{F}_q$ with rank r and support contained in the complement of S	1, 5
$T_1(m \times n, S, r)$	$\#T_1(m \times n, S, r)$	set of 0-1 matrices with exactly r 1's, no two of which lie in the same row or column, and with support contained in the complement of S	1, 5

2 Matrices with zeroes on the diagonal

In this section, we consider the problem of counting invertible matrices over $\mathbf{F}_q$ with zero diagonal. In Section 2.1 we recursively count full rank matrices of rectangular shape with all-zero diagonal and in Section 2.2 we recursively count square matrices by rank and number of zeroes on the diagonal. We solve the first recursion and obtain a closed form formula for the number of invertible matrices with zeroes on the diagonal. These numbers give an enumerative q-analogue of the derangements, i.e., dividing all factors of $q-1$ and setting $q=1$ in the result gives the number of derangements.

2.1 Recursion by size

For $1 \leq k \leq n$, denote by $f_{k,n}$ the number of $k \times n$ matrices A over $\mathbf{F}_q$ such that A has full rank k and such that $A_{ii} = 0$ for $1 \leq i \leq k$. We give a recursive proof of a simple explicit formula for $f_{k,n}$. In particular, we will have a formula for $f_{n,n}$, the number of invertible $n \times n$ matrices with zeroes on the diagonal.

Proposition 2.1 *For $1 \le k \le n$,*

$$f_{k,n} = q^{\binom{k-1}{2}}(q-1)^k \left(q^{-1} \sum_{i=0}^{k} (-1)^i \binom{k}{i} \frac{[n-i]_q!}{[n-k]_q!} \right)$$

is the number of $k \times n$ matrices of rank k with zeroes on the diagonal. In particular,

$$f_{n,n} = q^{\binom{n-1}{2}}(q-1)^n \left(q^{-1} \sum_{i=0}^{n} (-1)^i \binom{n}{i} [n-i]_q! \right)$$

is the number of invertible $n \times n$ matrices with zeroes on the diagonal.

Proof idea: We proceed recursively, building matrices up by adding one row at a time. This leads to the recursion

$$f_{k+1,n} = q^{k-1}(q-1)(f_{k,n} \cdot [n-k]_q - f_{k,n-1})$$

with initial values $f_{1,n} = q^{n-1} - 1$, from which the result follows. □

Remark 2.2 *In the expression for $f_{n,n}$, the $q = 1$ specialization of the alternating sum is*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} (n-i)! = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!},$$

which is the number of derangements of length n. The above proof does not "explain" this fact, but one can give more conceptual proofs based on the Bruhat decomposition of $\mathbf{GL}(n, q)$ (see (LLMPSZ, Sec. 2.2) for details) or on Proposition 5.1. ■

2.2 Recursion by rank

In this section, we use recursive methods to attack the problem of enumerating square matrices with a prescribed number of zeroes on the diagonal by rank. We use the following strategy: each $n \times n$ matrix can be inflated to q^{2n+1} different $(n+1) \times (n+1)$ matrices, and we count these by keeping careful track of what their rank is and how many zeroes they have on the diagonal. We are unable to obtain a closed formula for this recursion, but the we use the same proof technique successfuly in Sections 3 and 4.

Let $\mathrm{mat}_0(n, k, r)$ be the number of $n \times n$ matrices over $\mathbf{F}_q$ of rank r whose first k diagonal entries are zero (and the other diagonal entries may or may not be zero).

Proposition 2.3 *We have the following recursion:*

$$\mathrm{mat}_0(n{+}1, k{+}1, r{+}1) = \frac{1}{q} \mathrm{mat}_0(n{+}1, k, r{+}1) + (q^{r+1} - q^r) \mathrm{mat}_0(n, k, r{+}1) - (q^r - q^{r-1}) \mathrm{mat}_0(n, k, r)$$

with initial conditions

$$\mathrm{mat}_0(n, 0, r) = \frac{q^{\binom{r}{2}}(q-1)^r}{[r]_q!} \left(\prod_{i=0}^{r-1} [n-i]_q \right)^2.$$

Proof idea: We proceed by decomposing each $(n+1)\times(n+1)$ matrix A as $A = \begin{bmatrix} a & u \\ v & B \end{bmatrix}$ where a is an element of $\mathbf{F}_q$, u is a row vector over $\mathbf{F}_q$ of length n, and v is a column vector over $\mathbf{F}_q$ of length n, and B is an $n \times n$ matrix. We compute for each B the possible ranks of A, taking into account whether $a = 0$ and sum over all B to get the recursion. □

The preceding recursion works by reducing the number of zeroes required to lie on the diagonal. However, we can easily modify the proof to work only with matrices of all-zero diagonal.

Corollary 2.4 *For $r \geq 0$, the number $g_{n,r}$ of $n \times n$ matrices over $\mathbf{F}_q$ of rank r and with zero diagonal satisfies the recursion*

$$g_{n+1,r+1} = (q^n - q^{r-1})^2 g_{n,r-1} + (q^{2r+1} + q^{r+1} - q^r) g_{n,r+1} \\ + (2q^{n+r} - q^{2r} - q^{2r-1} - q^r + q^{r-1}) g_{n,r}$$

with initial conditions $g_{n,0} = 1$, $g_{n,-1} = 0$ and $g_{1,1} = 0$.

3 Symmetric and skew-symmetric matrices

A natural next step is to consider symmetric matrices, which are (at least morally) a q-analogue of involutions, suggesting the possibility of interesting combinatorial results. This also brings us closer to a speculation by Kontsevich (see Section 5). In this section, we begin by enumerating symmetric invertible matrices over $\mathbf{F}_q$ whose diagonal is all zero, a q-analogue of involutions with no fixed points. This leads to two very unintuitive facts: in Section 3.1, we show that the number of these matrices of size $2n$ is the same as the number of invertible symmetric matrices of size $2n-1$; in Section 3.2, we show that both of these numbers are equal to the number of invertible skew-symmetric matrices of the size $2n$. We give more refined but less beautiful versions of the first of these results in Section 4.

While extending the approach of Section 2.1 to the case of symmetric matrices seems impossible, the ideas of Section 2.2 can be adjusted to work in this context. The major complicating factor is that the bilinear form uBv that we worked with implicitly in Section 2.2 must be replaced with the quadratic form vBv^T. Quadratic forms behave very differently in even and odd characteristic, so we give the following *proviso*:

Remark 3.1 *Our proofs of the results in this section are only valid for q odd.* ■

Of course, some of the results still hold when q is even: for example, symmetric matrices of rank r with all-zero diagonal are equinumerous with skew-symmetric matrices of rank r over fields of characteristic 2 for the silly reason that they are exactly the same set of matrices. For a more thorough treatment of the case q even, see (Mac) and (Sta2).

3.1 Symmetric matrices with zeroes on the diagonal

Let $\mathrm{Sym}_0(n)$ denote the set of $n\times n$ symmetric matrices in $\mathbf{GL}(n,q)$ with zero diagonal and let $\mathrm{sym}_0(n) = \#\,\mathrm{Sym}_0(n)$. Similarly, let $\mathrm{S}(n,k)$ be the set of $n \times n$ symmetric matrices in $\mathbf{GL}(n,q)$ whose first k diagonal entries are zero and let $\mathrm{s}(n,k) = \#\,\mathrm{Sym}_0(n,k)$, so $\mathrm{S}(n,n) = \mathrm{Sym}_0(n)$.

In (Mac, Theorem 2), MacWilliams shows

Theorem *The number of symmetric invertible matrices (for any characteristic) is*

$$\mathrm{sym}(n) = q^{\binom{n+1}{2}} \prod_{j=1}^{\lceil n/2 \rceil} (1 - q^{1-2j}). \tag{3.2}$$

Observe that when n is even, $\mathrm{sym}(n-1)$ is a q-analogue of $(n-1)!!$, the number of fixed point-free involutions in $\mathfrak{S}_n$.

Theorem 3.3 *When n is even, the number of $(n-1) \times (n-1)$ symmetric invertible matrices is equal to the number of $n \times n$ symmetric invertible matrices with zero diagonal, i.e.,* $\mathrm{sym}(n-1) = \mathrm{sym}_0(n)$.

Proof idea: Naively, one might expect that about q^{-k} of all matrices in $\mathrm{Sym}(n)$ have first k diagonal entries equal to 0. We begin by showing that, remarkably, this estimate is actually exact when n is even. The proof of this lemma proceeds by decomposing a matrix as in Section 2.2 followed by the classification of symmetric bilinear forms over fields of odd characteristic (see (Wan, Theorem 1.22)). Thanks to this lemma and Equation (3.2), the proof is an easy induction. □

Note that the case for q even was done by MacWilliams (Mac, Theorems 2, 3) (see also Equation (4.3)).

3.2 Skew-symmetric matrices

In this section, we count invertible skew-symmetric matrices by rank, obtaining a q-analogue of fixed point-free "partial involutions." It is not clear *a priori* that there is any connection between these matrices and symmetric matrices, but we obtain as a consequence of the result that the when n is even, the number of $n \times n$ invertible skew-symmetric matrices is the same as the number of $(n-1) \times (n-1)$ invertible symmetric matrices (and so, by Theorem 3.3, also the same as the number of $n \times n$ invertible symmetric matrices with all-zero diagonal). After the first write-up of this paper we found that this was proven by Jones (Jon, Theorems 1.7, 1.7′, 1.8′, 1.9) using topological methods.

Proposition 3.4 *Let $\mathrm{sk}(n,r)$ be the number of $n \times n$ skew-symmetric matrices of rank r. When r is odd we have $\mathrm{sk}(n,r) = 0$ and when r is even we have*

$$\mathrm{sk}(n,r) = q^{r(r-2)/4}(1-q)^{r/2} \cdot \frac{[n]_q!}{[n-r]_q! \cdot [r]_q!!}.$$

In particular, when $r = n$ is even we have $\mathrm{sk}(n,n) = \mathrm{sym}(n-1)$.

One interesting observation is that this is a q-analogue of $\binom{n}{r}(r-1)!!$, the number of "partial involutions of rank r" with no fixed points, i.e., the number of pairs of an r-subset of $\{1, \ldots, n\}$ together with a fixed point-free involution on that set. Note that in (LLMPSZ, Sec. 3.3) we give another proof of $\mathrm{sk}(n,n) = \mathrm{sym}(n-1)$ via Schubert varieties.

Proof idea: We decompose the skew-symmetric matrices in the same way as in preceding sections to obtain the recurrence

$$\mathrm{sk}(n,r) = q^r\,\mathrm{sk}(n-1,r) + (q^{n-1} - q^{r-2})\,\mathrm{sk}(n-1,r-2),$$

with initial values $\mathrm{sk}(n,0) = 1$ and $\mathrm{sk}(n,1) = 0$, whose solution is the formula stated above. In the case $r = n$ is even, compare with Equation (3.2) to obtain $\mathrm{sk}(n,n) = \mathrm{sym}(n-1)$. □

4 Refined enumeration of symmetric matrices

In this section, we attack the problem of enumerating $n \times n$ symmetric matrices over $\mathbf{F}_q$ with zeroes on the diagonal by rank. Roughly speaking, we should expect this problem to be a q-analogue of counting fixed point-free involutions, or of "partial fixed point-free involutions" when we consider matrices of less than full rank. As in the preceding sections, we construct a recursion to count the desired objects. Our basic approach is the same as in Section 2.2. The main difference is that the symmetry of our matrices forces us to introduce a sort of parity condition depending on whether or not we can write a matrix in the form $M \cdot M^T$ for some other matrix M. The details on whether or not we can do this are different for odd and even characteristic. We begin by mentioning both cases to then restrict our attention and results to the odd case.

Remark 4.1 **(q even)** *It was shown by Albert (Alb, Thm. 7) that a symmetric matrix A in $\mathbf{GL}(n, q)$ can be factored in the form $A = M \cdot M^T$ for some matrix M in $\mathbf{GL}(n, q)$ if and only if A has at least one nonzero diagonal entry. Thus $\#\{A \in \mathrm{Sym}(n) \mid A = M \cdot M^T \text{ for some } M \in \mathbf{GL}(n, q)\} = \mathrm{sym}(n) - \mathrm{sym}_0(n)$. MacWilliams (Mac) gave an elementary proof of Albert's theorem and also calculated $\mathrm{sym}_0(n, r)$, the number of $n \times n$ symmetric matrices of rank r with zero diagonal, when q is even.*

Theorem (Mac, Thm. 3, Sec. III) *For q even, if $r = 2s + 1$ is odd then*

$$\mathrm{sym}_0(n, 2s + 1) = 0 \tag{4.2}$$

while if $r = 2s$ is even then

$$\mathrm{sym}_0(n, 2s) = \prod_{i=1}^{s} \frac{q^{2i-2}}{q^{2i} - 1} \prod_{i=0}^{2s-1} (q^{n-i} - 1). \tag{4.3}$$

Henceforth, we will always assume that q is odd. ■

For q odd, define $\psi \colon \mathbf{F}_q^\times \to \{+, -\}$ by $\psi(\delta) = +$ if and only if δ is a perfect square in $\mathbf{F}_q$. In other words, ψ is the Legendre symbol for $\mathbf{F}_q$. We can also extend ψ naturally to symmetric matrices using the following remark.

Remark 4.4 *By applying symmetric row and column reductions, every $n \times n$ symmetric matrix A of rank $r > 0$ can be written either in the form $A = M \cdot \mathrm{diag}(1^r, 0^{n-r}) \cdot M^T$ for some $M \in \mathbf{GL}(n, q)$ or in the form $M \cdot \mathrm{diag}(1^{r-1}, z, 0^{n-r}) \cdot M^T$ for some non-perfect square $z \in \mathbf{F}_q$ and some $M \in \mathbf{GL}(n, q)$.* ■

In the former case we say that A has **(quadratic) character** $\psi(A) = \psi(1) = +$ and in the latter case we say it has character $\psi(A) = \psi(z) = -$. Two notable special cases are that if $A \in \mathbf{GL}(n, q)$ then $\psi(A) = \psi(\det A)$, while if A is diagonal then $\psi(A) = +$ if and only if the product of the nonzero diagonal entries of A is a square in $\mathbf{F}_q$.

Let $\mathrm{sym}_0(n, k, r)$ be the number of $n \times n$ symmetric matrices with rank r and the first k diagonal elements equal to 0, with no other restrictions. Thus, we have for example that $\mathrm{sym}_0(n, n, r) = \mathrm{sym}_0(n, r)$ while $\mathrm{sym}_0(n, 0, r) = \mathrm{sym}(n, r)$ is the number of symmetric matrices of rank r with no other restrictions which has been calculated in (Mac, Thm. 2, Sect. III). Let $\mathrm{sym}_0^\psi(n, r, k)$ count only those matrices that have character ψ. We now give a recurrence for $\mathrm{sym}_0(n, k, r)$. We use this recurrence to enumerate invertible symmetric matrices over $\mathbf{F}_q$ with zero diagonal (Theorem 4.7), generalizing Theorem 3.3.

Proposition 4.5 *If r is odd, define $t = 0$ if $(-1)^{(r+1)/2}$ is a square in $\mathbf{F}_q$ and $t = 1$ otherwise. Then*

$$\operatorname{sym}_0^{\psi}(n+1,k+1,r+1) = \frac{1}{q}\operatorname{sym}_0^{\psi}(n+1,k,r+1)+$$
$$+(-1)^t\cdot\psi\cdot\left(\frac{1}{2}\operatorname{sym}_0(n,k,r)+\operatorname{sym}_0^{\psi}(n,k,r+1)\right)\cdot(q^{(r+1)/2}-q^{(r-1)/2}).$$

If r is even and $r > 0$, define $t = 0$ if $(-1)^{r/2}$ is a square in $\mathbf{F}_q$ and $t = 1$ otherwise. Then

$$\operatorname{sym}_0^{\psi}(n+1,k+1,r+1) = \frac{1}{q}\operatorname{sym}_0^{\psi}(n+1,k,r+1)-$$
$$-\frac{(-1)^t}{2}(\operatorname{sym}_0^{+}(n,k,r)-\operatorname{sym}_0^{-}(n,k,r))(q^{r/2}-q^{r/2-1}).$$

We have initial values

$$\operatorname{sym}_0^{\psi}(n+1,k+1,1) = \frac{1}{2}\operatorname{sym}_0(n+1,k+1,1) = \frac{q-1}{2}\sum_{i=0}^{n-k-1} q^i = \frac{q^{n-k}-1}{2},$$

$$\operatorname{sym}_0^{+}(n,0,2s+1) = \frac{1}{2}\operatorname{sym}(n,2s+1),$$

and

$$\operatorname{sym}_0^{+}(n,0,2s) = \frac{1}{2}\frac{q^s+(\psi(-1))^s}{q^s}\operatorname{sym}(n,2s).$$

Proof idea: As before, we proceed by building larger matrices by adding rows and columns to smaller matrices; for each $n \times n$ symmetric matrix B, we consider the $(n+1)\times(n+1)$ matrices of the form

$$A = \begin{bmatrix} a & v \\ v^T & B \end{bmatrix}$$

and analyze them (taking into account whether $a = 0$) to write down a recursion. The number of matrices A of a given rank associated to a matrix B now depends on the rank of B (as in Proposition 2.3) and also on its quadratic character. At important junctures we use Wan's result (Wan, Theorem 1.26) on the number of zeroes over $\mathbf{F}_q$ of certain quadratic forms. The $k = 0$ base cases are provided by (Mac, Thm. 2, Sect. III). □

We do not have a solution for this recurrence. However, we use it to obtain two partial results towards its solution:

Corollary 4.6 *We have*

$$\operatorname{sym}_0^{+}(n+1,,k+1,2s+1) = \operatorname{sym}_0^{-}(n+1,k+1,2s+1) = \frac{1}{2}\operatorname{sym}_0(n+1,k+1,2s+1),$$

and

$$\operatorname{sym}_0(n+1,k+1,2s)+\operatorname{sym}_0(n+1,k+1,2s+1) = \frac{1}{q^{k+1}}(\operatorname{sym}(n+1,2s)+\operatorname{sym}(n+1,2s+1)).$$

We also use Proposition 4.5 to obtain an explicit formula in the case of invertible symmetric matrices (i.e., when $r = n$). Let $\mathrm{s}(n, k) = \mathrm{sym}_0(n, k, n)$ be the number of invertible $n \times n$ symmetric matrices with first k diagonal elements are equal to 0, with no other restrictions. We use the recurrence in Proposition 4.5 to give a recurrence for this full rank case.

Theorem 4.7 *Let $\mathrm{s}(n, k)$ be the number of invertible $n \times n$ symmetric matrices with the first k diagonal elements equal to 0 and let $\mathrm{sym}(n)$ be the number of invertible $n \times n$ symmetric matrices with no other restrictions. We have*

$$\mathrm{s}(2m, k+1) = \frac{1}{q^{k+1}}\,\mathrm{sym}(2m),$$

and

$$\mathrm{s}(2m+1, k+1) = \frac{1}{q^{k+1}} \sum_{j=0}^{2m+1} (-1)^{\lceil j/2 \rceil} q^{\lfloor j/2 \rfloor (2m+1-\lfloor j/2 \rfloor)} (q-1)^j \binom{k+1}{j} \mathrm{sym}(2m+1-j).$$

Proof idea: Substitute $r = n$ into Proposition 4.5, sum up the two different characters, and iterate. $\square$

5 Polynomiality, q-analogues, and some open questions

So far, we have fixed sets of the form $S = \{(i,i) \mid 1 \le i \le k\}$, counted matrices over $\mathbf{F}_q$ with support avoiding S by rank, and done analogous counts for symmetric and skew-symmetric matrices. In this section, we briefly examine what happens when we enumerate matrices of given rank whose support avoids an arbitrary fixed set of entries.

5.1 q-analogues and the proof idea of Proposition 5.1

Fix $m, n \ge 1$, $r \ge 0$, and $S \subset \{(i,j) \mid 1 \le i \le m, 1 \le j \le n\}$. Let $T_q = T_q(m \times n, S, r)$ be the set of $m \times n$ matrices A over $\mathbf{F}_q$ with rank r and support contained in the complement of S. We consider the problem of computing $\#T_q$, the number of such matrices.

A first observation is that, holding m, n, r, S fixed and letting q vary, the function $\#T_q$ need not be polynomial in q. We have already seen this phenomenon in the case of symmetric matrices; for instance, setting $m = n = r$ to be an odd positive integer and $S = \{(i,i) \mid 1 \le i \le n\}$ we have from Equations (4.2) and (4.3) and Theorem 4.7 that $\#(T_q(n \times n, S, n) \cap \mathrm{Sym}(n)) = \mathrm{sym}_0(n)$ is equal to zero when q is even but is nonzero when q is odd. This lack of polynomiality also occurs in the not-necessarily symmetric case. Stembridge (Ste1, Section 7) showed that for $n = m = 7$, if S' is the complement of the incidence matrix of the Fano plane, then the number of invertible 7×7 matrices in $\mathbf{F}_q$ whose support avoids S' is given by two different polynomials depending on whether q is even or odd. (This is the smallest such example in the sense that $T_q(n \times n, S, n)$ is a polynomial if $n < 7$ for any set S, and if $n = 7$ and $\#S > 28$.)

A second observation is that we expect $\#T_q$ to be a q-analogue of a closely related problem for permutations. Specifically, let $T_1 = T_1(m \times n, S, r)$ be the set of 0-1 matrices with exactly r 1's, no two of which lie in the same row or column, and with support contained in the complement of S. The following proposition makes this precise.

Proposition 5.1 *Fix $m, n \geq 1$, $r \geq 0$, and $S \subset \{(i,j) \mid 1 \leq i \leq m, 1 \leq j \leq n\}$. Let $T_q = T_q(m \times n, S, r)$ be the set of $m \times n$ matrices A over $\mathbf{F}_q$ with rank r and support contained in the complement of S, and T_1 be the set of 0-1 matrices with exactly r 1's, no two of which lie in the same row or column, and with support contained in the complement of S. Then we have*

$$\#T_q \equiv \#T_1 \cdot (q-1)^r \pmod{(q-1)^{r+1}}.$$

In particular, for any infinite set of values of q for which $\#T_q$ is a polynomial in q we have that $(q-1)^r$ divides $\#T_q$ as a polynomial and that $\#T_q/(q-1)^r \mid_{q=1} = \#T_1$.

Proof idea: For each ℓ, identify $(\mathbf{F}_q^\times)^\ell$ with the group of invertible diagonal $\ell \times \ell$ matrices. Consider the action of $(\mathbf{F}_q^\times)^m \times (\mathbf{F}_q^\times)^n$ on T_q given by $(X, Y) \cdot A = XAY^{-1}$. For any $A \in T_q$, let G be the bipartite graph with vertices $v_1, \ldots, v_m, w_1, \ldots, w_n$ and an edge $v_i w_j$ if $A_{ij} \neq 0$. Then $(x_1, \ldots, x_m, y_1, \ldots, y_n) \in (\mathbf{F}_q^\times)^m \times (\mathbf{F}_q^\times)^n$ stabilizes A if and only if $x_i = y_j$ for all edges $v_i w_j$ of G. Thus, the size of the stabilizer of A is $(q-1)^{C(G)}$, where $C(G)$ is the number of connected components of G, and the size of the orbit of A is therefore $(q-1)^{m+n-C(G)}$.

Since A has rank r, at least r of the v_i and r of the w_i have positive degree. It follows that $C(G) \leq m+n-r$ with equality if and only if G consists of r disjoint edges, that is, when G is the graph associated to a matrix in T_1. It follows that the size of each orbit is $(q-1)^a$ for some $a \geq r$, and the number of orbits of size $(q-1)^r$ is $\#T_1$. □

Remark 5.2 *The technique in the proof of Proposition 5.1 is widely applicable to similar problems. For example, we can use it in the case of symmetric matrices (when q is odd) with rank $2s$ with zero diagonal. Here, the group $(\mathbf{F}_q^\times)^n$ of invertible diagonal matrices acts on the set of symmetric matrices by the rule $X \cdot A = XAX$ and we consider the graph G on n vertices $v_1, \ldots, v_n$ with edge $v_i v_j$ if and only if $A_{ij} \neq 0$. Proceeding in a similar way it is then possible to conclude that (looking modulo $(q-1)^{s+1}$) we have that symmetric matrices with zero diagonal are a q-analogue of "partial fixed point-free involutions."* ■

5.2 Polynomiality and a speculation of Kontsevich

As mentioned in Section 1, the question of the polynomiality of $\#T_q$ is related to a speculation from Kontsevich. We briefly provide some background on Stanley's (Sta2) reformulation of this speculation and on its relation to the polynomiality of $\#T_q$.

Let G be an undirected connected graph with edge set E, and form the polynomial ring $\mathbf{Z}[x_e \mid e \in E]$. We consider the polynomial $Q_G(x) = \sum_T \prod_{e \in T} x_e$, where the sum is over all spanning trees T of G. Let $g_G(q) = \#\{x \in \mathbf{F}_q^E \mid Q_G(x) \neq 0\}$. Kontsevich inquired whether for fixed G, $g_G(q)$ is a polynomial function in the parameter q.

Let $v_1, \ldots, v_n$ be the vertices of G and suppose that v_n is adjacent to all the other vertices. By the Matrix-Tree Theorem, one may conclude that $g_G(q)$ is the number of symmetric matrices in $\mathbf{GL}(n-1, q)$ such that the (i,j)-th entry is 0 whenever $i \neq j$ and v_i and v_j are not connected. Therefore, setting $S_G = \{(i,j) \mid i \neq j \text{ and } v_i v_j \notin E\}$ we have $g_G(q) = \#(T_q(n \times n, S_G, n) \cap \mathrm{Sym}(n))$.

Belkale and Brosnan showed in (BB) that Kontsevich's speculation is false by showing that the functions $g_G(q)$ are as complicated (in a very precise sense) as the functions counting the number of solutions over $\mathbf{F}_q$ of any variety defined over $\mathbf{Z}$. In addition, Stembridge and Schnetz in (Ste1) and (Sch) showed that $g_G(q)$ is a polynomial for graphs G with ≤ 12 and 13 edges respectively, and the latter found six non-isomorphic graphs with 14 edges such that $g_G(q)$ is not a polynomial in q. Given these results, it becomes an interesting problem to determine when $g_G(q)$ is a polynomial in q. Taken together with Proposition 5.1, they also suggest the following question:

Question 5.3 *For which families of sets S is $\#T_q(m \times n, S, r)$ a polynomial in q?*

Note that $\#T_q(m \times n, S, r)$ is invariant under permutations of rows and columns. For simplicity, we restrict the question to the case of square matrices. Below, we describe one class of sets S for which the answer is already known by the theory of q-rook numbers.

Let $\overline{S}$ denote the complement of the set S. We say that $S \subseteq [n] \times [n]$ is a **straight shape** if its elements form a Young diagram. Thus, to every integer partition λ with at most n parts and with largest part at most n (i.e., to each sequence of integers $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $n \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$) there is an associated set $S = S_\lambda$. We have that $\#S_\lambda = \sum \lambda_i = |\lambda|$ is the sum of the parts of λ. Similarly, if λ and μ are partitions such that $S_\mu \subseteq S_\lambda$ then we say that the set $S_\lambda \setminus S_\mu$ is a **skew shape** and we denote it by $S_{\lambda/\mu}$. Next we give three easy facts about straight and skew shapes.

Remark 5.4 (i) *Up to a rotation of $[n] \times [n]$, the complement $\overline{S_\lambda}$ of the straight shape S_λ is also a straight shape. However, $\overline{S_{\lambda/\mu}}$ is typically not a skew shape.*

(ii) *If $(i, j) \in S_\lambda$ then the rectangle $\{(s, t) \mid 1 \leq s \leq i, 1 \leq t \leq j\}$ is contained in S_λ. General skew shapes $S_{\lambda/\mu}$ do not have this property.*

(iii) *If $\lambda = (n, n-1, \ldots, 2, 1)$ and $\mu = (n-1, n-2, \ldots, 1, 0)$ are so-called "staircase shapes" then $S_{\lambda/\mu}$ is, up to rotation, the set of diagonal entries. Thus the value $\#T_q(n \times n, S_{\lambda/\mu}, n)$ is given in Proposition 2.1 while trivially $T_q(n \times n, \overline{S_{\lambda/\mu}}, n) = \#\{$invertible diagonal matrices$\} = (q-1)^n$.* ∎

Given a set $S \subseteq [n] \times [n]$, the r q**-rook number** of Garsia and Remmel (GR) is $R_r(S, q) = \sum_C q^{\mathrm{inv}(C,S)}$, where the sum is over all rook placements $C \in T_1(n \times n, \overline{S}, r)$ of r non-attacking rooks in S and where $\mathrm{inv}(C, S)$ is the number of squares in S not directly above (in the same column) or to the left (in the same row) of any placed rook.

The following result of Haglund shows that when $S = S_\lambda$, we have that $T_q(n \times n, S_\lambda, n)$ is a polynomial, and in fact is the product of a power of $q-1$ and a polynomial with nonnegative coefficients.

Theorem (Hag, Theorem 1) *For straight shapes S_λ, $\#T_q(n \times n, S_\lambda, r) = (q-1)^r q^{n^2-|\lambda|-r} R_r(\overline{S_\lambda}, q^{-1})$.*

Question 5.5 *The proof of the above theorem relies on Remark 5.4 (ii) and it does not immediately extend to skew shapes $\overline{S_{\lambda/\mu}}$. However, computations using Stembridge's Maple package* `reduce` *(Ste2) suggest that when S is a skew shape, $\#T_q$ is still a polynomial and that when S is the complement of a skew shape, $\#T_q$ is a power of $q-1$ times a polynomial with nonnegative coefficients. Is this true for all skew shapes and their complements?*

(Recall that any counter-examples satisfy $n = 7$ and $\#S \geq 28$ or $n \geq 8$.)

Question 5.6 *Haglund's theorem and the preceding question suggest similarities between $\#T_q$ for S and $\overline{S}$ that is reminiscent of the classical reciprocity of rook placements and rook numbers (see (Cho) for a short combinatorial proof). Dworkin (Dwo, Theorem 8.21) gave an analogue of this classical reciprocity for q-rook numbers $R_r(S, q)$ when $S = S_\lambda$. By Haglund's result, this implies a reciprocity formula relating $T_q(n \times n, S_\lambda, r)$ and $T_q(n \times n, \overline{S_\lambda}, r)$. Can this reciprocity be extended to skew or other shapes? If so, we could recover the formula for $f_{n,n}$ in Proposition 2.1 from the formula of its complement: $(q-1)^n$.*

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Row-strict quasisymmetric Schur functions

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Abstract. Haglund, Luoto, Mason, and van Willigenburg introduced a basis for quasisymmetric functions called the *quasisymmetric Schur function basis* which are generated combinatorially through fillings of composition diagrams in much the same way as Schur functions are generated through reverse column-strict tableaux. We introduce a new basis for quasisymmetric functions called the *row-strict quasisymmetric Schur function basis* which are generated combinatorially through fillings of composition diagrams in much the same way as Schur functions are generated through row-strict tableaux. We describe the relationship between this new basis and other known bases for quasisymmetric functions, as well as its relationship to Schur polynomials. We obtain a refinement of the omega transform operator as a result of these relationships.

Résumé. Haglund, Luoto, Mason, et van Willigenburg ont introduit une base pour les fonctions quasi-symétriques appelée *base des fonctions de Schur quasi-symétriques*, qui sont construites en remplissant des diagrammes de compositions, d'une manière très semblable à la construction des fonctions de Schur à partir des tableaux "column-strict" (ordre strict sur les colonnes). Nous introduisons une nouvelle base pour les fonctions quasi-symétriques appelée *base des fonctions de Schur quasi-symétriques "row-strict"*, qui sont construites en remplissant des diagrammes de compositions, d'une manière très semblable à la construction des fonctions de Schur à partir des tableaux "row-strict" (ordre strict sur les lignes). Nous décrivons la relation entre cette nouvelle base et d'autres bases connues pour les fonctions quasi-symétriques, ainsi que ses relations avec les polynômes de Schur. Nous obtenons un raffinement de l'opérateur oméga comme conséquence de ces relations.

Keywords: symmetric and quasisymmetric functions, omega operator, Schur functions

1 Introduction

Quasisymmetric functions have emerged as a powerful tool for investigating many diverse areas such as symmetric functions [2, 4], combinatorial Hopf algebras [1], discrete geometry [3], and representation theory [11, 13]. Quasisymmetric functions were introduced by Gessel as a source of generating functions for P-partitions [6], although they appeared in a different format in earlier work by Stanley [16]. Gessel developed many properties of quasisymmetric functions and applied them to solve a number of problems in permutation enumeration. Gessel also proved that they were dual to Solomon's descent algebra. This duality is further explored by Ehrenborg [5], Malvenuto and Reutenauer [14], and Thibon [19].

In [7], Haglund, Luoto, Mason, and van Willigenburg introduced a new basis for quasisymmetric functions called the *quasisymmetric Schur function basis* which are generated combinatorially through fillings of composition diagrams

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in much the same way as Schur functions are generated through reverse column-strict tableaux. Each quasisymmetric Schur function is a positive sum of Demazure atoms. In [7], it was shown that the quasisymmetric Schur functions refine the Schur functions in a way that respects the Schur function decomposition into Gessel's fundamental quasisymmetric functions. In [8], Haglund, Luoto, Mason, and van Willigenburg gave a refinement of the Littlewood-Richardson rule which proved that the product of a quasisymmetric Schur function and a Schur function expands positively as a sum of quasisymmetric Schur functions.

This paper was motivated by an attempt to extend the duality between column-strict tableaux and row-strict tableaux to quasisymmetric Schur functions. That is, let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be a partition of n. The diagram associated to λ (in English notation) consists of k rows of left-justified boxes, or *cells*, such that the i^{th} row from the top contains λ_i cells. A *reverse column-strict tableau T of shape λ* is a filling of the cells of λ with positive integers so that the rows are weakly decreasing and the columns are strictly decreasing. A *reverse row-strict tableau T of shape λ* is a filling of the cells of λ with positive integers so that the rows are strictly decreasing and the columns are weakly decreasing. Let $\mathcal{RCS}_\lambda$ ($\mathcal{RRS}_\lambda$) denote the set of all reverse column strict tableaux (reverse row strict tableaux) of shape λ. If T is a reverse column-strict tableau or a reverse row-strict tableau, we let $T(i,j)$ be the element in the cell which is in the i-th row of T, reading from top to bottom, and the j-th column of T, reading from left to right, and we let the weight, x^T, of T be defined as $x^T = \prod_{(i,j)\in\lambda} x_{T(i,j)}$. Then the Schur function $s_\lambda(x_1, x_2, \ldots)$ is defined as

$$s_\lambda(x_1, x_2, \ldots) = \sum_{T\in\mathcal{RCS}_\lambda} x^T. \tag{1}$$

If T is a reverse column-strict tableau or a reverse row-strict tableau of shape λ, we define the conjugate of T, T', to be the filled diagram of shape λ' which results by reflecting the cells of T across the main diagonal. Clearly T is a reverse column-strict tableaux if and only if T' is a reverse-row strict tableaux. Thus

$$s_{\lambda'}(x_1, x_2, \ldots) = \sum_{T\in\mathcal{RRS}_\lambda} x^T \tag{2}$$

where λ' is the transpose of the partition λ, often referred to as conjugate partition [9, 10, 18]. Moreover, if ω is the algebra isomorphism defined on the ring of symmetric functions Λ so that $\omega(h_n) = e_n$, where $h_n = h_n(x_1, x_2, \ldots)$ is n-th homogeneous symmetric function and $e_n = e_n(x_1, x_2, \ldots)$ is n-th elementary symmetric function, then it is well known that $\omega(s_\lambda(x_1, x_2, \ldots)) = s_{\lambda'}(x_1, x_2, \ldots)$.

The question that motivated this paper is whether we can find a duality like that expressed in (1) and (2) for quasisymmetric Schur functions. In this paper, we introduce a new basis for the quasisymmetric functions which we call *row-strict quasisymmetric Schur functions* which are generated combinatorially through fillings of composition diagrams in much the same way as Schur functions are generated through reverse row-strict tableaux. However, the process of conjugation becomes less transparent in the quasisymmetric setting since bases for quasisymmetric functions are typically indexed by compositions instead of partitions. That is, it is not enough to simply reflect across the main diagonal since this does not necesarily produce a left-justified diagram. In fact, the number of compositions which are rearrangements of a given partition is generally not equal to the number of compositions which are rearrangements of its transpose so that any relationship between these two collections must necessarily be more complex than a simple bijection. There is a refinement of the ω transformation which is defined on the space of quasisymmetric functions and we shall use this refinement to better understand the relationship between the compositions rearranging a partition and those rearranging its conjugate.

2 Symmetric and quasisymmetric functions

A *symmetric function* is a bounded degree formal power series $f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that $f(x)$ is fixed under the action of the symmetric group. We let Λ denote the ring of symmetric functions and Λ_n denote the space of homogeneous symmetric functions of degree n so that $\Lambda = \oplus_{n \geq 0} \Lambda_n$.

A *partition* of n is a weakly decreasing sequence of positive integers which sum to n. We write $|\lambda| = n$ and let $l(n) = k$ be the *length* of λ. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n, we say that a reverse column-strict tableau T is a *standard reverse column-strict tableau* if each of the numbers $1, 2, \ldots, n$ appear exactly once in T. Standard reverse row-strict tableaux are defined similarly. A reverse column-strict tableau can be converted to a standard reverse column-strict tableau by a procedure known as *standardization*. Let T be an arbitrary reverse column-strict tableau such that $x^T = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k}$. First replace (from right to left) the a_1 1s in T with the numbers $1, 2, \ldots, a_1$. Then replace the a_2 2s with the numbers $a_1 + 1, \ldots, a_1 + a_2$, and so on. The resulting diagram is a standard reverse column-strict tableau, called the *standardization* $std(T)$ *of* T. The standardization of a reverse row-strict tableau is defined analogously to that of a reverse column-strict tableau but with the entries replaced from bottom to top rather than right to left.

A *composition* $\alpha \vDash n$ of n is a sequence of positive integers which sum to n. Each composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is associated to the subset of $[n-1]$ given by $S(\alpha) = \{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \ldots \alpha_{k-1}\}$. Note that this is an invertible procedure. That is, if $P = \{s_1, s_2, \ldots, s_k\}$ is an arbitrary subset of $[n-1]$, then the composition $\boldsymbol{\beta}(P) = (s_1, s_2 - s_1, \ldots, s_k - s_{k-1})$ is precisely the composition such that $S(\boldsymbol{\beta}(P)) = P$. We deonte by $\tilde{\alpha}$ the *complement* of α, obtained by taking the composition corresponding to the complement of $S(\alpha)$. We will make use of the *refinement order* $\preceq$ on compositions which states that $\alpha \preceq \beta$ if and only if β is obtained from α by summing some of the consecutive parts of α. If α is a composition, let $\boldsymbol{\lambda}(\alpha)$ denote the partition obtained by arranging the parts of α in weakly decreasing order. We say that α is a *rearrangement* of the partition $\boldsymbol{\lambda}(\alpha)$. For example, $\boldsymbol{\lambda}(1, 3, 2, 1) = (3, 2, 1, 1)$.

A *quasisymmetric function* is a bounded degree formal power series $f(x) \in \mathbb{Q}[[x_1, x_2, \ldots]]$ such that for all compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the coefficient of $\prod x_i^{\alpha_i}$ is equal to the coefficient of $\prod x_{i_j}^{\alpha_i}$ for all $i_1 < i_2 < \ldots < i_k$. We let $Qsym$ denote the ring of quasisymmetric functions and $Qsym_n$ denote the space of homogeneous quasisymmetric functions of degree n so that $Qsym = \oplus_{n \geq 0} Qsym_n$.

A natural basis for $Qsym_n$ is the *monomial quasisymmetric basis*, given by the set of all M_α such that $\alpha \vDash n$ where

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

Gessel's *fundamental basis for quasisymmetric functions* [6] can be expressed by

$$F_\alpha = \sum_{\beta \preceq \alpha} M_\beta,$$

where $\beta \preceq \alpha$ means that β is a refinement of α.

The *descent set* $D(T)$ of a standard tableau is the set of all positive integers i such that $i + 1$ appears in a column weakly to the right of the column containing i. The following theorem provides describes the way a Schur function can be expressed as a positive sum of fundamental quasisymmetric functions.

Theorem 2.1 *[6] The Schur function s_λ can be written as a positive sum*

$$s_\lambda = \sum_\beta d_{\lambda\beta} F_\beta$$

of fundamental quasisymmetric functions, where $d_{\lambda\beta}$ is equal to the number of standard reverse column-strict tableaux T of shape λ and descent set $D(T)$ such that $\boldsymbol{\beta}(D(T)) = \beta$.

An extension of the classical ω transformation on symmetric functions defined in the introduction to the space of quasisymmetric functions appears in the work of Ehrenborg [5], Gessel [6], and Malvenuto-Reutenauer [14]. One can define this endomorphism on the fundamental quasisymmetric functions by $\omega(F_\alpha) = F_{rev(\tilde{\alpha})}$, where $rev(\tilde{\alpha})$ is the composition obtained by reversing the order of the entries in $\tilde{\alpha}$. Then ω is an automorphism of the algebra of quasisymmetric functions whose restriction to the space of symmetric function equals the classical ω transformation.

3 Quasisymmetric Schur functions and row-strict quasisymmetric Schur functions

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a composition of n. The diagram associated to α consists of l rows of left-justified boxes, or *cells*, such that the i^{th} row from the top contains α_i cells, as in the English notation. Given a composition diagram $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ with largest part m, a *column-strict composition tableau* (CSCT), F is a filling of the cells of α with positive integers such that the entries of F weakly decrease in each row when read from left to right, the entries in the leftmost column of F strictly increase when read from top to bottom, and F satisfies the column-strict triple rule.

Here we say that F satisfies the *column-strict triple rule* if when we supplement F by adding enough cells with zero valued entries to the end of each row so that the resulting supplemented tableau, $\hat{F}$, is of rectangular shape $l \times m$, then for $1 \le i < j \le l, \;\; 2 \le k \le m$:

$$(\hat{F}(j,k) \neq 0 \text{ and } \hat{F}(j,k) \geq \hat{F}(i,k)) \Rightarrow \hat{F}(j,k) > \hat{F}(i,k-1)$$

where $\hat{F}(i,j)$ denotes the entry of $\hat{F}$ that lies in the cell in the i-th row and j-th column.

Define the type of a column-strict composition tableau F to be the weak composition $w(F) = (w_1(F), w_2(F), \ldots)$ where $w_i(F) =$ the number of times i appears in F. The weight of F is $x^F = \prod_i x_i^{w_i(F)}$. A CSCT F with n cells is *standard* if $x^F = \prod_{i=1}^n x_i$. If T is a standard *CSCT*, then we define the descent set $D(T)$ of T to be the set of all i such that $i+1$ appears in a column weakly to the right of the column containing i.

Haglund, Luoto, Mason, and van Willigenburg [7] defined the quasisymmetric Schur function $\mathcal{S}_\alpha$ by

$$\mathcal{S}_\alpha = \sum_F x^F \tag{3}$$

where the sum runs over all column-strict composition tableaux of shape α. They showed that $\mathcal{S}_\alpha$ as α ranges over all compositions of n is a basis for the space $Qsym_n$. They also showed that for any partition λ of n,

$$s_\lambda = \sum_{\alpha : \boldsymbol{\lambda}(\alpha) = \lambda} \mathcal{S}_\alpha \tag{4}$$

and gave a combinatorial description of the expansion of $\mathcal{S}_\alpha$ in terms of the fundamental quasisymmetric function basis in the following proposition.

Proposition 3.1 *[7, Proposition 6.2] Let α, β be compositions. Then*

$$\mathcal{S}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta,$$

where $d_{\alpha\beta} =$ the number of standard column-strict composition tableaux T of shape α and $\boldsymbol{\beta}(D(T)) = \beta$.

$$F = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\,, \qquad \hat{F} = \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & 0 \\ \hline 3 & 2 \\ \hline 3 & 0 \\ \hline\end{array}\,, \qquad x^F = x_1x_2^3x_3^2$$

Fig. 1: An RSCT F of shape $(2,1,2,1)$ and weight $x_1x_2^3x_3^2$

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 3 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & 2 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1} 4 & 3 \\ \hline 4 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$$

Fig. 2: The row-strict quasisymmetric Schur function $\mathcal{RS}_{(2,1,2,1)}(x_1,x_2,x_3,x_4) = x_1x_2^3x_3^2 + x_1x_2^3x_3x_4 + x_1x_2^2x_3x_4^2 + x_1x_2^3x_4^2 + x_1x_2x_3^2x_4^2 + x_1x_3^3x_4^2 + x_2x_3^3x_4^2$

Next we define our row-strict version of the quasisymmetric Schur functions. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$ be a composition of n. Given a composition diagram α with largest part m, we define a *row-strict composition tableau* (RSCT), F to be a filling of the cells of α with positive integers such that the entries of F strictly decrease in each row when read from left to right, the entries in the leftmost column of F weakly increase when read from top to bottom, and F satisfies the row-strict triple rule.

Here we say that F satisfies the *row-strict triple rule* if when we supplement F by adding enough cells with zero valued entries to the end of each row so that the resulting supplemented tableau, $\hat{F}$, is of rectangular shape $l \times m$, then for $1 \le i < j \le l,\ \ 2 \le k \le m :$

$$(\hat{F}(j,k) > \hat{F}(i,k)) \Rightarrow \hat{F}(j,k) \ge \hat{F}(i,k-1).$$

This mirrors the definition of a composition tableau given in [7] and presented above, interchanging the roles of weak and strict. Continuing this analogy, define the type of a row-strict composition tableau F to be the weak composition $w(F) = (w_1(F), w_2(F), \ldots)$ where $w_i(F) =$ the number of times i appears in F. The weight associated to F is $x^F = \prod_i x_i^{w_i(F)}$. Also, a RSCT F with n cells is *standard* if $x^F = \prod_{i=1}^n x_i$. See Figure 1 for an example of a RSCT and its weight.

Definition 3.1 *Let α be a composition. Then the* row-strict quasisymmetric Schur function $\mathcal{RS}_\alpha$ *is given by $\mathcal{RS}_\alpha = \sum_T x^T$, where the sum is over all RSCT's T of shape α. See Figure 2 for an example.*

Proposition 3.2 *If α is an arbitrary composition, then $\mathcal{RS}_\alpha$ is a quasisymmetric polynomial.*

We shall see in Section 4.2 that the row-strict quasisymmetric Schur functions form a basis for quasisymmetric functions. The row-strict quasisymmetric Schur functions are a different basis from the quasisymmetric Schur functions even though they can be described through a similar process. For example, the transition matrix between the two bases is given in Figure 3 for $n = 4$ where each row gives the expansion of $\mathcal{RS}_\alpha$ in terms of the quasisymmetric Schur functions. This illustrates that their relationship is fairly complex.

$\mathcal{RS}_\alpha \setminus \mathcal{S}_\alpha$	4	31	13	22	211	121	112	1111
4	0	0	0	0	0	0	0	1
31	0	0	0	0	0	0	1	0
13	0	0	0	0	1	1	0	0
22	0	0	0	1	0	0	0	0
211	0	0	1	-1	0	1	0	0
121	0	0	0	1	0	-1	0	0
112	0	1	0	0	0	0	0	0
1111	1	0	0	0	0	0	0	0

Fig. 3: The transition matrix from row-strict quasisymmetric Schur functions to quasisymmetric Schur functions

3.1 Decomposing a Schur function into row-strict quasisymmetric Schur functions

Recall that a *reverse row-strict tableau* T of partition shape λ is a filling of the cells of the diagram of λ with positive integers such that the entries in each column weakly decrease from top to bottom and the entries in each row strictly decrease from left to right. Row-strict composition tableaux are related to reverse row-strict tableaux by a simple bijection, which is analogous to the bijection between column-strict composition tableaux and reverse column-strict tableaux [15].

The following map sends a reverse row-strict tableau T to a RSCT $\rho(T) = F$. We describe it algorithmically. Begin with the entries in the leftmost column of T and place them into the first column of F in weakly increasing order from top to bottom. After the first $k-1$ columns of T have been placed into F, place the entries from the k^{th} column of T into F, beginning with the largest. Place each entry e into the cell (i,k) in the highest row i such that (i,k) does not already contain an entry from T and the entry $(i,k-1)$ is greater than e. See Figure 4 for an example of the map ρ from a reverse row-strict tableau T to a RSCT $\rho(T) = F$.

Lemma 3.2 *The map ρ is a weight preserving bijection between the set of reverse row-strict tableaux of shape λ and the set of RSCT's of shape α where $\boldsymbol{\lambda}(\alpha) = \lambda$.*

Lemma 3.2 implies that each Schur function s_λ decomposes into a positive sum of row-strict quasisymmetric Schur functions indexed by compositions that rearrange the transpose of λ. That is,

$$s_\lambda = \sum_{\alpha:\lambda(\alpha)=\lambda'} \mathcal{RS}_\alpha \,.$$

An arbitrary Schur function can therefore be decomposed into either a sum of quasisymmetric Schur functions [7] or a sum of row-strict quasisymmetric Schur functions.

4 Properties of the row-strict quasisymmetric Schur functions

In order to develop several fundamental properties of the row-strict quasisymmetric Schur functions, we need to understand the behavior of the map ρ which interpolates between row-strict composition tableaux and reverse row-strict tableaux.

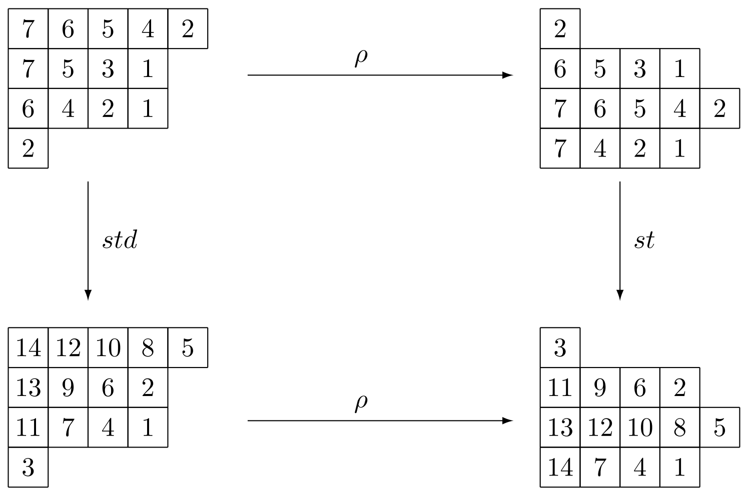

Fig. 4: The map ρ commutes with standardization.

4.1 Properties of the map ρ

Recall that a reverse row-strict tableau can be converted to a standard tableau by the standardization procedure described in Section 2. The standardization of a row-strict composition tableau F is defined similarly to that of a reverse row-strict tableau, where the entries are replaced beginning with the leftmost column and moving left to right, replacing entries within a column from bottom to top except for the entries in the leftmost column, which are replaced from top to bottom. The resulting standard composition tableau is called the *standardization* of F and denoted $st(F)$. See Figure 4 for an example. Note that this procedure does not alter the shape of F.

Proposition 4.1 *Standardization commutes with the map ρ in the sense that if T is an arbitrary reverse row-strict tableau then $st(\rho(T)) = (\rho(std(T))$.*

4.2 Transitions to classical quasisymmetric function bases

Each monomial in a row-strict quasisymmetric Schur function corresponds to a row-strict composition tableau whose weight corresponds to the non-zero exponents in the monomial. Consider the monomial $x_1^{\beta_1} x_2^{\beta_2} \cdots x_k^{\beta_k}$ with exponent composition $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$. The coefficient of this monomial in $\mathcal{RS}_\alpha$ is equal to the number of row-strict composition tableaux of shape α and weight β. By shifting the entries in a row-strict composition tableau appropriately, it is easy to see that

$$\mathcal{RS}_\alpha = \sum_\beta K(\alpha, \beta)^r M_\beta,$$

where $K(\alpha, \beta)^r$ is the number of row-strict composition tableaux of shape α and weight β.

Given a standard row-strict composition tableau F, its *transpose descent set* $D'(F)$ is the set of all i such that $i + 1$ appears in a column strictly to the left of the column containing i. Since the map ρ preserves the column sets of the diagram, the transpose descent set of a reverse row-strict standard Young tableau T (defined analogously) is equal to the transpose descent set of $\rho(T)$. To each transpose descent set $S = \{s_1, s_2, \dots s_k\}$ one associates a unique composition $\boldsymbol{\beta}(S) = (s_1, s_2 - s_1, \dots, s_k - s_{k-1}, n - s_k)$ whose successive parts are equal to the differences between consecutive elements of the set S and whose last part is given by the difference of n and the last element s_k in the set S.

We also recall the notion of *complementary compositions*. The *complement* $\tilde{\beta}$ to a composition $\boldsymbol{\beta}(S)$ arising from a subset $S \subseteq [n-1]$ is the composition obtained from the subset $S^c \subseteq [n-1]$. For example, the composition $\beta = (1, 4, 2)$ arising from the subset $S = \{1, 5\} \subseteq [7]$ has complement $\tilde{\beta} = (2, 1, 1, 2, 1)$ arising from the subset $S^c = \{2, 3, 4, 6, 7\}$.

Next we provide the analogue of Proposition 3.1 for row-strict quasisymmetric Schur functions.

Proposition 4.2 *Let α, β be compositions. Then*

$$\mathcal{RS}_\alpha = \sum_\beta d_{\alpha\beta} F_\beta$$

where $d_{\alpha\beta}$ is equal to the number of standard row-strict composition tableaux F of shape α and $\boldsymbol{\beta}(D'(F)) = \beta$.

One can use Proposition 4.2 to show that under a proper ordering of the compositions of n, the transition matrix between the basis of row-strict quasisymmetric Schur functions of degree n and the basis of fundamental quasisymmetric functions of degree n is an upper triangular matrix with 1's on the diagonal. This implies the following theorem.

Theorem 4.1 *The set $\{\mathcal{RS}_\alpha(\mathbf{x_k}) | \alpha \vDash n \text{ and } k \geq n\}$ forms a $\mathbb{Z}$-basis for $QSym_n(\mathbf{x_k})$, where $\mathbf{x_k} = x_1, x_2, \dots, x_k$.*

5 A linear endomorphism of QSym

An algebra endomorphism $\omega : \Lambda \to \Lambda$ on symmetric functions is defined by $\omega(e_n) = h_n,\ n \geq 1$. The definitions of e_λ and h_λ imply that $\omega(e_\lambda) = h_\lambda$ since ω preserves multiplication. The endomorphism is an involution and $\omega(s_\lambda) = s_{\lambda'}$, where λ' is the transpose of the partition λ, often referred to as conjugate partition [9, 10, 18].

Theorem 5.1 *The ω operator maps quasisymmetric Schur functions to row-strict quasisymmetric Schur functions. That is, $\omega(\mathcal{S}_\alpha(x_1, \dots, x_n)) = \mathcal{RS}_\alpha(x_n, \dots, x_1)$.*

To see this, apply ω to the decomposition of $\mathcal{S}_\alpha$ into a sum of fundamental quasisymmetric functions.

5.1 A notion of conjugation for compositions

The ω operator applied to a Schur function s_λ produces the Schur function $s_{\lambda'}$ indexed by the conjugate partition to λ. Since both the quasisymmetric Schur functions and the row-strict quasisymmetric Schur functions are generated by sums of monomials arising from fillings of composition diagrams, it is natural to seek a conjugation-like operation on composition diagrams.

This idea at first seems to be too much to ask since there cannot be a bijection between compositions that rearrange a given partition and compositions that rearrange its conjugate. Consider for example the partition $\lambda = (2, 1, 1)$. There are three compositions ($(2, 1, 1), (1, 2, 1)$ and $(1, 1, 2)$) which rearrange λ but only two compositions ($(3, 1)$ and $(1, 3)$) which rearrange $\lambda' = (3, 1)$. However, the ω operator can be used to collect the compositions which rearrange a given partition so that there exists a bijection between these collections and the collections corresponding to the conjugate partition. In particular, note that since ω sends a quasisymmetric Schur function to a row-strict quasisymmetric Schur function, a method for writing the quasisymmetric Schur functions in terms of their duals under ω and vice versa

$$F = \begin{array}{lllll} 2 & 1 & 1 & & \\ 4 & & & & \\ 5 & 5 & 5 & 3 & 1 \\ 6 & 3 & 2 & 2 & \end{array} \qquad \begin{array}{c} C_1 = \{1,3,5,5,6\} \\ C_2 = \{2,2,3,5\} \\ C_3 = \{1,1,4\} \\ C_4 = \{2\} \end{array} \qquad \phi(F) = \begin{array}{llll} 1 & & & \\ 3 & 2 & 1 & \\ 5 & 3 & 1 & \\ 5 & 2 & & \\ 6 & 5 & 4 & 2 \end{array}$$

Fig. 5: The map ϕ from a $CSCT$ to an $RSCT$

$$\begin{array}{lll} 1 & & \\ 4 & 3 & 2 \end{array} \longrightarrow \begin{array}{ll} 2 & 1 \\ 3 & \\ 4 & \end{array} \qquad \begin{array}{lll} 2 & & \\ 4 & 3 & 1 \end{array} \longrightarrow \begin{array}{ll} 1 & \\ 3 & 2 \\ 4 & \end{array}$$

Fig. 6: The compositions $(2,1,1)$ and $(1,2,1)$ are conjugate to $(1,3)$

would allow us to interpret the indexing compositions in much the same way as we interpret the indexing partitions for Schur functions and their images under ω.

Recall that the Schur functions can be expanded into sums of either quasisymmetric Schur functions or row-strict quasisymmetric Schur functions as follows:

$$\sum_{\boldsymbol{\lambda}(\alpha)=\lambda} \mathcal{S}_\alpha = s_\lambda = \sum_{\boldsymbol{\lambda}(\beta)=\lambda'} \mathcal{RS}_\beta \, .$$

This suggests that there must be a weight-preserving bijection between column-strict composition tableaux and row-strict composition tableaux which transposes the shape of the underlying partition.

Given an arbitrary column-strict composition tableau F, choose the largest entry in each column of F and construct the leftmost column of $\phi(F)$ by placing these entries (the collection C_1) in weakly increasing order from top to bottom. Then choose the second-largest entry from each column of F (ignoring empty cells) and insert this collection (C_2) into the new diagram by the following procedure. Place the largest entry e_i into the highest position in the second column so that the entry immediately to the left of e_i is strictly greater than e_i. Repeat this insertion with the next largest entry, considering only the unoccupied positions. Continue this procedure with the remainder of the entries in this collection from largest to smallest until the entire collection has been inserted. Then repeat the procedure for the third largest entry in each column (C_3) of F and continue inserting collections of entries until all entries of F have been inserted. The resulting diagram is $\phi(F)$. See Figure 5 for an example.

Proposition 5.1 *The map ϕ is a weight-preserving bijection between column-strict composition tableaux which rearrange a given partition and row-strict composition tableaux which rearrange the conjugate partition.*

6 An extension of dual Schensted insertion

Schensted insertion provides a method for inserting an arbitrary positive integer into an arbitrary column-strict tableau. This insertion process forms the foundation for the well-known Robinson-Schensted-Knuth (RSK) algorithm

$$F = \begin{array}{|c|c|c|}\hline 2 & 1 \\ \hline 2 \\ \hline 4 & 3 & 2 \\ \hline 4 & 2 \\ \hline 5 & 2 \\ \hline \end{array} \qquad \tilde{F} = \begin{array}{|c|c|c|c|}\hline 2 & 1 & 0 \\ \hline 2 & 0 \\ \hline 4 & 3 & 2 & 0 \\ \hline 4 & 2 & 0 \\ \hline 5 & 2 & 0 \\ \hline \end{array}$$

Fig. 7: $\mathrm{read}(\tilde{F}) = 002001032222445$

which produces a bijection between matrices with non-negative integer coefficients and pairs of reverse column-strict tableaux of the same shape. Mason [15] gave an extension of RSK algorithm to insert an integer into a column-strict composition tableaux and Haglund, Luoto, Mason and van Willigenburg [8] used this insertion procedure to prove the refinement of the Littlewood-Richardson rule which allows one to give a combinatorial interpretation to the coefficients that arise in the expansion the product of a Schur function times a quasisymmetric Schur function into a positive sum of quasisymmetric Schur functions.

In this section, we shall define an extension of dual Schensted insertion. This extension is used by Jeff Ferreira to give another refinement of the Littlewood-Richard rule which allows one to give a combinatorial interpretation to the coefficients that arise in the expansion the product of a Schur function times a row-strict quasisymmetric Schur functions into a positive sum of row-strict quasisymmetric Schur functions. We now describe our extension of the dual Schensted insertion to row-strict composition tableaux.

Dual Schensted insertion inserts an arbitrary positive integer into a reverse row-strict tableau by bumping entries from row to row. That is, given a reverse row-strict tableau T and a positive integer x, first set R equal to the first row of T. Let y be the largest entry in R less than or equal to x. Replace y by x in R. Set $x = y$ and set R equal to the next row down and repeat. If there is no such entry y which is less than or equal to x, place x at the end of row R and stop. The resulting figure is denoted $T \leftarrow x$. A more detailed exposition of Schensted insertion and its variations can be found in [17] or [12].

The following analogue of dual Schensted insertion provides a method for inserting a new cell into a RSCT. Given an arbitrary RSCT F, let $\mathrm{read}(F)$ be the *reading word* for F given by reading the entries of F by column from right to left, reading the columns from top to bottom. This ordering of the cells is called the *reading order* on the cells of F. The *modified reading word* $\mathrm{read}(\tilde{F})$ for F is given by appending a cell containing the entry 0 after the rightmost cell in each row to obtain $\tilde{F}$ and then recording the entries of $\tilde{F}$ in reading order. (See Figure 6 for an example.)

To insert an arbitrary positive integer x into F, scan $\mathrm{read}(\tilde{F})$ to find the first entry y less than or equal to x such that the entry immediately to the left of y in $\tilde{F}$ is greater than x. If such a y does not exist place x after the last entry smaller than or equal to x in the leftmost column, shifting the lower rows down by one, and stop. If $y = 0$ then replace y by x and stop. Otherwise, replace y by x in which case we say x *bumps* y and repeat the procedure using y instead of x and considering only the portion of $\mathrm{read}(\tilde{F})$ appearing after y. The resulting figure is denoted $F \leftharpoondown x$. See Figure 8 for an example. Note that x followed by all the elements which were bumped in the insertion procedure read in reading order must form a weakly decreasing sequence.

Lemma 6.1 *The insertion procedure $F \leftharpoondown x$ produces an RSCT.*

Our extension of the dual Schensted algorithm has a number of nice properties. For example, we can prove that the following theorem.

$$3 \rightarrow \begin{array}{|c|c|c|c|}\hline 2 & 1 & 0 \\ \cline{1-3} 2 & 0 \\ \cline{1-4} 4 & 3 & 2 & 0 \\ \hline 4 & 2 & 0 \\ \cline{1-3} 5 & 2 & 0 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 1 & 0 \\ \cline{1-3} 2 & 0 \\ \cline{1-4} 4 & \mathbf{3} & 2 & 0 \\ \hline 4 & 2 & 0 \\ \cline{1-3} 5 & 2 & 0 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 2 & 1 & 0 \\ \cline{1-3} 2 & 0 \\ \cline{1-4} 4 & 3 & 2 & 0 \\ \hline 4 & \mathbf{3} & 0 \\ \cline{1-3} 5 & 2 & 0 \\ \cline{1-3}\end{array} \qquad \begin{array}{|c|c|c|}\hline 2 & 1 \\ \cline{1-2} 2 \\ \cline{1-1} \mathbf{2} \\ \hline 4 & 3 & 2 \\ \hline 4 & 3 \\ \cline{1-2} 5 & 2 \\ \cline{1-2}\end{array}$$

$$\begin{array}{c} 3 \rightarrow \\ 0020010\mathbf{3}2222445 \end{array} \qquad 3 \rightarrow \mathbf{2}222445 \qquad 2 \rightarrow 222445 \qquad F \leftharpoondown 3$$

Fig. 8: The insertion procedure $F \leftharpoondown 3$

Theorem 6.2 *The insertion procedure on RSCT commutes with the reverse row insertion in the sense that* $\rho(T \leftarrow x) = \rho(T) \leftharpoondown x$.

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Kerov's central limit theorem for Schur-Weyl and Gelfand measures (extended abstract)

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Abstract. We show that the shapes of integer partitions chosen randomly according to Schur-Weyl measures of parameter $\alpha = 1/2$ and Gelfand measures satisfy Kerov's central limit theorem. Thus, there is a gaussian process Δ such that under Plancherel, Schur-Weyl or Gelfand measures, the deviations $\Delta_n(s) = \lambda_n(\sqrt{n}\, s) - \sqrt{n}\, \lambda^*_\infty(s)$ converge in law towards $\Delta(s)$, up to a translation along the x-axis in the case of Schur-Weyl measures, and up to a factor $\sqrt{2}$ and a deterministic remainder in the case of Gelfand measures. The proofs of these results follow the one given by Ivanov and Olshanski for Plancherel measures; hence, one uses a "method of noncommutative moments".

Résumé. Nous montrons que les formes des partitions d'entiers choisies aléatoirement sous les mesures de Schur-Weyl de paramètre $\alpha = 1/2$ et sous les mesures de Gelfand obéissent au théorème central limite de Kerov. Ainsi, il existe un processus gaussien Δ tel que sous les mesures de Plancherel, de Schur-Weyl ou de Gelfand, les déviations $\Delta_n(s) = \lambda_n(\sqrt{n}\, s) - \sqrt{n}\, \lambda^*_\infty(s)$ convergent en loi vers $\Delta(s)$, à une translation près le long de l'axe des abscisses pour les mesures de Schur-Weyl, et à un facteur $\sqrt{2}$ et un reste déterministe près dans le cas des mesures de Gelfand. Les preuves de ces résultats suivent celle donnée par Ivanov et Olshanski pour les mesures de Plancherel; ainsi, on utilise une "méthode de moments non commutatifs".

Keywords: Random partitions, representation theory of symmetric groups.

In this article, we investigate the **fluctuations of random partitions** chosen according to probability measures stemming from the representation theory of the symmetric groups. Given a group G and a (complex, finite-dimensional) representation V of G, the decomposition of V in irreducible components yields a probability measure $\mathbb{P}_V$ on the set $\widehat{G}$ of isomorphism classes of irreducible representations of G:

$$V = \bigoplus_{\lambda \in \widehat{G}} m_\lambda\, V^\lambda \quad \Rightarrow \quad \mathbb{P}_V[\lambda] = \frac{m_\lambda\, \dim V^\lambda}{\dim V} \tag{1}$$

When $G = \mathfrak{S}_n$ is the symmetric group of size n, the elements of $\widehat{\mathfrak{S}}_n$ can be labelled by integer partitions of size n, that is to say, non-increasing sequences of positive integers that sum to n; see *e.g.* [Ful97]. We shall denote by $\mathfrak{P}_n$ the set of partitions of size n, and a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ will be represented by its Young diagram, that is the array of n boxes with λ_1 boxes on the first row, λ_2 boxes on the second

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row, *etc.* Hence, any representation V_n of $\mathfrak{S}_n$ yields a model of random Young diagrams in $\mathfrak{P}_n$. For various families of representations $(V_n)_{n\in\mathbb{N}}$, one can establish limit theorems for the shapes of the Young diagrams chosen randomly according to the measures $\mathbb{P}_{V_n}$. In particular, let us consider the case when $V_n = \mathbb{C}\mathfrak{S}_n$ is the (left) regular representation of $\mathfrak{S}_n$; then, $m_\lambda = \dim V^\lambda$ for any partition λ, and $\mathbb{P}_n[\lambda] = (\dim V^\lambda)^2/n!$. The following result has been proved by Logan, Shepp, Kerov and Vershik, and a complete exposition can be found in [IO02]:

Theorem 1 *Let λ_n be a random partition of size n under the Plancherel measure $\mathbb{P}_n = \mathbb{P}_{\mathbb{C}\mathfrak{S}_n}$; we denote $s \mapsto \lambda_n(s)$ the upper boundary of the Young diagram rotated 45° (see §1), and $\lambda_n^*(s) = \lambda_n(s\sqrt{n})/\sqrt{n}$. As n goes to infinity, λ_n^* converges in probability and for the topology of uniform convergence on $\mathbb{R}$ towards the function*

$$\lambda_\infty^*(s) = \Omega(s) = \begin{cases} \frac{2}{\pi}\left(s\ \arcsin\left(\frac{s}{2}\right) + \sqrt{4-s^2}\right) & \text{if } |s| < 2, \\ |s| & \text{if } |s| \geq 2. \end{cases} \tag{2}$$

Moreover, on $[-2,2]$, $\Delta_n(s) = \sqrt{n}\,(\lambda_n^*(s) - \Omega(s))$ converges in law towards the generalized gaussian process

$$\Delta(s) = \frac{2}{\pi}\sum_{k\geq 2}\frac{\xi_k}{\sqrt{k}}\sin(k\theta) \quad \text{with } s = 2\cos\theta, \tag{3}$$

where the ξ_k's are standard independent gaussian variables.

The purpose of this paper is to describe a similar result of **gaussian concentration** for two other models of random partitions: the Schur-Weyl measures and the Gelfand measures (*cf.* §1). The limit shapes λ_∞^* for these models have already been computed, see for instance [Bia01] and [LS77]; so the true novelty of this paper consists in the central limit theorems, see Theorems 2 and 5. The situation can be summarized as follows:

- For Schur-Weyl measures $\mathbb{SW}_{n,c}$, everything happens as if the fluctuations $\sqrt{n}\,(\lambda_n^*(s) - \lambda_\infty^*(s))$ were those of the Plancherel measures, up to a translation of c along the x-axis.
- Similarly, for Gelfand measures $\mathbb{G}_n$, the fluctuations are again those of the Plancherel measures, but this time multiplied by a factor $\sqrt{2}$ and translated by a deterministic function along the y-axis.

Hence, *the same* generalized gaussian process Δ is involved in the description of the fluctuations of the random shapes under Schur-Weyl and Gelfand measures; this is a striking fact, and our main result.

To prove this, we shall follow and adapt the proof of Theorem 1 given by Ivanov and Olshanski in [IO02]; it is essentially a method of moments, and it involves the algebra of observables of diagrams, see §2. By using Śniady's theory of cumulants of observables, we shall first prove that irreducible characters $\chi^\lambda(k1^{n-k})$ under Schur-Weyl or Gelfand measures are jointly asymptotically gaussian (§3), and we shall compute explicitly their limit laws. Then, we will use exactly as in [IO02] the combinatorics of Chebyshev polynomials of the second kind to deduce from the asymptotics of characters the asymptotics of the shapes of the random partitions. Although most of our arguments can be found in the two papers [IO02, Ś06], new difficulties have arised in our asymptotic study. Thus, for Schur-Weyl measures, the calculations of linear functionals of the fluctuations involve hypergeometric identities; whereas for Gelfand measures, the problem resides mainly in the proof of the *joint* convergence of the renormalized character values, and one has to go beyond the setting of **asymptotic factorization** described in [Ś06], see Lemma 4.

This paper should be thought of as a extended abstract of [Mél10a] and [Mél10b]. In particular, we only present sketches of proofs, and we will omit most of the technical computations (for instance, the aforementioned hypergeometric identities in the case of Schur-Weyl measures). The author would like to thank M. Sage for showing him a proof of the second part of Lemma 4, and P. Biane and V. Féray for various comments and suggestions.

1 Schur-Weyl and Gelfand measures

A first family of representations of the symmetric groups is provided by **tensor powers of vector spaces**. If N is an integer, let us consider the space $W = (\mathbb{C}^N)^{\otimes n}$; it is endowed with a "diagonal" action of $\mathrm{GL}(N, \mathbb{C})$ on the left $g \cdot (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = g(v_1) \otimes g(v_2) \otimes \cdots \otimes g(v_n)$, and with an action of $\mathfrak{S}_n$ on the right by permutation of the letters: $(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \cdot \sigma = v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$. The algebras generated by these two actions are mutual full centralizers in $\mathrm{End}(W)$, whence a decomposition of W in direct sum of irreducible $(\mathrm{GL}(N, \mathbb{C}), \mathfrak{S}_n)$-bimodules:

$$(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\substack{\lambda \in \mathfrak{P}_n \\ \ell(\lambda) \leq N}} H^\lambda(N) \otimes_{\mathbb{C}} S^\lambda \tag{4}$$

Here, $H^\lambda(N)$ is the irreducible $\mathrm{GL}(N, \mathbb{C})$-module of highest weight λ, and S^λ denotes the irreducible Specht module of $\mathfrak{S}_n$ of type λ. The **Schur-Weyl measure** of parameters N and n is the measure on partitions of size n and length less than N associated to this decomposition:

$$\mathbb{P}_{N,n}[\lambda] = \frac{\dim H^\lambda(N) \times \dim S^\lambda}{N^n} = \left(\prod_{\square \in \lambda} 1 + \frac{c(\square)}{N}\right) \times \frac{n!}{\prod_{\square \in \lambda} h(\square)^2} \tag{5}$$

where $\{\square \in \lambda\}$ is the set of boxes of the Young diagram λ, $c(i,j) = i - j$ is the content of a box $\square = (i,j)$, and $h(i,j) = (\lambda_j - i) + (\lambda'_i - j) + 1$ is the hook length of the box, see [Ful97] for a proof of this formula. Since the term $n!/(\prod_{\square \in \lambda} h(\square)^2)$ is precisely the Plancherel measure $\mathbb{P}_n[\lambda]$, Schur-Weyl measures can be thought of as deformations of the Plancherel measures (fix n and let N go to infinity). With this point of view, an interesting scale to look at is when $\sqrt{n} \simeq cN$ with $c \geq 0$; with a slight abuse of notation(i), we shall denote $\mathbb{SW}_{n,c}$ the corresponding Schur-Weyl measure.

If $g \in \mathrm{GL}(N, \mathbb{C})$ has eigenvalues $x_1, \ldots, x_N$ and $\sigma \in \mathfrak{S}_n$ has cycle type $\mu \in \mathfrak{P}_n$, it is not very difficult to show that the bitrace of (g, σ) acting on $(\mathbb{C}^N)^{\otimes n}$ is given by the symmetric function $p_\mu(x_1, \ldots, x_N)$. As a consequence, if one denotes by χ^λ (respectively, ς^λ) the normalized (resp., non normalized) irreducible character of $\mathfrak{S}_n$ of type λ, and if λ is picked randomly according to the Schur-Weyl measure, then for any permutation σ of type μ:

$$\mathbb{SW}_{n,c}[\chi^\lambda(\sigma)] = \frac{1}{N^n} \sum_\lambda \dim H^\lambda(N) \times \varsigma^\lambda(\sigma) = \frac{\mathrm{bitr}(\mathrm{id}, \sigma)}{N^n} = \frac{p_\mu(1^N)}{N^n} = N^{\ell(\mu) - |\mu|} \tag{6}$$

(i) Indeed, the Schur-Weyl measure depends on the exact value of N, but for our asymptotic study, the estimate $\sqrt{n} \simeq cN$ will be sufficient.

Notice that the quantity $|\mu| - \ell(\mu)$ is not changed if one adds or removes parts of size 1 to the integer partition μ. In the following, we shall rather work with renormalized character values $\Sigma_\mu(\lambda)$, that are defined in the following way for any pair of partitions (λ, μ) of respective sizes n and k:

$$\Sigma_\mu(\lambda) = \begin{cases} n(n-1)\cdots(n-k+1)\,\chi^\lambda(\mu \sqcup 1^{n-k}) = n^{\downarrow|\mu|}\,\chi^\lambda(\mu \sqcup 1^{n-|\mu|}) & \text{if } n \geq k, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

With these conventions, and assuming that $\sqrt{n} = cN + O(1)$, Equation (6) becomes:

$$\mathbb{SW}_{n,c}[\Sigma_\mu] = c^{|\mu|-\ell(\mu)}\, n^{\frac{|\mu|+\ell(\mu)}{2}}\, (1 + O(n^{-1/2})). \tag{8}$$

This identity will turn out to be fundamental for the asymptotic analysis of Schur-Weyl measures.

If G is a finite group, recall that a **Gelfand model** for G is a representation in which each irreducible representation of G appears exactly one time. The Gelfand models of the symmetric groups have been studied for instance in [APR08], and they provide another interesting family of probability measures on the sets $\mathfrak{P}_n$, which we shall call **Gelfand measures** and denote by $\mathbb{G}_n$:

$$\mathbb{G}_n[\lambda] = \frac{\dim S^\lambda}{\sum_{\mu \in \mathfrak{P}_n} \dim S^\mu} = \frac{\dim S^\lambda}{I_n} \tag{9}$$

The normalization constant I_n is actually equal to the number of **involutions** of size n. More generally, the trace of a permutation σ in a Gelfand model G_n of $\mathfrak{S}_n$ is the number of "square roots" of σ, see Theorem 3.1 in [APR08]. As a consequence, for any permutation σ,

$$\mathbb{G}_n[\chi^\lambda(\sigma)] = \frac{\mathrm{card}\{\tau \in \mathfrak{S}_n \mid \sigma = \tau^2\}}{\mathrm{card}\{\tau \in \mathfrak{S}_n \mid \mathrm{id}_{[\![1,n]\!]} = \tau^2\}}. \tag{10}$$

The number I_n of involutions of size n is well known to be asymptotically equivalent to $\left(\frac{n}{\mathrm{e}}\right)^{\frac{n}{2}} \frac{\mathrm{e}^{\sqrt{n}-1/4}}{\sqrt{2}}$ — this follows from a saddle point analysis of the exponential generating function $\exp(x + x^2/2)$. On the other hand, for any partition μ, one can give an exact formula for the number of square roots of a permutation of cycle type $\mu \sqcup 1^{n-|\mu|}$, see Corollary 3.2 in [APR08]. Using these two facts and (10), one concludes that for any partition $\mu = 1^{m_1} 2^{m_2} \cdots s^{m_s}$:

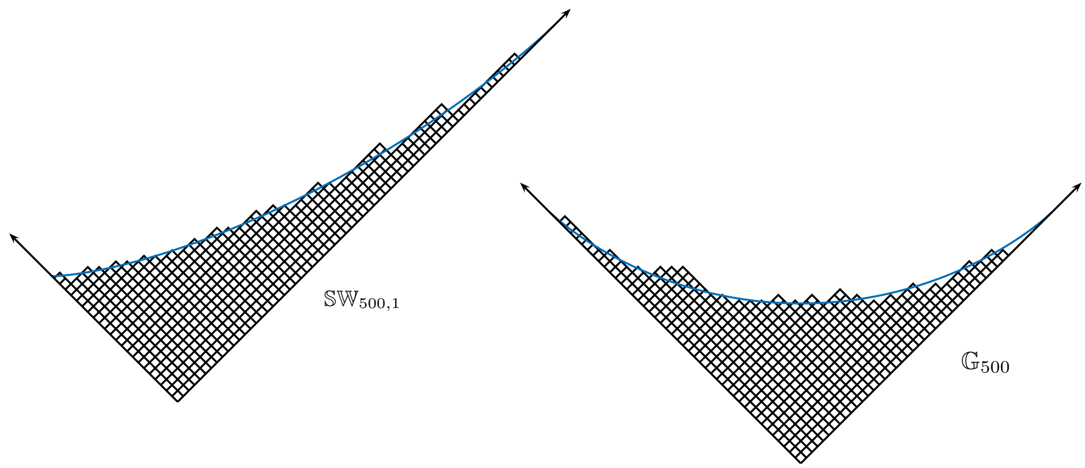
$$\mathbb{G}_n[\Sigma_\mu] = \left(\prod_{i=2}^{s} f(i, m_i) \right) n^{\frac{|\mu|+m_1(\mu)}{2}}\, (1 + O(n^{-1/2})), \tag{11}$$

where f is the function on pairs of non-negative integers defined by:

$$f(i,m) = \begin{cases} 0 & \text{if } i \text{ is even and } m \text{ is odd,} \\ \frac{m!}{m/2!} \left(\frac{i}{2}\right)^{m/2} & \text{if } i \text{ and } m \text{ are even,} \\ \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{m-2k!\,k!} \left(\frac{i}{2}\right)^k & \text{if } i \text{ is odd.} \end{cases} \tag{12}$$

Again, (11) is fundamental for the asymptotic analysis of Gelfand measures.

If λ is a Young diagram with longest rows at the bottom and boxes of area 2, let us rotate the figure 45^{o} counterclockwise, and consider the upper boundary $s \mapsto \lambda(s)$ of the planar shape — this is the usual russian convention for drawing Young diagrams. One obtains thus a Lipschitz function with constant 1, such that $\lambda(s) = |s|$ for $|s|$ big enough. The scaled version $\lambda^*(s) = \lambda(s\sqrt{n})/\sqrt{n}$ of this function is normalized so that $\int_{\mathbb{R}} \frac{\lambda^*(s)-|s|}{2}\,ds = 1$. We have drawn below $\lambda^*(s)$ when λ is a random partition of size $n = 500$ under the Schur-Weyl measure of parameter $c = 1$ (resp., under the Gelfand measure).



It is clear from the drawings that the scaled random partitions under Schur-Weyl measures $\mathbb{SW}_{n,c}$ and Gelfand measures $\mathbb{G}_n$ should have **limit shapes** Ω_c and Ω (figured in blue above). As we shall see hereafter, it is indeed the case, and $\Omega_0 = \Omega$ is the limit shape evoked in Theorem 1; the $\Omega_{c\neq 0}$'s are some deformations of this curve. In the following, we will mainly be interested in the scaled fluctuations:

$$\Delta_{n,c}(s) = \sqrt{n}\,(\lambda^*(s) - \Omega_c(s)) \quad \text{with } \lambda \sim \mathbb{SW}_{n,c} \tag{13}$$

$$\Delta_n(s) = \sqrt{n}\,(\lambda^*(s) - \Omega(s)) \qquad \text{with } \lambda \sim \mathbb{G}_n \tag{14}$$

and in §4, we will give a central limit theorem for these quantities.

To conclude this section, let us give some extra motivation for the asymptotic study of Schur-Weyl and Gelfand measures. First of all, recall that if w is a word of size n over an alphabet $[\![1, A]\!]$, then the **RSK algorithm** (*cf.* [Ful97]) associates to w a pair of Young tableaux $(P(w), Q(w))$ such that $P(w)$ is semistandard with entries in $[\![1, A]\!]$, $Q(w)$ is standard with entries in $[\![1, n]\!]$, and $P(w)$ and $Q(w)$ have same shape $\lambda(w) \in \mathfrak{P}_n$. Moreover, the first rows (resp., the first columns) of $\lambda(w)$ correspond to the lengths of the longest non-decreasing (resp., decreasing) subwords in w. That said, the Schur-Weyl measure of parameters N and n (resp., the Gelfand measure of parameter n) is exactly the image by $w \mapsto \lambda(w)$ of the uniform measure on words of size n over $[\![1, N]\!]$ (resp., of the uniform measure on involutions of size n). Consequently, our results can be restated in asymptotic combinatorial properties of words. Another motivation comes from **random matrix theory**: indeed, models of random partitions can be considered as discrete analogues of models of random matrices, and in this correspondence, Gelfand measures are related to the GOE and Wigner's law (see [BR01]), and Schur-Weyl measures are related to random covariance matrices and Marčenko-Pastur laws (*cf.* [Bia01]). Hence, asymptotic results on Gelfand and Schur-Weyl measures may provide a better understanding of some models from random matrix theory.

2 Observables of diagrams and partial permutations

If λ is a Young diagram, let us denote by $x_1 < y_1 < x_2 < y_2 < \cdots < y_{v-1} < x_v$ the interlacing sequences of local mimina and local maxima of the function $s \mapsto \lambda(s)$. The **interlacing moments** $\widetilde{p}_{k\geq 1}(\lambda)$ of λ are defined by:

$$\widetilde{p}_k(\lambda) = \sum_{i=1}^{v}(x_i)^k - \sum_{i=1}^{v-1}(y_i)^k = \int_{\mathbb{R}} s^k \sigma_\lambda''(s)\, ds \quad \text{with } \sigma_\lambda(s) = \frac{\lambda(s)-|s|}{2}. \tag{15}$$

The integral expression of $\widetilde{p}_k$ enables us to consider interlacing moments of more general planar shapes, for instance **continuous diagrams**, that is to say functions $s \mapsto \omega(s)$ such that ω is Lipschitz with constant 1 and equal to $|s|$ for $|s|$ big enough. On the other hand, $\widetilde{p}_1(\lambda) = 0$ for any (continuous) Young diagram, and the $\widetilde{p}_{k\geq 2}$'s generate a complex algebra of **observables of diagrams** which we shall denote $\mathscr{O}$, and which is freely generated by these $\widetilde{p}_k$'s:

$$\mathscr{O} = \mathbb{C}[\widetilde{p}_2, \widetilde{p}_3, \widetilde{p}_4, \ldots] \tag{16}$$

In [IO02], it is shown that $\mathscr{O}$ also contains the symbols Σ_μ introduced in §1, and that $\mathscr{O}$ is freely generated by $(\Sigma_k)_{k\geq 1}$, and linearly generated by $(\Sigma_\mu)_{\mu\in\mathfrak{P}}$. Moreover, $\mathscr{O}$ is graded either by the **weight** $\mathrm{wt}(\Sigma_\mu) = |\mu| + \ell(\mu)$ or by **Kerov's degree** $\deg_{\mathrm{K}}(\Sigma_\mu) = |\mu| + m_1(\mu)$, and with respect to these gradations, $\widetilde{p}_{k\geq 2}$ writes as:

$$\widetilde{p}_k = \sum_{\substack{\mu = 1^{m_1}2^{m_2}\cdots s^{m_s}\\ |\mu|+\ell(\mu)=k}} \frac{k^{\downarrow \ell(\mu)}}{\prod_{i\geq 1} m_i!} \prod_{i\geq 1} (\Sigma_i)^{m_i} + \left\{\substack{\text{observable of weight}\\ \text{smaller than } k-1}\right\} \tag{17}$$

$$= \sum_{j=0}^{\lfloor \frac{k-3}{2} \rfloor} \frac{k^{\downarrow j+1}}{j!}\, \Sigma_{k-1-2j}\, (\Sigma_1)^j + \begin{cases} \binom{k}{k/2}\, (\Sigma_1)^{k/2} & \text{if } k \text{ is even,} \\ 0 & \text{otherwise,} \end{cases} + \left\{\substack{\text{observable of Kerov}\\ \text{degree smaller than } k-1}\right\} \tag{18}$$

see [IO02, Proposition 3.7 and Equation 7.9]. Notice that by a change of basis between the $\widetilde{p}$'s and the Σ's, it is possible to evaluate the symbols Σ_μ on any *continuous* Young diagram ω.

In the following, we will need to know how to compute a product $\Sigma_\mu\, \Sigma_\nu$ and decompose it in the basis $(\Sigma_\lambda)_{\lambda\in\mathfrak{P}}$; in other words, we ask for the structure constants of $\mathscr{O}$ with respect to the basis of rescaled character values. Although there is no general formula for these coefficients, many things can be said about the symbols Σ_μ if one interprets them as elements of the algebra of **partial permutations**, see [IK99] for the definition of these objects. Thus, if one associates to $\Sigma_{(\mu_1,\ldots,\mu_r)}$ the formal sum of partial permutations

$$\sum_{a_{11}\neq a_{12}\neq \cdots \neq a_{r\mu_r}} \Big[(a_{11},\ldots,a_{1\mu_1})\cdots(a_{r1},\ldots,a_{r\mu_r}), \{a_{ij}\}_{1\leq i\leq r, 1\leq j\leq \mu_i}\Big] \tag{19}$$

then one gets an isomorphism between $\mathscr{O}$ and a commutative subalgebra $\mathscr{A}_\infty$ of the algebra $\mathscr{B}_\infty$ of partial permutations. This combinatorial interpretation of the symbols $\varSigma_\mu$ gives at least the terms of higher weight or higher Kerov degree of a product $\varSigma_\mu\,\varSigma_\nu$. In particular, if k and l are bigger than 2, then:

$$\varSigma_k\,\varSigma_l = \varSigma_{k,l} + \sum_{r\geq 1}\sum_{\substack{a_1+\cdots+a_r=k\\ b_1+\cdots+b_r=l}} \frac{kl}{r}\,\varSigma_{(a_1+b_1-1),\ldots,(a_r+b_r-1)} + \left\{\substack{\text{observable of weight}\\ \text{smaller than } k+l-2}\right\} \tag{20}$$

$$= \varSigma_{k,l} + \delta_{k,l}\,k\,\varSigma_{1^k} + \left\{\substack{\text{observable of Kerov}\\ \text{degree smaller than } k+l-1}\right\} \tag{21}$$

These expansions will be very useful for proving that renormalized characters are asymptotically gaussian under Schur-Weyl or Gelfand measures, and for computing the limit covariances.

In §1, we have computed the expectations of the observables $\varSigma_\mu$ under Schur-Weyl and Gelfand measures. Equation (8) implies that $\mathbb{SW}_{n,c}[f] = O(n^{\mathrm{wt}(f)/2})$ for any observable f; similarly, (11) implies that $\mathbb{G}_n[f] = O(n^{\deg_{\mathrm{K}}(f)/2})$ for any $f \in \mathscr{O}$. Using this with $f = \varSigma_k$ or $(\varSigma_k)^2$, one then gets by Bienaymé-Chebyshev inequality:

$$\forall k\geq 1,\ \frac{\varSigma_k(\lambda)}{n^{\frac{k+1}{2}}} \longrightarrow_{\mathbb{SW}_{n,c}} c^{k-1} \qquad ; \qquad \forall k\geq 1,\ \frac{\varSigma_k(\lambda)}{n^{\frac{k+1}{2}}} \longrightarrow_{\mathbb{G}_n} \delta_{k,1} \tag{22}$$

where the long right arrows indicate convergence in law (here, we have even convergence in probability). Moreover, a change of variables shows that if λ^* is the scaled version of a Young diagram λ, then $\widetilde{p}_k(\lambda) = n^{k/2}\,\widetilde{p}_k(\lambda^*)$ for any $k\geq 2$. Since $\varSigma_k$ is an observable of weight $k+1$, one has as a consequence $\varSigma_k(\lambda) = n^{(k+1)/2}\,\varSigma_k(\lambda^*) + O(n^{k/2})$, and (22) becomes

$$\forall k\geq 1,\ \varSigma_k(\lambda^*) \longrightarrow_{\mathbb{SW}_{n,c}} c^{k-1} \qquad ; \qquad \forall k\geq 1,\ \varSigma_k(\lambda^*) \longrightarrow_{\mathbb{G}_n} \delta_{k,1}. \tag{23}$$

Finally, (17) allows to recover the limits of the $\widetilde{p}_k(\lambda^*)$'s:

$$p_k(\lambda^*) \longrightarrow_{\mathbb{SW}_{n,c}} \sum_{i=1}^{\lfloor \frac{k}{2}\rfloor} \frac{k^{\downarrow 2i}}{(k-i)\,i!\,i-1!}\,c^{k-2i} \qquad ; \qquad p_k(\lambda^*) \longrightarrow_{\mathbb{G}_n} \begin{cases}\binom{k}{k/2} & \text{if } k \text{ is even,}\\ 0 & \text{if } k \text{ is odd.}\end{cases} \tag{24}$$

For continuous Young diagrams, convergence of all the observables (or equivalently, convergence of all the interlacing moments) is equivalent to the uniform convergence on $\mathbb{R}$. As a consequence, (24) implies the existence of limit shapes Ω_c under Schur-Weyl measures and $\Omega = \Omega_0$ under Gelfand measures. A lengthy calculation gives an explicit formula for these limit shapes, see [Bia01, p. 8]; in particular,

$$\Omega_c'(s\in[c-2,c+2]) = \frac{2}{\pi}\,\arcsin\left(\frac{s+c}{2\sqrt{1+sc}}\right) \qquad ; \qquad \Omega'(s\in[-2,2]) = \frac{2}{\pi}\,\arcsin\left(\frac{s}{2}\right). \tag{25}$$

The previous computations illustrate the use of observables of diagrams in the setting of asymptotic representation theory; for further details, we refer to [IO02]. It turns out that the same methods allow to treat the second order asymptotics, that is to say the asymptotics of fluctuations $\Delta_{n,c}(s)$ or $\Delta_n(s)$. Hence, in §3, we will show that the observables $\varSigma_k$ under Schur-Weyl or Gelfand measures converge in joint law towards a gaussian vector with explicit covariance matrix. Then, in §4, by using the combinatorics of $\mathscr{O}$, we will obtain the asymptotic gaussian behaviour of the fluctuations of the shapes of the random partitions.

3 Limiting distributions of the renormalized characters

The algebraic counterpart of Theorem 1 is the following result: under Plancherel measures, rescaled characters values $\Sigma_k(\lambda)/n^{k/2}$ with $k \geq 2$ converge jointly towards independent variables $\sqrt{k}\,\xi_k$, where the ξ_k's are standard gaussian variables of mean 0 and covariance 1. There is an analoguous result for Schur-Weyl and Gelfand measures, that can be stated as follows:

Theorem 2 *We fix a family of independent standard gaussian variables $(\xi_k)_{k\geq 2}$. As n goes to infinity, under Schur-Weyl measures $\mathbb{SW}_{n,c}$, the rescaled character values $X_{k,n,c} = \frac{\Sigma_k(\lambda)}{n^{k/2}} - c^{k-1}\, n^{1/2}$ converge in finite-dimensional laws towards the gaussian variables*

$$X_{k,\infty,c} = \sum_{r=0}^{k-2} \binom{k}{r} c^r \sqrt{k-r}\,\xi_{k-r}. \tag{26}$$

Similarly, under Gelfand measures $\mathbb{G}_n$, the rescaled character values $Y_{k,n} = \frac{\Sigma_k(\lambda)}{n^{k/2}}$ converge in joint law towards the gaussian variables

$$Y_{k,\infty} = e_k + \sqrt{2k}\,\xi_k, \tag{27}$$

where $e_k = 0$ if k is even, and $e_k = 1$ if k is odd.

In the following, $\mathfrak{Q}_r$ is the set of set partitions of $[\![1, r]\!]$, and if $\pi \in \mathfrak{Q}_r$ has $\ell(\pi)$ parts, then $\mu(\pi)$ is the Möbius function of π, that is to say $(-1)^{\ell(\pi)-1}\,(\ell(\pi)-1)!$. The **joint cumulant** of random variables $X_1, \ldots, X_r$ is defined by

$$k(X_1, \ldots, X_r) = \sum_{\pi\in\mathfrak{Q}_r} \mu(\pi) \prod_{\pi_j\in\pi} \mathbb{E}\left[\prod_{i\in\pi_j} X_i\right]. \tag{28}$$

To prove Theorem 2, one can study joint cumulants of rescaled character values: indeed, if all the joint cumulants of order $r \geq 3$ of the coordinates of a random vector converge to 0, and if there is a finite limit for the cumulants of order 1 and 2, then the random vector converges in law towards a *gaussian* vector whose coordinates have for means the limiting cumulants of order 1, and for covariances the limiting cumulants of order 2. That said, in the context of asymptotic representation theory of symmetric groups, a sufficient condition for the asymptotic gaussian behaviour of rescaled character values has been exhibited by P. Śniady in [Ś06]. Let $(\mathbb{P}_n)_{n\in\mathbb{N}}$ be a family of probability measures on the sets $\mathfrak{P}_n$ of integer partitions; we denote by $\mathbb{E}_n$ and k_n the corresponding expectations and cumulants for observables of diagrams, viewed as random variables if the diagrams are chosen according to the measures $\mathbb{P}_n$. Notice that any family of commuting permutations $\sigma_1, \ldots, \sigma_r$ can also be considered as a family of random variables *via* the maps $(\sigma_i, \lambda) \mapsto \chi^\lambda(\sigma_i)$.

Proposition 3 *The following assertions are equivalent:*

1. *For all positive integers $l_1, \ldots, l_r$,*
$$k_n(\Sigma_{l_1}, \ldots, \Sigma_{l_r})\, n^{-\frac{l_1+\cdots+l_r-r+2}{2}} = O(1). \tag{29}$$
2. *If $\sigma_{l_1}, \ldots, \sigma_{l_r}$ are disjoint cycles of respective lengths $l_1, \ldots, l_r$, then*
$$k_n(\sigma_{l_1}, \ldots, \sigma_{l_r})\, n^{\frac{l_1+\cdots+l_r+r-2}{2}} = O(1). \tag{30}$$

In that case known as asymptotic factorization property, the following limits, if they exist, are equal:

$$c_{l+1} = \lim_{n\to\infty} n^{-\frac{l+1}{2}}\, \mathbb{E}_n[\Sigma_l] = \lim_{n\to\infty} n^{\frac{l-1}{2}}\, \mathbb{E}_n[\sigma_l] \tag{31}$$

$$v_{k,l} = \lim_{n\to\infty} k_n(\Sigma_k, \Sigma_l)\, n^{-\frac{k+l}{2}} = \lim_{n\to\infty} k_n(\sigma_k, \sigma_l)\, n^{\frac{k+l}{2}} - kl\, c_{k+1}\, c_{l+1} + \sum_{\substack{a_1+\cdots+a_r=k\\ b_1+\cdots+b_r=l}} \frac{kl}{r}\, c_{a_1+b_1} \cdots c_{a_r+b_r} \tag{32}$$

the second identity being of course related to Equation (20)*. Then, the scaled and centered character values* $\frac{\Sigma_k}{n^{k/2}} - c_{k+1}\, n^{1/2}$ *converge in finite-dimensional laws towards a centered gaussian vector of covariance matrix* $(v_{k,l})_{k,l\geq 2}$.

Because of Equation (6), the Schur-Weyl measures satisfy trivially (30); this fact was already mentioned in [Ś06, Example 6]. The first part of Theorem 2 is therefore a simple consequence of Formula (32) — we also needed to perform a Gram-Schmidt orthogonalization. Unfortunately, we do not know whether the Gelfand measures have the asymptotic factorization property; it might be the case, but for instance, given 4 disjoint transpositions $\tau_1, \ldots, \tau_4$, Equation (10) only ensures that $k_n(\tau_1, \ldots, \tau_4)$ is a $O(n^{-9/2})$, and not a $O(n^{-5})$ as would be if Condition 30 were satisfied[(ii)]. Hence, extra technology is needed to establish the second part of Theorem 2, although the idea remains the same, namely, showing that all the cumulants of order $r \geq 3$ of the variables $Y_{k,n}$ vanish at infinity.

To begin with, let us notice that it is easy to demonstrate the simple (*non-joint*) convergence of the $Y_{k,n}$'s towards gaussian variables of respective means e_k and respective variances $2k$. Indeed, by induction on m, one sees from (21) that

$$(\Sigma_k)^m = \sum_{p=0}^{\lfloor \frac{m}{2} \rfloor} \frac{m!}{m-2p!\,p!} \left(\frac{k}{2}\right)^p \Sigma_{1^{pk}k^{m-2p}} + \left\{\substack{\text{observable of Kerov degree}\\ \text{smaller than } km-1}\right\} \tag{33}$$

for any $k \geq 2$. As a consequence, all the moments of the variables $Y_{k,n}$ have limits that can be explicitly computed using simply (11), and these limits are exactly the moments of the gaussian variables $Y_{k,\infty}$; since gaussian variables are characterized by their moments, it implies the non-joint convergence in law. Now, to obtain the *joint* convergence and the asymptotic independence, one has to prove the two following facts:

$$\forall r \geq 3,\ \forall l_1, \ldots, l_r \geq 2,\ k_n(\Sigma_{l_1}, \Sigma_{l_2}, \ldots, \Sigma_{l_r}) = o\left(n^{\frac{l_1+l_2+\cdots+l_r}{2}}\right) \tag{34}$$

$$\forall k, l \geq 2, \quad (k \neq l) \;\Rightarrow\; k_n(\Sigma_k, \Sigma_l) = o\left(n^{\frac{k+l}{2}}\right) \tag{35}$$

As $\deg_{\mathrm{K}}(\Sigma_{k\geq 2}) = k$ and $\mathbb{G}_n[f] = O(n^{\deg_{\mathrm{K}}(f)/2})$ for any observable f, the cumulants above are already known to be $O(n^{\frac{l_1+l_2+\cdots+l_r}{2}})$ or $O(n^{\frac{k+l}{2}})$; hence, one has to gain only one order of magnitude.

(ii) In fact, by using a more precise estimate of the expectations, one can show that this joint cumulant is indeed a $O(n^{-5})$, but these are heavy computations and we do not know how to generalize them.

In the algebra $\mathscr{O}$, let us define the **disjoint product** $\bullet$ by $\Sigma_\mu \bullet \Sigma_\nu = \Sigma_{\mu \sqcup \nu}$; this product is compatible with the weight of observables or with Kerov's degree. Then, one defines the **disjoint cumulant** and the **identity cumulant** of observables $X_1, \ldots, X_r$ as follows:

$$k^\bullet(X_1, \ldots, X_r) = \sum_{\pi \in \mathfrak{Q}_r} \mu(\pi) \prod_{\pi_j \in \pi} \mathbb{E}\left[\dot{\prod_{i \in \pi_j}} X_i\right] \tag{36}$$

$$k^{\mathrm{id}}(X_1, \ldots, X_r) = \sum_{\pi \in \mathfrak{Q}_r} \mu(\pi) \dot{\prod_{\pi_j \in \pi}} \left[\prod_{i \in \pi_j} X_i\right] \tag{37}$$

These new quantities enable us to decompose the standard cumulant of observables as a sum over set partitions

$$k(X_1, \ldots, X_r) = \sum_{\pi \in \mathfrak{Q}_r} k^\bullet(k^{\mathrm{id}}(X_{i \in \pi_1}), \ldots, k^{\mathrm{id}}(X_{i \in \pi_s})); \tag{38}$$

see [Ś06, Proposition 13]. In order to gain one order of magnitude in the estimate of $k(\Sigma_{l_1}, \ldots, \Sigma_{l_r})$, one can use this expansion in disjoint cumulants $k^\bullet_\pi$, and for each $k^\bullet_\pi$:

1. either bound the sum of the Kerov degrees of the identity cumulants by $l_1 + l_2 + \cdots + l_r - 1$;
2. or, in case the sum of the Kerov degrees is maximal and equal to $l_1 + \cdots + l_r$, use the properties of the set partition π and Equation (11) to get directly a bound on the disjoint cumulant $k^\bullet_\pi$.

The first method works for almost all set partitions π, thanks to a combinatorial interpretation of the identity cumulants of Σ_l's in the algebra of partial permutations, see [FM10, Lemma 19]; the remaining cases can be worked out by using a kind of Möbius inversion formula. More precisely:

Lemma 4 *The total Kerov degree of the identity cumulants involved in a disjoint cumulant $k^\bullet_\pi$ is smaller than $l_1 + l_2 + \cdots + l_r - 1$, unless π has only parts of size 1 and parts π_j of size 2 such that if $\pi_j = \{i_1, i_2\}$, then $\Sigma_{l_{i_1}} = \Sigma_{l_{i_2}}$.*

In that case, $k^\bullet_\pi$ is proportional to a disjoint cumulant $k^\bullet(\Sigma_{m_1}, \ldots, \Sigma_{m_s}, \Sigma_{1^{p_1}}, \ldots, \Sigma_{1^{p_t}})$, and for Gelfand measures, the higher term of this disjoint cumulant vanishes again, because of the following Möbius inversion formula. Let $(1, \ldots, 1, 2, \ldots, 2, \ldots, s, \ldots, s)$ be a sequence of r integers with $r_1 \geq 1$ integers 1, $r_2 \geq 1$ integers 2, etc. *If $\pi = \pi_1 \sqcup \pi_2 \sqcup \cdots \sqcup \pi_{\ell(\pi)}$ is in $\mathfrak{Q}_r$, we denote by r_{ij} the number of integers i that fall in π_j. Suppose that $s \geq 2$. Then, for any function F,*

$$\sum_{\pi \in \mathfrak{Q}_r} (-1)^{\ell(\pi)-1} \, (\ell(\pi) - 1)! \prod_{j=1}^{\ell(\pi)} \prod_{r_{ij} \geq 1} F(i, r_{ij}) = 0\,. \tag{39}$$

The proof of the second part goes by induction on the r_i's. As a consequence of Lemma 4 and of the previous discussion, all the disjoint cumulants in the expansion of a standard cumulant of observables Σ_l as in (34) or (35) have a smaller order of magnitude, which ends the proof of Theorem 2. As mentioned in the introduction, although Lemma 4 may seem rather innocent and technical, it is a key argument when Proposition 3 fails to apply and one wants nethertheless to establish the asymptotic gaussian behaviour of character values.

4 Central limit theorems for the shapes of the random partitions

Finally, let us explain how to translate Theorem 2 in terms of the fluctuations $\Delta_{n,c}(s)$ and $\Delta_n(s)$. It is clear that the moments

$$\int_{\mathbb{R}} (s-c)^k \left[\sqrt{n}\,(\lambda^*(s) - \Omega_c(s))\right] ds \qquad \text{and} \qquad \int_{\mathbb{R}} s^k \left[\sqrt{n}\,(\lambda^*(s) - \Omega(s))\right] ds \tag{40}$$

can be written in terms of the observables $\widetilde{p}_k(\lambda)$; then, using equations (17) or (18), one can express these moments in terms of the $X_{k,n,c}$'s or the $Y_{k,n}$'s, up to a negligible remainder. Thus:

$$\left\langle (s-c)^k \mid \Delta_{n,c} \right\rangle \simeq_{\mathbb{SW}_{n,c}} \frac{2}{k+1} \sum_{l=0}^{k-1} \left(\sum_{m=0}^{\lfloor \frac{l}{2} \rfloor} \binom{k+1}{m} \binom{k+1-2m}{l-2m} (-c)^{l-2m} \right) X_{k+1-l,n,c} \tag{41}$$

$$\left\langle s^k \mid \Delta_n \right\rangle \simeq_{\mathbb{G}_n} \frac{2}{k+1} \sum_{m=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k+1}{m} Y_{k+1-2m,n} \tag{42}$$

Then, it has been shown in [IO02] that these identities can be reversed by using the **Chebyshev polynomials of the second kind** $U_k(X)$, renormalized so that $U_k(2\cos\theta) = \frac{\sin(k+1)\theta}{\sin\theta}$. Hence:

$$\langle U_k(s-c) \mid \Delta_{n,c} \rangle \simeq_{\mathbb{SW}_{n,c}} \frac{2}{k+1} \sum_{l=0}^{k-1} \binom{k+1}{l} (-c)^l \, X_{k+1-l,n,c} \longrightarrow_{\mathbb{SW}_{n,c}} 2\sqrt{\frac{1}{k+1}}\, \xi_{k+1} \tag{43}$$

$$\langle U_k(s) \mid \Delta_n \rangle \simeq_{\mathbb{G}_n} \frac{2\, Y_{k+1,n}}{k+1} \longrightarrow_{\mathbb{G}_n} \frac{2e_{k+1}}{k+1} + 2\sqrt{\frac{2}{k+1}}\, \xi_{k+1} \tag{44}$$

The calculations are really the same as in [IO02, §7] for Gelfand measures; on the other hand, they are much more convoluted in the case of Schur-Weyl measures, and they involve various hypergeometric identities (actually, it is quite a miracle to obtain at the end such a simple expression for $\langle U_k(s-c) \mid \Delta_{n,c} \rangle$). As the Chebyshev polynomials U_k form an orthogonal basis, the same discussion as in [IO02, §9] gives us finally the convergence in law of the deviations in the space of distributions:

Theorem 5 *In the sense of distributions on the interval $[c-2, c+2]$, under the Schur-Weyl measures $\mathbb{SW}_{n,c}$, $\Delta_{n,c}(s)$ converges in law towards the generalized gaussian process*

$$\Delta(s-c), \tag{45}$$

where Δ is as in Theorem 1. Similarly, in the sense of distributions on the interval $[-2, 2]$, under the Gelfand measures $\mathbb{G}_n$, $\Delta_n(s)$ converges in law towards the generalized gaussian process

$$\frac{1}{2} - \frac{\sqrt{4-s^2}}{\pi} + \sqrt{2}\,\Delta(s). \tag{46}$$

The precaution "in the sense of distributions" is justified, because the infinite sum of random variables $\Delta(s)$ does not converge pointwise. However, it makes sense as a distribution — this is the same phenomenon as for the well-known gaussian free field. It is really an amazing fact that the same gaussian process is involved in the asymptotics of Plancherel, Schur-Weyl and Gelfand measures. Let us conclude our extended abstract with an open problem. We define the **β-Plancherel probability measures** by:

$$\mathbb{P}_{n,\beta}[\lambda] = \frac{(\dim S^\lambda)^\beta}{\sum_{\mu \in \mathfrak{P}_n} (\dim S^\mu)^\beta}, \quad \beta > 0. \tag{47}$$

They are discrete analogues of the β-ensembles of random matrices (see [BR01]), and when $\beta = 1$ or 2, one recovers the Gelfand measures and the Plancherel measures. One can conjecture that under β-Plancherel measures, the fluctuations $\sqrt{n}\,(\lambda^*(s) - \Omega(s))$ converge in law towards generalized gaussian processes, possibly of the form

$$f_\beta(s) + \sqrt{\frac{2}{\beta}}\;\Delta(s) \tag{48}$$

with f_β deterministic function on $[-2, 2]$. It would be interesting to prove such a general result.

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Bijective evaluation of the connection coefficients of the double coset algebra

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Abstract. This paper is devoted to the evaluation of the generating series of the connection coefficients of the double cosets of the hyperoctahedral group. Hanlon, Stanley, Stembridge (1992) showed that this series, indexed by a partition ν, gives the spectral distribution of some random matrices that are of interest in random matrix theory. We provide an explicit evaluation of this series when $\nu = (n)$ in terms of monomial symmetric functions. Our development relies on an interpretation of the connection coefficients in terms of locally orientable hypermaps and a new bijective construction between partitioned locally orientable hypermaps and some permuted forests.

Résumé. Cet article est dédié à l'évaluation des séries génératrices des coefficients de connexion des classes doubles (cosets) du groupe hyperoctaédral. Hanlon, Stanley, Stembridge (1992) ont montré que ces séries indexées par une partition ν donnent la distribution spectrale de certaines matrices aléatoires jouant un rôle important dans la théorie des matrices aléatoires. Nous fournissons une évaluation explicite de ces séries dans le cas $\nu = (n)$ en termes de monômes symétriques. Notre développement est fondé sur une interprétation des coefficients de connexion en termes d'hypercartes localement orientables et sur une nouvelle bijection entre les hypercartes localement orientables partitionnées et certaines forêts permutées.

Keywords: double coset algebra, connection coefficients, locally orientable hypermaps, forests

1 Introduction

In what follows, we denote by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)_\geq \vdash n$ an integer partition of n and $\ell(\lambda) = k$ the number of parts of λ. If $n_i(\lambda)$ is the number of parts of λ that are equal to i (by convention $n_0(\lambda) = 0$), then we write λ as $1^{n_1(\lambda)}\, 2^{n_2(\lambda)} \ldots$ and let $Aut_\lambda = \prod_i n_i(\lambda)!$. Also, if $\lambda \vdash n$, let $\lambda\lambda$ and 2λ be the partitions of $2n$ $(\lambda_1, \lambda_1, \lambda_2, \lambda_2, \ldots)$ and $(2\lambda_1, 2\lambda_2, \ldots)$ respectively. Let $[m] = \{1, \ldots, m\}$ and S_m be the symmetric group on m elements, *e.g.* on $[m]$, and let $\mathcal{C}_\lambda$ be the conjugacy class in S_m of permutations w with cycle type $type(w) = \lambda \vdash m$.

We look at **perfect pairings** of the set $[n] \cup [\widehat{n}] = \{1, \ldots n, \widehat{1}, \ldots, \widehat{n}\}$ which we view as fixed point free involutions in $S_{2n}([n] \cup [\widehat{n}])$. Note that for $f, g \in S_{2n}$, the disjoint cycles of the product $f \circ g$ have repeated lengths *i.e.* $f \circ g \in \mathcal{C}_{\lambda\lambda}$. Also, for $w \in S_{2n}$, let f_w be the pairing $(w^{-1}(1), w^{-1}(\hat{1})) \cdots (w^{-1}(n), w^{-1}(\hat{n}))$.

Let B_n be the hyperoctahedral group which we view as the centralizer in S_{2n} of the involution $f_\star = (1\widehat{1})(2\widehat{2}) \cdots (n\widehat{n})$. Then $|B_n| = 2^n n!$, and it is well known that the double cosets of B_n in S_{2n} are indexed by partitions ν of n, and consist of permutations $w \in S_{2n}$ such that the cycle type of $f_\star \circ f_w$ is $\nu\nu$ [8, Ch.

VII.2]. If we denote such double coset by K_ν and pick from it a fixed element w_ν, then let $b^\nu_{\lambda,\mu}$ be the number of ordered factorizations $u_1 \cdot u_2$ of w_ν where $u_1 \in K_\lambda$ and $u_2 \in K_\mu$. We provide a combinatorial formula for $b^\nu_{\lambda,\mu}$ when $\nu = (n) = n$ (say $w_{(n)} = (123\dots n)(\widehat{n}\,\widehat{n-1}\,\widehat{n-2}\dots\widehat{1})$) by interpreting these factorizations as **locally orientable (partitioned) unicellular hypermaps**.

Theorem 1.1 *Let $b^n_{\lambda,\mu}$ be the number of ordered factorizations $u_1 \cdot u_2$ of $w_{(n)}$ where $u_1 \in K_\lambda$ and $u_2 \in K_\mu$. If $p_\lambda(\mathbf{x})$ and $m_\lambda(\mathbf{x})$ are the power and monomial symmetric functions then*

$$\frac{1}{2^n n!}\sum_{\lambda,\mu\vdash n} b^n_{\lambda,\mu}\, p_\lambda(\mathbf{x})p_\mu(\mathbf{y}) = \sum_{\lambda,\mu\vdash n} Aut_\lambda Aut_\mu m_\lambda(\mathbf{x})m_\mu(\mathbf{y}) \sum_{\mathbf{A}\in M_{\lambda,\mu}} \frac{\mathcal{N}(\mathbf{A})}{\mathbf{A}!}\frac{(n-q-2r)!(n-p-2r)!}{(n+1-p-q-2r)!} \frac{p'!q'!\,(r-p')!\,(r-q')!}{2^{2r-p'-q'}}\prod_{i,j,k}\binom{i-1}{j,k,j+k}^{(P+Q)_{i,j,k}}\binom{i-1}{j,k,j+k-1}^{(P'+Q')_{i,j,k}} \tag{1}$$

Where, $M_{\lambda,\mu}$ is the set of 4-tuples $\mathbf{A} = (P, P', Q, Q')$ of tridimensional arrays of non negative integers with $p = |P| = \sum_{i,j,k\geq 0} P_{ijk} \neq 0, p' = \ell(\lambda) - p = |P'|, q = |Q|, q' = \ell(\mu) - q = |Q'|$, and

$$\begin{aligned} n_i(\lambda) &= \textstyle\sum_{j,k} P_{ijk} + P'_{ijk}, & n_i(\mu) &= \textstyle\sum_{j,k\geq 0} Q_{ijk} + Q'_{ijk},\\ r &= \textstyle\sum_{i,j,k}(j+k)(P_{ijk}+P'_{ijk}), & r &= \textstyle\sum_{i,j,k}(j+k)(Q_{ijk}+Q'_{ijk}),\\ q' &= \textstyle\sum_{i,j,k} j(P_{ijk}+P'_{ijk}), & p' &= \textstyle\sum_{i,j,k} j(Q_{ijk}+Q'_{ijk}). \end{aligned}$$

where $\mathbf{A}! = \prod_{i,j,k} P_{ijk}!\,P'_{ijk}!\,Q_{ijk}!\,Q'_{ijk}!$. And $\mathcal{N}(\mathbf{A}) = \sum_{t,u,v} tP_{tuv}$ if $q' = 0$, otherwise if $q' \neq 0$ then

$$\mathcal{N}(\mathbf{A}) = \frac{1}{q'}\sum_{t-2u-2v>0}\frac{tP_{tuv}}{t-2u-2v}\left[(t-2u-2v)\left(\frac{\delta_{p'\neq 0}}{p'}\sum_{i,j,k} jQ\sum_{i,j,k} jP' + \frac{\sum_{i,j,k}(i-1-2j-2k)Q\sum_{i,j,k} jP}{n-q-2r}\right) + u\left(\frac{\delta_{p'\neq 0}}{p'}\sum_{i,j,k}(i-2j-2k)P'\sum_{i,j,k} jQ' + \sum_{i,j,k}(i-2j-2k)Q'\frac{1+\sum_{i,j,k}(i-1-2j-2k)P}{n-q-2r}\right)\right].$$

Remark 1.2 *For some limit values of the 4-tuple $\mathbf{A}$, the summand definition in $\sum_{\mathbf{A}\in M_{\lambda,\mu}}\cdots$ on the RHS of Equation (1) is slightly different as detailed in Appendix 6 of the paper.*

1.1 Background on connection coefficients $b^\nu_{\lambda,\mu}$

By abuse of notation, let the double coset K_ν also represent the sum of its elements in the group algebra $\mathbb{C}S_{2n}$. Then K_ν form a basis of a commutative subalgebra of $\mathbb{C}S_{2n}$ (the *Hecke algebra* of the *Gelfand pair* (S_{2n}, B_n)) and one can check that $K_\lambda \cdot K_\mu = \sum_\nu b^\nu_{\lambda,\mu} K_\nu$. Thus, $\{b^\nu_{\lambda,\mu}\}$ are the **connection coefficients** of this double coset algebra. We use $Z_\lambda(\mathbf{x})$ to denote the **zonal polynomial** indexed by λ which can be viewed as an analogue of the Schur function s_λ (for more information on these polynomials see [8, Ch. VII]). In terms of p_μ: $s_\lambda(\mathbf{x}) = \sum_\mu z_\mu^{-1}\chi^\lambda_\mu p_\mu(\mathbf{x})$ where $z_\lambda = Aut_\lambda \prod_i i^{n_i(\lambda)}$, χ^λ_μ are the irreducible characters of the symmetric group; and $Z_\lambda(\mathbf{x}) = \frac{1}{|B_n|}\sum_{\mu\vdash n}\varphi^\lambda(\mu)p_\mu(\mathbf{x})$ where $\varphi^\lambda(\mu) = \sum_{w\in K_\mu}\chi^{2\lambda}_{type(w)}$.

In [5, Lemma 3.3] a formula for the connection coefficients in terms of χ^λ_μ was given:

$$b^\nu_{\lambda,\mu} = \frac{1}{|K_\nu|}\sum_{\beta\vdash n}\frac{1}{H_{2\nu}}\varphi^\beta(\nu)\varphi^\beta(\lambda)\varphi^\beta(\mu), \tag{2}$$

where $|K_\nu| = |B_n||\mathcal{C}_\nu|2^{n-\ell(\nu)}$ [2, Lemma 2.1], and $H_{2\lambda}$ is the product of all the *hook-lengths* of the partition 2λ.

Let $\Psi^\nu(\mathbf{x},\mathbf{y}) = \frac{1}{|B_n|}\sum_{\lambda,\mu} b^\nu_{\lambda,\mu}p_\lambda(\mathbf{x})p_\mu(\mathbf{y})$. So $\Psi^{(n)}$ is the LHS of (1). Equation (2) immediately implies that $\Psi^\nu(\mathbf{x},\mathbf{y}) = \frac{1}{|K_\nu|}\sum_{\lambda\vdash n}\frac{|B_n|}{H_{2\lambda}}\varphi^\lambda(\nu)Z_\lambda(\mathbf{x})Z_\mu(\mathbf{y})$. Moreover, if for an $n\times n$ matrix X we say that $p_k(X) = trace(X^k)$, then in [5, Thm. 3.5] it was shown that Ψ^ν is also the expectation of $p_\nu(XUYU^T)$ over U, where U are $n\times n$ matrices whose entries are independent standard normal random *real* variables and X, Y are arbitrary but fixed real symmetric matrices.

2 Combinatorial formulation

2.1 Unicellular locally orientable hypermaps

From a topological point of view, a **locally orientable hypermap** of n edges can be defined as a connected bipartite graph with black and white vertices. Each edge is composed of two half edges both connecting the two incident vertices. This graph is embedded in a locally orientable surface such that if we cut the graph from the surface, the remaining part consists of connected components called faces or cells, each homeomorphic to an open disk. The map can be represented as a ribbon graph on the plane keeping the incidence order of the edges around each vertex. In such a representation, two half edges can be parallel or cross in the middle. A crossing (or a twist) of two half edges indicates a change of orientation in the map and that the map is embedded in a non orientable surface (projective plane, Klein bottle,...). We say a hypermap is **rooted** if it has a distinguished half edge. In [2], it was shown that rooted hypermaps admit a natural formal description involving triples of perfect pairings (f_1, f_2, f_3) on the set of half edges where:

- f_3 associates half edges of the same edge,
- f_1 associates immediately successive (i.e. with no other half edges in between) half edges moving around the white vertices, and
- f_2 associates immediately successive half edges moving around the black vertices.

Formally we label each half edge with an element in $[n]\cup[\widehat{n}] = \{1,\dots,n,\widehat{1},\dots,\widehat{n}\}$, labelling the rooted half edge by 1. We then define (f_1, f_2, f_3) as perfect pairings on this set. Combining the three pairings gives the fundamental characteristics of the hypermap since:

- The cycles of $f_3\circ f_1$ give the succession of edges around the white vertices. If $f_3\circ f_1 \in \mathcal{C}_{\lambda\lambda}$ then the degree distribution of the white vertices is λ (counting only once each pair of half edges belonging to the same edge),
- The cycles of $f_3\circ f_2$ give the succession of edges around the black vertices. If $f_3\circ f_2 \in \mathcal{C}_{\mu\mu}$ then the degree distribution of the black vertices is μ (counting only once each pair of half edges belonging to the same edge),
- The cycles of $f_1\circ f_2$ encode the faces of the map. If $f_1\circ f_2\in\mathcal{C}_{\nu\nu}$ then the degree distribution of the faces is ν

In what follows, we consider the number $L^n_{\lambda,\mu}$ of rooted **unicellular**, or one-face, locally orientable hypermaps with face distribution $\nu = (n) = n^1$, white vertex distribution λ, and black vertex distribution μ.

Let f_1 be the pairing $(1\,\widehat{n})(2\,\widehat{1})(3\,\widehat{2})\dots(n\,\widehat{n-1})$ and $f_2 = f_\star = (1\,\widehat{1})(2\,\widehat{2})\dots(n\,\widehat{n})$. We have $f_1 \circ f_2 = (123\dots n)(\widehat{n}\,\widehat{n-1}\,\widehat{n-2}\dots\widehat{1}) \in \mathcal{C}_{(n)(n)}$. Then by the above description, one can show that

$$L^n_{\lambda,\mu} = |\ \{f_3 \text{ pairings in } S_{2n}([n]\cup[\widehat{n}]) \mid f_3\circ f_1 \in \mathcal{C}_{\lambda\lambda},\, f_3\circ f_2 \in \mathcal{C}_{\mu\mu}\}\ |\ . \tag{3}$$

Moreover, the following relation between $L^n_{\lambda,\mu}$ and $b^n_{\lambda,\mu}$ holds [2, Cor 2.3]

$$L^n_{\lambda,\mu} = \frac{1}{2^n n!} b^n_{\lambda,\mu}. \tag{4}$$

Thus we can encode the connection coefficients as numbers of locally orientable hypermaps.

Example 2.1 *Figure 1 depicts a locally orientable unicellular hypermap in $L^n_{\lambda,\mu}$ with $\lambda = 1^1 2^2 3^1 4^1$ and $\mu = 3^1 4^1 5^1$ (at this stage we disregard the geometric shapes around the vertices).*

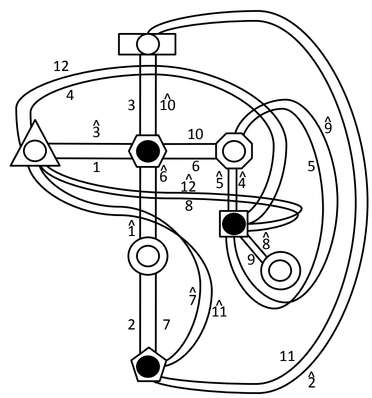

Fig. 1: A unicellular locally orientable hypermap

2.2 Partitioned locally orientable hypermaps

Next, we consider locally orientable hypermaps where we partition the white vertices (black resp.). In terms of the pairings, this means we "color" the cycles of $f_3 \circ f_1$ ($f_3 \circ f_2$ resp.) allowing repeated colors but imposing that the two cycles corresponding to each white (black resp.) vertex have the same color. The following definition in terms of set partitions of $[n]\cup[\widehat{n}]$ makes this more precise.

Definition 2.2 (Locally orientable partitioned hypermaps) *We consider the set $\mathcal{LP}^n_{\lambda,\mu}$ of triples of the form (f_3,π_1,π_2) where f_3 is a pairing on $[n]\cup[\widehat{n}]$, π_1 and π_2 are sets partitions on $[n]\cup[\widehat{n}]$ with blocks of even size and of respective types 2λ and 2μ (or* half types *λ and μ) with the constraint that π_i $(i=1,2)$ is stable by f_i and f_3. Any such triple is called a* **locally orientable partitioned hypermap** *of type (λ,μ). In addition, let $LP^n_{\lambda,\mu} = \mid \mathcal{LP}^n_{\lambda,\mu} \mid$.*

Remark 2.3 *The analogous notion of partitioned or colored map is common in the study of orientable maps. e.g. see [7],[4] for maps. Recently Bernardi in [1] extended the approach in [7] to find a bijection between locally orientable partitioned maps and orientable partitioned maps with a distinguished* planar submap. *As far as we know [1, Sect. 7] this technique does not extend to locally orientable hypermaps.*

Lemma 2.4 *The number of hat numbers in a block is equal to the number of non hat numbers*

Proof: If a non hat number i belongs to block π_1^k then $f_1(i) = \widehat{i-1}$ also belongs to π_1^k. The same argument applies to blocks of π_2 with $f_2(i) = \widehat{i}$. □

Example 2.5 *As an example, the locally orientable hypermap on Figure 1 is partitioned by the blocks:*

$$\begin{aligned} \pi_1 &= \{\{\widehat{12}, 1, \widehat{3}, 4, \widehat{7}, 8, \widehat{11}, 12\}; \{\widehat{1}, 2, \widehat{6}, 7, \widehat{8}, 9\}; \{\widehat{2}, 3, \widehat{10}, 11\}; \{\widehat{4}, 5, \widehat{5}, 6, \widehat{9}, 10\}\} \\ \pi_2 &= \{\{1, \widehat{1}, 3, \widehat{3}, 6, \widehat{6}, 10, \widehat{10}\};\ \{2, \widehat{2}, 7, \widehat{7}, 11, \widehat{11}\};\ \{4, \widehat{4}, 5, \widehat{5}, 8, \widehat{8}, 9, \widehat{9}, 12, \widehat{12}\}\} \end{aligned}$$

(blocks are depicted by the geometric shapes around the vertices, all the vertices belonging to a block have the same shape).

Let $\overline{R}_{\lambda,\mu}$ be the number of unordered partitions $\pi = \{\pi^1, \dots, \pi^p\}$ of the set $[\ell(\lambda)]$ such that $\mu_j = \sum_{i\in\pi^j} \lambda_i$ for $1 \le j \le \ell(\mu)$. Then for the monomial and power symmetric functions, m_λ and p_λ, we have: $p_\lambda = \sum_{\mu\succeq\lambda} Aut_\mu \overline{R}_{\lambda,\mu} m_\mu$. We use this to obtain a relation between $L^n_{\lambda,\mu}$ and $LP^n_{\lambda,\mu}$

Proposition 2.1 *For partitions $\rho, \epsilon \vdash n$ we have $LP^n_{\nu,\rho} = \sum_{\lambda,\mu} \overline{R}_{\lambda\nu}\overline{R}_{\mu\rho} L^n_{\lambda,\mu}$, where λ and μ are refinements of ν and ρ respectively.*

Proof: Let $(f_3, \pi_1, \pi_2) \in \mathcal{LP}^n_{\nu,\rho}$. If $f_3 \circ f_1 \in \mathcal{C}_{\lambda\lambda}$ and $f_3 \circ f_2 \in \mathcal{C}_{\mu\mu}$ then by definition of the set partitions we have that λ and μ are refinements of $type(\pi_1) = \nu$ and $type(\pi_2) = \rho$ respectively. Thus, we can classify the elements of $\mathcal{LP}^n_{\nu,\rho}$ by the cycle types of $f_3 \circ f_1$ and $f_3 \circ f_2$. *i.e.* $\mathcal{LP}^n_{\nu,\rho} = \bigcup_{\lambda,\mu} \mathcal{LP}^n_{\nu,\rho}(\lambda,\mu)$ where

$$\mathcal{LP}_{\nu,\rho}(\lambda,\mu) = \{(f_3, \pi_1, \pi_2) \in \mathcal{LP}^n_{\nu,\rho} \mid (f_3 \circ f_1, f_3 \circ f_2) \in \mathcal{C}_{\lambda\lambda} \times \mathcal{C}_{\mu\mu}\}.$$

If $LP^n_{\mu\rho}(\lambda,\mu) = |\mathcal{LP}^n_{\mu\rho}(\lambda,\mu)|$ then it is easy to see that $LP^n_{\mu,\rho}(\lambda,\mu) = \overline{R}_{\lambda\nu}\overline{R}_{\mu\rho} L^n_{\lambda\mu}$.

□

By the change of basis equation between p_λ and m_λ, this immediately relates the generating series Ψ^n and the generating series for $LP^n_{\lambda,\mu}$ in monomial symmetric functions. *i.e.* $\Psi^n(\mathbf{x}, \mathbf{y})$ is

$$\sum_{\lambda,\mu\vdash n} L^n_{\lambda,\mu} p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) = \sum_{\lambda,\mu\vdash n} Aut_\lambda Aut_\mu LP^n_{\lambda,\mu} m_\lambda(\mathbf{x}) m_\mu(\mathbf{y}). \tag{5}$$

Definition 2.6 *Let $\mathcal{LP}(\mathbf{A})$ be the set, of cardinality $LP(\mathbf{A})$, of partitioned locally orientable hypermaps with n edges where $\mathbf{A} = (P, P', Q, Q')$ are tridimensional arrays such that for $i, j, k \ge 0$:*

- *P_{ijk} (resp. P'_{ijk}) is the number of blocks of π_1 of half size i such that:*
  - *(i) either 1 belongs to the block or its maximum* **non-hat** *number is paired to a* **hat** *number by f_3 (resp. blocks of π_1 not containing 1 such that the maximum* **non-hat** *number of the block is paired to a* **non-hat** *number by f_3),*
  - *(ii) the block contains j pairs $\{t, f_3(t)\}$ where t is the maximum* **hat** *number of a block of π_2 such that $f_3(t)$ is also a* **hat** *number, and,*
  - *(iii) the block contains as a whole $j + k$ pairs $\{u, f_3(u)\}$ where both u and $f_3(u)$ are* **non-hat** *numbers.*
- *Q_{ijk} (resp. Q'_{ijk}) is the number of blocks of π_2 of half size i such that:*

(i) the maximum **hat** *number of the block is paired to a* **non-hat** *(resp.* **hat***) number by f_3,*
(ii) the block contains j pairs $\{t, f_3(t)\}$ where t is the maximum **non-hat** *number of a block of π_1 non containing 1 and such that $f_3(t)$ is also a* **non-hat** *number, and*
(iii) the block contains as a whole $j+k$ pairs $\{u, f_3(u)\}$ where both u and $f_3(u)$ are **hat** *numbers.*

As a direct consequence, for $LP(\mathbf{A})$ to be non zero $\mathbf{A}$ has to check the conditions of Theorem 1.1. Furthermore:

$$LP^n_{\lambda,\mu} = \sum_{\mathbf{A}\in M_{\lambda,\mu}} LP(\mathbf{A}) \tag{6}$$

Example 2.7 *The partitioned hypermap on Figure 1 belongs to $\mathcal{LP}(\mathbf{A})$ for $P = E_{4,1,0} + E_{3,0,1} + E_{2,0,0}$, $P' = E_{3,0,1}$, $Q = E_{5,0,1} + E_{4,1,0}$, $Q' = E_{3,0,1}$ where $E_{t,u,v}$, the elementary array with $E_{t,u,v} = 1$ and 0 elsewhere.*

One can notice that a hypermap is orientable if and only if $f_3(t)$ is a hat number when t is a non hat number (we go through each edge of the map in both directions and there are no changes of direction during the map traversal). As a result, a hypermap in $\mathcal{LP}(\mathbf{A})$ is orientable if and only if:

- $p' = q' = r = 0$ and
- $P_{ijk} = Q_{ijk} = 0$ if $j > 0$ and/or $k > 0$.

In this particular case, we have the following values for $\mathcal{N}(\mathbf{A})$ and $\mathbf{A}!$:

- $\mathcal{N}(\mathbf{A}) = \sum_{i,i,k} iP_{ijk} = \sum_t in_i(\lambda) = n$
- $\mathbf{A}! = \prod_i P_{i,0,0}!Q_{i,0,0}! = Aut_\lambda Aut_\mu$

If we denote $c^n_{\lambda,\mu}$ the number of such orientable maps, by Theorem 1.1, Equation 6, Lemma 5 and Relation 4 we recover the following combinatorial result from [9, Thm. 1]:

Corollary 2.8 *[9, Thm. 1]*

$$\sum_{\lambda,\mu\vdash n} c^n_{\lambda,\mu} p_\lambda(\mathbf{x}) p_\mu(\mathbf{y}) = n \sum_{\lambda,\mu\vdash n} \frac{(n-\ell(\lambda))!(n-\ell(\mu))!}{(n+1-\ell(\lambda)-\ell(\mu))!} m_\lambda(\mathbf{x}) m_\mu(\mathbf{y}). \tag{7}$$

Remark 2.9 *Note that $\{c^n_{\lambda,\mu}\}_{\lambda,mu}$ are better known as the connection coefficients of the symmetric group which count the number of ordered factorizations $w_1 \cdot w_2$ of the long cycle $(1, 2, \ldots, n)$ in S_n where $w_1 \in \mathcal{C}_\lambda$ and $w_2 \in \mathcal{C}_\mu$.*

2.3 Permuted forests and reformulation of the main theorem

We show that partitioned locally orientable hypermaps admit a nice bijective interpretation in terms of some recursive forests defined as follows:

Definition 2.10 (Rooted bicolored forests of degree A) *In what follows we consider the set $\mathcal{F}(\mathbf{A})$ of permuted rooted forests composed of:*

- *a bicolored identified ordered* **seed tree** *with a white root vertex,*
- *other bicolored ordered trees, called* **non-seed trees** *with either a white or a black root vertex,*
- *each vertex of the forest has three kind of ordered descendants :* **tree-edges** *(connecting a white and a black vertex),* **thorns** *(half edges connected to only one vertex) and* **loops** *connecting a vertex to itself. The two* extremities *of the loop are part of the ordered set of descendants of the incident vertex and therefore the loop can be intersected by thorns, edges and other loops as well.*

The forests in $\mathcal{F}(\mathbf{A})$ also have the following properties:

(i) the root vertices of the non-seed trees have at least one descending loop with one extremity being the rightmost descendant of the considered vertex,

(ii) the total number of thorns (resp. loops) connected to the white vertices is equal to the number of thorns (resp. loops) connected to the black ones,

(iii) there is a bijection between thorns connected to white vertices and the thorns connected to black vertices. The bijection between thorns will be encoded by assigning the same symbolic latin *labels $\{a, b, c, \ldots\}$ to thorns associated by this bijection,*

(iv) there is a mapping that associates to each loop incident to a white (resp. black) vertex, a black (resp. white) vertex v *such that the number of white (resp. black) loops associated to a fixed black (resp. white) vertex* v *is equal to its number of incident loops. We will use symbolic* greek *labels $\{\alpha, \beta, \ldots\}$ to associate loops with vertices except for the maximal loop of a root vertex* r *of the non-seed trees. In this case, we draw an arrow (- - - ->) outgoing from the root vertex* r *and incoming to the vertex associated with the loop. Arrows are non ordered, and*

(v) the ascendant/descendant structure defined by the edges of the forest and the arrows defined above is a tree structure rooted in the root of the seed tree.

Finally the degree $\mathbf{A}$ of the forest is given in the following way:

(vii) P_{ijk} (resp P'_{ijk}) counts the number of non root white vertices (including the root of the seed tree) (resp. white root vertices excluding the root of the seed tree) of degree i, with j incoming arrows and a total of $j + k$ loops.

(viii) Q_{ijk} (resp Q'_{ijk}) counts the number of non root black vertices (resp. black root vertices) of degree i, with j incoming arrows and total $j + k$ loops.

Example 2.11 *As an example, Figure 2 depicts two permuted forests. The one on the left is of degree $\mathbf{A} = (P, P', Q, Q')$ for $P = E_{4,1,0} + E_{3,0,1} + E_{2,0,0}$, $P' = E_{3,0,1}$, $Q = E_{5,0,1} + E_{4,1,0}$, and $Q' = E_{3,0,1}$ while the one on the right is of degree $\mathbf{A^{(2)}} = (P^{(2)}, P'^{(2)}, Q^{(2)}, Q'^{(2)})$ for $P^{(2)} = E_{7,0,3} + E_{4,1,0}$, $P'^{(2)} = \{0\}_{i,j,k}$, $Q^{(2)} = E_{7,0,2}$, and $Q'^{(2)} = E_{4,0,2}$.*

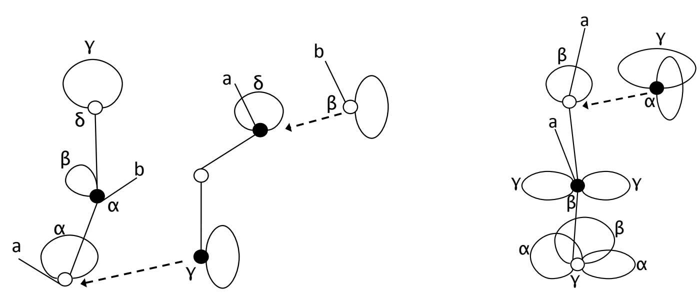

Fig. 2: Two Permuted Forests

Lemma 2.12 *Using the Lagrange theorem for implicit functions, one can show that $F(\mathbf{A})$ equals*

$$\frac{\mathcal{N}(\mathbf{A})}{\mathbf{A}!} \frac{(n-q-2r)!(n-p-2r)!}{(n+1-p-q-2r)!} \frac{p'!q'!\,(r-p')!\,(r-q')!}{2^{2r-p'-q'}} \prod_{i,j,k} \binom{i-1}{j,k,j+k}^{(P+Q)_{ijk}} \binom{i-1}{j,k,j+k-1}^{(P'+Q')_{ijk}}.$$

Reformulation of the main theorem
In order to show Theorem 1.1 the next sections are dedicated to the proof of the following stronger result:

Theorem 2.13 *There is a bijection* $\Theta_{\mathbf{A}} : \mathcal{LP}(\mathbf{A}) \to \mathcal{F}(\mathbf{A})$ *and so* $LP(\mathbf{A}) = F(\mathbf{A})$.

3 Bijection between partitioned locally orientable unicellular hypermaps and permuted forests

We proceed with the description of the bijective mapping $\Theta_{\mathbf{A}}$ between partitioned locally orientable hypermaps and permuted forests of degree $\mathbf{A}$. Let (f_3, π_1, π_2) be a partitioned hypermap in $\mathcal{LP}(\mathbf{A})$. The first step is to define a set of white and black vertices with labeled ordered half edges such that:

- each white vertex is associated to a block of π_1 and each black vertex is associated to a block of π_2,
- the number of half edges connected to a vertex is half the cardinality of the associated block, and
- the half edges connected to the white (resp. black) vertices are labeled with the non hat (resp. hat) integers in the associated blocks so that moving clockwise around the vertices the integers are sorted in increasing order.

Then we define an ascendant/descendant structure on the vertices. A black vertex b is the descendant of a white one w if the maximum half edge label of b belongs to the block of π_1 associated to w. Similar rules apply to define the ascendant of each white vertex except the one containing the half edge label 1.
If black vertex b^d (resp. white vertex w^d) is a descendant of white vertex w^a (resp. black vertex b^a) and has maximum half edge label m such that $f_3(m)$ is the label of a half edge of w^a (resp. b^a), i.e. $f_3(m^b)$ is a non hat (resp. hat) number, then we connect these two half edges to form an edge. Otherwise $f_3(m)$ is a hat (resp. non hat) number and we draw an arrow ($\dashrightarrow$) between the two vertices. Note that descending edges are ordered but arrows are not.

Lemma 3.1 *The above construction defines a tree structure rooted in the white vertex with half edge* 1.

Proof: Let black vertices b_1 and b_2 associated to blocks $\pi_2^{b_1}$ and $\pi_2^{b_2}$ be respectively a descendant and the ascendant of white vertex w associated to π_1^w. We denote by m^{b_1}, m^{b_2} and m^w their respective maximum half edge labels (hat, hat, and non hat) and assume $m^{b_1} \neq \widehat{n}$. As π_1^w is stable by f_1, then $f_1(m^{b_1})$ is a non hat number in π_1^w not equal to 1. It follows $m^{b_1} < f_1(m^{b_1}) \leq m^w < f_2(m^w)$. Then as $\pi_2^{b_2}$ is stable by f_2, it contains $f_2(m^w)$ and $f_2(m^w) \leq m^{b_2}$. Putting everything together yields $m^{b_1} < m^{b_2}$. In a similar fashion, assume white vertices w_1 and w_2 are descendant and ascendant of black vertex b. If we note m^{w_1}, m^{w_2} and m^b their maximum half edge labels (non hat, non hat, and hat) with $m^b \neq \widehat{n}$, one can show that $m^{w_1} < m^{w_2}$. Finally, as $f_1(\widehat{n}) = 1$, the black vertex with maximum half edge $\widehat{n}$ is descendant of the white vertex containing the half edge label 1. $\square$

Example 3.2 *Using the hypermap of Figure 1 we get the set of vertices and ascendant/descendant structure as described on Figure 3.*

Next, we proceed by linking half edges connected to the same vertex if their labels are paired by f_3 to form loops. If i and $f_3(i)$ are the labels of a loop connected to a white (resp. black) vertex such that neither i nor $f_3(i)$ are maximum labels (except if the vertex is the root), we assign the same label to the loop and the black (resp. white) vertex associated to the block of π_2 (resp. π_1) also containing i and $f_3(i)$.

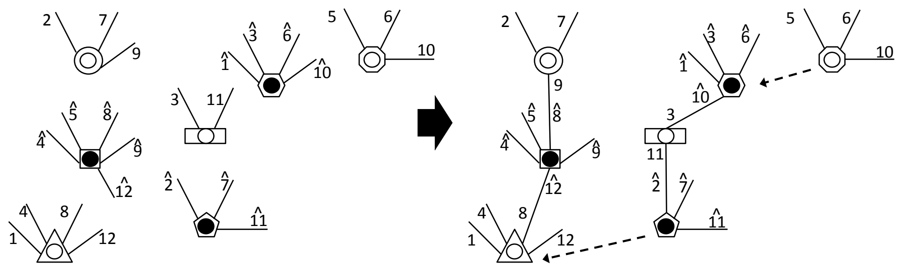

Fig. 3: Construction of the ascendant/descendant structure

As we assign at most one label from $\{\alpha, \beta, \ldots\}$ to a given vertex, several loops may share the same label.

Lemma 3.3 *The number of loops connected to the vertex labeled α is equal to its number of incoming arrows plus the number of loops labeled α incident to other vertices in the forest.*

Proof: The result is a direct consequence of the fact that in each block the number of hat/hat pairs is equal to the number of non hat/non hat pairs □

As a final step, we define a permutation between the remaining half edges (thorns) connected to the white vertices and the one connected to the black vertices. If two remaining thorns are paired by f_3 then these two thorns are given the same label from $\{a, b, \ldots\}$. All the original integer labels are then removed. We denote by $\widetilde{F}$ the resulting forest.

Example 3.4 *We continue with the hypermap from Figure 1 and perform the final steps of the construction as described on Figure 4.(Note that the geometric shapes are here for reference only, they do not play any role in the final object $\widetilde{F}$).*

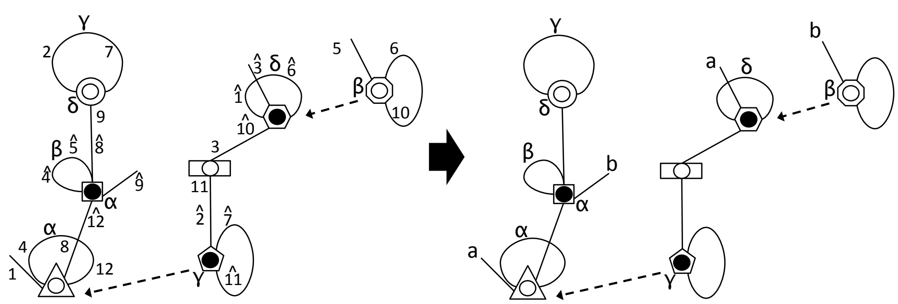

Fig. 4: Final steps of the permuted forest construction

As a direct consequence of definition 2.10, $\widetilde{F}$ belongs to $\mathcal{F}(A)$.

4 Proof of the bijection

We show that mapping $\Theta_{\mathbf{A}} : (f_3, \pi_1, \pi_2) \mapsto \widetilde{F}$ is indeed one-to-one.

4.1 Injectivity

We start with a forest $\widetilde{F}$ in $\mathcal{F}(\mathbf{A})$ and show that there is at most one triple (f_3, π_1, π_2) in $\mathcal{LP}(\mathbf{A})$ such that $\Theta_{\mathbf{A}}(f_3, \pi_1, \pi_2) = \widetilde{F}$. The first part is to notice that within the construction in $\Theta_{\mathbf{A}}$, the original integer label of the leftmost descendant (thorn, half loop or edge) of the root vertex of the seed tree is necessarily 1 (this root is the vertex containing 1 and the labels are sorted in increasing order from left to right).
Assume we have recovered the positions of integer labels $1, \widehat{1}, 2, \widehat{2}, \ldots, i$, for some $1 \leq i \leq n-1$, non hat number. Then four cases can occur:

- i is the integer label of a thorn of latin label a. In this case, $f_3(i)$ is necessarily the integer label of the thorn connected to a black vertex also labeled with a. But as the blocks of π_2 are stable by both f_3 and f_2 then $\widehat{i} = f_2(i)$ is the integer label of one of the descendants of the black vertex with thorn a. As these labels are sorted in increasing order, necessarily, $\widehat{i}$ labels the leftmost descendant with no recovered integer label.
- i is the integer label of a half loop of greek label α. Then, in a similar fashion as above $\widehat{i}$ is necessarily the leftmost unrecovered integer label of the black vertex with symbolic label α.
- i is the integer label of a half loop with no symbolic label (i.e, either i or $f_3(i)$ is the maximum label of the considered white vertex). Then, $\widehat{i}$ is necessarily the leftmost unrecovered integer label of the black vertex at the other extremity of the arrow outgoing from the white vertex containing integer label i.
- i is the integer label of an edge. $\widehat{i}$ is necessarily the leftmost unrecovered integer label of the black vertex at the other extremity of this edge.

Finally, using similar four cases for the black vertex containing the descendant with integer label $\widehat{i}$ and the fact that blocks of π_1 are stable by f_3 and f_1, the thorn, half loop or edge with integer label $i+1 = f_1(\widehat{i})$ is uniquely determined as well.

We continue with the procedure described above up until we fully recover all the original labels $[n] \cup [\hat{n}]$. According to the construction of $\widetilde{F}$, the knowledge of all the integer labels uniquely determines the blocks of π_1 and π_2. The pairing f_3 is as well uniquely determined by the loops, edges and thorns with same latin labels.

Example 4.1 *Assume the permuted forest $\widetilde{F}$ is the one on the right hand side of Figure 2. The steps of the reconstruction are summarized in Figure 5. We get that the unique triple (f_3, π_1, π_2) such that $\Theta_{\mathbf{A}}(f_3, \pi_1, \pi_2) = \widetilde{F}$ is:*

$$\begin{aligned}
f_3 &= (1\ 4)(\widehat{1}\ \widehat{8})(2\ 9)(\widehat{2}\ \widehat{3})(3\ \widehat{11})(\widehat{4}\ \widehat{10})(5\ 7)(\widehat{5}\ 6)(\widehat{6}\ 11)(\widehat{7}\ \widehat{9})(8\ 10) \\
\pi_1 &= \{\{\widehat{11}, 1, \widehat{1}, 2, \widehat{2}, 3, \widehat{3}, 4, \widehat{7}, 8, \widehat{8}, 9, \widehat{9}, 10\}; \{\widehat{4}, 5, \widehat{5}, 6, \widehat{6}, 7, \widehat{10}, 11\}\} \\
\pi_2 &= \{\{2, \widehat{2}, 3, \widehat{3}, 5, \widehat{5}, 6, \widehat{6}, 7, \widehat{7}, 9, \widehat{9}, 11, \widehat{11}\}; \{1, \widehat{1}, 4, \widehat{4}, 8, \widehat{8}, 10, \widehat{10}\}\}
\end{aligned}$$

4.2 Surjectivity

To prove that $\Theta_{\mathbf{A}}$ is surjective, we have to show that the reconstruction procedure of the previous section always finishes with a valid output.

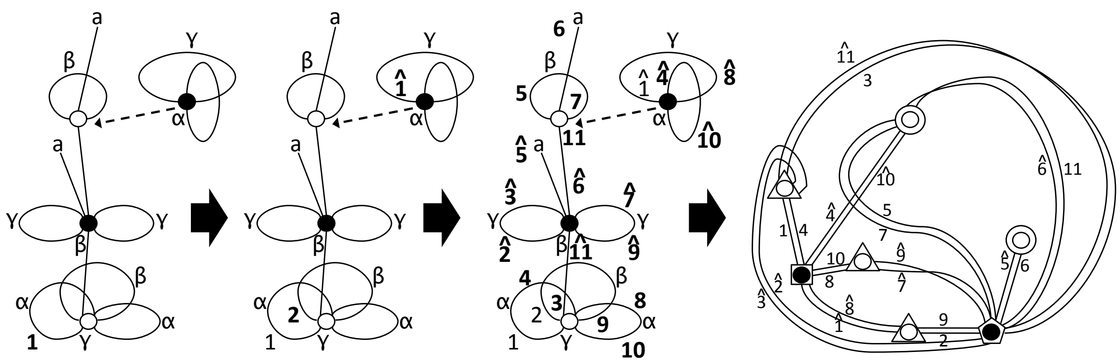

Fig. 5: Recovery of the integer labels and the partitioned map

Assume the procedure comes to an end at step i before all the integer labels are recovered (where i is for example non hat, the hat case having a similar proof). It means that we have already recovered all the labels of vertex v^i identified as the one containing $\widehat{i}$ (or $i+1$) prior to this step. This is impossible by construction provided v^i is not the root vertex of the seed tree since the number of times a vertex is identified for the next step is equal to its number of thorns, plus its number of edges, plus twice the number of loops that have the same greek label as v^i, plus twice the incoming arrows. Using Property *(iv)* of Definition 2.10, we have that the sum of the two latter numbers is twice the number of loops of v^i. As a consequence, the total number of times the recovering process goes through v^i is exactly (and thus never more than) the degree of v^i.

If v is the root vertex of the seed tree, the situation is slightly different due to the fact that we recover label 1 before we start the procedure. To ensure that the procedure does not terminate prior to its end, we need to show that the $\mid v \mid$-th time the procedure goes through the root vertex is right after all the labels of the forest have been recovered. Again, this is always true because:

- The last element of a vertex to be recovered is the label of the maximum element of the associated block. Consequently, all the elements of a vertex are recovered only when all the elements of the descending vertices (through both arrows and edges) are recovered.
- Property *(v)* of Definition 2.10 states that the ascendant/descendant structure involving both edges and arrows is a tree rooted in v. As a result, the procedure goes the v-th time through v only when all the elements of all the other vertices are recovered.

5 On proving Theorem 1.1 using Zonal polynomials

In the orientable case, one can use Schur symmetric functions and the irreducible characters of the symmetric group to prove the identity in Equation (7) (see [6]). This requires: (i) ($p_\lambda \to s_\mu$) the Murnaghan-Nakayama rule, (ii) ($s_\mu \to m_\nu$) finding the number of semistandard Young tableaux of hook shape $\lambda = a\,1^{n-a}$ and type μ which is just $\binom{\ell(\mu)-1}{r}$, and (iii) using inclusion exclusion. One could try to replicate this on Ψ^n and obtain an algebraic proof of Theorem 1.1. We show the outcome after step (i)' using [5, Cor. 5.2]. Steps (ii)' and (iii)' appear quite less tractable.

(i)' $\Psi^n(\mathbf{x},\mathbf{y}) = \frac{|B_n|}{|K_{(n)}|}\sum_{a\ge b\ge 1} \frac{\varphi^{(a,b,1^{n-a-b})}(n)}{H_{2(a,b,1^{n-a-b})}} Z_{(a,b,1^{n-a-b})}(\mathbf{x}) Z_{(a,b,1^{n-a-b})}(\mathbf{y}).$

6 Appendix: completion of the main formula

For Theorem 1.1 to be exact, the following additional definitions for some limit values of $\mathbf{A}$ are needed:

(i) If $q' \neq 0$, then there is at most one given (i_0, j_0, k_0) with $i_0 = 2(j_0 + k_0)$, for which $P_{i_0 j_0 k_0} = 1$ instead of 0. In that situation, we define $\mathcal{N}(\mathbf{A})\binom{i_0-1}{j_0,k_0,j_0+k_0}^{P_{i_0 j_0 k_0}} =$

$$\frac{j_0}{q'}\binom{i_0}{j_0,k_0}\left(\frac{\delta_{p'\neq 0}}{p'}\sum_{i,j,k}(i-2j-2k)P'\sum_{i,j,k} jQ' + \sum_{i,j,k}(i-2j-2k)Q'\frac{1+\sum_{i,j,k}(i-1-2j-2k)P}{n-q-2r}\right)$$

(ii) When $q' \neq 0$ and $n - q - 2r = 0$, which can only occur if $p = 1$ (we assume $P = E_{t,u,v}$, the elementary array with $E_{t,u,v} = 1$ and 0 elsewhere) the whole summand reduces to:

$$\frac{1}{\mathbf{A}!}\left[\delta_{(n-2r)p'\neq 0}\frac{(t-2u-2v)\sum_{j,k,l} jQ}{(n-2r)p'} + \delta_{p'\neq 0}u\frac{\sum_{i,j,k} jQ'}{p'q'} + \delta_{p'=0}\right]$$
$$\times\frac{(n-2r)!p'!q'!\,(r-p')!\,(r-q')!}{2^{2r-p'-q'}}\binom{t}{u,v,u+v}\prod_{i,j,l}\binom{i-1}{j,k,j+k}^{Q_{ijk}}\binom{i-1}{j,k,j+k-1}^{(P'+Q')_{ijk}}$$

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A topological interpretation of the cyclotomic polynomial

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Abstract. We interpret the coefficients of the cyclotomic polynomial in terms of simplicial homology.

Résumé. Nous donnons une interprétation des coefficients du polynôme cyclotomique en utilisant l'homologie simpliciale.

Keywords: Cyclotomic polynomial, higher-dimensional tree, matroid duality, oriented matroid, simplicial matroid, simplicial homology

1 Introduction

This paper studies the cyclotomic polynomial $\Phi_n(x)$, which is defined as the minimal polynomial over $\mathbb{Q}$ for any primitive n^{th} root of unity ζ in $\mathbb{C}$. It is monic, irreducible, and has degree given by the Euler phi function $\phi(n)$, with formula

$$\Phi_n(x) = \prod_{j \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta^j).$$

The equation $x^n - 1 = \prod_{d|n} \Phi_d(x)$ gives a recurrence showing that all coefficients of $\Phi_n(x)$ lie in $\mathbb{Z}$.

Although well-studied, the coefficients of $\Phi_n(x)$ are mysterious [2, 10, 11, 14, 15, 17, 29]. We offer here two interpretations for their magnitudes, as orders of cyclic groups. In the first interpretation (Corollary 5 below) this group is a quotient of the free abelian group $\mathbb{Z}[\zeta]$ by a certain full rank sublattice.

The second interpretation is topological, given by Theorem 1 below, as the torsion in the homology of a certain simplicial complex associated with a squarefree integer $n = p_1 \cdots p_d$. These simplicial complexes originally arose in the work of Bolker [6], reappeared in the work of Kalai [13] and Adin [1] on higher-dimensional matrix-tree theorems, and were shown to be connected with cyclotomic extensions in work of J. Martin and the second author [18]. We review these simplicial complexes briefly here in order to state the result; see Section 4 for more details.

Given a positive integer p, let K_p denote a 0-dimensional abstract simplicial complex having p vertices[(i)], which we will label by the residues

$$\{0 \bmod p,\ 1 \bmod p,\ \ldots,\ (p-1) \bmod p\}$$

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(i) Note that here K_p does *not* refer to a complete graph on p vertices; we hope that this causes no confusion.

for reasons that will become clear in a moment. Given primes $p_1, \dots, p_d$, let

$$K_{p_1,\dots,p_d} := K_{p_1} * \cdots * K_{p_d}$$

be the *simplicial join*, [21, §62], of $K_{p_1}, \dots, K_{p_d}$. This is a pure $(d-1)$-dimensional abstract simplicial complex, that may be thought of as the *complete d-partite complex* on vertex sets K_{p_1} through K_{p_d} of sizes $p_1, \dots, p_d$. The *facets* (maximal simplices) of $K_{p_1,\dots,p_d}$ are labelled by sequences of residues $(j_1 \bmod p_1, \dots, j_d \bmod p_d)$. Denoting the squarefree product $p_1 \cdots p_d$ by n, the Chinese Remainder Theorem isomorphism

$$\mathbb{Z}/p_1\mathbb{Z} \ \times \cdots \times \ \mathbb{Z}/p_d\mathbb{Z} \ \xrightarrow{\Xi} \ \mathbb{Z}/n\mathbb{Z} \tag{1}$$

allows one to label such a facet by a residue $j \bmod n$; call this facet $F_{j \bmod n}$. Then for any subset $A \subseteq \{0, 1, \dots, \phi(n)\}$, let K_A denote the subcomplex of $K_{p_1,\dots,p_d}$ which is generated by the facets $\{F_{j \bmod n}\}$ as j runs through the following set of residues:

$$A \cup \{\phi(n)+1,\ \phi(n)+2,\ \dots,\ n-1,\ n\}.$$

Our first main result interprets the magnitudes of the coefficients of $\Phi_n(x)$. Let $\tilde{H}_i(-;\mathbb{Z})$ denote reduced simplicial homology with coefficients in $\mathbb{Z}$.

Theorem 1 *For a squarefree positive integer $n = p_1 \cdots p_d$, with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$, one has*

$$\tilde{H}_i(K_{\{j\}};\mathbb{Z}) = \begin{cases} \mathbb{Z}/c_j\mathbb{Z} & \text{if } i = d-2, \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } c_j = 0, \\ 0 & \text{otherwise.} \end{cases}$$

We furthermore interpret topologically the signs of the coefficients in $\Phi_n(x)$. For this, we use oriented simplicial homology, and orient the facet $F_{j \bmod n}$ having $j \equiv j_i \bmod p_i$ for $i = 1, 2, \dots, d$ as

$$[F_j] = [F_{j \bmod n}] = [j_1 \bmod p_1, \dots j_d \bmod p_d].$$

Theorem 2 *Fix a squarefree positive integer $n = p_1 \cdots p_d$ with cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$. Then for any $j \neq j'$ such that $c_j, c_{j'} \neq 0$, one has $\tilde{H}_{d-1}(K_{\{j,j'\}};\mathbb{Z}) \cong \mathbb{Z}$, and any nonzero $(d-1)$-cycle $z = \sum_\ell b_\ell [F_\ell]$ in this homology group will have $b_j, b_{j'} \neq 0$, with*

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Example 3 We illustrate these theorems for $n = 15$. Here $d = 2, p_1 = 3, p_2 = 5$, and $\phi(n) = 2 \cdot 4 = 8$. The cyclotomic polynomial is

$$\begin{aligned}\Phi_{15}(x) &= 1 - x + x^3 - x^4 + x^5 - x^7 + x^8 \\ &= (+1) \cdot (x^0 + x^3 + x^5 + x^8) + (-1) \cdot (x^1 + x^4 + x^7) + 0 \cdot (x^2 + x^6).\end{aligned}$$

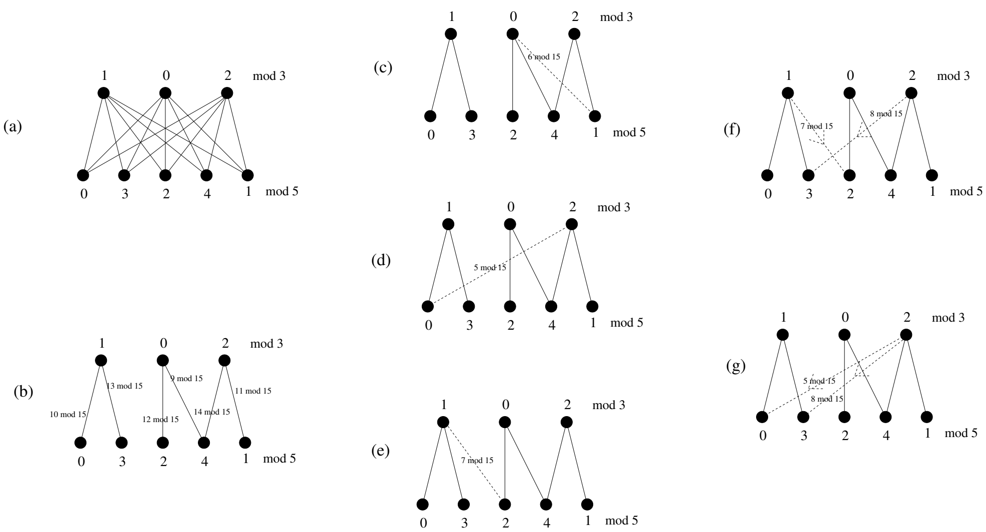

Fig. 1: The case of $\Phi_{15}(x)$

The complex $K_{p_1,p_2} = K_{3,5}$ is a complete bipartite graph with vertex sets labelled as in Figure 1(a). The subcomplex $K_\varnothing$ generated by the edges $F_{j \bmod 15}$ with $j \in \{\phi(n)+1, \phi(n)+2, \ldots, n-1\} = \{9, 10, 11, 12, 13, 14\}$ is the subgraph shown in Figure 1(b).

To see why the coefficient $c_6 = 0$ in $\Phi_{15}(x)$, one adds the edge $F_{6 \bmod 15}$ to the graph $K_\varnothing$, obtaining the graph $K_{\{6\}}$, shown in Figure 1(c), which has

$$\begin{aligned}\tilde{H}_0(K_{\{6\}};\mathbb{Z}) &= \mathbb{Z} = \mathbb{Z}/0\mathbb{Z}\\ \tilde{H}_1(K_{\{6\}};\mathbb{Z}) &= \mathbb{Z}.\end{aligned}$$

To see why the coefficients $c_5 = +1$ or $c_7 = -1$ have magnitude 1, one adds the edge $F_{5 \bmod 15}$ or $F_{7 \bmod 15}$ to the graph $K_\varnothing$, obtaining the graphs $K_{\{5\}}$ or $K_{\{7\}}$ shown in Figures 1(d) and 1(e), which have

$$\begin{array}{rcccl}\tilde{H}_0(K_{\{5\}};\mathbb{Z}) & = & 0 & = & \mathbb{Z}/(+1)\mathbb{Z}\\ \tilde{H}_0(K_{\{7\}};\mathbb{Z}) & = & 0 & = & \mathbb{Z}/(-1)\mathbb{Z}.\end{array}$$

To understand the signs of the coefficients, note first that, by convention, $\Phi_{15}(x)$ is monic, so the coefficient $c_8 = c_{\phi(n)} = +1$. Therefore any other coefficient c_j should have sign

$$\operatorname{sgn}(c_j) = \frac{\operatorname{sgn}(c_j)}{\operatorname{sgn}(c_8)} = -\frac{\operatorname{sgn}(b_8)}{\operatorname{sgn}(b_j)}$$

where $z = \sum_i b_i[F_i]$ is a nontrivial cycle in $K_{\{j,8\}}$, in which the edge $[F_j]$ is directed from the vertex $(j_1 \bmod 3)$ toward the vertex $(j_2 \bmod 5)$. As shown in Figures 1(f) and 1(g), the nontrivial cycle in $K_{\{7,8\}}$ has $[F_7], [F_8]$ oriented in the *same* direction, explaining why $c_7 = -1$, while the nontrivial cycle in $K_{\{5,8\}}$ has $[F_5], [F_8]$ oriented in the *opposite* direction, explaining why $c_5 = +1$.

The remainder of the paper is structured as follows. Section 2 describes our first interpretation for the cyclotomic polynomial, which applies much more generally to any monic polynomial in $\mathbb{Z}[x]$. Section 3 reviews some facts, underlying the main results, about duality of matroids, Plücker coordinates, and oriented matroids. Section 4 recalls results and establishes terminology on Kalai's higher dimensional spanning trees in a simplicial complex. Section 5 discusses further properties of the simplicial complex $K_{p_1,\ldots,p_d}$ whose subcomplexes appear in Theorem 1 and 2. Section 6 proves these theorems. We end with Section 7, where we discuss known properties of $\Phi_n(x)$ that manifest themselves topologically.

2 Coefficients of monic polynomials in $\mathbb{Z}[x]$

Our goal here is the first interpretation for the coefficients of $\Phi_n(x)$, which applies more generally to the coefficients of *any* monic polynomial $f(x)$ in $\mathbb{Z}[x]$. Recall that when $f(x)$ is of degree r, one has an isomorphism of $\mathbb{Z}$-modules

$$\begin{array}{rcl} \mathbb{Z}^r & \longrightarrow & \mathbb{Z}[x]/(f(x)) \\ (a_0, a_1, \ldots, a_{r-1}) & \longmapsto & \displaystyle\sum_{j=0}^{r-1} a_j \overline{x}^j. \end{array}$$

As notation, given a subset A of some abelian group, let $\mathbb{Z}A$ denote the collection of all $\mathbb{Z}$-linear combinations of elements of A.

Proposition 4 *For a monic polynomial $f(x) = \sum_{j=0}^{r} c_j x^j$ of degree r in $\mathbb{Z}[x]$, one has an isomorphism of abelian groups*

$$\big(\mathbb{Z}[x]/(f)\big) \,/\, \mathbb{Z}A \quad \cong \quad \mathbb{Z}/c_j\mathbb{Z}$$

where A is the subset of size r given as $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^r}\} \setminus \{\overline{x^j}\}$.

Proof: Consider the matrix in $\mathbb{Z}^{r\times(r+1)}$

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & -c_0 \\ 0 & 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -c_{r-1} \end{bmatrix}$$

whose columns express the elements of $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^r}\}$ uniquely in the $\mathbb{Z}$-basis $\{\overline{1}, \overline{x}, \overline{x^2}, \ldots, \overline{x^{r-1}}\}$ for $\mathbb{Z}[x]/(f)$. The $r \times r$ submatrix obtained by restricting this matrix to the columns indexed by A is equivalent by row and column permutations to a upper triangular matrix with diagonal entries $(1, 1, \ldots, 1, -c_j)$. Hence $\big(\mathbb{Z}[x]/(f)\big) \,/\, \mathbb{Z}A \cong \mathbb{Z}/c_j\mathbb{Z}$. $\square$

The special case where $f(x)$ is the cyclotomic polynomial $\Phi_n(x)$ leads to the following considerations. Fix once and for all a primitive n^{th} root of unity ζ.

Corollary 5 *The cyclotomic polynomial $\Phi_n(x) = \sum_{j=0}^{\phi(n)} c_j x^j$ has*

$$\mathbb{Z}[\zeta]/\mathbb{Z}A \cong \mathbb{Z}/c_j\mathbb{Z}$$

where $A = \{1, \zeta, \zeta^2, \ldots, \zeta^{\phi(n)}\} \setminus \{\zeta^j\}$.

Proof: Apply the previous proposition with $f(x) = \Phi_n(x)$ and $r = \phi(n)$, noting that the ring map $\mathbb{Z}[x] \to \mathbb{Z}[\zeta]$ sending x to ζ will also send x^j to ζ^j, and induce an isomorphism $\mathbb{Z}[x]/(\Phi_n(x)) \to \mathbb{Z}[\zeta]$. □

For later use (see the proof of Theorem 20), we note here that the set

$$P_n := \{\zeta^m\}_{m \in (\mathbb{Z}/n\mathbb{Z})^\times}$$

of all *primitive* n^{th} roots of unity within $\mathbb{Z}[\zeta]$ forms a $\mathbb{Z}$-basis whenever n is squarefree. This is a sharpening of an observation of Johnsen [12], who noted that P_n forms a $\mathbb{Q}$-basis of $\mathbb{Q}[\zeta]$ in the same situation.

Proposition 6 *When n is squarefree, the collection P_n of all primitive n^{th} roots of unity forms a $\mathbb{Z}$-basis for $\mathbb{Z}[\zeta]$.*

Proof: The result is easy when n is prime and can be deduced from the Chinese Remainder Theorem in the general case. See [22] for details. □

3 Duality of matroids or Plücker coordinates

We will need a version of the linear algebraic duality between Plücker coordinates for complementary Grassmannians $G(r, \mathbb{F}^n), G(n-r, \mathbb{F}^n)$, or equivalently, the duality between bases and cobases in coordinatized matroids.

Proposition 7 *Let $0 \leq r \leq n$. Let M and $M^\perp$ be matrices in $\mathbb{F}^{r \times n}$ and $\mathbb{F}^{(n-r) \times n}$, respectively, both of maximal rank, with the following property:* $\ker M$ *is equal to the row space of $M^\perp$, or equivalently,* $\ker M^\perp$ *is the row space of M. Then*

(i) there exists a scalar α in $\mathbb{F}^\times$ having the following property: for every $(n-r)$-subset T of $[n]$, with complementary set T^c,

$$\det\left(M\big|_{T^c}\right) = \pm\alpha \cdot \det\left(M^\perp\big|_T\right)$$

where $A\big|_J$ denotes the restriction of a matrix A to the subset of columns indexed by J, and the $\pm$ sign depends upon the set T.

(ii) if one furthermore assumes that $\mathbb{F} = \mathbb{Q}$, that M and $M^\perp$ have entries in $\mathbb{Z}$, and that there exists at least one $(n-r)$-subset T_0 for which $M\big|_{T_0^c}, M^\perp\big|_{T_0}$ are both invertible over $\mathbb{Z}$, then the scalar α above equals ± 1, and one has for every $(n-r)$-subset T,

$$\operatorname{coker}\left(M\big|_{T^c}\right) \cong \operatorname{coker}\left(M^\perp\big|_T\right).$$

Here, we are thinking of $\operatorname{coker} M$ *as signifying a map between powers of $\mathbb{Z}$.*

Proof: Both assertions can be reduced via row and column operations to the case where M takes the form $[I_r|A]$ for some r-by-$(n-r)$ matrix A, where they are easier to verify. See [22] for details. □

The proof of Theorem 2 will ultimately rely on the following statement about duality of *oriented matroids* for vectors in a vector space over an ordered field $\mathbb{F}$, such as $\mathbb{F} = \mathbb{Q}$.

Proposition 8 *Let $\mathbb{F}$ be an ordered field, M and let $M^\perp$ be matrices in $\mathbb{F}^{r\times n}$ and $\mathbb{F}^{(n-r)\times n}$ as in Proposition 7, that is, both of maximal rank, with $\ker M$ perpendicular to the row space of $M^\perp$. Let the vectors v_ℓ in $\mathbb{F}^r$ and $v_\ell^\perp$ in $\mathbb{F}^{n-r}$ be the ℓ^{th} columns of M and $M^\perp$. Let A be an $(r+1)$-subset of $\{1,2,\ldots,n\}$ such that the matrix $M\big|_A$ in $\mathbb{F}^{r\times(r+1)}$ has full rank r, with*

$$\sum_{\ell\in A} c_\ell v_\ell = 0 \tag{2}$$

the unique dependence among its columns, up to scaling. Then for any pair of nonzero coefficients $c_j, c_{j'} \neq 0$, the matrix $M^\perp\big|_{A^c\cup\{j,j'\}}$ in $\mathbb{F}^{(n-r)\times(n-r+1)}$ has full rank $n-r$, and the unique dependence among its columns, up to scaling,

$$\sum_{\ell\in A^c\cup\{j,j'\}} b_\ell v_\ell^\perp = 0, \tag{3}$$

will have both $b_j, b_{j'} \neq 0$, with

$$\frac{c_j}{c_{j'}} = -\frac{b_{j'}}{b_j}.$$

In particular, $c_j, c_{j'}$ have the same sign if and only if $b_j, b_{j'}$ have opposite signs.

Proof: The main observation here is that vectors in the row space of $M^\perp$ are *covectors* for $\{v_\ell^\perp\}$. See [22] for details.

□

4 Simplicial spanning trees

For a collection of subsets S of some vertex set V, let $\langle S\rangle$ denote the (abstract) simplicial complex S on V generated by S, that is, $\langle S\rangle \subset 2^V$ consists of all subsets of V contained in at least one subset from S. We recall the notion of a simplicial spanning tree in S, following Adin [1], Duval, Klivans and Martin [8], Kalai [13], and Maxwell [19].

Definition 9 Let S be the collection of facets of a pure k-dimensional (abstract) simplicial complex. Say that $R \subset S$ is an *S-spanning tree* if

(i) $\langle R\rangle$ contains the entire $(k-1)$-skeleton of $\langle S\rangle$,

(ii) $\tilde{H}_k(\langle R\rangle;\mathbb{Z}) = 0, and$

(iii) $\tilde{H}_{k-1}(\langle R\rangle;\mathbb{Z})$ is finite.

We point out here three well-known features of this definition.

Proposition 10 *Fix the collection of facets S of a pure k-dimensional simplicial complex.*

(i) Condition (i) in Definition 9 is equivalent to $\tilde{H}_k(\langle S\rangle,\langle R\rangle;\mathbb{Z}) = \mathbb{Z}^{|S\setminus R|}$.

(ii) Condition (ii) in Definition 9 is equivalent to $\tilde{H}_k(\langle R\rangle;\mathbb{Q}) = 0$.

(iii) All S-spanning trees R have the same cardinality, namely

$$|R| = |S| - \operatorname{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z}). \tag{4}$$

Proof: See [22] for details. □

The following key observation essentially goes back to work of Kalai [13, Lemma 2].

Proposition 11 *Fix a vertex set V and a collection of k-dimensional simplices S. Consider a collection of $(k+1)$-dimensional faces T of cardinality*

$$|T| := \operatorname{rank}_{\mathbb{Z}} \tilde{H}_k(\langle S \rangle; \mathbb{Z})$$

for which $T \cup \langle S \rangle$ forms a simplicial complex K, that is, all boundaries of faces in T lie in $\langle S \rangle$.

Then the following two assertions hold for any choice of an S-spanning tree R.

(i) The $|T| \times |T|$ matrix ∂ that represents the relative simplicial boundary map

$$\begin{array}{ccc} C_{k+1}(K, \langle R \rangle; \mathbb{Z}) & \to & C_k(K, \langle R \rangle; \mathbb{Z}) \\ \| & & \| \\ \mathbb{Z}^{|T|} & & \mathbb{Z}^{|S \setminus R|} \end{array}$$

is nonsingular if and only if $\tilde{H}_{k+1}(K; \mathbb{Q}) = 0$.

(ii) When the matrix ∂ is nonsingular, then $\operatorname{coker}(\partial) = \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$.

Proof: See [22] for details. □

Definition 12 Given a collection of k-simplices S, and an S-spanning tree R, say[(ii)] that R is *torsion-free* if Condition (iii) in Definition 9 is strengthened to the vanishing condition

(iv) $\tilde{H}_{k-1}(\langle R \rangle; \mathbb{Z}) = 0$.

Example 13 For example, when $\langle R \rangle$ is a contractible subcomplex of $\langle S \rangle$ then it satisfies Condition (ii) of Definition 9 as well as the vanishing condition (iv). If it furthermore satisfies Condition (i) of Definition 9, then R becomes a torsion-free S-spanning tree.

A frequent combinatorial setting where this occurs (such as in Proposition 15 below) is when S is the set of facets of a (pure) *shellable* [3] simplicial complex, and R is the subset of facets which are not fully attached along their entire boundaries during the shelling process.

Proposition 14 *Using the hypotheses and notation of Proposition 11, if one assumes in addition that R is torsion-free, assertion (ii) of Proposition 11 becomes the following assertion about (non-relative) homology:*

(ii) When the matrix ∂ is nonsingular, then $\operatorname{coker}(\partial) = \tilde{H}_k(K; \mathbb{Z})$

Proof: When R is torsion-free, the long exact sequence for the pair $(K, \langle R \rangle)$ shows that $\tilde{H}_k(K; \mathbb{Z}) \cong \tilde{H}_k(K, \langle R \rangle; \mathbb{Z})$. □

(ii) This condition on an S-spanning tree also plays an important role in [9] by Duval, Klivans and Martin.

5 More on the complete d-partite complex

It is well-known and easy to see that for a positive integer n having prime factorization $n = p_1^{e_1} \cdots p_d^{e_d}$ with $e_i \geq 1$, one always has $\Phi_n(x) = \Phi_{p_1 \cdots p_d}(x^{n/p_1 \cdots p_d})$. Thus it suffices to interpret the coefficients of cyclotomic polynomials for squarefree n.

In this section, we fix such a squarefree $n = p_1 \cdots p_d$, and discuss further properties of the simplicial complexes $K_{p_1,\ldots,p_d}$, defined in Section 1, appearing in Theorems 1 and 2.

Proposition 15 *The $(d-2)$-dimensional skeleton of $K_{p_1,\ldots,p_d}$ is shellable, with*

$$\tilde{H}_{d-2}(K_{p_1,\ldots,p_d};\mathbb{Z}) = \mathbb{Z}^{n-\phi(n)}.$$

Proof: To show that the $(d-2)$-skeleton is shellable, we note the following three facts: (i) zero-dimensional complexes are all trivially shellable, (ii) joins of shellable complexes are shellable [24, Sec. 2], and (iii) skeleta of (pure) shellable simplicial complexes are shellable [5, Corollary 10.12]. Having shown that this skeleton is shellable, it therefore has only top homology; see, for example [3, Appendix]. This homology is free abelian, with rank the absolute value of its reduced Euler characteristic, namely

$$\left|\sum_{i\geq -1}(-1)^i \operatorname{rank}_{\mathbb{Z}}(C_i)\right| = \left|\sum_{i\geq -1}(-1)^i \sum_{\substack{I\subsetneq\{1,2,\ldots,d\}\\ |I|=i+1}} \prod_{i\in I} p_i\right| = \left|\sum_{I\subsetneq\{1,2,\ldots,d\}} (-1)^{|I|-1} \prod_{i\in I} p_i\right|$$
$$= |(p_1-1)\cdots(p_d-1) - p_1\cdots p_d| = |\phi(n)-n|.$$

□

As noted in the introduction, the Chinese Remainder Theorem isomorphism (1) identifies elements of $\mathbb{Z}/n\mathbb{Z}$ with the $(d-1)$-dimensional simplices of $K_{p_1,\ldots,p_d}$. Lower dimensional faces of $K_{p_1,\ldots,p_d}$ can also be identified as cosets of subgroups within $\mathbb{Z}/n\mathbb{Z}$, but we will use this identification sparingly in this paper. For the sake of writing down oriented simplicial boundary maps, choose the following orientation on the simplices of $K_{p_1,\ldots,p_d}$, consistent with the orientation of facets preceding Theorem 2: choose the oriented $(\ell-1)$-simplex $[j_{i_1} \bmod p_{i_1}, \ldots, j_{i_\ell} \bmod p_{i_\ell}]$ with $i_1 < \ldots < i_\ell$ as a basis element of $C_{\ell-1}(K_{p_1,\ldots,p_d};\mathbb{Z})$. The following simple observation was the crux of the results in [18].

Proposition 16 *If one identifies the indexing set $\mathbb{Z}/n\mathbb{Z}$ for the columns of the boundary map*

$$C_{d-1}(K_{p_1,\ldots,p_d};\mathbb{Z}) \to C_{d-2}(K_{p_1,\ldots,p_d};\mathbb{Z}) \tag{5}$$

with the set $\mu_n := \{\zeta^j\}_{j\in\mathbb{Z}/n\mathbb{Z}}$ of all n^{th} roots of unity, then every row of this boundary map represents a $\mathbb{Q}$-linear dependence on μ_n.

Proof: A row in this boundary map is indexed by an oriented $(d-2)$-face, which has the form

$$[j_1 \bmod p_1, \ldots, j_k \widehat{\bmod} \, p_k, \ldots, j_d \bmod p_d]$$

for some $j_k \in \{0,1,\ldots,p_k-1\}$ and $1 \leq k \leq d$. This row will contain mostly zeroes. Its non-zero entries are all $(-1)^{k-1}$, and lie in the columns indexed by those ζ^j having $j \equiv j_i \bmod p_i$ for $i \neq k$, and $j \bmod p_k$ arbitrary. These exponents j are exactly those lying in one coset of the subgroup $p_1 \cdots \hat{p_k} \cdots p_d\mathbb{Z}/n\mathbb{Z}$ within $\mathbb{Z}/n\mathbb{Z}$. Summing ζ^j over j in such a coset gives zero. □

Example 17 Let $n = 15$ as in Example 3, and consider the matrix for the simplicial boundary map $C_1(K_{3,5};\mathbb{Z}) \to C_0(K_{3,5};\mathbb{Z})$. One of its rows is indexed by the 0-face $[2 \bmod 5]$ and this row has exactly three nonzero entries, all equal to $(-1)^0 = +1$. To see these signs, we rewrite $[2 \bmod 5]$ in three ways, all of which involve deleting the first entry out of two in an oriented 1-face:

$$[2 \bmod 5] = [\widehat{0 \bmod 3}, 2 \bmod 5] = [\widehat{1 \bmod 3}, 2 \bmod 5] = [\widehat{2 \bmod 3}, 2 \bmod 5].$$

The columns corresponding to these three 1-faces are indexed by the roots of unity ζ^{12}, ζ^7, and ζ^2, respectively. Summing these up with coefficients of positive one, we get

$$1 \cdot \zeta^{12} + 1 \cdot \zeta^7 + 1 \cdot \zeta^2 = \zeta^2(\zeta^{10} + \zeta^5 + 1),$$

which is the sum of ζ^j over j lying in a coset of $5\mathbb{Z}/15\mathbb{Z}$, and hence is zero.

Definition 18 Assume that n is squarefree and let T denote any set of $n - \phi(n)$ columns of the boundary map (5). Identify the complementary set T^c of $\phi(n)$ columns with a subset of the n^{th} roots-of-unity μ_n. Create a subcomplex of $K_{p_1,\ldots,p_d}$ by including its entire $(d-2)$-skeleton and attaching the subset of $(d-1)$-faces indexed by T. We denote this subcomplex as $K[T]$.

With this definition in mind, we will make use of an interesting feature of this labelling of the boundary map and the set P_n of primitive n^{th} roots of unity, noted already in [18, Remark 5]. For this next result, we let P_n^c denote the $(n - \phi(n))$-element subset of μ_n indexed by the n^{th} roots of unity which are not primitive.

Proposition 19 *Let n be a squarefree integer and P_n^c be as above. Then the subcomplex $K[{P_n}^c]$ of $K_{p_1,\ldots,p_d}$ is contractible.*

Proof: Observe that the primitive roots in $\mathbb{Z}/n\mathbb{Z}$ are exactly those elements which do not vanish modulo p_i for $i = 1, \ldots, d$. Tracing through the labelling of the $(d-1)$-faces via Ξ, we obtain the description

$$K[{P_n}^c] = \bigcup_{i=0}^{d} \mathrm{star}_{K_{p_1,\ldots,p_d}}(0 \bmod p_i),$$

where $\mathrm{star}_\Delta(v)$ denotes the *simplicial star* of the vertex v inside a simplicial complex Δ. Furthermore, each intersection of these stars is nonempty and contractible, because it is the star of another face: for $I \subset [d]$,

$$\bigcap_{i \in I} \mathrm{star}_{K_{p_1,\ldots,p_d}}(0 \bmod p_i) \quad = \quad \mathrm{star}_{K_{p_1,\ldots,p_d}}\left(\{0 \bmod p_i\}_{i \in I}\right).$$

A standard nerve lemma [4, Theorem 10.6] then shows that $K[{P_n}^c]$ itself is contractible. □

Theorem 20 *Let n be a squarefree integer and T be a subset of μ_n of size $n - \phi(n)$. Let $K[T]$ be the subcomplex of $K_{p_1,\ldots,p_d}$ of Definition 18. Then*

$$\tilde{H}_i(K[T];\mathbb{Z}) \cong \begin{cases} \mathbb{Z}[\zeta]/\mathbb{Z}T^c & \text{if } i = d-2, \\ \mathbb{Z} & \text{if both } i = d-1 \text{ and } \mathrm{rank}_{\mathbb{Z}}(\mathbb{Z}T^c) < \phi(n), \\ 0 & \text{otherwise.} \end{cases}$$

where $\mathbb{Z}T^c$ is the sublattice $\mathbb{Z}$-spanned by the roots-of-unity $T^c \subset \mu_n$.

Proof: Choose any $\mathbb{Z}$-basis for $\mathbb{Z}[\zeta]$. Let M in $\mathbb{Z}^{\phi(n)\times n}$ be the matrix that expresses the n^{th} roots of unity μ_n in this basis. We construct a particular matrix $M^\perp$ to accompany M as in Proposition 7 part (ii). Consider the collection S of all $(d-2)$-faces in the complete d-partite complex $K_{p_1,\ldots,p_d}$. The complex $\langle S\rangle$ generated by S is therefore the $(d-2)$-skeleton of $K_{p_1,\ldots,p_d}$. Proposition 15 implies that $\langle S\rangle$ is shellable, and that it has $\mathrm{rank}_{\mathbb{Z}}\tilde{H}_{d-2}(\langle S\rangle;\mathbb{Z}) = n-\phi(n)$. Therefore, we are in the situation of Example 13, implying that there exists a torsion-free S-spanning tree R, and any such R will have $|S\setminus R| = n-\phi(n)$.

Our candidate for the matrix $M^\perp$ in $\mathbb{Z}^{(n-\phi(n))\times n}$ is the restriction of the boundary map from (5) to its rows indexed by $S\setminus R$. Proposition 16 shows that the rows of $M^\perp$ are all perpendicular to the rows of M. Now choose T, T^c so that T^c indexes the set P_n of primitive n^{th} roots of unity. Proposition 6 implies that the maximal minor $M\big|_{T^c}$ of M is invertible over $\mathbb{Z}$, while Proposition 19 implies that the maximal minor $M^\perp\big|_T$ of $M^\perp$ is invertible over $\mathbb{Z}$. Thus $M, M^\perp$ satisfy the hypotheses of Proposition 7 part (ii), and combining this with Proposition 14 gives the assertion of the theorem. $\square$

6 Proof of Theorems 1 and 2

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1: Let $T^c = \{1,\zeta,\zeta^2,\ldots,\zeta^{\phi(n)}\}\setminus\{\zeta^j\}$ so that we have the equality of complexes $K[T] = K[\{\zeta^{\phi(n)+1},\zeta^{\phi(n)+2},\ldots,\zeta^{n-1}\}\cup\{j\}] = K_j$. The theorem then follows from Theorem 20 and Corollary 5. $\square$

Proof of Theorem 2: We prove Theorem 2 by applying Proposition 8 to the matrices $M, M^\perp$ in the proof of Theorem 20, with $A = \{1,\zeta,\zeta^2,\ldots,\zeta^{\phi(n)}\}$. The dependence (2) among the columns of $M|_A$ has the same coefficients (up to scaling) as the cyclotomic polynomial, and the dependence (3) among the columns of $M^\perp\big|_{A^c\cup\{j,j'\}}$ has the same coefficients (up to scaling) as a nonzero cycle $z=\sum_\ell b_\ell[F_\ell]$ in $\tilde{H}_{d-1}(K_{\{j,j'\}};\mathbb{Z})$. $\square$

7 Concordance with known properties of $\Phi_n(x)$

Here are some results about $\Phi_n(x)$ that manifest themselves topologically. See [22] for details.

1. The two maps $\mathbb{Z}/n\mathbb{Z}\to\mathbb{Z}/n\mathbb{Z}$ that send $\overline{j}$ to $\overline{-j}$ and send $\overline{j}$ to $\overline{j+1}$ generate a dihedral group of simplicial automorphisms of $K_{p_1,p_2,\ldots,p_d}$. One such automorphism sends the subcomplex $K_{\{j\}}$ to $K_{\{\phi(n)-j\}}$, and the subcomplex $K_{\{j,\phi(n)\}}$ to $K_{\{0,\phi(n)-j\}}$, explaining the symmetry $c_j = c_{\phi(n)-j}$ in $\Phi_n(x)$.

2. The fact that $\Phi_{2n}(x) = \Phi_n(-x)$ when n is odd manifests itself topologically as follows: the subcomplex $K_{\{j\}}$ whose homology interprets the coefficient of x^j for $\Phi_{2n}(x)$ is homotopy-equivalent to the suspension of the corresponding complex for $\Phi_n(x)$. Furthermore, there is a similar suspension relation between the complexes that predict the coefficients' signs.

3. When $d = 2$ so $n = p_1 p_2$ is the product of only two primes, all the subcomplexes $K_{\{j\}}$ of K_{p_1,p_2} are graphs. Hence their $(d-2)$-dimensional homology is torsion-free. It follows that the only nonzero coefficients of $\Phi_n(x)$ are ± 1, agreeing with a well-known old observation of Migotti [20]. The explicit expansion of $\Phi_{p_1 p_2}(x)$ is given in Elder [10], Lam and Leung [16], and Lenstra [17].

4. In contrast to above, when $d \geq 3$ and the p_i's are odd primes, $\Phi_n(x)$ often has coefficients with absolute value ≥ 2. For example, $\Phi_{105}(x)$ has coefficient -2 on x^7 and x^{41}. The 2-dimensional subcomplexes $K_{\{7\}}$ and $K_{\{41\}}$, whose 1-homology equals $\mathbb{Z}/2\mathbb{Z}$, turn out to be surprisingly nontrivial. For example, neither one can be collapsed down to a real projective plane.

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Generalized permutohedra, h-vectors of cotransversal matroids and pure O-sequences (extended abstract)

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Abstract. Stanley has conjectured that the h-vector of a matroid complex is a pure O-sequence. We will prove this for cotransversal matroids by using generalized permutohedra. We construct a bijection between lattice points inside a r-dimensional convex polytope and bases of a rank r transversal matroid.

Résumé. Stanley a conjecturé que le h-vecteur d'un complexe matroide est une pure O-séquence. Nous allons le prouver pour les matroides cotransversal en utilisant generalized permutohedra. Nous construisons une bijection entre les points du réseau intérieur d'un polytope convexe r-dimensions et les bases d'un matroide transversal r-rang.

Keywords: generalized permutohedra, Stanley's conjecture, h-vector, matroid, cotransversal, bipartite, matching, polytope, pure O-sequence

1 Introduction

Matroids, simplicial complexes and their h-vectors are all interesting objects that are of great interest in algebraic combinatorics and combinatorial commutative algebra. An ***order ideal*** is a finite collection X of monomials such that, whenever $M \in X$ and N divides M, then $N \in X$. If all maximal monomials of X have the same degree, then X is pure. A pure O-sequence is the vector, $h = (h_0 = 1, h_1, ..., h_t)$, counting the monomials of X in each degree. The following conjecture by Stanley has motivated a great deal of research on h-vectors of matroid complexes:

Conjecture 1.1 *The h-vector of a matroid is a pure O-sequence.*

The above conjecture has been proven for cographic matroids by both Merino (2001) and Chari (1997). It also has been proven for lattice-path matroids by Schweig (2010). Lattice path matroids are special cases of cotransversal matroids, and we will prove the conjecture for cotransversal matroids. We would also like to note that there has been plenty of interesting results related to this conjecture: Boij et al. (2010),Chari (1995),Hausel and Sturmfels (2002),Hibi (1989),Stokes (2009).

We prove the conjecture for cotransversal matroids by associating a polytope to each transversal matroid called ***transversalhedron***. This polytope is closely related to a particular ***generalized permutohedron***. The lattice points inside this polytope (excluding the points on the coordinate hyperplanes $x_i = 0$) will be in bijection with bases of the matroid, and the set of lattice points inside this polytope will naturally induce the pure order ideal we are looking for.

In Section 2, we will go over properties of transversal matroids. In Section 3, we go over properties of generalized permutohedra. In Section 4, we define the transversalhedra and "good" lattice points inside each Minkowski cell. We also give a bijection between bases of $\mathcal{M}$ and "good" lattice points inside the corresponding transversalhedron. In Section 5, we state our main result and give an example.

This is an extended abstract. Proofs and more details are given in Oh (2010).

2 Preliminaries on matroids

In this Section, we will provide some notations and tools on transversal matroids that we are going to use throughout the paper. We will assume basic familiarity with matroid theory. Throughout the paper, unless stated otherwise, a matroid $\mathcal{M}$ will be a rank r matroid over the ground set $[n] := \{1, \cdots, n\}$.

An element i of a base B is ***internally active*** if $(B \setminus i) \cup j$ is not a base for any $j < i$. An element $e \notin B$ is ***externally active*** if $(B \cup e) \setminus j$ is a not a base for all $j > e$. Given a base B, we denote $ep_{\mathcal{M}}(B)$ to be the number elements $e \notin B$ that are not externally active. We call such elements as ***externally passive*** elements of B.

The h-vector of a matroid is defined as the h-vector of its corresponding independent complex. Rather than working with the definition of the h-vector directly, we will use the following characterization:

Lemma 2.1 (Björner (1992)) *Let $(h_0, \cdots, h_r)$ be the h-vector of a matroid $\mathcal{M}$. For $0 \le i \le r$, h_i is the number of bases of $\mathcal{M}$ with $r - i$ internally active elements.*

Remark 2.2 *The way we will view h_i in this paper is to count the number of bases in the dual-matroid of $\mathcal{M}$ with i externally passive elements.*

Our main result in this paper is that the h-vector of a cotransversal matroid is a pure O-sequence.

Now let's go over the basics of ***transversal matroids***. We are going to be looking at a subgraph of the complete bipartite graph $K_{n,r}$. Denote the vertices on the left with $S = \{1, \cdots, n\}$ and on the right with $J = \{1, \cdots, r\}$. Let $\mathcal{A}$ be a family $(A_1, A_2, \cdots, A_r)$ of subsets of the set S. Then the bipartite graph $\mathcal{G}(\mathcal{A})$ associated with $\mathcal{A}$ has vertex set $S \cup J$ and its edge set is $\{(x, j) | x \in S, j \in J \text{ and } x \in A_j\}$. Given a subgraph T of this graph, let $lt(T)$ denote the set of left vertices covered by edges of T, and let $rt(T)$ denote the set of right vertices covered by edges of T. So we would have $lt(T) \subseteq S$ and $rt(T) \subseteq J$. The collection of $lt(T)$ for all maximal matchings of $\mathcal{G}(\mathcal{A})$ form the set of bases of a matroid. We denote this matroid by $\mathcal{M}(\mathcal{A})$. If $\mathcal{M}$ is an arbitrary matroid and $\mathcal{M} \cong \mathcal{M}(\mathcal{A})$ for some family $\mathcal{A}$ of sets, then we call $\mathcal{M}$ a transversal matroid and $\mathcal{A}$ a ***presentation*** of $\mathcal{M}$. The sets $A_1, \cdots, A_m$ are called ***members***.

In Figure 1, we have a presentation of a family $(\{1, 2, 6, 7, 8, 9\}, \{3, 4, 5, 6, 7, 8, 9\})$.

The ***type*** of $i \in S$ will be defined as the set of vertices of J connected to i in $\mathcal{G}(\mathcal{A})$, and will be denoted by $\phi(i)$. Using this definition, $C_{\mathcal{M},I} \subseteq S$ is defined to be collection of elements of S that have type I. We will denote $l_{\mathcal{M},I}$ to be the cardinality of $C_{\mathcal{M},I}$. Type of a subset $H \subseteq S$ will be given as the multiset obtained by collecting the types for each element of H, and will be denoted by $\phi(H)$. We put a total ordering on subsets $I \subseteq [r]$ by the following rule:

1. if $|I| < |I'|$, then $I \prec I'$ and,
2. if $|I| = |I'|$, then $I \prec I'$ if I is smaller in lexicographical order.

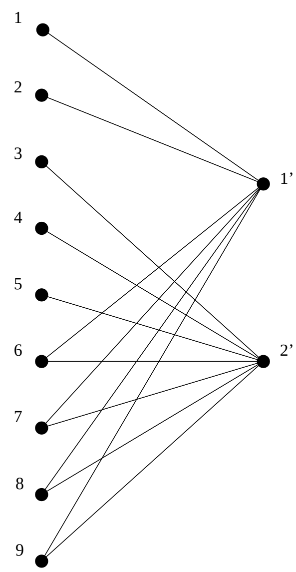

Fig. 1 – Bipartite graph defining $\mathcal{M}$

Then we relabel the set S such that if $\phi(i) \prec \phi(j)$ then $i < j$. Denote the sets $I \subseteq [r]$ that satisfy $l_{\mathcal{M},I} > 0$ as $I_1 \prec I_2 \prec \cdots \prec I_m$. Then we can express the type of each subset $H \subseteq S$ as a sequence $(a_1, \cdots, a_m)$, where a_i encodes the number of times I_i appears in the collection. This will be called the ***type sequence*** of H. In case of Figure 1, we would have $I_1 = \{1\}, I_2 = \{2\}, I_3 = \{1,2\}$. The type sequence of $\{4,8\}$ would be $(0,1,1)$, since $\phi(4) = \{2\}$ and $\phi(8) = \{1,2\}$.

For notational convenience, we will denote a set I occurring q-times in a collection by I^q. For example, the collection $\{\{1\},\{1\},\{1\},\{2\},\{2\},\{1,2\},\{1,2\}\}$ will be expressed as $\{\{1\}^3,\{2\}^2,\{1,2\}^2\}$. Since we are viewing these collections as multisets, whenever we do a set minus, we will delete only one occurrence of I from the collection per times I appears in the set being negated with. For example, $\{\{1\}^3,\{2\}^2,\{1,2\}^2\} \setminus \{\{1\},\{2\}^2\} = \{\{1\}^2,\{1,2\}^2\}$.

Definition 2.3 *Given a sequence $a = (a_1, \cdots, a_m)$, we denote $\mathcal{I}^a$ to be the collection $I_1^{a_1}, \cdots, I_m^{a_m}$. If $\mathcal{I}^a$ satisfies the Hall's condition, (i.e. if union of any t sets of the collection has cardinality at least t), then we say that a is* **valid**. *We will say that a is* **maximal** *if $\sum_i a_i = r$. If a is a type sequence of a base of $\mathcal{M}$, then it is called the* **base sequence** *of $\mathcal{M}$.*

A maximal valid sequence is a base sequence of $\mathcal{M}$ if and only if $a_i \leq l_{\mathcal{M},I_i}$ for all i. Since the type sequences and their corresponding collections carry the same information, we will say that a type of a subset is valid or maximal if its corresponding sequence is.

Now let's fix a base B of $\mathcal{M}$ and study its externally passive elements. We define $ep_{\mathcal{M}}(\phi(B))$ to be the minimum among all bases having the same type. In other words, we look at the number of externally passive elements in the first base having type $\phi(B)$. We also define $ep_{\mathcal{M}}^{\phi(B)}(B)$ to be $ep_{\mathcal{M}}(B) - ep_{\mathcal{M}}(\phi(B))$. So we can rewrite this as:

$$ep_{\mathcal{M}}(B) = ep_{\mathcal{M}}(\phi(B)) + ep_{\mathcal{M}}^{\phi(B)}(B).$$

Let's look at the example given in Figure 1. We saw that base $\{4,8\}$ has type sequence $(0,1,1)$. If we look at Table 1, we can see that $ep_{\mathcal{M}}(\{4,8\}) = 5$ and that $\{3,6\}$ also has the same type sequence. So in this case, we would have $ep_{\mathcal{M}}(\phi(\{4,8\})) = ep_{\mathcal{M}}(\{3,6\}) = 2$ and hence $ep_{\mathcal{M}}^{\phi(\{4,8\})}(\{4,8\}) = 5-2 = 3$.

We will define $EP_{\mathcal{M}}(\phi(B))$ as the collection of $I \subseteq [r]$ where there exists some $I' \subseteq [r]$ such that:

- $I \prec I'$ and
- $\phi(B) \setminus \{I'\} \cup \{I\}$ is a maximal valid type.

Given a base $B \in \mathcal{M}$, the element $e \notin B$ is an externally passive element only if $\phi(e)$ is an element of $EP_{\mathcal{M}}(\phi(B))$.

Lemma 2.4 *Let $\mathcal{M}$ be a transversal matroid and a be the type sequence for a base $B \in \mathcal{M}$. We can compute $ep_{\mathcal{M}}(\phi(B))$ and $ep_{\mathcal{M}}^{\phi(B)}(B)$ as:*

- *first set both of them to 0,*
- *for each $I_i \in EP_{\mathcal{M}}(\phi(B))$, add $l_{\mathcal{M},I_i} - a_i$ to $ep_{\mathcal{M}}(\phi(B))$ and 0 to $ep_{\mathcal{M}}^{\phi(B)}(B)$,*
- *for each $I_i \notin EP_{\mathcal{M}}(\phi(B))$, add 0 to $ep_{\mathcal{M}}(\phi(B))$ and $s - a_i$ to $ep_{\mathcal{M}}^{\phi(B)}(B)$, where the largest element of $B \cap C_{\mathcal{M},I_i}$ is the s-th element in $C_{\mathcal{M},I_i}$.*

Corollary 2.5 *Let $\mathcal{M}$ be a transversal matroid and a be the type sequence for a base $B \in \mathcal{M}$. Then $ep_{\mathcal{M}}(B) = \sum_{I_i \in EP_{\mathcal{M}}(\mathcal{I}^a)} (l_{\mathcal{M},I_i} - a_i) + ep_{\mathcal{M}}^{\phi(B)}(B)$.*

Let's look back at the example from Figure 1 and Table 1. $EP_{\mathcal{M}}(\phi(\{4,8\}))$ is going to be $\{I_1\}$. So $ep_{\mathcal{M}}(\phi(\{4,8\})) = l_{\mathcal{M},I_1} - a_1 = 2 - 0 = 2$, which coincides with our previous observation that $ep_{\mathcal{M}}(\phi(\{4,8\})) = ep_{\mathcal{M}}(\{3,6\}) = 2$. Since 4 is the 2nd element of $C_{\mathcal{M},I_2}$ and 8 is the 3rd element of $C_{\mathcal{M},I_3}$, we have $ep_{\mathcal{M}}^{\phi(\{4,8\})}(\{4,8\}) = (2-1) + (3-1) = 3$.

3 Generalized permutohedra

In this Section, we review the generalized permutohedra. The contents related to generalized permutohedra follows that of Postnikov (2009).

Definition 3.1 (Postnikov (2009)) *Let d be the dimension of the Minkowski sum $P_1 + \cdots + P_m$. A* **Minkowski cell** *in this sum is a polytope $B_1 + \cdots + B_m$ of dimension d where B_i is the convex hull of some subset of vertices of P_i. A* **mixed subdivision** *of the sum is the decomposition into union of Minkowski cells such that interSection of any two cells is their common face. A mixed subdivision is* **fine** *if for all cells $B_1 + \cdots + B_m$, all B_i are simplices and $\sum dim B_i = d$.*

Remark 3.2 *All mixed subdivisions in our paper, unless otherwise stated, will be referring to fine mixed subdivisions.*

We will use the term ***Minkowski face*** to be the sum $B_1 + \cdots + B_m$ that has dimension $\leq d$. Let $G \subseteq K_{m,r+1}$ be a bipartite graph with no isolated vertices. We label the vertices of G by $1, \cdots, m, 0', 1', \cdots, r'$ and call $1, \cdots, m$ the ***left vertices*** and $[0, r'] := 0', 1', \cdots, r'$ the ***right vertices***. Let us associate this graph with the collection $\mathcal{I}_G$ of subsets $I_1, \cdots, I_m \subseteq [0, r] := \{0, 1, \cdots, r\}$ such that $j \in I_i$ if and only if (i, j') is an edge of G. The generalized permutohedron $P_G(y_1, \cdots, y_m)$ is defined as the Minkowski sum

$$P_G(y_1, \cdots, y_m) = y_1\Delta'_{I_1} + \cdots + y_m\Delta'_{I_m},$$

where Δ'_I is defined to be to be the convex hull of points e_i for $i \in I$ and y_i are nonnegative integers.

Proposition 3.3 (Postnikov (2009)) *Let $H_1, \cdots, H_r \subset [0, r]$. The following conditions are equivalent:*

1. *For any distinct $i_1, \cdots, i_k$, we have $|H_{i_1} \cup \cdots \cup H_{i_k}| \geq k + 1$.*
2. *For any $j \in [0, r]$, there is a system of distinct representatives in $H_1, \cdots, H_r$ that avoids j.*

The above condition is called the **dragon marriage condition***.*

Definition 3.4 (Postnikov (2009)) *Let us say that a sequence of nonnegative integers $(a_1, \cdots, a_m)$ is a G-***draconian sequence** *if $\sum a_i = r$ and, for any subset $\{i_1 < \cdots < i_k\} \subseteq [m]$, we have $|I_{i_1} \cup \cdots \cup I_{i_k}| \geq a_{i_1} + \cdots + a_{i_k} + 1$. Equivalently, if the sequence $I_1^{a_1}, \cdots, I_m^{a_m}$ satisfies the dragon marriage condition.*

One important property of generalized permutohedra is that fine Minkowski cells can be described by spanning trees of G. For a sequence of nonempty subsets $\mathcal{J} = (J_1, \cdots, J_m)$, let $G_{\mathcal{J}}$ be the graph with edges (i, j') for $j \in J_i$.

Lemma 3.5 (Postnikov (2009)) *Each fine mixed cell in a mixed subdivision of $P_G(y_1, \cdots, y_m)$ has the form $y_1\Delta'_{J_1} + \cdots y_m\Delta'_{J_m}$, for some sequence of nonempty subsets $\mathcal{J} = (J_1, \cdots, J_m)$ in $[0, r]$ such that $G_{\mathcal{J}}$ is a spanning tree of G.*

Given a spanning tree $T \subseteq G$, we denote $\prod'_T$ to be the corresponding Minkowski cell. We can say a bit more about the lattice points in each $\prod'_T$:

Proposition 3.6 (Postnikov (2009)) *Any lattice point of a fine Minkowski cell $\prod'_{G_{\mathcal{J}}}$ in $P_G(y_1, \cdots, y_m)$ is of form $p_1 + \cdots + p_m$ where p_i is a lattice point in $y_i\Delta'_{J_i}$.*

Given any subgraph T in G, define the ***left degree vector*** $ld(T) = (d_1, \cdots, d_m)$ and the ***right degree vector*** $rd(T) = (d'_0, \cdots, d'_r)$ where d_i, d'_j is the degree of the vertex i, j' in T minus 1. The following proposition is stated in the proof of Theorem 11.3 in Postnikov (2009).

Proposition 3.7 (Postnikov (2009)) *Let us fix a fine mixed subdivision $\{\prod'_{T_1}, \cdots, \prod'_{T_s}\}$ of the polytope $P_G(y_1, \cdots, y_m)$. Then the map $\prod'_{T_i} \to ld(T_i)$ is a bijection between fine cells $\prod'_{T_i}$ in this subdivision and G-draconian sequences.*

For two spanning trees T and T' of G, let $U(T, T')$ be the directed graph which is the union of edges of T and T' with edges of T oriented from left to right and edges of T' oriented from right to left. A directed ***cycle*** is a sequence of directed edges $(i_1, i_2), (i_2, i_3), \cdots, (i_{k-1}, i_k), (i_k, i_1)$ such that all $i_1, \cdots, i_k$ are distinct.

Lemma 3.8 (Postnikov (2009)) *For two spanning trees T, T', the corresponding Minkowski cells can be in the same mixed subdivision only if $U(T, T')$ has no directed cycles of length ≥ 4.*

We will say that T, T' are ***compatible*** if it satisfies the condition of Lemma 3.8, and ***incompatible*** if not.

4 Transversalhedra

In this Section, we construct a polytope from a transversal matroid. The lattice points inside this polytope (excluding the coordinate hyperplanes) will give a pure order ideal that we are looking for in Stanley's conjecture. We define the ***transversalhedron*** of $\mathcal{M}$ to be

$$P_{\mathcal{M}} = \sum_{I \subseteq [r]} l_{\mathcal{M},I} \Delta_{\{0\} \cup I},$$

where Δ_J for $J \subseteq [0,r]$ is defined as:

- if $0 \in J$, the convex hull of origin and e_i for $i \in J \cap \{1, \cdots, r\}$,
- if $0 \notin J$, the convex hull of e_i for $i \in J \cap \{1, \cdots, r\}$.

Under the projection map that sends values of the 0-coordinate to 0, the generalized permutohedron $P'_{\mathcal{M}} := \sum_{I \subseteq [r]} l_{\mathcal{M},I} \Delta'_{\{0\} \cup I}$ gets sent to a transversalhedron. This projection map is actually a bijection between (lattice) points $(x_0, x_1, \cdots, x_r)$ of $P'_{\mathcal{M}}$ and $(x_1, \cdots, x_r)$ of $P_{\mathcal{M}}$ since $P'_{\mathcal{M}}$ is on a hyperplane $x_0 + x_1 + \cdots x_r = n$.

Denote the bipartite graph defining $\mathcal{M}$ to be $G_{\mathcal{M}}$. Identify all vertices on the left side of $G_{\mathcal{M}}$ having the same type, and add a vertex to the right side labeled 0 that is connected to all vertices of the left side, to get a bipartite graph $\overline{G_{\mathcal{M}}}$. Recall that we relabeled the ground set of $\mathcal{M}$ such that if $\phi(i) \prec \phi(j)$, then we have $i < j$. The i-th vertex on the left side is associated with i-th subset I that has $l_{\mathcal{M},I} > 0$ with respect to ordering on all subsets of $[r]$ given in Section 2.

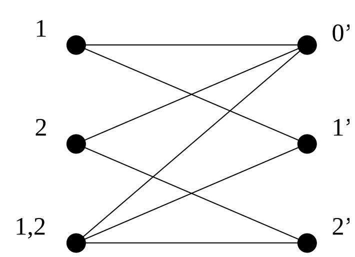

Fig. 2 – $\overline{G_{\mathcal{M}}}$ corresponding to the bipartite graph given in Figure 1.

Recall that each fine Minkowski cell $\prod'_{(\overline{G_{\mathcal{M}}})_{\mathcal{J}}}$ of $P'_{\mathcal{M}}$ can be described by $l_{\mathcal{M},I_1}\Delta'_{J_1} + \cdots + l_{\mathcal{M},I_m}\Delta'_{J_m}$. It corresponds to a fine Minkowski cell in $P_{\mathcal{M}}$ given by $l_{\mathcal{M},I_1}\Delta_{J_1} + \cdots + l_{\mathcal{M},I_m}\Delta_{J_m}$, and we will denote this cell by $\prod_{\mathcal{J}}$.

Lemma 4.1 *Maximal valid type sequences of $\mathcal{M}$ are exactly $\overline{G_{\mathcal{M}}}$-draconian sequences.*

So we immediately get the following result by using Proposition 3.7.

Corollary 4.2 *Let us fix a fine mixed subdivision of $P_{\mathcal{M}}$. Then the fine Minkowski cells of this subdivision are in bijection with the maximal type sequences of $\mathcal{M}$.*

Using this result, we define the ***type*** of a fine Minkowski cell as $\mathcal{I}^a$, where a is the $\overline{G_{\mathcal{M}}}$-draconian sequence of the cell. Since the polytope $P_{\mathcal{M}}$ contains the origin and the coordinate hyperplanes at the boundary, if we define the ***degree*** of a lattice point to be the sum of its coordinates, the degree is nonnegative for all lattice points inside the polytope. We define x_I to be $\sum_{i\in I} x_i$. The facets of this polytope are given by the coordinate hyperplanes $x_i = 0$ and hyperplanes $x_I = \sum_{I\cap H\neq\emptyset} l_{\mathcal{M},H}$. We will refer to facets that do not come from coordinate hyperplanes as the ***nontrivial boundary*** of the polytope.

Let's study the lattice points inside the Minkowski cells of $P_{\mathcal{M}}$. We have the following result by Proposition 3.6.

Corollary 4.3 *Let us fix a fine mixed subdivision of $P_{\mathcal{M}}$. Then any lattice point of a fine Minkowski cell $\prod_{\mathcal{J}}$ is of form $p_1 + \cdots + p_m$ where p_i is a lattice point of $l_{\mathcal{M},I_i}\Delta_{J_i}$.*

Inside the mixed subdivision of our transversalhedron, some lattice points are contained in several Minkowski cells. We want to decide which cell takes ownership.

Definition 4.4 *Let $\prod_{\mathcal{J}}$ be a fine mixed cell of a transversalhedron. We will say that a lattice point of $l\Delta_{J_i}$ is* **good** *if it satisfies:*

- *when $0 \in J_i$, it is not on $l\Delta_{J_i\setminus\{j\}}$ for $j \in J_i \setminus \{0\}$,*
- *when $0 \notin J_i$, it is not on $l\Delta_{J_i\setminus\{j\}}$ for $j \in J_i \setminus \{t_{i,0}\}$, where $t_{i,0}$ is the unique element of J_i such that any path from an element of J_i to 0 must pass through $t_{i,0}$.*

Using the notations of the previous corollary, we call a lattice point in a cell $\prod_{\mathcal{J}}$ to be good if for all i, p_i is a good lattice point of $l_{\mathcal{M},I_i}\Delta_{J_i}$.

A fine Minkowski cell, whose type $\mathcal{I}^a$ is not a base type of $\mathcal{M}$ (happens when $a_i > l_{\mathcal{M},I_i}$ for some i), does not contain any good lattice points.

Proposition 4.5 *Fix a fine mixed subdivision in $P_{\mathcal{M}}$. Let p be a lattice point of $P_{\mathcal{M}}$ not on any of the coordinate hyperplanes $x_i = 0$. Then p is a good lattice point of exactly one fine Minkowski cell.*

Hence regardless of which fine mixed subdivision we use, the good lattice points of $P_{\mathcal{M}}$ are going to be the lattice points not on any of the coordinate hyperplanes.

Assuming we are given a fine mixed subdivision of $P_{\mathcal{M}}$, we will now construct a bijection between bases of $\mathcal{M}$ of type $\mathcal{I}^a$ and good lattice points in fine Minkowski cells of $P_{\mathcal{M}}$ of type $\mathcal{I}^a$. The number of good lattice points in each $l_{\mathcal{M},I_i}\Delta_{J_i}$ is equal to $\binom{|C_{\mathcal{M},I_i}|}{|J_i|-1}$. So we can construct a bijection between good lattice points of $l_{\mathcal{M},I_i}\Delta_{J_i}$ and $\binom{C_{\mathcal{M},I_i}}{|J_i|-1}$ for each i. This determines a bijection between good lattice points of cells of type I^a and bases of type I^a for each maximal valid type a, and it results in a bijection between good lattice points of $P_{\mathcal{M}}$ and bases of $\mathcal{M}$. Figure 3 shows an example of such construction. Given such a bijection, we can label each good lattice point of $P_{\mathcal{M}}$ with $B \in \mathcal{M}$ and define $d_{\mathcal{M}}(B)$ as the degree of the corresponding lattice point.

The main goal we want to acheive with our bijection is to relate $ep_{\mathcal{M}}(B)$ with $d_{\mathcal{M}}(B)$. The key idea is to divide $d_{\mathcal{M}}(B)$ into two parts as we did for $ep_{\mathcal{M}}(B)$. Let's define $d_{\mathcal{M}}(\phi(B))$ as the minimum degree of all good lattice points inside the cell, of which the lattice point labeled B is a good lattice point inside. And let's define $d_{\mathcal{M}}^{\phi(B)}(B)$ as $d_{\mathcal{M}}(B) - d_{\mathcal{M}}(\phi(B))$. Then we can write:

$$d_{\mathcal{M}}(B) = d_{\mathcal{M}}(\phi(B)) + d_{\mathcal{M}}^{\phi(B)}(B).$$

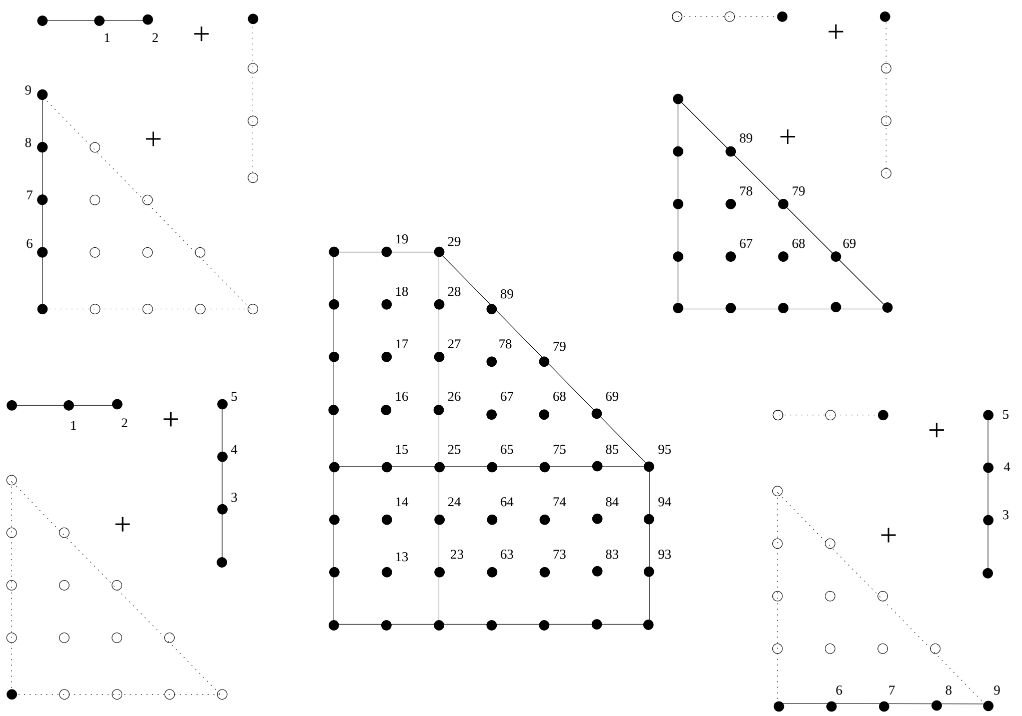

Fig. 3 – How the bijection between bases of $\mathcal{M}$ and good lattice points of $P_{\mathcal{M}}$ is constructed.

Lemma 4.6 *Let $\mathcal{M}$ be a transversal matroid, $P_{\mathcal{M}}$ the corresponding transversalhedron and fix a fine mixed subdivision of $P_{\mathcal{M}}$. Then there is a bijection between good lattice points in cell $\prod_{\mathcal{J}}$ of type $\mathcal{I}^a$ (we have $a = (|J_1| - 1, \cdots, |J_m| - 1)$) and bases of type $\mathcal{I}^a$ such that we can compute $d_{\mathcal{M}}(\phi(B))$ and $d_{\mathcal{M}}^{\phi(B)}(B)$ as:*

- *first set both of them to 0,*
- *for each J_i such that $0 \notin J_i$, add $l_{\mathcal{M},I_i}$ to $d_{\mathcal{M}}(\phi(B))$ and 0 to $d_{\mathcal{M}}^{\phi(B)}(B)$,*
- *for each J_i such that $0 \in J_i$, add a_i to $d_{\mathcal{M}}(\phi(B))$ and $s - a_i$ to $d_{\mathcal{M}}^{\phi(B)}(B)$, where the largest element of $B \cap C_{\mathcal{M},I_i}$ is the s-th element in $C_{\mathcal{M},I_i}$.*

Let's look at an example. The transversalhedron for $\mathcal{M}$ in Figure 1 is given in Figure 5. Take a look at cell with type sequence $(0, 1, 1)$. The corresponding sum $2\Delta_{J_1} + 3\Delta_{J_2} + 4\Delta_{J_3}$ and the corresponding spanning tree of $\overline{G_{\mathcal{M}}}$ is given in Figure 4. Each good lattice point of the cell corresponds to choosing a good lattice point in $2\Delta_{J_1}, 3\Delta_{J_2}, 4\Delta_{J_3}$. A base of this type corresponds to choosing $0, 1, 1$ element from $C_{\mathcal{M},I_1}, C_{\mathcal{M},I_2}, C_{\mathcal{M},I_3}$. Now let's take a look at $d_{\mathcal{M}}(\{4, 8\}) = 7$. Using the above lemma, we can check that $d_{\mathcal{M}}(\phi(\{4, 8\})) = d_{\mathcal{M}}(\{3, 6\}) = l_{\mathcal{M},I_1} + a_2 + a_3 = 4$ and $d_{\mathcal{M}}^{\phi(\{4,8\})}(\{4, 8\}) = 0 + (2-1) + (3-1) = 3$ and their sum equals 7.

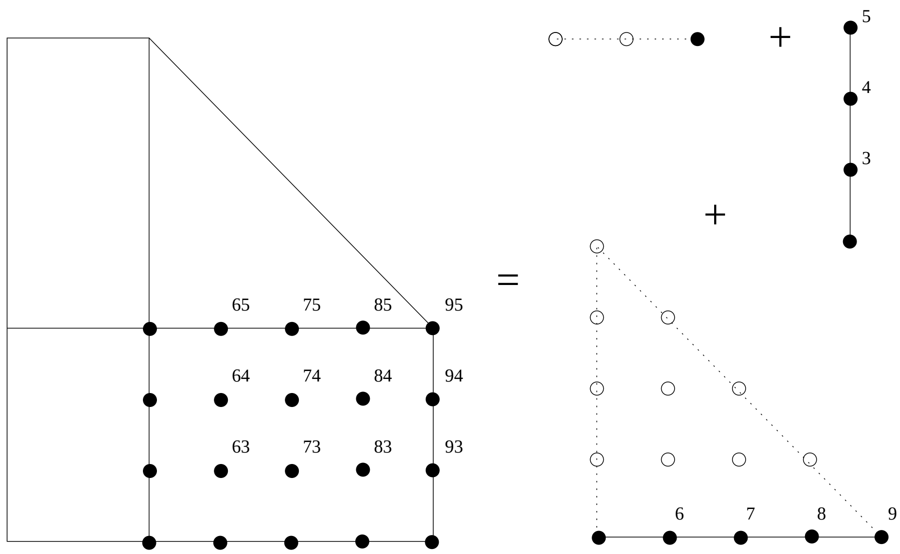

Fig. 4 – A cell of type $(0,1,1)$, its spanning tree and the corresponding sum $2\Delta_{J_1} + 3\Delta_{J_2} + 4\Delta_{J_3}$.

5 Main Result

The degree sequence obtained by the externally passive degree $ep_{\mathcal{M}}$ is the h-vector of $\mathcal{M}^*$ (Remark 2.2). The degree sequence obtained by the lattice degree $d_{\mathcal{M}}$ is by construction a pure O-sequence. Showing that these two degree sequences are the same would imply Stanley's conjecture in the cotransversal case. So our goal is to show that there is a bijection between bases of $\mathcal{M}$ and good lattice points in $P_{\mathcal{M}}$ such that $ep_{\mathcal{M}}(B) = d_{\mathcal{M}}(B) - r$ for all bases $B \in \mathcal{M}$. One should notice a similarity in the decomposition of $ep_{\mathcal{M}}$ given in Lemma 2.4 and the decomposition of $d_{\mathcal{M}}$ given in Lemma 4.6. All we have to show is that there is a bijection that makes these two decompositions essentially the same. That bijection comes from a fine mixed subdivision of $P_{\mathcal{M}}$ called the ***canonical subdivision***, defined in Oh (2010).

Lemma 5.1 *Let $\mathcal{M}$ be a transversal matroid and $P_{\mathcal{M}}$ be its transversalhedron. Look at a Minkowski cell $\prod_{\mathcal{J}}$ of type $\mathcal{I}^a$ inside the canonical subdivision of the transversalhedron. And let $l_{\mathcal{M},I_1}\Delta_{J_1} + \cdots + l_{\mathcal{M},I_m}\Delta_{J_m}$ be the corresponding Minkowski sum. We have $0 \notin J_i$ if and only if $I_i \in EP_{\mathcal{M}}(\mathcal{I}^a)$.*

By combining Lemma 2.4, Lemma 4.6 and Lemma 5.1, we have $d_{\mathcal{M}}^{\phi(B)}(B) = ep_{\mathcal{M}}^{\phi(B)}(B)$. Using Lemma 5.1 and Lemma 4.6, we get:

$$d_{\mathcal{M}}(\phi(B)) = \sum_{I_i \in EP_{\mathcal{M}}(\mathcal{I}^a)} l_{\mathcal{M},I_i} + \sum_{I_i \notin EP_{\mathcal{M}}(\mathcal{I}^a)} a_i.$$

Combining this with Lemma 2.4, we get $d_{\mathcal{M}}(\phi(B)) - r = ep_{\mathcal{M}}(\phi(B))$.

Proposition 5.2 *Given a transversal matroid $\mathcal{M}$ of rank r, we look at the canonical mixed subdivision of $P_{\mathcal{M}}$. For each base type, there exists a bijection between good lattice points in fine Minkowski cells of that type and bases of that type. And for all bases $B \in \mathcal{M}$, we have $ep_{\mathcal{M}}(B) = d_{\mathcal{M}}(B) - r$.*

For each good lattice point at $(c_1, \cdots, c_r)$, let's make a monomial ${x_1}^{c_1-1} \cdots {x_r}^{c_r-1}$. Then we get a pure order ideal of which Stanley's conjecture is asking for.

Proposition 5.3 *Let $\mathcal{M}$ be a cotransversal matroid. We denote $\mathcal{M}^*$ for the dual matroid, which is in this case a transversal matroid. For each good lattice point $(c_1, \cdots, c_r)$ in $P_{\mathcal{M}^*}$, take a monomial ${x_1}^{c_1-1} \cdots {x_r}^{c_r-1}$ to form a collection X. Then X is a pure order ideal and its degree sequence equals the h-vector of $\mathcal{M}$.*

This implies Stanley's conjecture for cotransversal matroids.

Theorem 5.4 *The h-vector of a cotransversal matroid is a pure O-sequence. In other words, Stanley's conjecture is true for cotransversal matroids.*

We will end with an example regarding our main result. We look at a transversal matroid $\mathcal{M}$ given by the bipartite graph in Figure 1. Then $C_{\mathcal{M},\{1\}} = \{1,2\}, C_{\mathcal{M},\{2\}} = \{3,4,5\}, C_{\mathcal{M},\{1,2\}} = \{6,7,8,9\}$. So our transversalhedron is given by $2\Delta_{\{1\}} + 3\Delta_{\{2\}} + 4\Delta_{\{1,2\}}$. The cells are given by base sequences $(1,1,0), (1,0,1), (0,1,1), (0,0,2)$. Then one can check that $d_{\mathcal{M}}(B) - 2 = ep_{\mathcal{M}}(B)$ in Figure 5 and the Table below. To get the h-vector of $\mathcal{M}^*$ using the polytope, we look at the degree sequence obtained by counting how many good lattice points (lattice points not on any coordinate hyperplanes) are on each diagonal. This gives us the sequence $(1,2,3,4,5,6,6,5)$, and one can check from the table that this is indeed the h-vector of $\mathcal{M}^*$ (the degree sequence of $ep_{\mathcal{M}}$).

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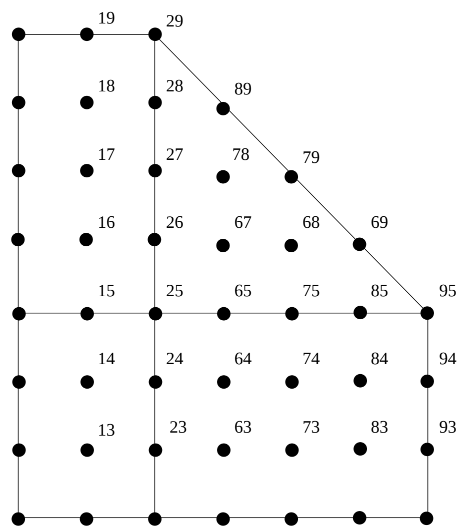

B	Type sequence	$ep_{\mathcal{M}}(B)$
13	(1,1,0)	0
14	(1,1,0)	1
15	(1,1,0)	2
16	(1,0,1)	3
17	(1,0,1)	4
18	(1,0,1)	5
19	(1,0,1)	6
23	(1,1,0)	1
24	(1,1,0)	2
25	(1,1,0)	3
26	(1,0,1)	4
27	(1,0,1)	5
28	(1,0,1)	6
29	(1,0,1)	7
36	(0,1,1)	2
37	(0,1,1)	3
38	(0,1,1)	4
39	(0,1,1)	5
46	(0,1,1)	3
47	(0,1,1)	4
48	(0,1,1)	5
49	(0,1,1)	6
56	(0,1,1)	4
57	(0,1,1)	5
58	(0,1,1)	6
59	(0,1,1)	7
67	(0,0,2)	5
68	(0,0,2)	6
69	(0,0,2)	7
78	(0,0,2)	6
79	(0,0,2)	7
89	(0,0,2)	7

Fig. 5 – Canonical mixed subdivision of the transversalhedron $P_{\mathcal{M}}$ and the table of $B \in \mathcal{M}$

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Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and Tropical Oriented Matroids

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Abstract. Develin and Sturmfels showed that regular triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ can be thought of as tropical polytopes. Tropical oriented matroids were defined by Ardila and Develin, and were conjectured to be in bijection with all subdivisions of $\Delta_{n-1} \times \Delta_{d-1}$. In this paper, we show that any triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ encodes a tropical oriented matroid. We also suggest a new class of combinatorial objects that may describe all subdivisions of a bigger class of polytopes.

Résumé. Develin et Sturmfels ont montré que les triangulations de $\Delta_{n-1} \times \Delta_{d-1}$ peuvent être considérées comme des polytopes tropicaux. Les matroïdes orientés tropicaux ont été définis par Ardila et Develin, et ils ont été conjecturés être en bijection avec les subdivisions de $\Delta_{n-1} \times \Delta_{d-1}$. Dans cet article, nous montrons que toute triangulation de $\Delta_{n-1} \times \Delta_{d-1}$ encode un matroïde orienté tropical. De plus, nous proposons une nouvelle classe d'objets combinatoires qui peuvent décrire toutes les subdivisions d'une plus grande classe de polytopes.

Keywords: triangulation, product of simplices, tropical pseudohyperplane arrangement, tropical oriented matroid

1 Introduction

Studying triangulations of product of simplices is a very active field of research and there have been numerous results being tied to many different fields ([1], [3],[4],[7],[5],[8],[10],[12],[14],[16]).

In [6], Develin and Sturmfels showed that regular triangulations can be thought as tropical polytopes. Tropical polytopes are essentially tropical hyperplane arrangements. Ardila and Develin defined tropical oriented matroids, that generalize tropical hyperplane arrangements [2]. And they conjectured that tropical oriented matroids are essentially the same as subdivisions of product of simplices. In oriented matroid theory, it is a very well known result that realizable oriented matroids come from hyperplane arrangements and oriented matroids in general come from pseudo-sphere arrangements. They showed that a tropical oriented matroid encodes a subdivision. They also showed that a triangulation of $\Delta_{n-1} \times \Delta_2$ enocodes a tropical oriented matroid. In this paper, we provide a strong evidence for the conjecture, by showing that a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ encodes a tropical oriented matroid.

In section 2, we go over the basics of triangulations of $\Delta_{n-1} \times \Delta_{d-1}$, fine mixed subdivisions of $n\Delta_{d-1}$ and develop some tools. In section 3, we go over the definition of tropical oriented matroids.

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In section 4, we show that the collection of trees in a fine mixed subdivision of $n\Delta_{d-1}$ satisfies the elimination property. In section 5, we suggest a new class of objects that may describe all subdivisions of a generalized permutohedra.

This is an extended abstract. Proofs and more details are given in [9].

2 Triangulations of $\Delta_{n-1} \times \Delta_{d-1}$ and Fine Mixed Subdivisions of $n\Delta_{d-1}$

Each full-dimensional simplex in a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ can be described by a spanning tree of the bipartite graph $K_{n,d}$. To see this, we label the vertices of Δ_{n-1} with $[n]$ and vertices of Δ_{d-1} with $[d]$, then each vertex of $\Delta_{n-1} \times \Delta_{d-1}$ corresponds to an edge of $K_{n,d}$. We will say that in $K_{n,d}$, the vertices corresponding to Δ_{n-1} are on the left side and the vertices corresponding to Δ_{d-1} are on the right side. The vertices of each subpolytope in $\Delta_{n-1} \times \Delta_{d-1}$ determine a subgraph of $K_{n,d}$. We use $(A_1, \cdots, A_n)$ where $A_1, \cdots, A_n \subseteq [d]$, to denote a subgraph of $K_{n,d}$ that has edges (i, j) for each $j \in A_i$.

Via the Cayley trick, one can think of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$ as a fine mixed subdivision of $n\Delta_{d-1}$ [15]. We will first go over the basics of fine mixed subdivisions, then state some properties that will be useful for our purpose.

Definition 2.1 ([11]) *Let r be the dimension of the Minkowski sum $P_1 + \cdots + P_n$. A* **Minkowski cell** *in this sum is a polytope $B_1 + \cdots + B_n$ of dimension r where B_i is the convex hull of some subset of vertices of P_i. A* **mixed subdivision** *of the sum is the decomposition into union of Minkowski cells such that intersection of any two cells is their common face.*

We define the simplex Δ_{d-1} as the convex hull of points $(1, 0, \cdots, 0), (0, 1, \cdots, 0), \cdots, (0, \cdots, 1)$ in $\mathbb{R}^d$. In this paper, we use the following lemma as a definition of the ***fine mixed subdivition***, and we will only consider the fine mixed subdivisions of $\Delta_{d-1} + \cdots + \Delta_{d-1}$.

Lemma 2.2 ([15]) *A mixed subdivision is fine if and only if, for each mixed cell $B = B_1 + \cdots + B_n$ in this subdivision, all B_i are simplices and $\sum dimB_i = dimB$.*

The lemma tells us that each fine cell $B_1 + \cdots + B_n$ is isomorphic to the direct product $B_1 \times \cdots \times B_n$ of simplices. Let I_i be the set of vertices of B_i. We think of each cell as a subgraph $(I_1, \cdots, I_n)$ and this is a spanning tree [11].

Remark 2.3 *The above lemma also tells us that if we take $J_i \subseteq I_i, J_i \neq \emptyset$ for each i, then $(J_1, \cdots, J_n)$ encodes a face of this cell. From now on, we will use the subgraph of $K_{n,d}$ and its corresponding face interchangeably. That is, a face $(J_1, \cdots, J_n)$ means a face $\Delta_{J_1} + \cdots + \Delta_{J_n}$.*

To avoid confusion with the tropical oriented matroid terminology, we call the 0-dimensional faces as ***topes***. For two trees T and T' of $K_{n,d}$, let $U(T, T')$ be the directed graph which is the union of edges of T and T' with edges of T oriented from left to right and edges of T' oriented from right to left. A directed ***cycle*** is a sequence of directed edges $(i_1, i_2), (i_2, i_3), \cdots, (i_{k-1}, i_k), (i_k, i_1)$ such that all $i_1, \cdots, i_k$ are distinct. Now we can say exactly which set of spanning trees describes a fine mixed subdivision of $n\Delta_{d-1}$.

Theorem 2.4 ([13],[1]) *A collection of subgraphs $T_1, \cdots, T_k$ of $K_{n,d}$ encodes a fine mixed subdivision of $n\Delta_{d-1}$ if and only if:*

1. *Each T_i is a spanning tree of $K_{n,d}$.*

2. *For each T_i and each edge e of T_i, either $T_i \setminus e$ has an isolated vertex or there is another T_j containing $T_i \setminus e$.*

3. *For any pair i, j of $[n]$, there is no cycle in $U(T_i, T_j)$.*

Given any subgraph T of $K_{n,d}$, define the ***left degree vector (LDV)*** $ld(T) = (d_1 - 1, \cdots, d_n - 1)$ where d_i is the degree of the vertex $i \in [n]$ on the left side of T. Similarly, define the ***right degree vector (RDV)*** $ld(T) = (d_1 - 1, \cdots, d_r - 1)$ where d_i is the degree of the vertex $i \in [d]$ on the right side of T. The following proposition is a special case of a statement in the proof of Theorem 11.3 in [11].

Proposition 2.5 ([11]) *Fix a fine mixed subdivision of $n\Delta_{d-1}$. Let $T_1, \cdots, T_s$ be the collection of cells. Then the map $T_i \to ld(T_i)$ is a bijection between fine cells in this subdivision and the set of sequences $(a_1, \cdots, a_n)$ satisfying $\sum a_i = d-1$ and $a_i \geq 0$ for all $i \in [n]$. The same holds for the map $T_i \to rd(T_i)$.*

The reason we are interested in LDV and RDV is because LDV governs the shape of the cell and RDV governs the location of the cell.

Given $n\Delta_{d-1}$ and $i \in [d]$, we call the facet opposite to vertex i as the ***i-facet***. A simplex in a plane, whose edge has the length n, can be subdivided by upper and lower unit triangles that are congruent to each other. In higher dimension, although there is no analogue for the lower triangles, there is one for the upper triangles. It is just the collection of simplices whose edges are of length 1 and whose vertices have integer coordinates. We call these simplices the ***unit simplices***. We express the ***location*** of a unit simplex as $(a_1, \cdots, a_d)$, where $a_i \in \mathbb{Z}$ stands for the distance between the i-facet and the unit simplex. We also have the relation that $\sum_i a_i = n - 1$. See Figure 1 for an example. The following lemma is a direct consequence of Lemma 14.9 of [11].

Lemma 2.6 *Each cell $T = (T_1, \cdots, T_n)$ in the fine mixed subdivision of $n\Delta_{d-1}$ contains exactly one unit simplex. The location of such simplex is equal to $rd(T)$.*

An example of this phenomenon is given in Figure 1.

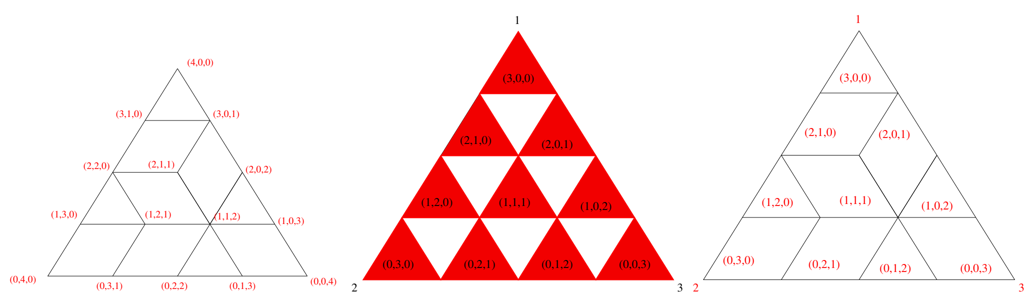

Fig. 1: The number of i's in a tope for each $i \in [d]$ describes the position of the tope. RDV describes the position of the unit simplex that the cell contains.

3 Tropical Oriented Matroids

In this section, we will review the definition of tropical hyperplane arrangements and tropical oriented matroids that were defined in [2].

Definition 3.1 *The* **tropical semiring** *is given by the real numbers $\mathbb{R}$ together with the operations of tropical addition $\oplus$ and tropical multiplication $\odot$ defined by $a \oplus b = max(a, b)$ and $a \odot b = a + b$.*

For convenience, we will work in the tropical projective $(d-1)$-space $\mathbb{TP}^{d-1}$, given by modding out by tropical scalar multiplication. In this space, ***tropical hyperplanes*** are given by the vanishing locus of $\bigoplus c_i \odot x_i$, where the vanishing locus is defined to be the set of points where $max(c_1 + x_1, \cdots, c_d + x_d)$ is achieved at least twice.

Given an arrangement $H_1, \cdots, H_n$ in $\mathbb{TP}^{d-1}$, let $v_i = (v_{i1}, \cdots, v_{id})$ be the vertex of the hyperplane H_i, for all i. The ***type*** of a point $x \in \mathbb{TP}^{d-1}$ is the n-tuple $(A_1, \cdots, A_n)$, where $A_i \subseteq [d]$ is the set of indices j for which $x_j - v_{ij}$ is maximal. Since all points in a face of the arrangement have the same type, that type is called the type of the face.

Definition 3.2 *An (n, d)-type is an n-tuple $A = (A_1, \cdots, A_n)$ of nonempty subsets of $[d] := \{1, \cdots, d\}$. The sets $A_1, \cdots, A_n$ are called the* **coordinates** *of A.*

One should keep in mind that these types will correspond to trees coming from the faces of a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$.

Definition 3.3 ([2]) *Given two (n, d)-types A and B, the* **comparability graph** *$CG_{A,B}$ has vertex set $[d]$. For $1 \le i \le n$, we draw an edge between j and k for each $j \in A_i$ and $k \in B_i$. That edge is undirected if $j, k \in A_i \cap B_i$, and it is directed $j \to k$ otherwise.*

Definition 3.4 ([2]) *A* **semidigraph** *is a graph with some undirected edges and some directed edges. A* **directed path** *from a to b in a semidigraph is a collection of vertices $v_0 = a, v_1, \cdots, v_k = b$ and a collection of edges $e_1, \cdots, e_k$, at least one of which is directed, such that e_i is either a directed edge from v_{i-1} to v_i or an undirected edge connecting the two. A* **directed cycle** *is a directed path with identical endpoints. A semidigraph is* **acyclic** *if it has no directed cycles.*

Here is the definition of ***refinement*** of a type that fits our needs. The reason that this definition is enough will be explained at the end of this section.

Definition 3.5 *The* **refinement** *of a type $A = (A_1, \cdots, A_n)$ is a type whose i-th coordinate is a nonempty subset of A_i for all i.*

Remark 3.6 *In fact, this definition is only good for our purpose (i.e. for fine mixed subdivitions), and the correct way to define the refinement for any mixed subdivision is as follows:*

Definition 3.7 ([2]) *The* **refinement** *of a type $A = (A_1, \cdots, A_n)$ with respect to an ordered partition $P = (P_1, \cdots, P_r)$ of $[d]$ is $A_P = (A_1 \cap P_{m(1)}, \cdots, A_n \cap P_{m(n)})$ where $m(i)$ is the largest index for which $A_I \cap P_{m(i)}$ is non-empty. A refinement A_P is* **total** *if all of its entries are singletons.*

Now we define the tropical oriented matroid which is our main subject of this extended abstract.

Definition 3.8 ([2]) *A* **tropical oriented matroid** *M (with parameters (n, d)) is a collection of (n, d)-types which satisfy the following four axioms:*

- *Boundary : For each $j \in [d]$, the type* $\mathbf{j} := (j, \cdots, j)$ *is in M.*
- *Elimination : If we have two types A and B in M and a position $j \in [n]$, then there exists a type C in M with $C_j = A_j \cup B_j$, and $C_k \in \{A_k, B_k, A_k \cup B_k\}$ for all $k \in [n]$.*
- *Comparability : The comparability graph $CG_{A,B}$ of any two types A and B in M is acyclic.*
- *Surrounding : If A is a type in M, then any refinement of A is also in M.*

Theorem 3.9 ([2]) *The types of the vertices of a tropical oriented matroid M with parameters (n, d) describe a set of spanning graphs defining a mixed subdivision of $n\Delta_{d-1}$.*

They proposed the following three conjectures:

1. There is a one-to-one correspondence between the set of all subdivision of $\Delta_{n-1} \times \Delta_{d-1}$ and a tropical oriented matroid with parameters (n, d).
2. The dual of a tropical oriented matroid with parameters (n, d) is a tropical oriented matroid with parameters (d, n).
3. Every tropical oriented matroid can be realized by an arrangement of tropical pseudo-hyperplanes.

Before we end this section, we explain why Definition 3.5 of refinement is enough for our purposes.

Lemma 3.10 *Let $A = (A_1, \cdots, A_n)$ be a (n, d)-type of a tropical oriented matroid, such that if one views this type as a subgraph of $K_{n,d}$, then it does not contain a cycle. Choose any $i \in [d]$ such that $|A_i| > 1$. Then choose any $k \in A_i$. Let A' be obtained from A by deleting k from A_i. Then surrounding axiom tells us that A' is in this tropical oriented matroid.*

Proof: Let Z be the union of all A_j such that $k \in A_j$ and $j \neq i$. Let W be the union of rest of A_j's. Then $Z \cap W = \{k\}$ since otherwise, we get a cycle in A. So let our ordered partition be $(W^c \cup \{k\}, W \setminus \{k\})$. Then we get A' from A by a refinement as given in Definition 3.7. $\square$

This is a more natural way to think of the surrounding axiom for our purpose, since all types coming from a fine mixed subdivision of $n\Delta_{d-1}$ have no cycles and satisfy this property, as can be seen from Remark 2.3. Whenever we use this property (or Remark 2.3), we will refer to this as the surrounding property.

4 Elimination Property

Fix a fine mixed subdivision of $n\Delta_{d-1}$. Let $\mathcal{M}$ denote the collection of trees coming the subdivision. We are going to show that this is a tropical oriented matroid. Although we don't use it, our proof is heavily motivated from the topological representation conjecture that a mixed subdivision of $n\Delta_{d-1}$ can be viewed as a tropical pseudo-hyperplane arrangement.

Roughly, the elimination property can be thought as existence of a very nice path between two types A and B. In particular, if $A_i = B_i$, we want a path such that its i-th coordinate is always equal to $A_i = B_i$. We are going to use induction based on an index defined for each pair of types, called ***rank***. Throughout the examples given in the section, for convenience, we are going to write sets such as $\{1, 2, 3\}$ by 123.

Also, recall that we call the 0-dimensional faces in a fine mixed subdivision of $n\Delta_{d-1}$ as ***topes***, instead of vertices, to avoid confusion with the tropical oriented matroid terminology.

Here is a motivation for the definition of the rank. Assume we are given a fine mixed subdivision of $n\Delta_2$ and let A and B be two types such that $A_i = B_i$. Let's look at the corresponding tropical pseudo-hyperplane arrangement. We are going to consider the case when $A_i = B_i = \{2\}$ and this is illustrated in Figure 2. Assume we are given some path between A and B such that for some types along this path, the i-th coordinate is not equal to A_i. Let C and D be the first and last points at which the path intersects the i-th tropical pseudo-hyperplane. Then C and D are both on the boundary of the region $\{2\}$ with respect to the i-th tropical pseudo-hyperplane. If we know that there is a nice path between these two points on this boundary, then we can lift this path a little bit to get a path inside the $\{2\}$-region by using the surrounding property.

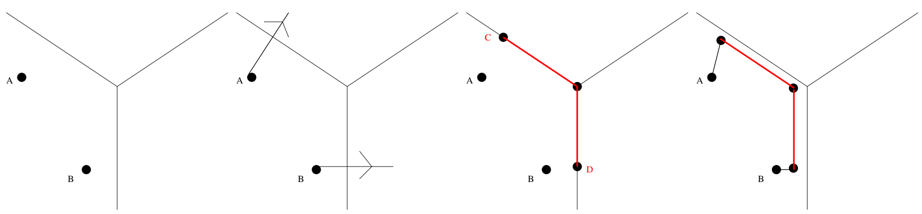

Fig. 2: Rank is a good index for proving elimination property

Definition 4.1 *For each pair of types $A, B \in \mathcal{M}$, the* **rank** *of the pair (A, B) is defined as $(r_1, \cdots, r_n)$ where for each $i \in [d]$, $r_i = min(|A_i|, |B_i|) - 1$. This is going to be denoted by $rk(A, B)$. For any $\alpha = (a_1, \cdots, a_n)$ and $\beta = (b_1, \cdots, b_n)$, we write $\alpha \geq \beta$ if we have $a_i \geq b_i$ for all $i \in [n]$. Similarly we write $\alpha > \beta$ if the inequality is strict in at least one coordinate.*

For example, $rk((123, 2), (3, 123)) = (0, 0)$ and $rk((123, 45), (3, 125)) = (0, 1)$.
So $rk((123, 2), (3, 123)) < rk((123, 45), (3, 125))$.

Definition 4.2 *We will say that two types A and B are* **adjacent** *if A and B are different in exactly one coordinate and also differ by one element in it. A* **path** *between two types is a sequence of types $A = C^0 \to C^1 \to C^2 \to \cdots \to C^{q-1} \to C^q = B$ such that each C^t is adjacent to C^{t-1} and C^{t+1}. The* **length** *of the path is given by q. Given a path, we say that coordinate i is* **strong** *if:*

1. *in that coordinate, after some some element is deleted, no element gets added.*
2. *for all t, we have $A_i \cup B_i \supseteq C_i^t$.*
3. *if an element j was added, then it does not get deleted later. This implies that $C_i^t \subseteq A_i \cup B_i$ for all t.*

A **strong path** *between types A and B is a path that is strong in every coordinates.*

A strong path is a path such that in each coordinate, it changes like

$$123 \to 1234 \to 12345 \to 1245 \to 145.$$

The reason we are interested in strong paths is because it is enough to find a strong path between any two types A and B to prove the elimination property for $\mathcal{M}$.

Lemma 4.3 *If there is a strong path between any two types A and B, then elimination holds.*

Notice that in the example of a strong path above, the cardinality of each set is bounded below by $min(|A_i|, |B_i|)$. When we are looking for a strong path between A and B, we do not consider all types. We only consider the types where the cardinality is bounded below by $rk(A, B)$.

Definition 4.4 *For each $\alpha = (a_1, \cdots, a_n)$, $\mathcal{Q}_\alpha$ is defined as the collection of types $(A_1, \cdots, A_n)$ such that $|A_i| > a_i$ for all $i \in [n]$.*

We use $\Delta(A, B)$ to denote $\sum_i (|A_i \setminus B_i| + |B_i \setminus A_i|)$. Then any path between A and B has length at least $\Delta(A, B)$. The length of a strong path between A and B is equal to $\Delta(A, B)$. We are later going to show that we can transform a lengthwise-shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$ to a strong path. So we want to show that given any types A and B in $\mathcal{Q}_\alpha$, there is a path connecting them in $\mathcal{Q}_\alpha$.

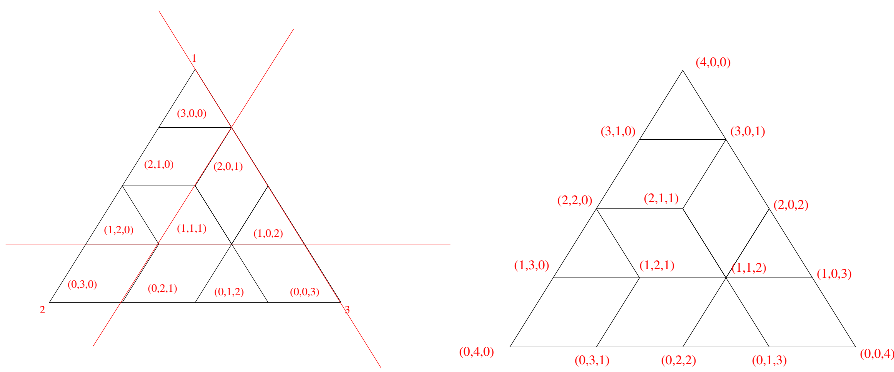

Fig. 3: How $\mathcal{S}^*_{1,0,1}$ looks like.

If we consider only the cells of $\mathcal{Q}_\alpha$, we are basically putting a cardinality restriction on the LDV's. Given an (n, d)-type A such that all $i \in [d]$ appears in A, we define A^T as a type with parameters (d, n), that has $i \in A^T_j$ if and only if $j \in A_i$. We say that A^T is the ***dual*** of A. We can take the dual of any type that is not on the boundary of $n\Delta_{d-1}$. Dual of a cell is a cell, LDV becomes RDV, and the cardinality restriction on RDV is easy to view.

Remark 4.5 *Define $\mathcal{Q}^*_\alpha$ to be the collection of types in the fine dual mixed subdivision (i.e. mixed subdivision of $d\Delta_{n-1}$ coming from the same triangulation $\Delta_{d-1} \times \Delta_{n-1}$) such that it contains strictly more than a_i number of i's for each $i \in [n]$. Then a cell is in $\mathcal{Q}_\alpha$ if and only if its dual is in $\mathcal{Q}^*_\alpha$. And two cells are adjacent (i.e. sharing a common facet) in $\mathcal{Q}_\alpha$ if and only if their duals are adjacent in $\mathcal{Q}^*_\alpha$.*

Due to Lemma 2.6, the unit simplices in these cells are exactly the ones that are inside a subsimplex of $d\Delta_{n-1}$. We will denote this subsimplex as $\mathcal{S}^*_\alpha$. Although we will not define $\mathcal{S}_\alpha$, we will keep the star in the notation to emphasize the fact that $\mathcal{S}^*_\alpha$ is in $\mathcal{Q}^*_\alpha$. Take a look at the first picture of Figure 3. Cells in $\mathcal{Q}^*_{1,0,1}$ are the two rhombi and the simplex having RDV $(1, 1, 1), (2, 0, 1)$ and $(1, 0, 2)$. Also, $\mathcal{S}^*_{1,0,1}$ is the length 2 simplex surrounded by the red lines. Before we prove that $\mathcal{Q}_\alpha$ is connected, we need the following well known results in integer programming.

Theorem 4.6 ([18],Theorem 5.20) *A matrix Y is called* **totally unimodular** *if each square submatrix of Y has determinant equal to $0, +1$ or -1. Let Y be a totally unimodular $m \times n$ matrix and let $b \in \mathbb{Z}^m$. Then the polyhedron $P := \{x | Yx \leq b\}$ has integer vertices.*

Lemma 4.7 ([17],p.279) *A matrix Y is an* **interval matrix** *if it is a $\{0, 1\}$-matrix and each row of Y has 1's consecutively. Then Y is also totally unimodular.*

We want to show that the matrix defining a fine cell is totally unimodular. To do this, we need a way to describe the matrix defining a fine cell. Note that $n\Delta_{d-1}$ lives on the plane $x_1 + \cdots + x_d = n$. Let us project it onto the plane $x_d = 0$. Denote the image as $n\Delta'_{d-1}$, which lives in $\mathbb{R}^{d-1}$. The projection does not change any fine mixed subdivision structure.

Lemma 4.8 *Let T be a fine mixed cell of $n\Delta'_{d-1}$. For any edge e of T that is not connected to a leaf on the left side, we assign a facet F_e of T by deleting the edge from T. Let us denote by I_e the set of vertices on the right side which are not connected to d in $T \setminus e$. The equation of F_e is given by $\sum_{j \in I_e} x_j = c$ for some $c \in \mathbb{Z}$.*

Corollary 4.9 *A matrix Y defining a fine mixed cell T in $n\Delta_{d-1}$ is totally unimodular.*

Proof: Each cell is a polytope, and it is defined by a matrix inequality as in Theorem 4.6(although may not unimodular in priori). From the way the projection was defined, it is enough to show that the matrix Y' defining a cell in $n\Delta'_{d-1}$ is totally unimodular. If there are two rows in Y' such that their support sets are incomparable, but not disjoint, the previous lemma tells us that there is a cycle in T of length ≥ 4. So the support sets of any pair of rows are either comparable or disjoint. After some reordering of the columns, this becomes an interval matrix. Lemma 4.7 implies that Y' is totally unimodular. □

Using this, we are going to show that when $\mathcal{S}^*_\alpha$ is a length 2 simplex, $\mathcal{Q}^*_\alpha$ is connected.

Lemma 4.10 *Let A and B be two cells in $\mathcal{Q}^*_\alpha$ such that $\alpha = (a_1, \cdots, a_n)$ and $\sum a_i = n-2$. Then there is a path in $\mathcal{Q}^*_\alpha$ from A to B, consisting of cells and their facets.*

Proof: Any tope in $\mathcal{S}^*_\alpha$ contains at least a_i number of i's. And any tope that is not on the i-facet of $\mathcal{S}_\alpha$ contains at least $a_i + 1$ number of i's. Now choose any tope C in $\mathcal{S}^*_\alpha$. Let T be a cell that contains C and intersects with the interior of $\mathcal{S}^*_\alpha$.

We can view $T \cap \mathcal{S}^*_\alpha$ as the solution space of inequalities defining the cell T and inequalities of the form $x_i \geq a_i \in \mathbb{Z}$. If we rewrite these inequalities in terms of $Yx \leq b$, then b is an integer vector. And Y is a totally unimodular matrix due to Corollary 4.9. We know that this intersection is non-empty, full-dimensional and bounded by $\mathcal{S}_\alpha$. Theorem 4.6 tells us that the solution space is a full-dimensional integer polytope. Hence T contains at least d topes of $\mathcal{S}^*_\alpha$ such that for each i, there is at least one tope not on the i-facet of $\mathcal{S}^*_\alpha$. If some tope of T contains k number of i's then T also contains at least k number of i's. So T is in $\mathcal{Q}^*_\alpha$.

Now let A and B be any two cells of $\mathcal{Q}^*_\alpha$. They share at least one tope in $\mathcal{S}^*_\alpha$. We can draw a path near this tope inside $\mathcal{S}^*_\alpha$ that starts at A, ends at B and goes through only the cells and their facets. From what we proved just before, all cells that this path goes through are cells of $\mathcal{Q}^*_\alpha$. □

Corollary 4.11 *Pick any $\alpha = (a_1, \cdots, a_n)$ and let A and B be two types in $\mathcal{Q}_\alpha$. Then there is a path connecting them.*

Now we are ready to prove that elimination holds.

Proposition 4.12 *Elimination property holds for $\mathcal{M}$, a collection of trees coming from a fine mixed subdivision of $n\Delta_{d-1}$.*

Proof: Let us dedicate $l_{A,B}$ to be the length of a shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$. It is well defined by Corollary 4.11. We are going to show that there is a strong path between A and B by induction, decreasing $rk(A, B)$ and then increasing $l_{A,B}$.

When $rk(A, B)$ is maximal (i.e. $\sum_i rk(A, B)_i = d - 1$), A and B have to be spanning trees. Since Proposition 2.5 tells us that $A = B$, the claim is obvious in this case. The claim is also obvious when $l_{A,B} = 0$, since $\Delta(A, B) \leq l_{A,B}$. So assume for the sake of induction, that we know there is a strong path between any pair D, E such that

- $rk(D, E) > rk(A, B)$ or
- $rk(D, E) = rk(A, B)$ and $l_{D,E} < l_{A,B}$.

Let $A = C^0 \to A' = C^1 \to \cdots \to C^{l_{A,B}} = B$ be a shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$. Notice that $A' \in \mathcal{Q}_{rk(A,B)}$ implies $rk(A', B) \geq rk(A, B)$. Then the induction hypothesis tells us that there is a strong path between A' and B. Replace $A' \to \cdots \to B$ with the strong path between A' and B, then we still get a shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$. Now we are going to do a case-by-case analysis on how $A \to A'$ looks like.

1. If an element of $B_i \setminus A_i$ is added to the i-th coordinate, or if $A_i \supset B_i$ and an element of $A_i \setminus B_i$ is deleted from i-th coordinate, then this path is a strong path between A and B.

2. Consider the case when some element $q \notin B_i \setminus A_i$ is added to the i-th coordinate. We are going to show that this case cannot happen. Let $C^t \to C^{t+1}$ be the first pair of types where q gets deleted from the i-th coordinate. Look at the path $A' = C^1 \to \cdots \to C^t$. Any type C among this path should satisfy $|C_i| \geq min(|C_i^1|, |C_i^t|) > min(|A_i|, |B_i|)$. Even after we delete q from the i-th coordinate for all types in this path, they are still in $\mathcal{Q}_{rk(A,B)}$. So we may replace $A' \to \cdots \to C^t$ with a path in $\mathcal{Q}_{rk(A,B)}$ that is strictly shorter. We get a contradiction since $A \to \cdots \to B$ is a shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$.

3. The remaining case is when some element q is deleted from the i-th coordinate where $A_i \not\supset B_i$. We are going to show that we may ignore this case. Let $C^t \to C^{t+1}$ be the first pair of types where some element q' gets added to the i-th coordinate. Such t exists since $A_i \not\supset B_i$. Notice that $C^{t+1} \in \mathcal{Q}_{rk(A,B)}$ implies $rk(A, C^{t+1}) \geq rk(A, B)$. Then induction hypothesis tells us that we have a strong path between A and C^{t+1}. We can replace $A \to \cdots \to C^{t+1}$ with this strong path between A and C^{t+1}. Then we get a path $A \to A' \to \cdots \to B$ that is a shortest path between A and B in $\mathcal{Q}_{rk(A,B)}$. As before, replace $A' \to \cdots \to B$ with a strong path between A' and B, then we get a path that falls into one of the previous cases.

So induction tells us that the claim is true.

□

We will roughly sketch how the process works. Let's assume that when going from A to A', the i-th coordinate changed. If the i-th coordinate of the path changes like

$$123 \to 1235 \to \cdots \to 14,$$

induction hypothesis on the length tells us that $A' \to \cdots \to B$ can be replaced with a strong path of same length. So now the i-th coordinate of the path changes like

$$123 \to 1235 \to 12345 \to 1245 \to 124 \to 14.$$

Using the surrounding property, we can get

$$123 \to 123 \to 1234 \to 124 \to 124 \to 14.$$

Then we get a redundant type in this path, so it is not a shortest-length path.

If the path changes like

$$123 \to 23 \to \cdots \to 14,$$

induction hypothesis tells us that $A' \to \cdots \to B$ can be replaced with a strong path of same length. So now the path changes like

$$123 \to 23 \to 234 \to 1234 \to 134 \to 14.$$

Induction hypothesis on the length tells us there is a strong path between 123 and 234, and we can replace this part to get

$$123 \to 1234 \to 234 \to 1234 \to 134 \to 14.$$

So for proof purposes, we could ignore the case when an element in an incomparable coordinate was deleted going from A to A'.

Corollary 4.13 *Given a collection of all trees in a triangulation of $\Delta_{n-1} \times \Delta_{d-1}$, it forms a tropical oriented matroid.*

5 Further Remarks

Tropical oriented matroids are in bijection with mixed subdivisions of $n\Delta_{d-1}$. Unimodular oriented matroids are in bijection with mixed subdivisions of a zonotope, where any edge used in the summand is an edge of Δ_{d-1}. There happens to be a natural class of polytopes that contains these two polytopes at the same time, which is called the ***generalized permutohedra*** [11]. The trees coming from faces of a fine mixed subdivision of a generalized permutohedra are also (n, d)-types, so this suggests that the general framework would be similar.

The surrounding property and the comparability property still hold for generalized permutohedra. In the proof of the elimination property for $n\Delta_{d-1}$ case, all we needed was the connectivity of $\mathcal{Q}_\alpha$. And this seems to be a property that generalized permutohedra would also have, since the fact that RDV encodes the position of the cell is still true for generalized permutohedra. Boundary axiom can be modified, in the sense that the boundary topes have to be the vertices defining the convex hull of a generalized permutohedron. Also we have to impose that every covector is a subgraph of the graph G defining the generalized permutohedron. Below is our definition of the ***generalized tropical oriented matroid (G-TOM)***:

Definition 5.1 *Let $P = P_G(y_1, \dots, y_n) = y_1\Delta_{I_1} + \cdots + y_n\Delta_{I_n}$ be a generalized permutohedron, where Δ_{I_i}'s are faces of Δ_{d-1} and $y_i \geq 0$ for all i. A collection $\mathcal{M}_{\mathcal{P}}$ of (n, d)-types is called a* **generalized tropical oriented matroid** *of P if it satisfies the following conditions:*

- *Subgraph : Every graph representing an (n, d)-type $A \in \mathcal{M}_{\mathcal{P}}$ is a subgraph of G,*
- *Boundary : If A is a tope and its RDV is unique among those satisfying the first condition, then $A \in \mathcal{M}_{\mathcal{P}}$,*
- *Surrounding : Same as tropical oriented matroids,*
- *Comparability : Same as tropical oriented matroids,*
- *Elimination : Same as tropical oriented matroids.*

And our question is:

Question 5.2 *Given a generalized permutohedron P_G, is there a bijection between the mixed subdivisions of P_G and G-TOM's?*

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Stable rigged configurations and Littlewood–Richardson tableaux

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Abstract. For an affine algebra of nonexceptional type in the large rank we show the fermionic formula depends only on the attachment of the node 0 of the Dynkin diagram to the rest, and the fermionic formula of not type A can be expressed as a sum of that of type A with Littlewood–Richardson coefficients. Combining this result with theorems of Kirillov–Schilling–Shimozono and Lecouvey–Okado–Shimozono, we settle the $X = M$ conjecture under the large rank hypothesis.

Résumé. Pour une algèbre affine de type nonexceptionnel de grand rang nous prouvons que la formule fermionique dépend seulement du voisinage du noeud 0 dans le diagramme de Dynkin, et également que la formule fermionique en type autre que A peut être exprimée comme combinaison de celles de type A avec des coefficients de Littlewood–Richardson. Combinant ce résultat avec des théorèmes de Kirillov–Schilling–Shimozono et de Lecouvey–Okado–Shimozono, nous résolvons la conjecture $X = M$ lorsque le rang est grand.

Keywords: affine crystals, rigged configurations, Littlewood–Richardson tableaux, fermionic formula

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1 Introduction

Let $\mathfrak{g}$ be an affine Lie algebra and I the index set of its Dynkin nodes. Let $\mathfrak{g}_0$ be the classical subalgebra of $\mathfrak{g}$, namely, the finite-dimensional simple Lie algebra whose Dynkin nodes are given by $I_0 := I \setminus \{0\}$ where the node 0 is taken as in [10]. Let $U'_q(\mathfrak{g})$ be the quantized enveloping algebra associated to $\mathfrak{g}$ without the degree operator. Among finite-dimensional $U'_q(\mathfrak{g})$-modules there is a distinguished family called Kirillov-Reshetikhin (KR) modules, which have nice properties such as $T(Q,Y)$-systems, fermionic character formulas, and so on. See for instance [1, 9, 14, 20] and references therein. In [7, 6], assuming the existence of the crystal basis $B^{r,s}$ ($r \in I_0, s \in \mathbb{Z}_{>0}$) of a KR module we defined the one-dimensional (1-d) sum

$$X_{\lambda,B}(q) = \sum_{b \in B} q^{D(b)}$$

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where the sum is over I_0-highest weight vectors in $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_m,s_m}$ with weight λ and D is a certain $\mathbb{Z}$-valued function on B called the energy function (see e.g. (3.9) of [6]), and conjectured that X has an explicit expression M called the fermionic formula ($X = M$ conjecture). This conjecture is settled in full generality if $\mathfrak{g} = A_n^{(1)}$ [13], when $r_j = 1$ for all j if $\mathfrak{g}$ is of nonexceptional affine types [27], and when $s_j = 1$ for all j if $\mathfrak{g} = D_n^{(1)}$ [26]. It should also be noted that recently the existence of KR crystals for nonexceptional affine types was settled [21, 23] and their combinatorial structures were clarified [2].

Another interesting equality related to X is the $X = K$ conjecture by Shimozono and Zabrocki [29, 28] that originated from the study of certain q-deformed operators on the ring of symmetric functions. Suppose $\mathfrak{g}$ is of nonexceptional type. If the rank of $\mathfrak{g}$ is sufficiently large, X does not depend on $\mathfrak{g}$ itself, but only on the attachment of the affine Dynkin node 0 to the rest of the Dynkin diagram. See Table 1. Let $X^{\Diamond}_{\lambda,B}(q)$ ($\Diamond = \varnothing, \square, \square\square, \begin{smallmatrix}\square\\ \square\end{smallmatrix}$) denote the 1-d sum for $\mathfrak{g}$ of kind $\Diamond$. Then the $X = K$ conjecture, which

Tab. 1:

$\Diamond$	$\mathfrak{g}$ of kind $\Diamond$
$\varnothing$	$A_n^{(1)}$
$\square$	$D_{n+1}^{(2)}, A_{2n}^{(2)}$
$\square\square$	$C_n^{(1)}$
$\begin{smallmatrix}\square\\ \square\end{smallmatrix}$	$A_{2n-1}^{(2)}, B_n^{(1)}, D_n^{(1)}$

has been settled in [28, 18, 19], states that if $\Diamond \neq \varnothing$, the following equality holds.

$$X^{\Diamond}_{\lambda,B}(q) = q^{-\frac{|B|-|\lambda|}{|\Diamond|}} \sum_{\mu \in \mathcal{P}^{\Diamond}_{|B|-|\lambda|}, \eta \in \mathcal{P}^{\square}_{|B|}} c^{\eta}_{\lambda\mu} X^{\varnothing}_{\eta,B}(q^{\frac{2}{|\Diamond|}}) \tag{1}$$

Here $|B| = \sum_{i=1}^m r_i s_i$, $\mathcal{P}^{\Diamond}_N$ is the set of partitions of N whose diagrams can be tiled by $\Diamond$, and $c^{\eta}_{\lambda\mu}$ is the Littlewood–Richardson coefficient. Note also that $\overline{X}^{\Diamond}_{\lambda,B}(q)$ in [19] is related to our $X^{\Diamond}_{\lambda,B}(q)$ by $\overline{X}^{\Diamond}_{\lambda,B}(q) = X^{\Diamond}_{\lambda,B}(q^{-1})$.

If we believe the $X = M$ conjecture, we have the right to expect exactly the same relation on the M side under the same assumption of the rank. This is what we wish to clarify in this paper. Namely, if $\mathfrak{g}$ is one of nonexceptional affine type and the rank is sufficiently large, we show the fermionic formula depends only on the symbol $\Diamond$, denoted by $M^{\Diamond}(\lambda, \mathbf{L}; q)$, and if $\Diamond \neq \varnothing$ we have

$$M^{\Diamond}(\lambda, \mathbf{L}; q) = q^{-\frac{|\mathbf{L}|-|\lambda|}{|\Diamond|}} \sum_{\mu \in \mathcal{P}^{\Diamond}_{|\mathbf{L}|-|\lambda|}, \eta \in \mathcal{P}^{\square}_{|\mathbf{L}|}} c^{\eta}_{\lambda\mu} M^{\varnothing}(\eta, \mathbf{L}; q^{\frac{2}{|\Diamond|}}). \tag{2}$$

Here $\mathbf{L} = (L_i^{(a)})_{a \in I_0, i \in \mathbb{Z}_{>0}}$ is a datum such that $L_i^{(a)}$ counts the number of $B^{a,i}$ in B and $|\mathbf{L}| = \sum_{a \in I_0, i \in \mathbb{Z}_{>0}} a i L_i^{(a)}$.

The proof of (2) proceeds as follows. We first rewrite the fermionic formula as

$$M^{\Diamond}(\lambda, \mathbf{L}; q) = \sum_{(\nu^\bullet, J^\bullet) \in \mathrm{RC}^{\Diamond}(\lambda, \mathbf{L})} q^{c(\nu^\bullet, J^\bullet)}$$

by introducing the notion of stable rigged configurations. c is a certain bilinear form on the rigged configurations called charge (see (2.11) of [22]). We then construct for $\diamondsuit \neq \varnothing$ a bijection

$$\Psi : \mathrm{RC}^{\diamondsuit}(\lambda, \mathbf{L}) \longrightarrow \bigsqcup_{\mu \in \mathcal{P}^{\diamondsuit}_{|\mathbf{L}|-|\lambda|}, \eta \in \mathcal{P}^{\square}_{|\mathbf{L}|}} \mathrm{RC}^{\varnothing}(\eta, \mathbf{L}) \times LR^{\eta}_{\lambda\mu},$$

where $LR^{\eta}_{\lambda\mu}$ is the set of Littlewood–Richardson skew tableaux of shape η/λ and weight μ (see, e.g., [4]). Roughly speaking, the bijection Ψ proceeds as follows. When the rank is sufficiently large, there exists k such that the a-th configuration $\nu^{(a)}$ is the same for $a = k, k+1, \ldots$. As opposed to the KKR algorithm [11] that removes a box from $\nu^{(a)}$ starting from $a = 1$, we perform a similar algorithm starting from the largest a. If we continue this procedure until all boxes are removed from $\nu^{(a)}$ for sufficiently large a, we can regard this as a rigged configuration of type A. Reflecting this sequence of procedures we can also define a recording tableau, that is shown to be a Littlewood–Richardson skew tableau. This map can be reversed at each step, and therefore defines a bijection.

Finally we show

$$c(\nu^{\bullet}, J^{\bullet}) = c(\nu'^{\bullet}, J'^{\bullet}) - \frac{|\mathbf{L}| - |\lambda|}{|\diamondsuit|}$$

where $(\nu'^{\bullet}, J'^{\bullet})$ is the first component of the image of $(\nu^{\bullet}, J^{\bullet})$ by Ψ. We note that the two equalities (1) and (2) together with the result of [13] imply

$$X^{\diamondsuit}_{\lambda,B}(q) = M^{\diamondsuit}(\lambda, \mathbf{L}; q)$$

for $\diamondsuit \neq \varnothing$ and therefore settle the $X = M$ when $\mathfrak{g}$ is of nonexceptional type and the rank is sufficiently large.

Let us summarize the combinatorial bijections that are relevant to our paper as the following schematic diagram:

$$\begin{tikzcd}
\{\text{type } \mathfrak{g} \text{ path}\} \arrow[d, leftrightarrow, "\text{(b)}"'] & & \{\text{type } A^{(1)}_n \text{ path}\} \times \mathrm{LR} \arrow[d, leftrightarrow, "\text{(a)}"] \\
\{\text{type } \mathfrak{g} \text{ RC}\} \arrow[rr, leftrightarrow, "\Psi"] & & \{\text{type } A^{(1)}_n \text{ RC}\} \times \mathrm{LR}
\end{tikzcd}$$

Here "path" stands for the highest weight elements of $\bigotimes_i B^{r_i,s_i}$ and "RC" stands for the rigged configurations. Our bijection Ψ, that exists when the rank is large, corresponds to the bottom edge. Bijection (a), which we call type $A^{(1)}_n$ RC-bijection, is established in full generality in the papers [11, 12, 13]. Algorithms for bijection (b) are known explicitly in the following cases:

- $(B^{1,1})^{\otimes L}$ type paths for all nonexceptional algebras $\mathfrak{g}$ [24],
- $\bigotimes B^{r_i,1}$ type paths for $\mathfrak{g} = D^{(1)}_n$ [26],
- $\bigotimes B^{1,s_i}$ type paths for all nonexceptional algebras $\mathfrak{g}$ [27].

For the cases that the bijection (b) is established, our bijection Ψ thus gives the combinatorial bijection between the set of type $\mathfrak{g}$ paths and the product set of the type $A_n^{(1)}$ paths and the Littlewood–Richardson skew tableaux. We refer to [28] for related combinatorial problems.

We expect that the bijection (b) exists in full generality even without the large rank hypothesis. It will give a combinatorial proof of the $X = M$ conjecture. Furthermore, it also gives an essential tool for the study of a tropical integrable system known as the box-ball system (see e.g., [3, 5, 8]) which is a soliton system defined on the paths and is supposed to give a physical background for the $X = M$ identities. More precisely, the rigged configurations are identified with the complete set of the action and angle variables for the type $A_n^{(1)}$ box-ball system [15] (see [17] for a generalization to type $D_n^{(1)}$). It is also interesting to note that by introducing a tropical analogue of the tau functions in terms of the charge $c(\nu^\bullet, J^\bullet)$, the initial value problem for the type $A_n^{(1)}$ box-ball systems is solved in [16, 25]. Therefore the construction of the bijection (b) in full generality will be a very important future problem.

Organization of the present paper is as follows. In section 2 we recall minimal facts about the rigged configurations. In section 3 we define the algorithm. In section 4 we describe main properties of the algorithm. Section 5 is devoted for a nontrivial example.

This paper is an extended abstract of the original paper [22].

2 Stable rigged configurations

In order to define the algorithm, we prepare minimal facts from the rigged configurations for nonexceptional algebras of rank n. The rigged configurations are the following set of data: $\mathbf{L} = (L_i^{(a)})_{a\in I_0, i\in\mathbb{Z}_{>0}}$ that appears in introduction, together with

$$(\nu^\bullet, J^\bullet) = \{(\nu^{(1)}, J^{(1)}), (\nu^{(2)}, J^{(2)}), \cdots (\nu^{(n)}, J^{(n)})\}$$

where $\nu^{(a)} = (\nu_1^{(a)}, \nu_2^{(a)}, \ldots, \nu_{l^a}^{(a)})$ $(1 \le a \le n)$ is positive integer sequence (called configuration) and $J^{(a)} = (J_1^{(a)}, J_2^{(a)}, \ldots, J_{l^a}^{(a)})$ is integer sequence associated with each entry of $\nu^{(a)}$ (called riggings). Here we have to impose some conditions on these sets that depend on the specific choice of the algebra. However we do not need to prepare full version of the definition. In fact, it is shown in [22] that the rigged configurations for algebras of sufficiently large rank takes a simplified structure. Let us assume that the rank n of the algebra is very large. Then we can show that there is some large $N(\ll n)$ such that there exists $N' \ll N$ with the property $\nu^{(N')} = \nu^{(N'+1)} = \cdots = \nu^{(N)}$ holds. According to [22], we can ignore details of $(\nu^{(a)}, J^{(a)})$ $(N < a)$ and we have to only think about the rest of the rigged configurations. The vacancy number $p_i^{(a)}$ $(a \le N)$ is defined as

$$p_i^{(a)} = \sum_{k\in\mathbb{Z}_{>0}} L_k^{(a)} \min(i,k) + Q_i^{(a-1)} - 2Q_i^{(a)} + Q_i^{(a+1)}$$

where $Q_i^{(a)} = \sum_j \min(i, \nu_j^{(a)})$. In our setting, we have $\nu^{(a)} \in \mathcal{P}^\diamondsuit$ and $p_i^{(a)} = 0$ for $N \le a$. For $a \le N$, we require the following inequalities:

$$0 \le J_i^{(a)} \le p_{\nu_i^{(a)}}^{(a)}, \qquad (\forall a, i).$$

We call such $(\nu^\bullet, J^\bullet)$ under large rank limit *stable rigged configurations*. In the original arguments in [22], we make precise estimates on the rank n such that our procedure is possible. In the present note, we will rely on the result of such estimate and forget about any technical difficulties related with $(\nu^{(a)}, J^{(a)})$ for $a \approx n$. For a stable rigged configuration $(\nu^\bullet, J^\bullet)$ one can define the weight λ as

$$\lambda_a = \sum_{b \geq a, i \in \mathbb{Z}_{>0}} iL_i^{(b)} + |\nu^{(a-1)}| - |\nu^{(a)}|,$$

where λ_a is the length of the a-th row of λ when identified with the Young diagram. For $(\nu^\bullet, J^\bullet)$ we denote it by $\mathrm{wt}\,(\nu^\bullet, J^\bullet)$. The stable rigged configurations depend only on the choice of $\diamondsuit$ (if we ignore the information near n). We will denote the set of the stable rigged configurations as $\mathrm{RC}^{\diamondsuit}(\lambda, \mathbf{L})$.

3 The bijection

The goal of this section is to give definitions of our main algorithms Ψ and its inverse $\tilde{\Psi}$. Roughly speaking, the algorithms consist of two parts: the one is box removing or adding procedure on the rigged configurations, and the other one is to create a kind of recording tableau T which eventually generates the LR tableaux. We will divide the definition according to this distinction. During definition, we choose a large integer N as in the previous section. We remark that more precise estimate on the rank n is possible (see [22]).

Definition 1 *The map* δ_l

$$\delta_l : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), k\},$$

is defined by the following algorithm. Here l is one of lengths of rows of $\nu^{(N)}$.

(i) Choose one of length l rows of $\nu^{(N)}$. Then choose rows of $\nu^{(a)}$ $(a < N)$ recursively as follows. Suppose that we have chosen a row of $\nu^{(a)}$. Find the shortest singular rows of $\nu^{(a-1)}$ whose length is equal to or longer than the chosen row of $\nu^{(a)}$. If there is no such row, set $k = a$ and stop. Otherwise choose one of such singular rows and continue. If the process does not stop, set $k = 1$.

(ii) $\nu'^\bullet$ is obtained by removing one box from the right end of each chosen row at Step (i).

(iii) The new riggings $J'^\bullet$ are defined as follows. For the rows that are not changed in Step (ii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using $\nu'^\bullet$.

Definition 2 *The map* Ψ

$$\Psi : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\}$$

is defined as follows. As the initial condition, set T = Young diagram that represents the weight of $(\nu^\bullet, J^\bullet)$. Let h_i denote the height of the i-th column (counting from left) of the partition $\nu^{(N)}$ and let $l = \nu_1^{(N)}$.

(i) We will apply δ_l for h_l times. Each time when we apply δ_l, we recursively redefine $(\nu^\bullet, J^\bullet)$ and T as follows. Assume that we have done δ_l^{i-1} and obtained $\{(\nu^\bullet, J^\bullet), T\}$. Let us apply δ_l one more time:

$$\delta_l : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), k\},$$

Using the output, do the following. Define new $(\nu^\bullet, J^\bullet)$ to be $(\nu'^\bullet, J'^\bullet)$. Define new T by putting i on the right of the k-th row of the previous T.

(ii) Recursively apply $\delta_{l-1}^{h_{l-1}}, \dots, \delta_2^{h_2}, \delta_1^{h_1}$ by the same procedure as in Step (i). Then the final outputs $(\nu'^{\bullet}, J'^{\bullet})$ and T give the image of Ψ.

Now we are going to give the description of the algorithm $\tilde{\Psi}$ which will be shown to be the inverse of Ψ. Again we shall forget about the procedures near n (ignore information for $N < a$).

Definition 3 *The map $\tilde{\delta}_k$*

$$\tilde{\delta}_k : (\nu^{\bullet}, J^{\bullet}) \longmapsto (\nu'^{\bullet}, J'^{\bullet})$$

is defined by the following algorithm. Here the integer k should satisfy $k \leq N$.

(i) Starting from $\nu^{(k)}$, choose rows of $\nu^{(a)}$ ($k < a$) recursively as follows. To initialize the process, let us tentatively assume that we have chosen an infinitely long row of $\nu^{(k-1)}$. Suppose that we have chosen a row of $\nu^{(a-1)}$. Find the longest singular rows of $\nu^{(a)}$ whose length does not exceed the length of the chosen row of $\nu^{(a-1)}$. If there is no such row, suppose that we have chosen a length 0 row of $\nu^{(a)}$ and continue. Otherwise choose one of such singular rows and continue.

(ii) $\nu'^{\bullet}$ is obtained by adding one box to each chosen row in Step (i). If the length of the chosen row is 0, create a new row at the bottom of the corresponding partition $\nu^{(a)}$.

(iv) The new riggings $J'^{\bullet}$ are defined as follows. For the rows that are not changed in Step (ii), take the same riggings as before. Otherwise set the new riggings equal to the corresponding vacancy numbers computed by using $\nu'^{\bullet}$.

Definition 4 *The map $\tilde{\Psi}$*

$$\tilde{\Psi} : \{(\nu^{\bullet}, J^{\bullet}), T\} \longmapsto (\nu'^{\bullet}, J'^{\bullet})$$

is defined as follows.

(i) Let h_1 be the largest integer contained in T. For h_1 do the following procedure. Among all h_1, find the rightmost one and fix. Repeat the same procedure for $h_1 - 1, h_1 - 2, \dots, 2, 1$. Call these fixed h_1 integers of T the first group. Remove all members of the first group from T and do the same procedure for the new T. Call the integers that are fixed this time the second group. Repeat the same procedure recursively until all integers of T are grouped. Let the total number of groups be l, the cardinality of the i-th group be h_i and the position of the letter j contained in the i-th group be the $k_{i,j}$-th row (counting from top of T).

(ii) The output of $\tilde{\Psi}$ is defined as follows:

$$(\nu'^{\bullet}, J'^{\bullet}) = \tilde{\delta}_{k_{l,1}} \cdot \cdots \cdot \tilde{\delta}_{k_{2,1}} \tilde{\delta}_{k_{2,2}} \cdots \tilde{\delta}_{k_{2,h_2}} \tilde{\delta}_{k_{1,1}} \tilde{\delta}_{k_{1,2}} \cdots \tilde{\delta}_{k_{1,h_1}} (\nu^{\bullet}, J^{\bullet}).$$

4 Main properties

The crux of the combinatorics is contained in the following two theorems on the well-definedness of both maps Ψ and $\tilde{\Psi}$, which are proved in [22].

Theorem 1 *Assume that $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit$. Suppose that the rank n is sufficiently large. Then the map Ψ*

$$\Psi : (\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\}$$

is well-defined. More precisely, $(\nu'^\bullet, J'^\bullet) \in \mathrm{RC}^\varnothing$ and the LR tableau $T \in LR^\eta_{\lambda\mu}$ satisfy the following properties:

$$\lambda = \mathrm{wt}(\nu^\bullet, J^\bullet), \qquad \mu = \nu^{(N)}, \qquad \eta = \mathrm{wt}(\nu'^\bullet, J'^\bullet).$$

Theorem 2 *Assume that $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\varnothing$ and T is the LR tableau that satisfy the following three properties: $T \in LR^\eta_{\lambda\mu}$ where $\lambda, \mu \in \mathcal{P}^\diamondsuit$ and $\eta = \mathrm{wt}(\nu^\bullet, J^\bullet)$. Then the map $\tilde{\Psi}$;*

$$\tilde{\Psi} : \{(\nu^\bullet, J^\bullet), T\} \longmapsto (\nu'^\bullet, J'^\bullet),$$

is well-defined. More precisely, we have $(\nu'^\bullet, J'^\bullet) \in \mathrm{RC}^\diamondsuit$, $\mathrm{wt}(\nu'^\bullet, J'^\bullet) = \lambda$ and $\nu'^{(N)} = \mu$.

By construction, δ and $\tilde{\delta}$ are mutually inverse procedure. Therefore the above theorems imply the following main theorem.

Theorem 3 *Assume that $(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit$. Suppose that the rank n is sufficiently large. Then Ψ gives a bijection between the $\mathrm{RC}^\diamondsuit$ and the product set of $\mathrm{RC}^\varnothing$ and the LR tableaux as follows:*

$$\begin{aligned} \Psi :&(\nu^\bullet, J^\bullet) \longmapsto \{(\nu'^\bullet, J'^\bullet), T\}, \\ &(\nu^\bullet, J^\bullet) \in \mathrm{RC}^\diamondsuit(\lambda, \mathbf{L}), \quad \{(\nu'^\bullet, J'^\bullet), T\} \in \mathrm{RC}^\varnothing(\eta, \mathbf{L}) \times LR^\eta_{\lambda\mu}, \end{aligned}$$

where λ, μ, η satisfy the following properties:

$$\lambda = \mathrm{wt}(\nu^\bullet, J^\bullet), \qquad \mu = \nu^{(N)}, \qquad \eta = \mathrm{wt}(\nu'^\bullet, J'^\bullet).$$

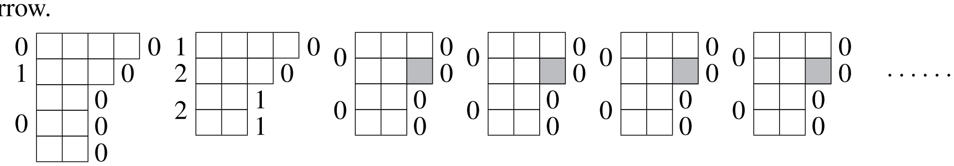
The inverse procedure is given by $\Psi^{-1} = \tilde{\Psi}$.

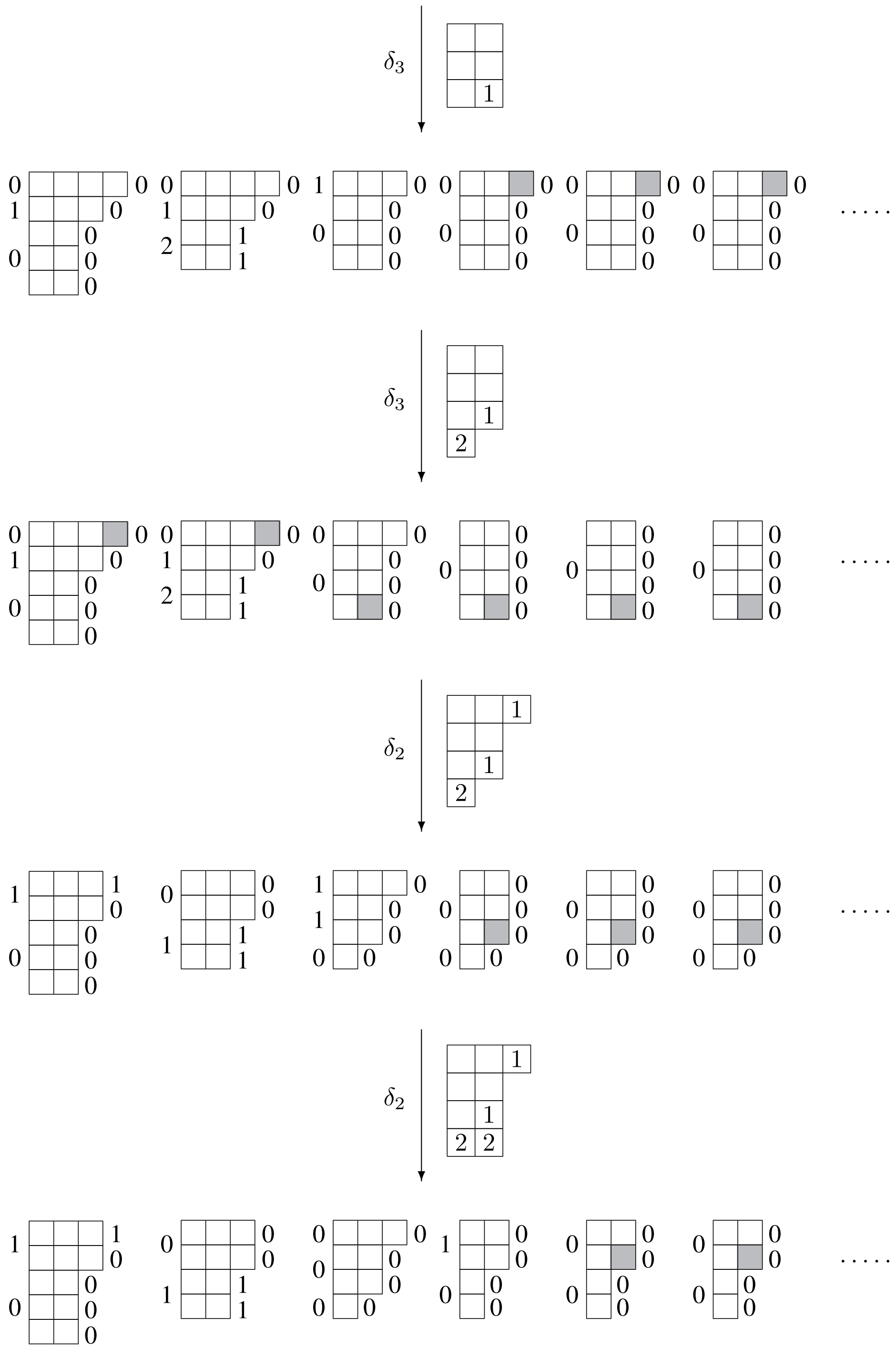
5 Example

Let us consider the special case of the bijection Ψ where the bijection [24, 27] between the rigged configurations and the tensor products of crystals is also available. Consider the following element of the tensor product $(B^{1,3})^{\otimes 3} \otimes (B^{1,2})^{\otimes 2} \otimes (B^{1,1})^{\otimes 2}$ of type $D^{(1)}_n$ ($n \geq 8$) crystals:

$$p = \boxed{1\,|\,1\,|\,1} \otimes \boxed{2\,|\,\bar{1}\,|\,\bar{1}} \otimes \boxed{1\,|\,2\,|\,\bar{2}} \otimes \boxed{2\,|\,3} \otimes \boxed{2\,|\,\bar{2}} \otimes \boxed{\bar{2}} \otimes \boxed{2}.$$

Due to Theorem 8.6 of [27] all the isomorphic elements under the combinatorial R-matrices correspond to the same rigged configuration. Then the map Ψ proceeds as follows. In the following diagrams, the first rigged configuration corresponds to the above p. Here, we put the vacancy numbers (resp. riggings) on the left (resp. right) of the corresponding rows. The gray boxes represent the boxes to be removed by each δ indicated on the left of each arrow. The corresponding recording tableau T is given on the right of each arrow.




δ_3
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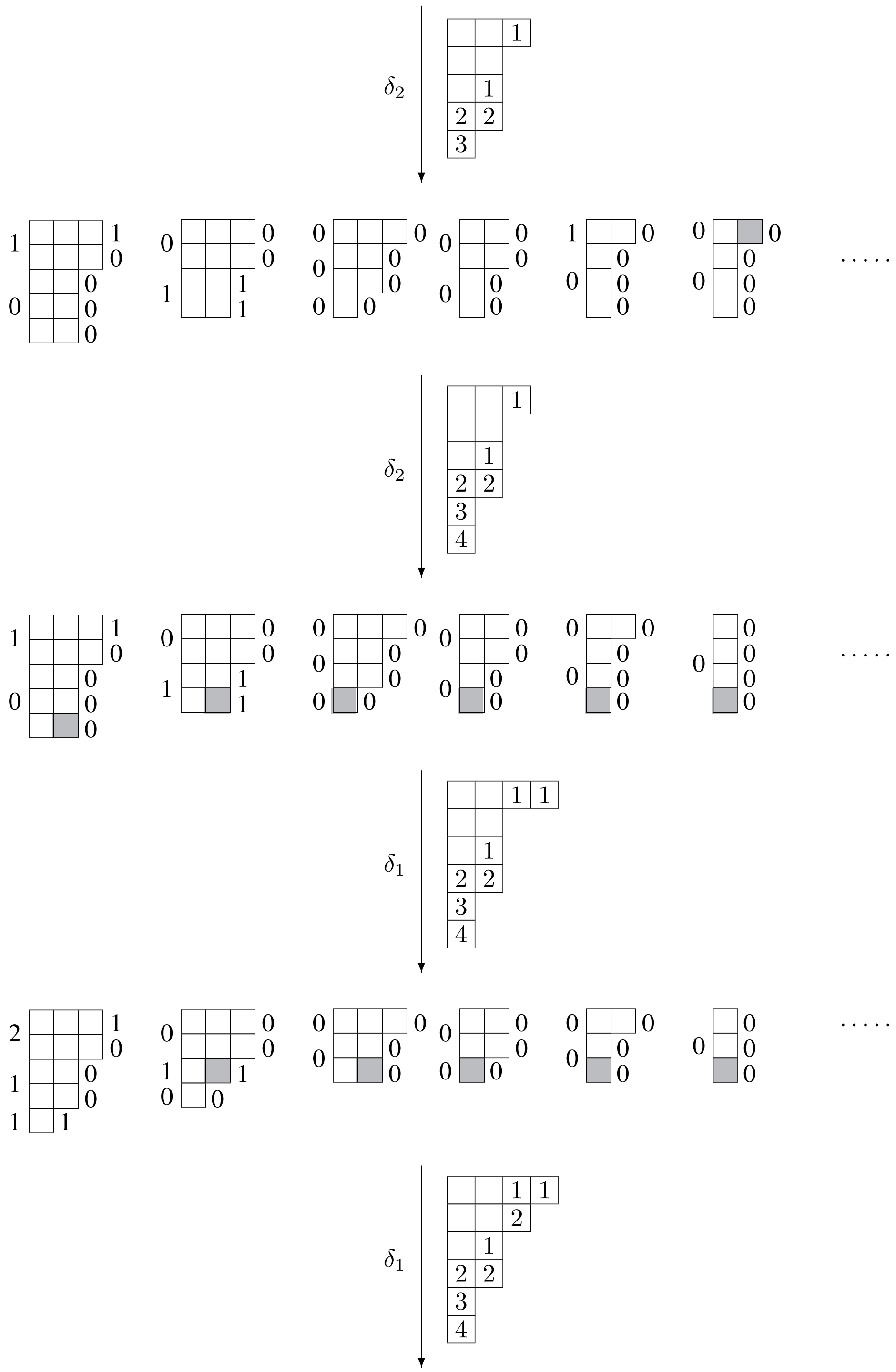

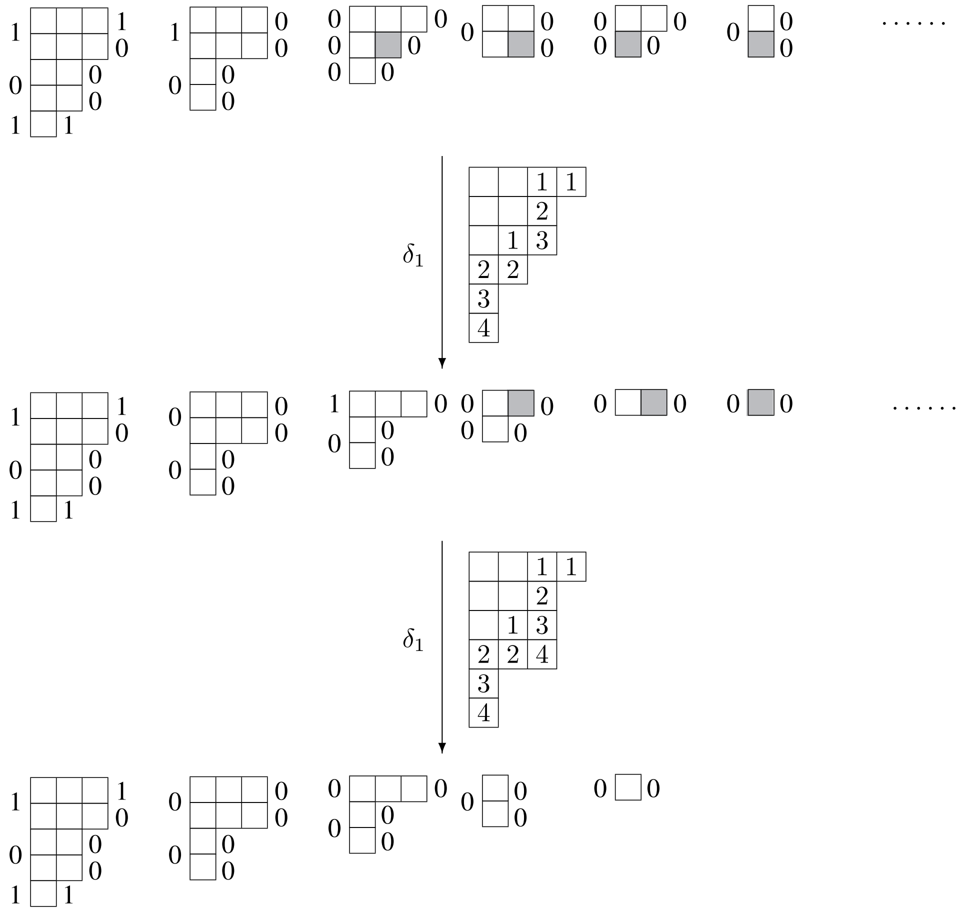

The final rigged configuration and T of the above diagrams give the image of Ψ. Under the bijection [12] the final rigged configuration corresponds to the following element:

$$p' = \boxed{1}\boxed{1}\boxed{1} \otimes \boxed{2}\boxed{2}\boxed{2} \otimes \boxed{1}\boxed{3}\boxed{3} \otimes \boxed{4}\boxed{4} \otimes \boxed{3}\boxed{5} \otimes \boxed{4} \otimes \boxed{6}.$$

Remark 1 *As for an example of $\tilde{\Psi}$, one should read the above example in the reverse order. More precisely, reverse all arrows and apply $\tilde{\delta}_4$, $\tilde{\delta}_3$, $\tilde{\delta}_2$, $\tilde{\delta}_1$, $\tilde{\delta}_6$, $\tilde{\delta}_5$, $\tilde{\delta}_4$, $\tilde{\delta}_1$, $\tilde{\delta}_4$, $\tilde{\delta}_3$ in this order.*

Remark 2 *Let p and p' as in the example in this section and consider them as elements of $D_8^{(1)}$. If we apply the involution σ at Section 5.3 of [19], we have*

$$\sigma(p) = \boxed{\bar{8}}\boxed{\bar{8}}\boxed{\bar{8}} \otimes \boxed{8}\boxed{8}\boxed{\bar{7}} \otimes \boxed{6}\boxed{\bar{8}}\boxed{\bar{6}} \otimes \boxed{\bar{7}}\boxed{\bar{6}} \otimes \boxed{6}\boxed{\bar{6}} \otimes \boxed{7} \otimes \boxed{\bar{7}}.$$

Then p' coincides with the I_0-highest element corresponding to $\sigma(p)$. We expect that the same relation holds for arbitrary image of Ψ.

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How often do we reject a superior value?

— Extended abstract —

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Abstract. Words $a_1 a_2 \dots a_n$ with independent letters a_k taken from the set of natural numbers, and a weight (probability) attached via the geometric distribution pq^{i-1} $(p+q=1)$ are considered. A consecutive record (motivated by the analysis of a skip list structure) can only advance from k to $k+1$, thus ignoring perhaps some larger (=superior) values. We investigate the number of these rejected superior values. Further, we study the probability that there is a single consecutive maximum and show that (apart from fluctuations) it tends to a constant.

Résumé. On considère des mots $a_1 a_2 \dots a_n$ formés de lettres à valeurs entières, tirées de façon indépendante avec une distribution géométrique pq^{i-1} $(p+q=1)$. Un record $k+1$ est dit consécutif si la lettre précédente est k. la notion est motivée par des considérations algorithmiques. Les autres records sont rejetés. Nous étudions le nombre de records rejetés. Nous étudions aussi la probabilité qu'il y ait un seul maximum consécutif, et montrons qu'elle converge vers une constante, à certaines fluctuations près.

Keywords: combinatorics on words, records, generating functions, Rice's method, q-series

1 Introduction

We consider words $a_1 a_2 \dots a_n$ with letters a_k taken from the set of natural numbers, and a weight (probability) attached to it by saying that the letter $i \in \mathbb{N}$ occurs with probability pq^{i-1} $(p+q=1)$ and that the letters are independent. The parameter $\mathcal{K}(a_1 a_2 \dots a_n)$, which we call the number of *weak consecutive records*, has proved to be essential in the analysis of a skip list structure Louchard and Prodinger (2006).(i) The word is scanned from left to right, and assuming that the current record (maximum) is value k, any letter different from k, $k+1$ is ignored. If, however, the symbol scanned is one of these, we call it a weak consecutive record, and set the value of the current maximum to it. So, the current record either stays at k or advances to the next value $k+1$. The skip list version assumes that the first letter of the word defines the first record.

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(i) In order to learn more about this structure, the interested reader is invited to consult the earlier paper Louchard and Prodinger (2006), which also appeared in this journal; it also contains many pointers to some interesting earlier papers.

For the sake of clarity, we consider the word $1\,3\,1\,1\,2\,4\,3\,5\,3\,5\,1\,4\,1\,2\,3\,4\,6\,5\,1$ and underline each consecutive weak maximum: $\underline{1}\,3\,\underline{1}\,\underline{1}\,\underline{2}\,4\,\underline{3}\,5\,\underline{3}\,5\,1\,\underline{4}\,2\,1\,3\,\underline{4}\,6\,\underline{5}\,1$. The number of underlined symbols (9 in this case) is the parameter $\mathcal{K}$ of interest.

In Louchard and Prodinger (2006), the average of the parameter $\mathcal{K}(n)$ was shown to be (with $Q = 1/q$, $L = \log Q$)

$$\mathbb{E}\mathcal{K}(n) = 1 + (Q+1) \sum_{j=1}^{n} \binom{n}{j} \frac{(-1)^{j-1}(q;q)_{j-1} p^{j+1} q^j}{1-q^{j+1}},$$

which was also evaluated asymptotically.

Theorem 1 *[Old theorem] The expectation of the $\mathcal{K}(n)$-parameter is asymptotic to*

$$\mathbb{E}\mathcal{K}(n) \sim (Q+1)\log_Q n + \frac{(Q+1)\gamma}{L} + \frac{Q+1}{L}\log(p) - (Q+1)\alpha - \frac{(1+q)^2}{2pq} + 1 + \delta(\log_Q n).$$

The constant α is given by

$$\alpha = \sum_{i\geq 1} \frac{q^i}{1-q^i};$$

$\delta(x)$ is a small periodic function. Its Fourier coefficients could be given in principle.

In Oliver and Prodinger (2009), we investigated the parameter $\mathcal{M}$, which is the *maximum of the underlined values.* Now, clearly, for that, we do not need to underline repetitions of the current maximum, as in the instance of the $\mathcal{K}$-parameter. So, when our current maximum is $k-1$, we ignore all letters different from k, and when it occurs (with probability pq^{k-1}) we set the current maximum to k. We obtained explicit and asymptotic enumerations:

Theorem 2 *[Old theorem] The average and variance of the $\mathcal{M}(n)$-parameter are asymptotically given by*

$$\mathbb{E}\mathcal{M}(n) \sim \log_Q n - \alpha + \frac{\log(p)}{L} + \frac{\gamma}{L} + \frac{1}{2} + \delta_E(\log_Q n),$$

$$\mathbb{V}\mathcal{M}(n) \sim \frac{\pi^2}{6L^2} + \frac{1}{12} - \beta + \delta_V(\log_Q n).$$

Here, $\delta_{\cdot}(x)$ is an unspecified periodic function of period 1 and small amplitude. Its Fourier coefficients could be computed in principle. The poles come from the residues at $z = \chi_k = \frac{2\pi i k}{L}$. These expansions follow from explicit expressions for the first and second (factorial) moment:

$$\frac{1}{p} + \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^{j+1} \frac{(q)_{j-1} q^j p^{j+1}}{1-q^{j+1}}$$

and

$$\frac{2q}{p^2} + 2\sum_{j=1}^{n} \binom{n}{j} (-1)^{j+1} (q)_{j-1} p^{j+1} q^j \left[-\frac{1}{1-q^{j+1}} \sum_{m=1}^{j-1} \frac{q^m}{1-q^m} + \frac{1}{(1-q^{j+1})^2} \right].$$

□

Now, we ignore repetitions and thus speak about about consecutive records (consecutive maxima). This is in contrast to ordinary records (left-to-right maxima), since the latter would advance to any larger value $> k$, not just to $k+1$.

In the present paper, which is a companion paper to Oliver and Prodinger (2009), we study the consecutive records further. In particular, we want to count the number of values $> k+1$ that would be accepted when looking for ordinary records, but must be rejected in the consecutive maximum instance. Here is an example to clarify this.

$$\overset{\vee}{\mathbf{1}}\,\overset{\vee}{\underset{\bullet}{3}}\,1\,1\,\mathbf{2}\,1\,\mathbf{3}\,3\,1\,\overset{\vee}{\underset{\bullet}{5}}\,1\,\overset{\vee}{\underset{\bullet}{7}}\,1\,2\,2\,3\,1\,\mathbf{4}\,1\,\underset{\bullet}{6}\,\mathbf{5}\,2\,5\,5\,3\,1\,1\,2\,1\,4\,\overset{\vee}{\underset{\bullet}{8}}\,1\,5\,\mathbf{6}\,2\,2\,3\,\underset{\bullet}{8}\,1\,1\,2\,5\,1\,5\,1\,3\,3\,1\,5\,1\,\mathbf{7}\,1$$

The consecutive records are printed in boldface; the ordinary records are marked by $\vee$, and the superior values that are neglected when scanning the word for consecutive records are marked by $\bullet$.

We consider this parameter in the next section, assuming random words of length n. It will turn out that both, expectation and variance, are of order $\log n$.

Some fifteen years ago, it was observed by several authors that the probability of a single winner (= a single maximum) in a word of length n does not tend to a limit, but rather oscillates around a certain value. Here are some papers about this Eisenberg et al. (1993); Baryshnikov et al. (1995); Bruss and O'Cinneide (1990); Pakes and Steutel (1997); Brands et al. (1994); Qi and Wilms (1997); Kirschenhofer and Prodinger (1996); Louchard and Prodinger (2006), and we apologize if we should have left out some relevant paper.

Now the maximum is also the left-to-right maximum (record). We investigate in the last section of this paper the analogous question related to consecutive records. Assume that k is the consecutive maximum, we consider the probability that after this consecutive record has been established, no further k's are read. Note, however, that earlier k's are possible, since they might have been ignored. $\mathbf{2}4115242\mathbf{3}121232\mathbf{4}11$ has consecutive single maximum 4, but earlier some 4's have been read (and rejected).

We encounter a similar phenomenon: there is a limiting value, but there are (tiny) oscillations around it. This constant comes out as a (quite complicated) series.

We use (standard) notation from q-analysis: $(x;q)_n = \prod_{i=0}^{n-1}(1-xq^i)$ and $(x;q)_\infty = \prod_{i\ge 0}(1-xq^i)$. Note that $(x;q)_n = (x;q)_\infty/(xq^n;q)_\infty$, and the latter form makes sense also for n a complex number. For several identities related to such quantities (Euler's partition identities, Heine's transformation formula, etc.), we refer to Andrews et al. (1999).

Furthermore, to say it again, we use $Q = 1/q$ and $L = \log Q$.

Our method is to set up (ordinary) generating functions, where z marks the length of the word and u the additional parameter. Moments are obtained by differentiation. It turns out that in this kind of problems the substitution $z = \frac{w}{w-1}$ makes everything much nicer. Eventually, one can translate everything back into the z-world:

$$\sum_{n\ge 1} a_n w^n = \sum_{n\ge 1} z^n \sum_{k=0}^{n-1} a_{k+1}(-1)^{k+1}\binom{n-1}{k}.$$

For the asymptotic evaluation, we use a contour integral representation of alternating sums ("Rice's method").

2 The difference between ordinary maxima and consecutive maxima

Since in the present setting the consecutive maximum can only advance by $+1$, we will miss (reject) some values that are larger than that. In this section, we count the number of rejected values.

Let $c_k(z,u)$ be the generating function where $[z^n u^j]c_k(z,u)$ is the probability that a random word of length n has consecutive maximum equal to k, and j better values have been rejected.

A recursion

$$c_k(z,u) = c_{k-1}(z,u)\frac{zpq^{k-1}}{1-z[1+(u-1)q^{k+1}-pq^k]} + \frac{zpq^{k-1}}{1-z[1+(u-1)q^{k+1}-pq^k]}.$$

It holds for $k \geq 1$, and we assume that $c_0 = 0$. It is good to use an abbreviation:

$$\lambda_k := 1 - z[1+(u-1)q^{k+1}-pq^k].$$

This follows from taking the instance with consecutive maximum $k-1$, and attaching the letter k, followed by an arbitrary sequence of letters different from $k+1$; the large ones are marked by u. The last terms reflects the situation that k is the first letter of the word.

Then

$$\frac{c_k(z,u)\lambda_1\dots\lambda_k}{z^k p^k q^{\binom{k}{2}}} = \frac{c_{k-1}(z,u)\lambda_1\dots\lambda_{k-1}}{z^{k-1}p^{k-1}q^{\binom{k-1}{2}}} + \frac{\lambda_1\dots\lambda_{k-1}}{z^{k-1}p^{k-1}q^{\binom{k-1}{2}}}.$$

This first order recursion can now be solved by summation:

$$c_k(z,u) = \frac{z^k p^k q^{\binom{k}{2}}}{\lambda_1\dots\lambda_k}\sum_{j=0}^{k-1}\frac{\lambda_1\dots\lambda_j}{z^j p^j q^{\binom{j}{2}}} = \sum_{j=0}^{k-1}\frac{z^{k-j}p^{k-j}q^{\binom{k}{2}-\binom{j}{2}}}{\lambda_{j+1}\dots\lambda_k}.$$

Expectations

The generating function of the expectations is obtained by differentiation:

$$d_k(z) := \frac{\partial}{\partial u}c_k(z,u)\Big|_{u=1},$$

and is finally given by

$$\sum_{k\geq 1} d_k(z),$$

since we must take any final consecutive maximum into account. Now

$$d_k(z) = \sum_{j=0}^{k-1}\frac{z^{k-j}p^{k-j}q^{\binom{k}{2}-\binom{j}{2}}}{\lambda_{j+1}\dots\lambda_k\big|_{u=1}}\sum_{i=j+1}^{k}\frac{zq^{i+1}}{1-z(1-pq^i)}$$

and

$$\lambda_{j+1}\dots\lambda_k\big|_{u=1}=(1-w)^{j-k}\prod_{i=j+1}^{k}(1-wpq^i)=(1-w)^{j-k}(wpq^{j+1};q)_{k-j}.$$

Rewriting in w-notation:

$$d_k(z)=\sum_{0\le j<i\le k}\frac{w^{k-j}(-1)^{k-j-1}p^{k-j}q^{\binom{k}{2}-\binom{j}{2}}}{(wpq^{j+1};q)_{k-j}}\frac{wq^{i+1}}{1-wpq^i}.$$

Summing up leads after a lengthy computation to:

$$\sum_{k\ge1}d_k(z)=\frac{1}{p}\sum_{j\ge0}\sum_{i\ge1}q^{\binom{i}{2}}\frac{(-pq^{j+1}w)^{i+1}}{(wpq^{j+1};q)_i}.$$

Note (this will be discussed later):

$$\sum_{h\ge1}\frac{(-q^dw)^hq^{\binom{h}{2}}}{(w;q)_h}=-\sum_{n\ge1}q^d\frac{(q;q)_{n+d-1}}{(q;q)_d}w^n. \tag{1}$$

Therefore

$$\sum_{k\ge1}d_k(z)=-\frac{1}{p}\sum_{n\ge1}(q;q)_{n-1}(wpq)^{n+1}\frac{1}{1-q^{n+1}}.$$

Hence the coefficient of w^n in this expression is

$$\frac{p^{n-1}q^n}{1-q^n}(q;q)_{n-2}$$

for $n\ge2$ and 0 otherwise. And now we rewrite this in terms of coefficients z^n:

$$[z^n]\sum_{j\ge1}d_j=(-1)^n[w^n](1-w)^{n-1}\sum_{j\ge1}d_j=\sum_{k=1}^{n-1}\binom{n-1}{k}(-1)^{k+1}\frac{p^kq^{k+1}}{1-q^{k+1}}(q;q)_{k-1}.$$

Second factorial moments

Let

$$f_k(z):=\frac{\partial^2}{\partial u^2}c_k(z,u)\Big|_{u=1}.$$

We have

$$\frac{\partial^2}{\partial u^2}\lambda_{j+1}\dots\lambda_k\big|_{u=1}=\frac{2}{(1-w)^{j-k}(wpq^{j+1};q)_{k-j}}\sum_{j+1\le h\le i\le k}\frac{w^2q^{h+i+2}}{(1-wpq^h)(1-wpq^i)}.$$

Combined with the rest,

$$f_k(z) = \sum_{0\le j<h\le i\le k} (-pw)^{k-j} q^{\binom{k}{2}-\binom{j}{2}} \frac{2}{(wpq^{j+1};q)_{k-j}} \frac{w^2 q^{h+i+2}}{(1-wpq^h)(1-wpq^i)}.$$

And this must be summed (the long computation is not shown here):

$$\sum_{k\ge1} f_k(z) = -\frac{2q}{p^2} \sum_{N\ge3} \frac{(q;q)_{N-2}(wpq)^N}{1-q^N} \sum_{n=1}^{N-2} \frac{1}{1-q^n}.$$

The coefficient of w^N is

$$-\frac{2(q;q)_{N-2}p^{N-2}q^{N+1}}{1-q^N} \sum_{n=1}^{N-2} \frac{1}{1-q^n}$$

for $N \ge 1$, otherwise 0. Now we rewrite this:

$$[z^n] \sum_{k\ge1} f_k(z) = \sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \frac{2(q;q)_{j-1}p^{j-1}q^{j+2}}{1-q^{j+1}} \sum_{m=1}^{j-1} \frac{1}{1-q^m}.$$

Let us summarize what we found so far:

Theorem 3 *The expectation and the second factorial moment of the number of superior elements that we reject when we consider the consecutive maxima instead of the true maxima, assuming random words of length n, are given by*

$$\sum_{k=1}^{n-1} \binom{n-1}{k} (-1)^{k+1} \frac{p^k q^{k+1}}{1-q^{k+1}} (q;q)_{k-1}$$

and

$$\sum_{j=1}^{n-1} \binom{n-1}{j} (-1)^j \frac{2(q;q)_{j-1}p^{j-1}q^{j+2}}{1-q^{j+1}} \sum_{m=1}^{j-1} \frac{1}{1-q^m}.$$

$\square$

In various places of these computations, the formula (1) was used. It can be shown using Heine's transform Andrews (1976); a proof is included in the full paper.

Asymptotics

The main ingredient is *Rice's formula* Flajolet and Sedgewick (1995) which allows to write an alternating sum as a contour integral:

$$\sum_{k=1}^{n} \binom{n}{k} (-1)^k f(k) = -\frac{1}{2\pi i} \int_{\mathcal{C}} \frac{n!\Gamma(-z)}{\Gamma(n+1-z)} f(z)dz.$$

The positively oriented curve $\mathcal{C}$ encircles the poles $1, 2, \ldots, n$ and no others. (The lower summation index 1 could be replaced by another constant, say a.) Changing the contour, there are extra residues, which have to be taken into account with the opposite sign, and they constitute the asymptotic expansion of the sum. In our instance, this residue is at 0. There are also contributions from poles at $\frac{2\pi ik}{L}$, with $L = \log Q$, $Q = \frac{1}{q}$, which we do not compute explicitly. When one collects them, they constitute a (tiny) periodic oscillation.

The function $f(z)$ extrapolates the sequence $f(k)$. This is often obvious, but sometimes requires some work to provide such a function. For our expected value, this discussion leads to

$$\sum_{k=1}^{n-1}\binom{n-1}{k}(-1)^{k+1}\frac{p^k q^{k+1}}{1-q^{k+1}}(q;q)_{k-1} = \frac{1}{2\pi i}\int_{\mathcal{C}}\frac{(n-1)!\Gamma(-z)}{\Gamma(n-z)}\frac{p^z q^{z+1}}{1-q^{z+1}}(q;q)_{z-1}dz.$$

It is not *a priori* clear what $(q;q)_{z-1} = \dfrac{(q;q)_z}{1-q^z}$ should be. However, note that

$$(q;q)_z = \frac{(q;q)_\infty}{(q;q^{z+1})_\infty}.$$

The expansion

$$(q;q)_z \sim 1 - \alpha Lz + \frac{\alpha^2+\beta}{2}L^2z^2$$

with

$$\alpha = \sum_{j\ge1}\frac{q^j}{1-q^j} \quad \text{and} \quad \beta = \sum_{j\ge1}\frac{q^j}{(1-q^j)^2}$$

is not hard to obtain. Providing this expansion, the rest can be done by a computer, and the expectation comes out as

$$\frac{q}{p}\log_Q n + \frac{q}{p}\log(p) - \frac{q}{p}\alpha + \frac{q\gamma}{pL} - \frac{q(1+q)}{2p^2}.$$

For the second factorial moment, things are more involved, and we must find the continuation of

$$\sum_{m=1}^{j-1}\frac{1}{1-q^m} = j-1+\sum_{m=1}^{j-1}\frac{q^m}{1-q^m}.$$

But

$$\sum_{m=1}^{j-1}\frac{q^m}{1-q^m} = \sum_{m=1}^{j}\frac{q^m}{1-q^m} - \frac{q^j}{1-q^j} = \alpha - \sum_{m\ge1}\frac{q^{m+j}}{1-q^{m+j}} - \frac{q^j}{1-q^j},$$

and thus we may take

$$z-1+\alpha-\sum_{m\ge1}\frac{q^{m+z}}{1-q^{m+z}} - \frac{q^z}{1-q^z}.$$

Near $z=0$, this function can be expanded as

$$z-1+\beta Lz - \frac{1}{Lz} + \frac{1}{2} - \frac{Lz}{12}.$$

Again, the rest can be done by a computer, and the second factorial moment can be obtained. It is not displayed here. From this, the variance is computed by adding the expectation and subtracting the square of the expectation. We summarize the results.

Theorem 4 *The expectation and the variance of the number of superior elements that we reject when we consider the consecutive maxima instead of the true maxima, assuming random words of length n, are given by*

$$\text{expectation} \sim \frac{q}{p}\log_Q n + \frac{q}{p}\log(p) - \frac{q}{p}\alpha + \frac{q\gamma}{pL} - \frac{q(1+q)}{2p^2} + \delta_{.}(\log_Q n).$$

and

$$\begin{aligned}\text{variance} \sim{}& \frac{q\log_Q n}{p^2}\\ &+ \frac{q\log(p)}{p^2 L} - \frac{q\alpha}{p^2} + \frac{q\gamma}{p^2 L} - \frac{q^2\beta}{p^2} + \frac{q^2\pi^2}{6p^2L^2} - \frac{2q^2}{p^2 L}\\ &+ \frac{q(q^3-6+16q^2+q)}{12p^4} + \delta_{.}(\log_Q n).\end{aligned}$$

Here, $\delta_{.}(x)$ denotes an unspecified tiny periodic function of period 1. It might differ in different places. The leading term in the variance of order $\log n$ has no fluctuating component, since they cancel out. □

3 Probability of a single (consecutive) winner

Exact enumeration

Recall that the generating function of words with consecutive maximum $= k$ is given by

$$b_k(z) = c_k(z,1) = \sum_{j=0}^{k-1} \frac{q^{\binom{k}{2}-\binom{j}{2}}(-pw)^{k-j}}{(wpq^{j+1};q)_{k-j}}.$$

Then

$$e_k(z) = (1+b_k(z))\frac{zp^k}{1-z(1-pq^k-pq^{k+1})} = -\left(1+\sum_{j=0}^{k-1}\frac{q^{\binom{k}{2}-\binom{j}{2}}(-pw)^{k-j}}{(wpq^{j+1};q)_{k-j}}\right)\frac{wp^k}{1-wp(1+q)q^k}$$

is the generating function where the maximum is $= k+1$, and it is a single consecutive maximum. This must be summed over $k+1 \ge 1$, to get the generating function of words with a single consecutive maximum (details are in the full paper):

$$\begin{aligned}\sum_{k\ge0} e_k(z) &= -\sum_{m\ge0}(1+q)^m(wp)^{m+1}\sum_{j\ge0}q^{j(m+1)}\left(1-\sum_{n\ge1}q^m\frac{(q;q)_{n+m-1}}{(q;q)_m}(wpq^{j+1})^n\right)\\ &=: -A+B.\end{aligned}$$

Then

$$A = \sum_{m\ge0}(1+q)^m(wp)^{m+1}\frac{1}{1-q^{m+1}}$$

and

$$B = \frac{1}{q}\sum_{m\geq 0}(1+q)^m \sum_{n\geq 1}\frac{(q;q)_{n+m-1}}{(q;q)_m}(wpq)^{m+1+n}\frac{1}{1-q^{m+1+n}}.$$

The coefficient of w^N in A is $\frac{p^N(1+q)^{N-1}}{1-q^N}$ for $N \geq 1$ and 0 otherwise.

The coefficient of w^N in B is

$$p^N q^{N-1}\frac{(q;q)_{N-2}}{1-q^N}\sum_{m=0}^{N-2}\frac{(1+q)^m}{(q;q)_m}$$

for $N \geq 2$ and 0 otherwise.

Therefore we get an explicit expression.

Theorem 5 *The probability that a random word of length n has a single consecutive maximum is given (exactly) by*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{p^{k+1}(1+q)^k}{1-q^{k+1}} + \sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}p^{k+1}q^k\frac{(q;q)_{k-1}}{1-q^{k+1}}\sum_{m=0}^{k-1}\frac{(1+q)^m}{(q;q)_m}.$$

□

Asymptotics

The first sum is $O(1/n)$, so we concentrate on the second one. The continuation of

$$\sum_{m=0}^{k-1}\frac{(1+q)^m}{(q;q)_m}$$

isn't as easy as before, since we cannot push it to infinity, and pulling out dominant terms isn't too obvious either. So we proceed in a different way (computation not shown in this abstract):

$$\sum_{m=0}^{n-1}\frac{(1+q)^m}{(q;q)_m} = \frac{1}{(q;q)_\infty}\sum_{j\geq 0}\frac{(-1)^j q^{\binom{j+1}{2}}}{(q;q)_j}\frac{1-(1+q)^n q^{jn}}{1-(1+q)q^j}.$$

Therefore we continue

$$p^{k+1}q^k\frac{(q;q)_{k-1}}{1-q^{k+1}}\sum_{m=0}^{k-1}\frac{(1+q)^m}{(q;q)_m}$$

via

$$p^{z+1}q^z\frac{(q;q)_{z-1}}{1-q^{z+1}}\frac{1}{(q;q)_\infty}\sum_{j\geq 0}\frac{(-1)^j q^{\binom{j+1}{2}}}{(q;q)_j}\frac{1-(1+q)^z q^{jz}}{1-(1+q)q^j}.$$

At $z = 0$, this function is regular, with the value

$$\frac{1}{(q;q)_\infty} \sum_{j \ge 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{(q;q)_j} \frac{j - \log_Q(1+q)}{1-(1+q)q^j}.$$

This follows from

$$\frac{1-(1+q)^z q^{jz}}{1-q^z} \sim \frac{1-(1+z\log(1+q))(1-Ljz)}{Lz} \sim j - \log_Q(1+q), \qquad (z \to 0).$$

So, at $z = 0$, there is ultimately a simple pole, and, apart from the complicated value, the negative residue is one.

Theorem 6 *The probability that the consecutive record is single, is asymptotically, as $n \to \infty$, given by*

$$\frac{1}{(q;q)_\infty} \sum_{j \ge 0} \frac{(-1)^j q^{\binom{j+1}{2}}}{(q;q)_j} \frac{j - \log_Q(1+q)}{1-(1+q)q^j} + \delta_\cdot(\log_Q n),$$

with a periodic function of period 1 and tiny oscillation $\delta_\cdot(x)$. $\square$

One could also consider a random variable, namely the number of times the consecutive maximum occurs, and compute moments. However, we decided not to go that route here.

Acknowledgements

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Tableaux and plane partitions of truncated shapes (extended abstract)

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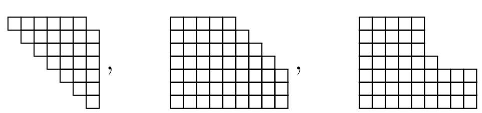
Abstract. We consider a new kind of straight and shifted plane partitions/Young tableaux — ones whose diagrams are no longer of partition shape, but rather Young diagrams with boxes erased from their upper right ends. We find formulas for the number of standard tableaux in certain cases, namely a shifted staircase without the box in its upper right corner, i.e. truncated by a box, a rectangle truncated by a staircase and a rectangle truncated by a square minus a box. The proofs involve finding the generating function of the corresponding plane partitions using interpretations and formulas for sums of restricted Schur functions and their specializations. The number of standard tableaux is then found as a certain limit of this function.

Résumé. Nous considérons un nouveau type de partitions planes, ou de tableaux de Young, droits ou décalés, obtenus en privant leurs diagrammes de certaines cellules en haut à droite, et dans certains cas nous trouvons des formules d'énumération pour les tableaux standard. Les preuves impliquent le calcul de la fonction génératrice pour les partitions planes correspondantes, en utilisant des interprétations et des formules pour les sommes de fonctions de Schur restreintes et leurs spécialisations. Le nombre de tableaux standard est alors obtenu comme une certaine limite de cette fonction.

Keywords: plane partitions, tableaux, truncated shapes, hook formulas, Schur functions

1 Introduction

In this paper we find product formulas for special cases of a new type of tableaux and plane partitions, ones whose diagrams are not straight or shifted Young diagrams of integer partitions. The diagrams in question are obtained by removing boxes from the north-east corners of a straight or shifted Young diagram and we say that the shape has been truncated by the shape of the boxes removed. We discover formulas for the number of tableaux of specific truncated shapes: shifted staircase truncated by one box in Theorem 1, rectangle truncated by a staircase shape in Theorem 2 and rectangle truncated by a square minus a box in Theorem 3; these shapes are illustrated as



The proofs rely on several steps of interpretations, their details can be found in [Pan10]. Plane partitions of truncated shapes are interpreted as (tuples of) SSYTs, which translates the problem into specializations of sums of restricted Schur functions. The number of standard tableaux is found as a polytope volume and then a certain limit of these specializations (generating function for the corresponding plane partitions). The computations involve, among others, complex integration and the Robinson-Schensted-Knuth correspondence.

The consideration of these objects started after R. Adin and Y. Roichman asked for a formula for the number of linear extensions of the poset of triangle-free triangulations, which are equivalent to standard tableaux of shifted staircase shape with upper right corner box removed, [AR]. We find and prove the formula in question as Theorem 1.

Theorem 1 *The number of shifted standard tableaux of shape $\delta_n \setminus \delta_1$ is equal to*

$$g_n \frac{C_n C_{n-2}}{2C_{2n-3}},$$

where $g_n = \frac{\binom{n+1}{2}!}{\prod_{0 \le i < j \le n}(i+j)}$ is the number of shifted staircase tableaux of shape $(n, n-1, \dots, 1)$ and $C_m = \frac{1}{m+1}\binom{2m}{m}$ is the $m-$th Catalan number.

Theorem 2 *The number of standard tableaux of truncated straight shape $\underbrace{(n, n, \dots, n)}_{m} \setminus \delta_k$ (assume $n \le m$), is*

$$\left(mn - \binom{k+1}{2}\right)! \times \frac{f_{(n-k-1)^m}}{(m(n-k-1))!} \times \frac{g_{(m,m-1,\dots,m-k)}}{((k+1)m - \binom{k+1}{2})!} \frac{E_1(k+1, m, n-k-1)}{E_1(k+1, m, 0)}, \tag{1}$$

where $E_1(r,p,s) = \begin{cases} \prod_{r<l<2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2 \le l \le r} \frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2 \rfloor}}, & r \text{ even}, \\ \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1, p, s), & r \text{ odd}. \end{cases}$

Theorem 3 *The number of standard truncated tableaux of shape $n^m \setminus (k^{k-1}, k-1)$ is*

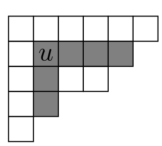
$$\frac{(nm - k^2 + 1)! f_{(m^{n-k})}}{(m(n-k))!} \frac{f_{((m-k)^k)}}{((m-k)k)!} \frac{(kn - k^2 - k + 1)!(k(m-k))!}{(mk + kn - 2k^2 - k + 2)!}.$$

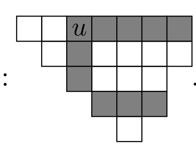
We will also exhibit connections with boxed plane partitions, as the generating function we use is the same as the volume generating function for boxed plane partitions. Computer evidence suggests that most truncated shapes do not have product-type formulas.

In [AKR] Adin, King and Roichman have independently found a formula for the case of shifted staircase truncated by one box and rectangle truncated by a square minus a box by methods different from the methods developed here.

2 Definitions

We will refer the reader to [Sta99] and [Mac95] for the basic facts and definitions regarding Young tableaux and symmetric functions, which we will use in this paper. Recall the hook-length formulas for the number f_λ of straight standard Young tableaux (SYT) of shape λ and g_λ for shifted tableaux:

$f_\lambda = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u}$, hook h_u: 

; $g_\lambda = \frac{|\lambda|!}{\prod_u h_u}$ hook h_u: 

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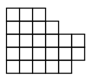
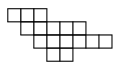
We are now going to define our main objects of study. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ and $\mu = (\mu_1, \mu_2, \ldots)$ be integer partitions, such that $\lambda_i \geq \mu_i$. A **straight diagram of truncated shape** $\lambda \setminus \mu$ is a left justified array of boxes, such that row i has $\lambda_i - \mu_i$ boxes. If λ has no equal parts we can define a **shifted diagram of truncated shape** $\lambda \setminus \mu$ as an array of boxes, where row i starts one box to the right of the previous row $i-1$ and has $\lambda_i - \mu_i$ number of boxes.

We define standard tableaux and plane partitions of truncated shape the usual way except this time they are fillings of truncated diagrams. A **standard truncated tableaux** of shape $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with the integers from 1 to $|\lambda| - |\mu|$, such that the entries across rows and down columns are increasing and each number appears exactly once. A **plane partition of truncated shape** $\lambda \setminus \mu$ is a filling of the corresponding truncated diagram with integers such that they weakly decrease along rows and down columns. For example,

$D_1 =$ , $D_2 =$ , $T_1 = \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & 4 & & \\ \hline & 3 & 5 & 7 & 8 \\ \hline & & 6 & 9 & \\ \hline \end{array}$, $T_2 = \begin{array}{|c|c|c|c|c|} \hline 5 & 5 & 3 & & \\ \hline & 4 & 2 & 2 & 1 \\ \hline & & 2 & 1 & \\ \hline \end{array}$

D_1 is a diagram of straight truncated shape $(6,6,6,6,5) \setminus (3,2)$; D_2 — of shifted truncated shape $(8,7,6,2) \setminus (5,2)$; T_1 and T_2 are respectively a standard tableaux, and a plane partition (PP) of shifted truncated shape $(5,4,2) \setminus (2)$.

We will denote the staircase partition of size k by $\delta_k = (k, k-1, k-2, \ldots, 1)$. Denote by $T[i,j]$ the entry in the box with coordinate (i,j) in the diagram of T with i being the row number and j – the column. Define **the generating function** for plane partitions of any shape D as

$$F_D(q) = \sum_{T:\mathrm{sh}(T)=D} q^{\sum_{(i,j)\in D} T[i,j]}. \tag{2}$$

3 A bijection with skew SSYT

We will consider a map between truncated plane partitions and skew Semi-Standard Young Tableaux which will enable us to enumerate them using Schur functions.

As a basic setup for this map we first consider truncated shifted plane partitions of staircase shape $\delta_n \setminus \delta_k$. Let T be such a plane partition. Let $\lambda^j = (T[1,j], T[2,j], \ldots, T[n-j,j])$ - the sequence of numbers in the jth diagonal of T.

Let P be a reverse skew semi-standard Young tableaux (SSYT) of shape λ^1/λ^{n-k}, such that the entries filling the subshape λ^j/λ^{j+1} are equal to j, i.e. it corresponds to the sequence $\lambda^{n-k} \subset \lambda^{n-k-1} \subset \cdots \subset \lambda^1$. The fact that this is all well defined follows from the inequalities that the $T[i,j]$'s satisfy by virtue of T being a plane partition, since then λ^i/λ^{i+1} will be horizontal strip.

Define $\phi(T) = P$, ϕ is the bijection in question. Given a reverse skew tableaux P of shape $\lambda \setminus \mu$ and entries smaller than n we can obtain the inverse shifted truncated plane partition $T = \phi^{-1}(P)$ as $T[i,j] = \max(s|P[i,s] \geq j)$; if no such entry of P exists let $s = 0$.

For example we have $T = \begin{array}{|c|c|c|c|c|} \hline 8 & 7 & 6 & 5 & \\ \hline & 7 & 5 & 4 & 3 \\ \hline & & 5 & 3 & 2 \\ \hline & & & 3 & 1 \\ \hline & & & & 1 \\ \hline \end{array} \xrightarrow{\phi} P = \begin{array}{|c|c|c|c|c|c|c|c|} \hline & & & & & 3 & 2 & 1 \\ \hline & & & 3 & 2 & 1 & 1 & \\ \hline 3 & 3 & 2 & 1 & 1 & & & \\ \hline 2 & 1 & 1 & & & & & \\ \hline 1 & & & & & & & \\ \hline \end{array}$, since $\lambda^1 = (8,7,5,3,1)$, $\lambda^2 = (7,5,3,1)$, $\lambda^3 = (6,4,2)$ and $\lambda^4 = (5,3)$. Notice also that

$$\sum T[i,j] = \sum P[i,j] + |\lambda^{n-k}|(n-k). \tag{3}$$

The map ϕ can be extended to any truncated shape, then the image will be tuples of SSYTs with certain restrictions. For the purposes of this paper we will extend it to truncated plane partitions of shape $(n^m) \setminus \delta_k$ as follows. Let T be a plane partition of shape $n^m \setminus \delta_k$ and assume that $n \leq m$ (otherwise we can reflect about the main diagonal). Let $\lambda = (T[1,1], T[2,2], \ldots, T[n,n])$, $\mu = (T[1, n-k], T[2, n-k+1], \ldots, T[k+1, n])$ and let T_1 be the portion of T above and including the main diagonal, hence of shifted truncated shape $\delta_n \setminus \delta_k$, and T_2 the transpose of the lower portion including the main diagonal, a shifted PP of shape $(m, m-1, \ldots, m-n+1)$.

Extend ϕ to T as $\phi(T) = (\phi(T_1), \phi(T_2))$. Here $\phi(T_2)$ is a SSYT of at most n rows (shape λ) and filled with $[1, \ldots, m]$ the same way as in the truncated case. As an example with $n = 5, m = 6, k = 2$ we have

$$T = \begin{array}{|c|c|c|c|c|} \hline 5&5&4& \multicolumn{2}{c}{} \\ \cline{1-3} 5&5&4&4& \multicolumn{1}{c}{} \\ \cline{1-4} 4&4&3&3&2 \\ \hline 4&3&2&2&2 \\ \hline 3&2&2&1&1 \\ \hline 2&1&1&1&1 \\ \hline \end{array} \Rightarrow T_1 = \begin{array}{ccccc} 5&5&4&& \\ &5&4&4& \\ &&3&3&2 \\ &&&2&2 \\ &&&&1 \end{array},\ T_2 = \begin{array}{cccccc} 5&5&4&4&3&2 \\ &5&4&3&2&1 \\ &&3&2&2&1 \\ &&&2&1&1 \\ &&&&1&1 \end{array} \Rightarrow \quad \phi(T) = \left(\begin{array}{ccccc} &&&&2 \\ &&&&1 \\ &&2&& \\ 2&2&&& \\ 1&&&& \end{array} , \begin{array}{ccccc} 6&6&5&4&2 \\ 5&4&3&2&1 \\ 4&3&1&& \\ 3&1&&& \\ 2&&&& \end{array} \right)$$

Proposition 1 *The map ϕ is a bijection between shifted truncated plane partitions T of shape $\delta_n \setminus \delta_k$ filled with nonnegative integers and (reverse) skew semi-standard Young tableaux with entries in $[1, \ldots, n-k-1]$ of shape λ/μ with $l(\lambda) \leq n$ and $l(\mu) \leq k+1$. Moreover, $\sum_{i,j} T[i,j] = \sum_{i,j} P[i,j] + |\mu|(n-k)$. Similarly ϕ is also a bijection between truncated plane partitions T of shape $n^m \setminus \delta_k$ and pairs of SSYTs (P, Q), such that $\mathrm{sh}(P) = \lambda/\mu$, $\mathrm{sh}(Q) = \lambda$ with $l(\lambda) \leq n$, $l(\mu) \leq k+1$ and P is filled with $[1, \ldots, n-k-1]$, Q with $[1, \ldots, m]$. Moreover, $\sum T[i,j] = \sum P[i,j] + \sum Q[i,j] - |\lambda| + |\mu|(n-k)$.*

4 Schur function identities

We will now consider the relevant symmetric function interpretation arising from the map ϕ. Substitute the entries $1, \ldots$ in the skew SSYTs in the image with respective variables $x_1, \ldots$ and $z_1, \ldots$. The idea is to evaluate the resulting expressions at certain finite specializations for x and z(e.g. $x = (q, q^2, \ldots, q^{n-k-1})$ and $z = (1, q, q^2, \ldots, q^{m-1})$) to obtain generating functions for the sum of entries in the truncated plane partitions which will later allow us to derive enumerative results.

For the case of shifted truncated shape $\delta_n \setminus \delta_k$ we have the corresponding sum

$$S_{n,k}(x;t) = \sum_{\lambda, \mu | l(\lambda) \leq n, l(\mu) \leq k+1} s_{\lambda/\mu}(x_1, \ldots, x_{n-k-1}) t^{|\mu|}, \tag{4}$$

and for the straight truncated shape $n^m \setminus \delta_k$

$$D_{n,m,k}(x;z;t) = \sum_{\lambda, \mu | l(\lambda) \leq n, l(\mu) \leq k+1} s_\lambda(z) s_{\lambda/\mu}(x) t^{|\mu|}. \tag{5}$$

We need to find formulas when $x_i = 0$ for $i > n-k-1$ and $z_i = 0$ for $i > m$. Keeping the restriction $l(\mu) \leq k+1$ we have that $s_{\lambda/\mu}(x) = 0$ if $l(\lambda) > n$ and this allows us to drop the length restriction on λ in both sums.

From now on the different sums will be treated separately. Consider another set of variables $y = (y_1, \ldots, y_{k+1})$ which together with $(x_1, \ldots, x_{n-k-1})$ form a set of n variables. Using the determinantal formula for the Schur functions, namely that

$$s_\nu(u_1, \ldots, u_p) = \frac{a_{\nu+\delta_p}(u)}{a_{\delta_p}(u)} = \frac{\det[u_i^{\nu_j+p-j}]_{i,j=1}^p}{\det[u_i^{p-j}]_{i,j=1}^p}, \tag{6}$$

in Cauchy's identity for the sum of Schur functions we obtain

$$\begin{aligned}&\sum_{\lambda,\mu} s_{\lambda/\mu}(x_1, \ldots, x_{n-k-1}) a_{\mu+\delta_{k+1}}(y_1, \ldots, y_{k+1}) t^{|\mu|} \\ &\quad = \prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \prod_{i<j\le k+1} \frac{y_i-y_j}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i}\end{aligned} \tag{7}$$

We extract the sum $S_{n,k}(x;t) = \sum_{\lambda,\mu,l(\mu)\le k+1} s_{\lambda/\mu}(x) t^{|\mu|}$ as the coefficient of y^0 in the following sum

$$[y^0] \left(\sum_{\lambda,\mu} s_{\lambda/\mu}(x) a_{\mu+\delta_{k+1}}(y_1, \ldots, y_{k+1}) t^{|\mu|} A(y^{-1}) \right) = (k+1)! S_{n,k}(x;t), \tag{8}$$

where $A(u) = \sum_\nu s_\nu(u) a_{\delta_p}(u) = \prod \frac{1}{1-u_i} \prod_{i<j} \frac{(u_i-u_j)}{1-u_iu_j}$. Since for any doubly infinite series $f(y)$, we have $[y^0]f(y) = \frac{1}{2\pi i}\int_C f(y)y^{-1}dy$, we obtain the following.

Proposition 2 *We have that*

$$\begin{aligned}S_{n,k}(x;t) =& \frac{(-1)^{\binom{k+1}{2}}}{(k+1)!} \prod \frac{1}{1-x_i} \prod_{i<j\le n-k-1} \frac{1}{1-x_ix_j} \\ & \frac{1}{(2\pi i)^{k+1}} \int_T \prod_{i<j\le k+1} \frac{(y_i-y_j)^2}{1-t^2y_iy_j} \prod_{i,j} \frac{1}{1-x_ity_j} \prod \frac{1}{1-ty_i} \prod \frac{1}{y_i-1} \prod_{i<j} \frac{1}{y_iy_j-1} dy_1 \cdots dy_{k+1},\end{aligned}$$

where $T = C_1 \times C_2 \times \cdots C_p$ and $C_i = \{z \in \mathbb{C} | |z| = 1+\epsilon_i\}$ for $\epsilon_i < |t^{-1}| - 1$.

In the case of **straight shapes** we use Cauchy's identity for the sum of products of Schur functions to obtain

$$\sum_\lambda s_\lambda(z) s_{\lambda/\mu}(x) = \prod \frac{1}{1-z_ix_j} \sum_\mu s_\mu(z). \tag{9}$$

Since the length restriction on λ becomes redundant when $x = (x_1, \ldots, x_{n-k-1})$ and $l(\mu) \le k+1$ we have that

Proposition 3 $D_{n,m,k}(x,z;t) = \prod_{i,j} \frac{1}{1-x_iz_j} \left(\sum_{\nu | l(\nu)\le k+1} s_\nu(zt) \right).$

For the purpose of enumeration of SYTs we will use this formula as it is. Even though there are formulas, e.g. of Gessel and King, for the sum of Schur functions of restricted length in the form of determinants or infinite sums, they would not give the enumerative answer any more easily.

5 A polytope volume as a limit

Plane partitions of specific shape (truncated or not) of size N can be viewed as integer points in a cone in $\mathbb{R}^N$. Let D be the diagram of a plane partition T with $N = |D|$, the coordinates of $\mathbb{R}^N$ are indexed by the boxes present in T. Then

$$C_D = \{(\cdots, x_{i,j}, \cdots) \in \mathbb{R}^N_{\geq 0} : \ [i,j] \in D, \ x_{i,j} \leq x_{i,j+1} \text{ if } [i,j{+}1] \in D, x_{i,j} \leq x_{i+1,j} \text{ if } [i{+}1,j] \in D\}$$

is the corresponding cone. Let $P(C)$ be the section of a cone C in $\mathbb{R}^N_{\geq 0}$ with the hyperplane $H = \{x | \sum_{[i,j]\in D} x_{i,j} = 1\}$. It is easy to see that

$$Vol_{N-1}(P_D) = \frac{\#T : SYT, \mathrm{sh}(T) = D}{N!} Vol(\Delta_N). \tag{10}$$

The following lemma, see [Pan10] for proof, helps determine the volume and thus the number of standard tableaux of shape D.

Lemma 1 *Let P be a $(d-1)$-dimensional rational polytope in $\mathbb{R}^d_{\geq 0}$, such that its points satisfy $a_1 + \cdots + a_d = 1$ for $(a_1, \ldots, a_d) \in P$, and let*

$$F_P(q) = \sum_n \sum_{(a_1,\ldots,a_d)\in nP\cap\mathbb{Z}^d} q^{a_1+a_2+\cdots+a_d}.$$

We have that the $(d-1)$-dimensional volume of P is

$$Vol_{d-1}(P) = (\lim_{q\to 1}(1-q)^d F_P(q)) Vol(\Delta_d),$$

where Δ_d is the $(d-1)$-dimensional simplex.

Notice that if $P = P(C_D)$ for some shape D, then

$$F_P(q) = \sum_n \sum_{a\in nP\cap\mathbb{Z}^N} q^n = \sum_{a\in C_D\cap\mathbb{Z}^N} q^{|a|} = \sum_{T:PP,\mathrm{sh}(T)=D} q^{\sum T[i,j]} = F_D(q).$$

Using (10) and this Lemma we get the key fact to enumerating standard tableaux of truncated shapes using evaluations of symmetric functions.

Proposition 4 *The number of standard tableaux of shape D is equal to*

$$N! \lim_{q\to 1}(1-q)^N F_D(q).$$

6 Shifted staircase truncated by a box: proof of Theorem 1

We are now going to use Propositions 2 and 4 to find the number of standard shifted tableaux of truncated shape $\delta_n \setminus \delta_1$. Numerical results show that a product formula for the general case of truncation by δ_k does not exist.

First we will evaluate the integral in Proposition 2 by iteration of the Residue theorem at the possible poles, first with respect to y_1, then y_2. We obtain

$$S_{n,1}(x;t) = \frac{1}{2} \prod \frac{1}{1-x_i} \prod_{1\leq i<j\leq n-k-1} \frac{1}{1-x_i x_j} \sum_{i\geq 0} \frac{1}{1-t^2}(1+u_i)^2 u_i^{n-3} \prod_{j\neq i} \frac{1}{u_i - u_j} \prod_{j\geq 0} \frac{1}{1-u_i u_j},$$

where $u_i = tx_i$ with $x_0 = 1$. Using the partial fraction version of the determinantal formula (6) for $s_{s-p+1}(u)$ we can simplify the sum above as

$$\sum_{i=0}^{n-k-1} \frac{u_i^s}{\prod(u_i-u_j)} \prod \frac{1}{1-u_iu_j} = \sum_{p\geq 0} h_{s-n+k+1+p}(u)h_p(u) = c_{s-n+k+1}(u), \tag{11}$$

where $c_i = \sum_{n\geq 0} h_n h_{n+i}$. We then have the new formulas

$$S_{n,1}(x;t) = \prod \frac{1}{1-x_i} \prod_{1\leq i<j\leq n-k-1} \frac{1}{1-x_ix_j} \frac{1}{1-t^2}(c_1(u)+c_0(u)), \tag{12}$$

Proof of Theorem 1: We will use Proposition 4 and the formula (12). For the shape $D = \delta_n \setminus \delta_1$ we have,

$$F_D(q) = \sum_{T\,|\,\mathrm{sh}(T)=D} q^{\sum T[i,j]} = \sum_{P=\phi(T)} q^{(n-1)|\mu|+\sum P[i,j]} = S_{n,1}(q,q^2,\ldots,q^{n-2};q^{n-1}).$$

In order to find $\lim_{q\to 1}(1-q)^{\binom{n+1}{2}-1}F_D(q)$, using the formula (12), we need to find $\lim_{q\to 1}(1-q)^{2n-3}c_s(u)$ for $u = (q^{n-1},\ldots,q^{2n-3})$. Let $c_s(x;y) = \sum_l h_l(x)h_{l+s}(y)$ where $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_m)$. We have that

$$\begin{aligned} c_s(x;y) &= \sum_l \sum_{i_1\leq\cdots\leq i_l; j_1\leq\cdots\leq j_{l+s}} x_{i_1}\cdots x_{i_l}y_{j_1}\cdots y_{j_{s+l}} \\ &= \sum_p h_s(y_1,\ldots,y_p) \sum_{P:(1,p)\to(n,m)} (-1)^{m+n-p-\#P} \sum_l h_l((xy)_P), \end{aligned}$$

where the sum runs over all fully ordered collections of lattice points P in between $(1,p)$ to (n,m) and $(xy)_P = (x_{i_1}y_{j_1},\ldots)$ for $(i_1,j_1),\ldots \in P$ and the (-1)s indicate the underlying inclusion-exclusion process. We also have that

$$\sum_l h_l((xy)_P) = \frac{1}{\prod_{(i,j)\in P}(1-x_iy_j)}.$$

The degree of $1-q$ dividing the denominators after substituting $(x_i,y_j) = (u_i,u_j) = (q^{n-2+i},q^{n-2+j})$ for the evaluation of $c_s(u)$ is equal to the number of points in P. $\#P$ is maximal when the lattice path is from $(1,1)$ to $(n-1,n-1)$ and is saturated, so $\max(\#P) = 2(n-1)-1 = 2n-3$. The other summands will contribute 0 when multiplied by the larger power of $(1-q)$ and the limit is taken. For each maximal path we have $\{i+j|(i,j)\in P\} = \{2,\ldots,2n-2\}$ and the number of these paths is $\binom{2n-4}{n-2}$, so

$$(1-q)^{2n-3}c_s(q^{n-1},\ldots,q^{2n-3}) = \binom{2n-4}{n-2}\prod_{i=2}^{2n-2}\frac{1-q}{1-q^{2(n-2)+i}} + (1-q)...,$$

where the remaining terms are divisible by $1-q$, hence contribute 0 when the limit is taken.

Now we can proceed to compute $\lim_{q\to 1}(1-q)^N S_{n,1}(q^1, q^2, \ldots, ; q^{n-1})$. Putting all these together we have that

$$\lim_{q\to 1}(1-q)^{\binom{n+1}{2}-1} S_{n,1}(q^1, \ldots, q^{n-2}; q^{n-1})$$
$$= \prod_{0\le i<j\le n-2} \frac{1}{i+j} \frac{1}{2(n-1)} 2\binom{2n-4}{n-2} \prod_{i=2}^{2n-2} \frac{1}{2n-4+i} = \frac{g_{\delta_{n-2}}}{\binom{n-1}{2}!} \frac{1}{(n-1)} \binom{2n-4}{n-2} \frac{(2n-3)!}{(4n-6)!},$$

where $g_{n-2} = \frac{\binom{n-1}{2}!}{\prod_{0\le i<j\le n-2}(i+j)}$ is the number of shifted staircase tableaux of shape $(n-2, \ldots, 1)$. After algebraic manipulations we arrive at the desired formula. $\square$

7 Rectangle truncated by staircase: proof of Theorem 2

We will compute the number of standard tableaux of straight truncated shape $D = n^m \setminus \delta_k$. By Propositions 1 and the definition of $D_{n,m,k}(x; z; t)$ in equation (5) we have that

$$F_{n^m\setminus\delta_k}(q) = D_{n,m,k}(q, q^2, \ldots, q^{n-k-1}; 1, q, \ldots, q^{m-1}; q^{n-k}).$$

Using the simplified almost-product type formula for $D_{n,m,k}$ from Proposition 3, the number of standard tableaux of shape $n^m \setminus \delta_k$ by Proposition 4 is

$$\lim_{q\to 1}(1-q)^{nm-\binom{k+1}{2}} F_D(q) \tag{13}$$
$$= \lim_{q\to 1}\left(\prod_{i=1,j=0}^{n-k-1,m-1} \frac{1-q}{1-q^{i+j}} (1-q)^{m(k+1)-\binom{k+1}{2}} \Big(\sum_{\nu | l(\nu)\le k+1} s_\nu(q^{n-k}, q^{n-k+1}, \ldots, q^{m-1+n-k}) \Big) \right).$$

We are thus going to compute the last factor.

Lemma 2 *Let $p \ge r$ and $N = rp - \binom{r}{2}$. Then for any s we have*

$$\lim_{q\to 1}(1-q)^N \sum_{\lambda | l(\lambda)\le r} s_\lambda(q^{1+s}, \ldots, q^{p+s}) = \frac{g_{(p,p-1,\ldots,p-r+1)}}{N!} \frac{E_1(r,p,s)}{E_1(r,p,0)},$$

where

$$E_1(r,p,s) = \prod_{r<l<2p-r+2} \frac{1}{(l+2s)^{r/2}} \prod_{2\le l\le r} \frac{1}{((l+2s)(2p-l+2+2s))^{\lfloor l/2 \rfloor}}$$

for r even and $E_1(r,p,s) = \frac{((r-1)/2+s)!}{(p-(r-1)/2+s)!} E_1(r-1,p,s)$ when r is odd and g_λ is the number of shifted SYTs of shape λ.

Proof: Consider the Robinson-Schensted-Knuth (RSK) correspondence between SSYTs with no more than r rows filled with $x_1, \ldots, x_p$ and symmetric $p \times p$ integer matrices A. The limit on the number of rows translates through Schensted's theorem to the fact that there are no $m+1$ nonzero entries in A with coordinates $(i_1, j_1), \ldots, (i_{r+1}, j_{r+1})$, s.t. $i_1 < \cdots < i_{r+1}$ and $j_1 > \cdots > j_{r+1}$ (i.e. a decreasing subsequence of length $r+1$ in the generalized permutation corresponding to A). Let $\mathcal{A}$ be the set of such matrices. Let $\mathcal{A}, \subset \mathcal{A}$ be the set of $0-1$ matrices satisfying this condition, we will refer to them as allowed configurations. Notice that $A \in \mathcal{A}$ if and only if $B \in \mathcal{B}$, where $B[i,j] = \begin{cases} 1, \text{ if } A[i,j] \neq 0 \\ 0, \text{ if } A[i,j] = 0 \end{cases}$.

We thus have that

$$\begin{aligned}\sum_{\lambda | l(\lambda) \leq r} s_\lambda(x_1, \ldots, x_p) &= \sum_{A \in \mathcal{A}} \prod_i x_i^{A[i,i]} \prod_{i>j} (x_i x_j)^{A[i,j]} \\ &= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \Big(\sum_{a_{i,i}=1}^{\infty} x_i^{a_{i,i}} \Big) \prod_{i>j:B[i,j]=1} \Big(\sum_{a_{i,j}=1}^{\infty} (x_i x_j)^{a_{i,j}} \Big) \\ &= \sum_{B \in \mathcal{B}} \prod_{i:B[i,i]=1} \frac{x_i}{1-x_i} \prod_{i>j:B[i,j]=1} \frac{x_i x_j}{1 - x_i x_j}.\end{aligned}$$

Notice that B cannot have more than N nonzero entries on or above the main diagonal. No diagonal $i+j=l$ (i.e. the antidiagonals) can have more than r nonzero entries on it because of the longest decreasing subsequence condition. Also if $l \leq r$ or $l > 2p-r+1$, the total number of points on such diagonal are $l-1$ and $2p-l+1$ respectively. Since B is also symmetric the antidiagonals $i+j=l$ will have $r-1$ entries if $l \equiv r-1 (\mathrm{mod} 2)$ and r is odd. Counting the nonzero entries on each antidiagonal on or above the main diagonal gives always exactly N in each case for the parity of r and p.

If B has less than N nonzero entries, then

$$\lim_{q \to 1} (1-q)^N \prod_{i:B[i,i]=1} \frac{q^{i+1}}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}}{1-q^{i+j+2s}} =$$

$$\lim_{q \to 1} (1-q)^{N-|B|>0} \prod_{i:B[i,i]=1} \frac{q^{i+1}(1-q)}{1-q^{i+s}} \prod_{i>j:B[i,j]=1} \frac{q^{i+j+2s}(1-q)}{1-q^{i+j+2s}} = 0,$$

so such Bs won't contribute to the final answer.

Consider now only Bs with maximal possible number of nonzero entries (i.e. N), which forces them to have exactly r (or $r-1$) nonzero entries on every diagonal $i+j=l$ for $r < l \leq 2p-r+1$ and all entries in $i+j \leq r$ and $i+j > 2p-r+1$.

If r is even, then there are no entries on the main diagonal when $r < l < 2p-r+2$ and so there are $r/2$ terms on each diagonal $i+j=l$. Thus every such B contributes the same factor when evaluated at $x = (q^{1+s}, \ldots)$:

$$E_q(r,p,s) := \prod_{r<l<2p-r+2} \frac{q^{(l+2s)r/2}}{(1-q^{l+2s})^{r/2}} \prod_{2 \leq l \leq r} \frac{q^{(l+4s+2p-l+2)\lfloor l/2 \rfloor}}{((1-q^{l+2s})(1-q^{2p-l+2+2s}))^{\lfloor l/2 \rfloor}}.$$

If r is odd, then the entries on the main diagonal will all be present with the rest being as in the even case with $r-1$, so the contribution is

$$E_q(r,p,s) := \prod_{\frac{r+1}{2} \leq i \leq p - \frac{r+1}{2}+1} \frac{q^{i+s}}{1-q^{i+s}} E_q(r-1,p,s).$$

Let now M be the number of such maximal Bs in $\mathcal{A}_0$. The final answer after taking the limit is $ME_1(r,p,s)$, where we have that $E_1(r,p,s) = \lim_{q\to 1}(1-q)^N E_q(r,p,s)$ as defined in the statement of the lemma.

In order to find M observe that the case of $s=0$ gives

$$\lim_{q\to 1}(1-q)^N \sum_{\lambda | l(\lambda)\leq r} s_\lambda(q^1,\ldots,q^p) = ME_1(r,p,0), \tag{14}$$

on one hand. On the other hand via the bijection ϕ we have that

$$\sum_{\lambda | l(\lambda)\leq m} s_\lambda(q^1,\ldots,q^n) = \sum_T q^{\sum T[i,j]},$$

where the sum on the right goes over all shifted plane partitions T of shape $(p, p-1, \ldots, p-r+1)$. Multiplying by $(1-q)^N$ and taking the limit on the right hand side gives us, by the inverse of Proposition 4, $\frac{1}{N!}$ times the number of standard shifted tableaux of that shape. This number is well known and is $g_{(p,p-1,\ldots,p-r+1)} = \frac{N!}{\prod_u h_u}$, where the product runs over the hooklengths of all boxes on or above the main diagonal of (p^r, r^{p-r}). Putting all this together gives

$$ME_1(r,p,0) = \frac{g_{(p,p-1,\ldots,p-r+1)}}{N!}.$$

Solving for M we obtain the final answer:

$$\frac{g_{(p,p-1,\ldots,p-r+1)}}{N!}\frac{E_1(r,p,s)}{E_1(r,p,0)}. \qquad \square$$

Proof of Theorem 2: Take the limit in equation (13), by using Lemma 2 with $r = k+1$, $s = n-k-1$ and $p = m-2$. $\square$

8 Truncation by almost squares: proof of Theorem 3

We will apply our methods to count standard tableaux, whose shape is a rectangle truncated by a square without a corner. Specifically, let $D = n^m \setminus (k^{k-1}, k-1)$ be such a truncated rectangle. As in the previous case we can assume that $n < m$. Let $2k \leq n+1$. For any plane partition T of shape D, let $p = T[k, n-k+1]$ be the value of the entry in that missing square corner. Because of the row and column inequalities, T is in bijection with T' of straight truncated shape $n^m \setminus \delta_{k-1}$, where the $T'[i,j] = T[i,j]$ for $(i,j)\in D$ and $T'[i,j] = T[k, n-k+1] = p$ for the values in the extra boxes. For example,

$$T = \begin{array}{|c|c|c|c|c|c|c|}\hline 6&6&5&4\\ \cline{1-4} 6&5&4&3\\ \cline{1-5} 5&3&3&2&2\\ \hline 4&3&2&2&2&1&1\\ \hline 3&2&2&1&1&1&1\\ \hline 2&1&1&1&1&1&1\\ \hline\end{array} \quad\longleftrightarrow\quad T' = \begin{array}{|c|c|c|c|c|c|c|}\hline 6&6&5&4&2\\ \cline{1-6} 6&5&4&3&2&2\\ \hline 5&3&3&2&2&2&2\\ \hline 4&3&2&2&2&1&1\\ \hline 3&2&2&1&1&1&1\\ \hline 2&1&1&1&1&1&1\\ \hline\end{array}.$$

Then the generating function $F_{n^m\setminus(k^{k-1},k-1)}(q)$ is obtained, using equation 9, from

$$H_{n,m,k}(z;x;t) = \sum_p \sum_\lambda s_\lambda(z) s_{\lambda/(p^k)}(x) t^p = \prod \frac{1}{1-z_i x_j} \sum_p s_{(p^k)}(z) t^p, \tag{15}$$

by substituting $x = (q, q^2, \ldots, q^{n-k})$, $z = (1, q, \ldots, q^{m-1})$ as in the case of the rectangle truncated by a staircase $n^m \setminus \delta_{k-1}$. For the value of t in this case, since $\sum T[i,j] = \sum T'[i,j] - p(\binom{k+1}{2} - 1)$ to account for the extra squares in T', we have $t^p = q^{|\mu|(n-k)-p(\binom{k+1}{2}-1)} = (q^{k(n-k)-\binom{k+1}{2}+1})^p$. The number of standard truncated tableaux of this shape will then be given by

$$\lim_{q\to 1}(1-q)^{nm-k^2+1} H_{n,m,k}(1, q, \ldots, q^{m-1}; q, q^2, \ldots, q^{n-k}; q^{k(n-k)-\binom{k+1}{2}+1}). \tag{16}$$

We will now evaluate the sum of $s_{(p^k)}$ over p and the relevant limit.

Lemma 3 *Let* $f_q(v) = v^{\binom{k}{2}} \prod_{i\le k<j}(vq^{m-i} - q^{m-j}) = \sum a_i(q) v^i$, *then*

$$\sum_p s_{(p^k)}(1, \ldots, q^{m-1}) t^p = \prod_{m-k\le i<m; 0\le j<m-k} \frac{1}{(q^i - q^j)} \sum_i a_i(q) \frac{1}{1-q^i t}. \tag{17}$$

If $t = q^s$, *then*

$$\lim_{q\to 1}(1-q)^{mk-k^2+1} \sum_p s_{(p^k)}(1, \ldots, q^{m-1})(q^s)^p = \prod_{i=m-k;j=0}^{m-1;m-k-1} \frac{1}{(j-i)} \frac{(\binom{k}{2}+s)!(k(m-k))!}{(mk-\binom{k+1}{2}+s+1)!}$$

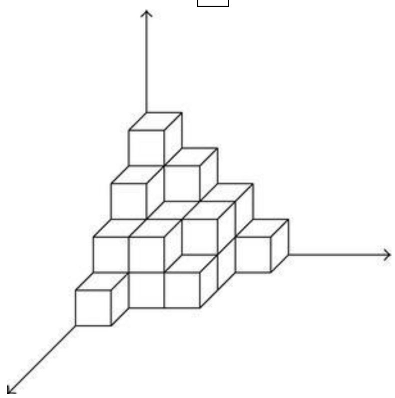
Proof outline: We apply the determinantal formula for the Schur functions (6) to $s_{(p)}(1, \ldots, q^{m-1})$, which in this case reduces to a $m \times m$ Vandermonde determinant. and hence a polynomial in $v = q^p$. Summing over p gives the first part of the lemma. Taking the limit of the resulting expression reduces the computation to certain integral of $f(v)$. □

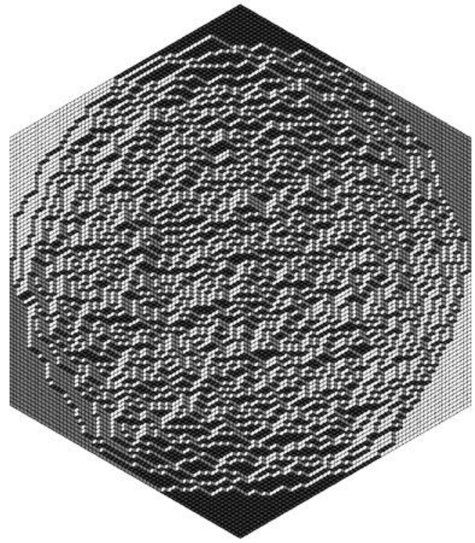
Proof of Theorem 3: For the number of standard tableaux of shape D take $\lim_{q\to 1}(1-q)^{nm-k^2+1} F_D(q)$, using the formula for $F_D(q)$ from (16), formula (15), and applying the second part of the Lemma with $s = k(n-k) - \binom{k+1}{2} + 1$. We can simplify the products by viewing them as products of hook-lengths and invoking the hook-length formula for the respective shape. □

9 Boxed plane partitions

The boxed plane partition corresponding to a plane partition T of shape D is a 3D diagram, where the base in the xy plane is D and on top of each square $[i,j] \in D$ there are $T[i,j]$ unit cubes. Its volume is the number of cubes, i.e. $\sum T[i,j]$, so our generating function $F_D(q)$ is the same as the volume generating function for boxed plane partitions of shape D. If $D = n^n$ and $T[1,1] = n$, as $n \to \infty$, such boxed plane partitions approximate stepped surfaces, as the picture on the right shows.

$$T = \begin{array}{|c|c|c|c|}\hline 4 & 3 & 2 & 1 \\ \hline 3 & 2 & 2 & \\ \cline{1-3} 2 & 2 & 1 & \\ \cline{1-3} 1 & & & \\ \cline{1-1}\end{array} \quad \leftrightarrow$$





Using the methods developed so far, in particular equation (9), we can easily derive volume generating functions for boxed plane partitions with certain restrictions.

Proposition 5 *The average value under the volume statistic of boxed plane partitions of base* (n^n) *and whose corner is at a fixed value* m*, i.e.* $T[1,n] = m$*, is given by*

$$\frac{\sum_{T:\mathrm{sh}(T)=n^n;T_{1n}=m} q^{\sum T_{ij}}}{\sum_{T:\mathrm{sh}(T)=n^n} q^{\sum T_{ij}}} = \prod_{i=1}^{n}(1-q^{n-1+i}) \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q .$$

More generally, the average value under the volume statistic of boxed partitions on (n^n), whose $n-k$'th diagonal is fixed at a certain value μ, i.e. $(T[1,n-k+1], T[2,n-k+2],\ldots) = \mu$, is

$$\frac{\sum_{T:\mathrm{sh}(T)=n^n;(T[1,n-k+1],\ldots,T[k,n])=\mu} q^{\sum T[i,j]}}{\sum_{\mathrm{sh}(T)=n^n} q^{\sum T[i,j]}} = \prod_{0<i<n+1, n-k<j<n+1} \frac{1}{1-q^{i+j-1}} \times$$
$$\times q^{(n-k+1)|\mu|} \frac{\prod_{0<i<j<n+1}(q^{\mu_i+n-i}-q^{\mu_j+n-j})}{\prod_{0<i<j<n+1}(q^j-q^i)} \frac{\prod_{0<i<j<k+1}(q^{\mu_i+k-i}-q^{\mu_j+k-j})}{\prod_{0\leq i<j\leq k}(q^j-q^i)}.$$

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Adjacent transformations in permutations

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Abstract. We continue a study of the equivalence class induced on S_n when one is permitted to replace a consecutive set of elements in a permutation with the same elements in a different order. For each possible set of allowed replacements, we characterise and/or enumerate the set of permutations reachable from the identity. In some cases we also count the number of equivalence classes.

Résumé. Nous étudions dans cet article les classes d'équivalence sur les permutations obtenues en remplaçant un ensemble consécutif de valeurs par ces même valeurs mais dans un ordre différent. Nous étudions l'ensemble des remplacements possibles de longueur 3 et pour chacun d'entre eux caractérisons et énumérons les permutations de la classe de l'identité. Pour certains ensembles, nous calculons de même le nombre de classes d'équivalence.

Keywords: permutation patterns, equivalence classes, integer sequences, Catalan numbers, sorting permutations

1 Introduction

In [LPRW10], the authors consider the equivalence class induced on S_n when one is permitted to replace a consecutive set of elements in a permutation with the same elements in a different order, where the sequence removed and the sequence replacing it must each *match as a pattern* any sequence in a given *replacement set*, which is a subset of some S_k. Here *match as a pattern* means to have all the same order relations. For instance, if the given replacement set is $\{123, 132\}$, then 12453 is equivalent to 14235, because we can perform the replacement $\mathbf{124}53 \rightarrow 14253$ followed by $14\mathbf{253} \rightarrow 14235$.

We write $\#\mathrm{Classes}(n; \{A\})$ to denote the number of equivalence classes into which S_n is divided by the replacement set A. We use $\mathcal{C}_n(A)$ to denote the equivalence class of the identity, and $c_n(A)$ its size.

In [LPRW10], the authors restricted their attention to replacement sets selected from S_3, in which each replacement can be viewed as fixing one of the three elements and exchanging the other two. We look at sets of patterns, such as $\{123, 231\}$, which do not meet this condition, and were therefore excluded from their paper. We focus on the characterization and enumeration of the class containing the identity permutation, and indeed with our new results we now know the enumeration of this class in all cases.

One of the earliest references for transformations in permutations is in Knuth's *Art of Computer Programming* [Knu73] where we find the idea of sorting a permutation by passage through a stack, illustrated by a chain of railway wagons which can be interchanged by the instrument of a spur of track.

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Because in the present case we are almost always concerned with the question of whether or not a given permutation can be restored to the identity, it is perhaps not too fanciful to imagine a similar context, where we are examining whether a permutation can be sorted, with some local (mechanical or logical) circuits but under global control.

We have divided our paper into sections based on the number of patterns contained in the replacement set, and in an appendix we give a summary table of $c_n(A)$ for all replacement sets A, both those enumerated in the present work, and those known previously.

Two operations familiar from the literature of permutation patterns, *reversal* and *complementation*, have a natural role in this context. It is clear that two permutations are equivalent with respect to a given replacement set if and only if their reversals (complements) are equivalent with respect to the reversal (complement) of the replacement set. Thus a result cited in the summary table may actually be the reversal (R), or reverse complement (RC), of another entry. In particular, since 123 is its own reverse complement, one can always try applying reverse-complementation to one class containing 123 to obtain another one.

2 Replacement sets of size 2

2.1 $123 \leftrightarrow 231$

Definition 2.1 *Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation, σ satisfies the* position parity *condition if for all $i \in \{1 \dots n\}$, then, taking the set $\{\sigma_1 \dots \sigma_i\}$ in increasing order, the rank of σ_i is congruent to i (mod 2).*

Example 2.1 *The permutation $\sigma = 45123$ satisfies position parity, because $\sigma_1 = 4$ is the smallest element of $\{4\}$, $\sigma_2 = 5$ is the 2nd-smallest element of $\{4, 5\}$, $\sigma_3 = 1$ is the smallest element of $\{4, 5, 1\}$, $\sigma_4 = 2$ is the 2nd-smallest element of $\{4, 5, 1, 2\}$ and $\sigma_5 = 3$ is the 3rd-smallest element of $\{4, 5, 1, 2, 3\}$. The permutation 4612375 also satisfies position parity, but 21 does not.*

Proposition 2.1 *Let $\sigma \in S_n$. Then $\sigma \in \mathcal{C}_n(123, 231)$ if and only if it satisfies position parity. Moreover it is possible to step from the identity permutation to σ using only steps of form $123 \to 231$.*

Proof: Let $\sigma = \sigma_1 \dots \sigma_n$ be a permutation satisfying position parity. To obtain σ from the identity, begin by placing the rightmost element, σ_n. By position parity, σ_n must have the same parity as n, and since it occupies position σ_n in the identity, it needs to move an even number of positions, $n - \sigma_n$. We move it forward two steps at a time, using only moves $123 \to 231$ (where σ_n plays the role of 1), which we can do because the identity permutation has all its elements in increasing order. We call the permutation which results $\sigma^{(n)}$; the element in position n is the correctly-placed σ_n, and the elements in the other $n - 1$ positions are in increasing order.

Now do the same thing to place σ_{n-1} in position $n - 1$, obtaining a permutation $\sigma^{(n-1)}$, and in general at step k a permutation $\sigma^{(n-k+1)}$, in which the final k elements are correctly placed, and the first $n - k$ elements are in increasing order, from which it follows that the next element to be placed, σ_{n-k}, occupies the position corresponding to its rank in the set $\{\sigma_1 \dots \sigma_{n-k}\}$ written in increasing order. So it has to move an even number of positions and we place it with moves $123 \to 231$. At the nth step, we have obtained the permutation $\sigma = \sigma^{(0)}$.

Conversely, let $\sigma \in \mathcal{C}_n(123, 231)$. A case study proves that position parity is kept under the transformations $123 \leftrightarrow 231$. As σ can be reached from the identity which has position parity this concludes the proof. □

Remark 2.1 *Let $\sigma \in \mathcal{C}_n(123, 231)$, then the proof gives us a canonical path to obtain σ from the identity, using only steps of form $123 \rightarrow 231$: we start by placing σ_n, then σ_{n-1}, etc.*

Corollary 2.1 *The number of permutations in the equivalence class of the identity on n elements is $c_n(123, 231) = \lceil \frac{1}{2} \rceil \cdot \lceil \frac{2}{2} \rceil \dots \lceil \frac{n}{2} \rceil = 1^2 \cdot 2^2 \dots \lceil \frac{n}{2} \rceil$ (n terms).*

Proof: We will place elements from right to left. Let $\sigma \in \mathcal{C}_n(123, 231)$, then the rank of σ_n in $\{\sigma_1 \dots \sigma_n\}$ is of the same parity as n, which gives $\lceil \frac{n}{2} \rceil$ choices for σ_n, $\lceil \frac{n-1}{2} \rceil$ choices for σ_{n-1}, and so forth. $\square$

3 Replacement sets of size 3

3.1 $123 \leftrightarrow 213 \leftrightarrow 231$

We will characterize and enumerate $\mathcal{C}_n(123, 213, 231)$ in Proposition 3.1 and Corollary 3.1. We will first need the following two lemmas.

Lemma 3.1 *Let σ be a permutation of size n such that n is immediately to the right of $n-1$. Then σ can be obtained from the identity using transformations $123 \leftrightarrow 213 \leftrightarrow 231$.*

Proof: Let τ be the pattern obtained by deleting $n-1$ and n from σ. We will first construct $\tau(n-1)n$, then position $n-1$ and n appropriately. Because $(n-1)$ and n are the two largest elements in the permutation, they can always be moved together as far as desired to the left using $123 \rightarrow 231$, or to the right using $231 \rightarrow 123$, as long as they remain together.

So, beginning with the identity permutation, first advance $(n-1)$ and n so that they are just to the right of τ_1. Now, letting $n-1$ play the role of 3, and using either $123 \rightarrow 213$ or $213 \rightarrow 123$ as appropriate, τ_1 can be moved one position to the left (if it is not already at the leftmost position). And $(n-1)$ and n can then be repositioned so that they remain just to its right. Repeat until τ_1 is in the leftmost position.

Now reposition $(n-1)$ and n just to the right or τ_2 and, as before, advance τ_2 to where it belongs, in the second position. Continue until we have built up $\tau(n-1)n$, one element at a time. Since $(n-1)$ and n remain free to move as a block, advance them as necessary to obtain the permutation σ. $\square$

Lemma 3.2 *Given a permutation $\alpha ijk\sigma\gamma$ such that the elements of σ are all less than k and $k < i < j$, it is possible to move to any permutation $\alpha i\sigma' kjl\sigma''\gamma$ using transformations $123 \leftrightarrow 213 \leftrightarrow 231$, where $\sigma = \sigma' \cup \sigma'' \cup \{l\}$ and $l = max\ \sigma - \sigma'$.*

Proof: We begin by applying $231 \rightarrow 213$ to ijk to obtain $\alpha ikj\sigma\gamma$. Now, if $kj\sigma$ is considered as a permutation having n elements, k and j represent the largest elements, $n-1$ and n. But Lemma 3.1 says that any permutation having n immediately following $n-1$ is equivalent to any other, and we can therefore obtain $\sigma' kjl\sigma''$ from $kj\sigma$, and therefore $\alpha i\sigma' kjl\sigma''\gamma$ from $\alpha ikj\sigma\gamma$. $\square$

Proposition 3.1 *Let σ be a permutation of size n. Then $\sigma \in \mathcal{C}_n(123, 213, 231)$ if and only if $n-1$ is to the left of n in σ.*

Proof: Since $n-1$ is to the left of n in the identity, and since in each of the patterns 123, 213, and 231, the relative order of the two largest elements is preserved, then $n-1$ must remain to the left of n.

Now suppose that σ is of size n and has $n-1$ to the left of n. We will show that σ can be reached from the identity. If $n-1$ is immediately to the left of n, we have the result by Lemma 3.1. If not, decompose σ as $\alpha(n-1)\beta n\gamma$. By Lemma 3.1, we can reach $\alpha(n-1)nk\beta'\gamma$, where $k = \max\beta$ and $\beta' = \beta - \{k\}$.

Now, to obtain σ, we proceed by induction on the number of right-to-left maxima of β, using Lemma 3.2. Let $k_1, k_2 \ldots$ be the right-to-left maxima of β, such that $\beta = \beta^{(1)} k_1 \beta^{(2)} k_2 \ldots$ and set $\beta^{(-i)} = \beta^{(i)}\, \beta^{(i+1)}\, k_{i+1} \ldots$. Then we have $\alpha\,(n-1)\, n\, k\, \beta'\, \gamma = \alpha\,(n-1)\, n\, k_1\, \beta^{(-1)}\, \gamma$. By Lemma 3.2, from $\alpha\,(n-1)\, n\, k_1\, \beta^{(-1)}\, \gamma$ we can reach $\alpha\,(n-1)\, \beta^{(1)}\, k_1\, n\, k_2\, \beta^{(-2)}\, \gamma$. Setting $\alpha' = \alpha\,(n-1)\, \beta^{(1)}$ we can apply again Lemma 3.2 to reach $\alpha\,(n-1)\, \beta^{(1)}\, k_1\, \beta^{(2)}\, k_2\, n\, k_3\, \beta^{(-3)}\, \gamma$, and so we can reach σ by applying Lemma 3.2 as many times as there are right-to-left maxima in β. □

Corollary 3.1 *The number of permutations of size n equivalent to the identity is $c_n(123, 213, 231) = \frac{n!}{2}$.*

3.2 $123 \leftrightarrow 132 \leftrightarrow 231$

Proposition 3.2 *Let σ be a permutation of size n. Then $\sigma \in \mathcal{C}_n(123, 132, 231)$ if and only if the left-to-right minima of σ are all in odd positions.*

Proof: The identity has only one left-to-right minimum, namely in position 1, which is an odd position. Now consider applying the permitted transfomations to a permutation. A move $123 \rightarrow 132$ neither adds nor removes a left-to-right minimum. However, a move $123 \rightarrow 231$ may either displace a left-to-right minimum or create a new one, but in either case does so two places to the right of an existing left-to-right minimum (and, conversely, the reverse move might either displace or create a left-to-right minimum two places to the left). Exactly the same is true for moves $132 \leftrightarrow 231$. Therefore repeated application of these rules to a permutation (such as the identity) which has all its left-to-right minima in odd positions can never introduce a left-to-right minimum in an even position.

Now we need to show that all such permutations can be obtained beginning with the identity. Let σ be a permutation with all left-to-right minima in odd positions. We will show by induction that we can obtain the permutations $\sigma^{(k)}$ in which the final k elements match those of σ, and the remaining $n-k$ elements are in increasing order. When $k = 0$, we have the identity.

Suppose we have obtained from the identity a permutation τ in which the final k elements are the same as those of σ and in which the first $n-k$ elements are in increasing order. We will place σ_{n-k} in position $n-k$. If σ_{n-k} is the smallest (and therefore leftmost) of the first $n-k$ elements of τ, then σ_{n-k} is a left-to-right minimum of σ, and therefore $n-k$ is odd. Since $\sigma_{n-k} = \tau_1$ and the first $n-k$ elements of τ are increasing, we can move it to position $n-k$ by repeated moves of type $123 \rightarrow 231$. If σ_{n-k} is not the smallest of the first $n-k$ elements of τ, then $\sigma_{n-k} = \tau_i$ with $i > 1$. If i is of the same parity as $n-k$, it is easy to move σ_{n-k} using $123 \rightarrow 231$. Otherwise, we place $\sigma_{n-k} = \tau_i$ in position $n-k-1$, then place τ_{i-1} in position $n-k-2$. At this point, $\tau_{i-1}\tau_i\tau_{n-k}$ forms a pattern 123, so we can use $123 \rightarrow 132$ to place $\tau_i = \sigma_{n-k}$ in position $n-k$. Finally, we can use $231 \rightarrow 123$ (letting τ_{i-1} play the role of 1) to return τ_{i-1} to position $i-1$, so that the initial $n-k-1$ elements are again in increasing order. The resulting permutation matches σ in the final $k+1$ positions, and has the initial $n-k-1$ elements in increasing order. □

3.3 $123 \leftrightarrow 231 \leftrightarrow 312$

Definition 3.1 *For convenience we will refer to the permutations in equivalence classes of size 1 as* isolated permutations. *That is, an isolated permutation is carried to nothing under any of the permitted transformations, equivalently, it does not contain any pattern in the replacement set.*

Proposition 3.3 *Let σ be a permutation of size n. Then $\sigma \in \mathcal{C}_n(123, 231, 312)$ if and only if σ is an even permutation which is not isolated.*

Proof: The transformations $123 \leftrightarrow 213 \leftrightarrow 312$ conserve the parity of the number of inversions, so each class contains only permutations of the same parity. In particular, if $\sigma \in \mathcal{C}_n(123, 231, 312)$ then σ is even.

Conversely suppose that σ is an even permutation which is not isolated. Let K be the equivalence class containing σ and let τ be an element of K with a minimal number of inversions. We will show that τ is the identity. We know that τ does not contain a 231 or a 312, because replacing these by 123 reduces the number of inversions. Since K does not consist of a single isolated permutation, it is possible to make a move from τ, so it contains a 123.

Suppose there is an index i such that $\tau_{i-2} < \tau_{i-1} < \tau_i > \tau_{i+1} > \tau_{i+2}$. As τ avoids 231, $\tau_{i-1}\tau_i\tau_{i+1}$ must be a 132, and so $\tau_{i-2}\tau_{i-1}\tau_i\tau_{i+1}$ is a 1243. Since $1243 \to 2413 \to 2134$, we can move from τ to $\tau' = \tau_1 \dots \tau_{i-3}\tau_{i-1}\tau_{i-2}\tau_{i+1}\tau_i\tau_{i+2} \dots \tau_n$, which has the same number of inversions as τ. But $\tau_{i+1}\tau_i\tau_{i+2}$ forms a 231, and so using $231 \to 123$ we can obtain a permutation with strictly fewer inversions, contradicting the minimality of τ.

Similarly, if there is an index i such that $\tau_{i-2} > \tau_{i-1} > \tau_i < \tau_{i+1} < \tau_{i+2}$, and using the fact that τ avoids 312, we can again obtain a permutation with strictly fewer inversions than τ.

We will say that τ satisfies property $P(i,k)$ if $\tau_i < \tau_{i+1} < \dots < \tau_{i+k-1} < \tau_{i+k}$, $\tau_{i-1} > \tau_i$ and $\tau_{i+k} > \tau_{i+k+1}$. Suppose that there exists an index i and a $k \geq 2$ such that τ satisfies $P(i,k)$. Since τ avoids 312, $\tau_{i-1}\tau_i\tau_{i+1}$ forms a 213, and since τ avoids 231, $\tau_{i+k-1}\tau_{i+k}\tau_{i+k+1}$ forms a 132. Thus $\tau_{i+k-2}\tau_{i+k-1}\tau_{i+k}\tau_{i+k+1}$ forms a 1243. Since $1243 \to 2413 \to 2134$, we can move from τ to $\tau' = \tau_1 \dots \tau_{i+k-3}\tau_{i+k-1}\tau_{i+k-2}\tau_{i+k+1}\tau_{i+k}\tau_{i+k+2} \dots \tau_n$, which has the same number of inversions as τ, and satisfies $P(i, k-2)$. By induction this allows us to assume that τ satisfies $P(i,k)$ for $k = 2$ or $k = 3$. First, suppose $k = 2$, so that $\tau_{i-1}\tau_i\tau_{i+1}\tau_{i+2}\tau_{i+3}$ forms a 21354. Via $21354 \to 25134 \to 12534 \to 12345$, we can obtain a permutation with fewer inversions, a contradiction. Then supoose $k = 3$, so that $\tau_{i-1}\tau_i\tau_{i+1}\tau_{i+2}\tau_{i+3}\tau_{i+4}$ forms a 213465. Again we can obtain a permutation with fewer inversions, via $213465 \to 241365 \to 246135 \to 246513 \to 462513 \to 461253 \to 146253 \to 145623 \to 156423 \to 156234 \to 125634 \to 123564 \to 123456$.

Recall that τ contains some 123. Let j be the smallest index with $\tau_j < \tau_{j+1} < \tau_{j+2}$. Then either $j = 1$ or $\tau_{j-1} > \tau_j$ (for otherwise we could have used $j-1$). Let m be the largest integer with $\tau_j < \tau_{j+1} < \dots < \tau_{j+m-1} < \tau_{j+m}$. By construction, $m \geq 2$, and $j + m = n$ or $\tau_{j+m+1} < \tau_{j+m}$ (by the maximality of m).

If $j \neq 1$ and $j + m \neq n$ then τ satisfies $P(j,m)$, contrary to our assumption.

Now suppose that $j \neq 1$ and $j + m = n$. Then we know that $\tau_{j-1} > \tau_j < \tau_{j+1}$, and since τ avoids 312, $\tau_{j-1}\tau_j\tau_{j+1}$ forms a 213. If $j - 1 = 1$ then τ has only one inversion, which is impossible as τ is even. So $j - 2 \geq 1$. If $\tau_{j-2} > \tau_{j-1}$ then the index j satisfies $\tau_{j-2} > \tau_{j-1} > \tau_j < \tau_{j+1} < \tau_{j+2}$, which we know is not the case. Thus $\tau_{j-2} < \tau_{j-1} > \tau_j$ and since τ avoids 231, $\tau_{j-2}\tau_{j-1}\tau_j$ forms a 132. If $j - 2 = 1$ then τ again has one inversion, which is impossible. Thus $j - 3 \geq 1$ and $\tau_{j-3} > \tau_{j-2}$, otherwise the index $j - 3$ contradicts the minimality of j. Furthermore, $\tau_{j-3} < \tau_{j-1}$ because τ has no 312, and so $\tau_{j-3}\tau_{j-2}\tau_{j-1}\tau_j\tau_{j+1}\tau_{j+2}$ forms a 214356. Now we can use $214356 \to 214635 \to 213465$, which by an argument above can be mapped to 123456, thus producing a permutation with fewer inversions than τ.

We show symmetrically that the case $j = 1$ and $j + m \neq n$ is impossible. The only other possibility is that $j = 1$ and $j + m = n$, which means that τ is the identity, and we are done. □

Thanks to the preceding proposition, we have a characterization of the set $C_n(123, 231, 312)$. Instead of enumerating this set we compute the complementary one among even permutations $S^e(n)$. This set contains all even permutations that avoid patterns 123, 231 and 312. We will denote by $S_n(123, 231, 312)$ the set of permutations of size n that avoid patterns 123, 231 and 312. Following article [KM05], denote by $s_e(n; i_1; \ldots; i_k)$ (resp. $s_o(n; i_1; \ldots; i_k)$) the number of even (resp. odd) permutations $\pi \in S_n(123, 231, 312)$ such that $\pi_1\pi_2 \ldots \pi_k = i_1 i_2 \ldots i_k$. Consider the transformation rot on permutations that consists in left multiplying the permutation by $n\,1\,2\,\ldots\,n{-}1$. This transformation rot is a bijection of $S_n(123, 231, 312)$. Moreover when the size of a permutation is even, the transformation rot flips the parity and we have $s_o(2n; a{+}1) = s_e(2n; a)$ and $s_e(2n; a{+}1) = s_o(2n; a)$. When the size of permutation is odd rot preserves parity and we have $s_o(2n{+}1; a{+}1) = s_o(2n{+}1; a)$ and $s_e(2n{+}1; a{+}1) = s_e(2n{+}1; a)$.

Proposition 3.4 *For all* n, x *such that* $1 \leq x \leq n$*:*

$$\begin{aligned}
s_o(2n; 1, 2x) &= s_e(2n; 1, 2x) \text{ if } x < n \\
s_o(2n; 1, 2n) &= s_e(2n; 1, 2n) + (-1)^n \\
s_o(2n+1; 1, 2x) + s_o(2n+1; 1, 2x+1) &= s_e(2n+1; 1, 2x) + s_e(2n+1; 1, 2x+1) \text{ if } x > 1 \\
s_o(2n+1; 1, 2) + s_o(2n+1; 1, 3) &= s_e(2n+1; 1, 2) + s_e(2n+1; 1, 3) + (-1)^{n+1}
\end{aligned}$$

Proof: The proof is by induction on the size of the permutations. When the result holds for permutations of size n, we we call this property $\mathcal{H}_n$. Clearly $\mathcal{H}_3$ holds : $s_o(3; 1, 2) + s_o(3; 1, 3) = 1$ and $s_e(3; 1, 2) + s_e(3; 1, 3) = 0$ because 123 is excluded and 132 is odd.

Proof for permutations of size $2n + 2$ Suppose $\mathcal{H}_{2n+1}$ holds. We prove $\mathcal{H}_{2n+2}$.

We prove the result $s_o(2n{+}2; 1, 2x) = s_e(2n{+}2; 1, 2x)$ by induction on x. In fact as $s_o(2n{+}2; 1, 2) = s_e(2n + 2; 1, 2) = 0$ the property is true for $x = 1$. Suppose it has been verified for $x < x_0$. We call this property $\mathcal{K}_{x_0}$.

The equality $s_o(2n{+}2; 1, 2x_0) = s_o(2n{+}2; 1, 2x_0{-}2) + s_o(2n{+}2; 1, 2x_0, 2) + s_o(2n{+}2; 1, 2x_0, 3) - s_o(2n + 2; 1, 2x_0 - 2, 2n + 2) - s_o(2n + 2; 1, 2x_0 - 2, 2n + 1)$ holds and is easily proved by considering that permutations σ in $S_o(2n + 2; 1, 2x_0)$ that do not have 2 or 3 as the third element are in one-to-one correspondence with those in $S_o(2n + 2; 1, 2x_0 - 2)$ that do not have $2n + 2$ or $2n + 1$ as the third element by rotating twice (i.e. decreasing by 2) every element of σ except its first one. Moreover, $s_o(2n + 2; 1, 2x_0{-}2, 2n + 2) = 0$ and $s_o(2n + 2; 1, 2x_0{-}2, 2n + 1) = 0$ as the first 3 elements of these permutations are in increasing order.

As $s_o(2n+2; 1, i, j) = s_o(2n+1; i-1, j-1) = s_o(2n+1; 1, 2n+2+j-i)$ for $j < i$ the preceding equality can be written as:
$s_o(2n+2; 1, 2x_0) = s_o(2n+2; 1, 2x_0-2) + s_o(2n+1; 1, 2n+4-2x_0) + s_o(2n+1; 1, 2n+5-2x_0)$.

We also have the same equality with s_e instead of s_o.

By property $\mathcal{K}_{x_0}$ we have $s_o(2n{+}2; 1, 2x_0{-}2) = s_e(2n{+}2; 1, 2x_0{-}2)$. If $x_0 < n{+}1$, then by property $\mathcal{H}_{2n+1}$ we have $s_o(2n{+}1; 1, 2n{+}4{-}2x_0) + s_o(2n{+}1; 1, 2n{+}5{-}2x_0) = s_e(2n{+}1; 1, 2n{+}4{-}2x_0) + s_e(2n + 1; 1, 2n + 5 - 2x_0)$. Hence $s_o(2n + 2; 1, 2x_0) = s_e(2n + 2; 1, 2x_0)$.

If $x_0 = n{+}1$, then by property $\mathcal{H}_{2n+1}$ we have $s_o(2n{+}1; 1, 2) + s_o(2n{+}1; 1, 3) = s_e(2n{+}1; 1, 2) + s_e(2n + 1; 1, 3) + (-1)^{n+1}$. Hence $s_o(2n + 2; 1, 2n + 2) = s_e(2n + 2; 1, 2n + 2) + (-1)^{n+1}$.

Proof for permutations of size $2n+3$: Suppose $\mathcal{H}_{2n+2}$ holds. We prove $\mathcal{H}_{2n+3}$.

$$s_o(2n+3;1,2x)+s_o(2n+3;1,2x+1)=\sum_{j=1}^{2x-2} s_o(2n+2;2x-1,j)+\sum_{j=1}^{2x-1} s_o(2n+2;2x,j)$$
$$=\sum_{j=1}^{2x-2} s_o(2n+2;1,2n+2+j-2(x-1))+\sum_{j=1}^{2x-1} s_e(2n+2;1,2n+2+j-(2x-1))$$
$$=\sum_{j=2n+5-2x}^{2n+2} (s_o(2n+2;1,j)+s_e(2n+2;1,j))+s_e(2n+2;1,2n+4-2x)$$

If $x>1$ as $2n+4-2x$ is even and $2n+4-2x<2n+2$ by induction we have $s_e(2n+2;1,2n+4-2x)=s_o(2n+2;1,2n+4-2x)$. Thus we finally have if $x>1$:

$$\begin{aligned} & s_o(2n+3;1,2x)+s_o(2n+3;1,2x+1) \\ = & \sum_{j=2n+5-2x}^{2n+2} (s_o(2n+2;1,j)+s_e(2n+2;1,j))+s_o(2n+2;1,2i+4-2x) \\ = & s_e(2n+3;1,2x)+s_e(2n+3;1,2x+1) \end{aligned}$$

If $x=1$ the same calculus leads to a difference of $(-1)^{n+2}$. □

Theorem 3.1 *The number of even permutations $s_e(n)$ and the number of odd permutations $s_o(n)$ in $S_n(123,231,312)$ satisfy the following equations :*

$$\begin{aligned} s_o(2k) &= s_e(2k)=E_{2k-1}/2 \\ s_o(2k+1)+s_e(2k+1)=E_{2k} \quad , & \quad s_o(2k+1)=s_e(2k+1)+(2k+1)(-1)^k \end{aligned}$$

Proof: By [KM05], $|S_n(123,231,312)|=E_{n-1}$ where E_n is the n- Euler number (Sloane A000111). In particular $s_o(2k+1)+s_e(2k+1)=E_{2k}$.

When the size of permutation is even, the transformation rot flips the parity hence $s_o(2k)=ks_e(2k;1)+ks_o(2k;1)=s_e(2k)$. But $s_o(2k)+s_e(2k)=E_{2k-1}$ so $s_o(2k)=s_e(2k)=E_{2k-1}/2$

When the size of the permutation is odd, transformation rot preserves parity. Permutations in $S_{2k+1}(123,231,312)$ can be grouped in sets of size $2k+1$ of permutations of the same parity which are equivalence classes of the relation rot. Note that in each class, only one permutation begins with 1. Hence, $s_o(2k+1)=(2k+1)s_o(2k+1;1)$ and $s_e(2k+1)=(2k+1)s_e(2k+1;1)$. But $s_o(2k+1;1)=\sum_{i=2}^{2k+1} s_o(2k+1;1,i)$ and Proposition 3.4 concludes that $s_o(2k+1)=s_e(2k+1)+(2k+1)(-1)^k$. □

Corollary 3.2 *The number $c_n(123,213,231)$ of permutations of size n equivalent to the identity is $c_{2k}(123,213,231)=\frac{(2k)!-E_{2k-1}}{2}$ and $c_{2k+1}(123,213,231)=\frac{(2k+1)!-E_{2k}+(2k+1)(-1)^k}{2}$*

4 Replacement sets of size 4

4.1 $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231$

Lemma 4.1 *Let $\sigma\in S_n$ have $\sigma_n=n$. Then $\sigma\in\mathcal{C}_n(123,132,213,231)$.*

Proof: Beginning with the identity, we will show by induction on $k < n-1$ that we can reach a permutation $\sigma^{(k)}$ in which the first k elements are the same as those of σ and the rest are in increasing order. The identity is $\sigma^{(0)}$. Suppose that we have reached $\sigma^{(k)}$. If $\sigma^{(k+1)} = \sigma^{(k)}$, we are done. Otherwise we use $123 \to 231$ as often as necessary to advance σ_{k+1} (playing the role of 2), together with the element which follows it, to positions $k+1$ and $k+2$. (We know that there is a following element as $\sigma_n = n$.) Next we use $231 \to 213$ to switch the following element (playing the role of 3) to position $k+3$, when we can begin a chain of $132 \to 123$ to restore it to its original location. □

Proposition 4.1 *Let $\sigma \in S_n$. Then $\sigma \in \mathcal{C}_n(123, 132, 213, 231)$ if and only if σ does not begin with n.*

Proof: No permutation $\sigma \in \mathcal{C}_n(123, 132, 213, 231)$ can begin with n because n can only participate in a transformation as a 3, and none of the transformations places a 3 in the leftmost position.

Conversely, let σ be a permutation which does not begin with n, and let τ be the pemutation obtained from σ by deleting the n. By Lemma 4.1, we can obtain τn from the identity. To obtain σ, we merely need to reposition the n, which we can do using $123 \to 132$ and $213 \to 231$ as necessary. □

Corollary 4.1 $c_n(123, 132, 213, 231) = n! - (n-1)! = (n-1) * (n-1)!$

4.2 $123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 321$

Lemma 4.2 *Let $\sigma \in S_n$ have $\sigma_1 = 1$. Then $\sigma \in \mathcal{C}_n(123, 132, 231, 321)$.*

Proof: We will show that we can obtain successively the permutations $\sigma^{(k)}$ in which the last k elements match σ and the first $n-k$ elements are in increasing order. The identity is $\sigma^{(0)}$. Suppose that we have obtained $\sigma^{(k)}$. By using $123 \to 231$ repeatedly, we can place $\sigma_{n-k} = \sigma_i^{(k)}$ in position $n-k$ or in position $n-k-1$. We remark that $i \neq 1$ because $\sigma_1^{(k)} = 1 = \sigma_1 \neq \sigma_{n-k}$. If σ_{n-k} has arrived in position $n-k$ then we have constructed $\sigma^{(k+1)}$. If conversely it is in position $n-k-1$, then by using $123 \to 231$ repeatedly, we can move $\sigma_{i-1}^{(k)}$ to position $n-k-2$, then we can use $123 \to 132$ on the three elements $\sigma_{i-1}^{(k)}\ \sigma_i^{(k)}\ \sigma_{n-k}^{(k)}$, which moves $\sigma_i^{(k)} = \sigma_{n-k}$ to position $n-k$. Finally we use $231 \to 123$, with $\sigma_{i-1}^{(k)}$ in the role of 1, as often as necessary to return $\sigma_{i-1}^{(k)}$ to its original location, thus obtaining $\sigma^{(k+1)}$. □

Proposition 4.2 *$\sigma \in \mathcal{C}_n(123, 132, 231, 321)$ if and only if 1 occupies an odd position in σ.*

Proof: In the identity, 1 occupies an odd position, and since 1 can only participate in a transformation as the smallest element, none of the possible transformations can change the parity of the position occupied by 1. Conversely, let σ be a permutation having the element 1 in odd position, and let τ be the permutation obtained from σ by deleting the 1. By Lemma 4.2, we can obtain 1τ from the identity. Then we can use $123 \to 231$ or $132 \to 321$ as necessary to move the element 1 to its correct position. □

Corollary 4.2 *The number of permutations equivalent to the identity under $123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 321$ is* $c_n(123, 132, 231, 321) = (n-1)!\lceil \frac{n}{2} \rceil$.

4.3 $123 \leftrightarrow 132 \leftrightarrow 231 \leftrightarrow 312$

Proposition 4.3 *The only permutations of length n not belonging to $\mathcal{C}_n(123, 132, 231, 312)$ are isolated permutations, i.e. belong to classes of size 1.*

Proof: Suppose σ belongs to a class K of size greater than 1, and let τ be a permutation in K having the smallest number of inversions. We will show that τ is the identity. First, τ cannot contain any 132, 231 or 312, because such a string could be replaced by 123, yielding a permutation with a strictly smaller number of inversions. But K contains at least two permutations, therefore some replacement in τ is possible, and therefore τ contains a 123.

Now if τ is not the identity, because τ has no 231 or 132, τ must consist of a decreasing sequence followed by an increasing sequence. Consider the string at the junction of these two parts, $x1bc$, where $x > 1 < b < c$ (as τ is not the identity and contains a 123, x, b and c exist). Note that $x < b$ because τ contains no 312, so we have $1 < x < b < c$. Now we make the transformations $x1bc \to xc1b \to 1xcb \to 1xbc$, arriving at a permutation with one fewer inversion than τ. Therefore τ can only be the identity. $\square$

Proposition 4.4 *The permutations which are not in* $\mathcal{C}_n(123, 132, 231, 312)$ *are those which are obtained from* $D_n = n\,(n{-}1)\dots 1$ *by taking an element other than* 2 *and placing it at the end.*

Proof: By proposition 4.3, the permutations not in $\mathcal{C}_n(123, 132, 231, 312)$ are isolated, i.e. contain no 123, 132, 231 or 312. Because they have no 132 or 231, they must be a descending sequence followed by an increasing sequence. Because they have no 123, the increasing sequence can have length at most 2. Therefore they look like D_n with one element (possibly 1) relocated to the end. But the relocated element cannot be 2 because the permutations contain no 312. $\square$

Corollary 4.3 *The number of permutations in* $\mathcal{C}_n(123, 132, 231, 312)$ *is* $c_n(123, 132, 231, 312) = n! - (n-1)$. *Moreover* $\#\text{Classes}(n; \{123, 132, 231, 312\}) = n$.

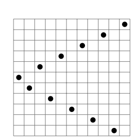
4.4 $123 \leftrightarrow 132 \leftrightarrow 312 \leftrightarrow 321$

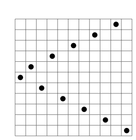
Proposition 4.5 *The only permutations of length n not belonging to* $\mathcal{C}_n(123, 132, 312, 321)$ *are isolated.*

Proof: Suppose σ belongs to a class K of size greater than 1, and let τ be a permutation in K having the smallest number of inversions. We will show that τ is the identity. First, τ cannot contain any 132, 312 or 321, because such a string could be replaced by 123, yielding a permutation with a strictly smaller number of inversions. But K contains at least two permutations, therefore some replacement in τ is possible, and therefore τ contains a 123. Now if τ is not the identity, it must either contain a 123 followed by a descent, or a 123 preceded by a descent. In the first case we have $abcx$ with $a < b < c$ and $b > x$ (because τ avoids 132), and we can make the transformations $abcx \to acbx \to axbc$, arriving at a permutation with two fewer inversions. In the second case we have $xabc$ with $a < b < c$ and $a < x < b$ (τ avoids 312), and we can make the transformations $xabc \to xcba \to cbxa \to caxb \to axcb \to axbc$, arriving at a permutation with one fewer inversion. Therefore τ can only be the identity. $\square$

Proposition 4.6 *The permutations not belonging to* $\mathcal{C}_n(123, 132, 312, 321)$ *are just the two wedge alternations open on the right as illustrated below.*

Proof: By proposition 4.5, the permutations not equivalent to the identity are those which contain no 123, 132, 312 or 321. Because there is no 123 or 321, the permutations must alternate ascents and descents. But because there is no 132 or 312, each descent must set a new left-to-right minimum, and each ascent a new left-to-right maximum. $\square$





Corollary 4.4 *The number of permutations in* $\mathcal{C}_n(123, 132, 312, 321)$ *is* $c_n(123, 132, 312, 321) = n! - 2$. *Moreover* $\#\text{Classes}(n; \{123, 132, 312, 321\}) = 3$.

4.5 $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$

Proposition 4.7 *The only permutations not equivalent to the identity under* $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 321$ *are the isolated permutations.*

Proof: Suppose σ belongs to a class K of size greater than 1, and let τ be a permutation in K having the smallest number of inversions. We will show that τ is the identity. First, τ cannot contain any 132, 213 or 321, because such a string could be replaced by 123, yielding a permutation with a strictly smaller number of inversions. But K contains at least two permutations, therefore some replacement in τ is possible, and therefore τ contains a 123.

Now if τ is not the identity, it must either contain a 123 followed by a descent, or a 123 preceded by a descent. In the first case we have $abcx$ with $a < b < c$ and $b > x$ (τ avoids 132), and we can make the transformations $abcx \to acbx \to axbc$, arriving at a permutation with two fewer inversions. In the second case we have $xabc$ with $a < b < c$ and $x > b$ (τ avoids 213), and we can make the transformations $xabc \to xbac \to abxc$, arriving at a permutation with two fewer inversions. Therefore τ can only be the identity. □

Proposition 4.8 *The only permutations not belonging to* $\mathcal{C}_n(123, 132, 213, 321)$ *are the alternating permutations in which the elements in odd positions form a decreasing sequence, and the elements in even positions form also a decreasing sequence.*

Proof: By Proposition 4.7, the permutations are those with none of the four patterns. Because there is no 123 or 321, these permutations must be alternating; because there is no 123, 132 or 213, two elements which are situated two positions apart must be decreasing. □

Proposition 4.9 *The number of permutations in* $\mathcal{C}_n(123, 132, 213, 321)$ *is* $c_n(123, 132, 213, 321) = n! - c(\lfloor \frac{n-1}{2} \rfloor) - c(\lfloor \frac{n}{2} \rfloor)$ *, where* $c(n)$ *is the* n*th Catalan number.*

Proof: We will denote the number of isolated permutations by d_n; as a result of proposition 4.7 we know that $c_n(123, 132, 213, 321) = n! - d_n$. The isolated permutations are the permutations which are alternating, and such that the sequences $(\sigma_{2k})_k$ and $(\sigma_{2k+1})_k$ are decreasing. It follows that $d_n = a_n + b_n$, where a_n is the number of isolated permutations of size n with n in position 1, and b_n is the number of isolated permutations of size n with n in position 2.

Let σ be an isolated permutation of size n with n in position 1. We set $\alpha_k = n - \sigma_{2k+1} - k$ for $k \geq 0$. Since $(\sigma_{2k+1})_k$ decreases and $\sigma_1 = n$, $(\alpha_k)_k$ is non decreasing and $\alpha_0 = 0$. Moreover, because σ is alternating and $(\sigma_{2k+1})_k$ and $(\sigma_{2k})_k$ are decreasing, $\forall k \geq 1$, $\sigma_{2k} < \sigma_{2k+1}$ and $\sigma_s < \sigma_{2k+1}$ if $2k + 2 \leq s \leq n$, so $\sigma_{2k+1} > n - 2k$ and $\alpha_k < k$ if $k \geq 1$.

If we represent the points (k, α_k) thus obtained, the result is a non decreasing sequence of $\lceil n/2 \rceil$ points situated strictly below the diagonal, except for $\alpha_0 = 0$. We can verify that this map is bijective with such sequences of points, and thus with the set of Dyck paths of length $\lceil n/2 \rceil - 1 = \lfloor \frac{n-1}{2} \rfloor$: connect each point to the next with a sequence of rightward steps followed by a sequence of upward steps, and then rotate the resulting diagram accordingly Figure 1. Thus $a_n = c(\lfloor \frac{n-1}{2} \rfloor)$,

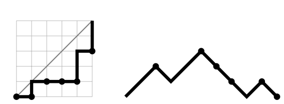


Fig. 1: $\sigma = 12\ 10\ 11\ 6\ 9\ 5\ 8\ 3\ 7\ 2\ 4\ 1$

by the well-known Catalan enumeration of Dyck paths. Now let σ be an isolated permutation with $\sigma_2 = n$, and set $\alpha_k = n+1-\sigma_{2k}-k$ for $k \geq 1$. As $(\sigma_{2k})_k$ is decreasing and $\sigma_2 = n$, $(\alpha_k)_k$ is non decreasing and $\alpha_1 = 0$. Furthermore, as σ is alternating and $(\sigma_{2k})_k$ and $(\sigma_{2k+1})_k$ are decreasing, $\forall k \geq 1$, $\sigma_{2k-1} < \sigma_{2k}$ and $\sigma_s < \sigma_{2k}$ if $2k+1 \leq s \leq n$, so $\sigma_{2k} > n+1-2k$ and $\alpha_k < k$ if $k \geq 1$.

If we represent the points (k, α_k) thus obtained, the result is an increasing sequence of $\lfloor n/2 \rfloor$ points situated strictly below the diagonal. Again, this map is bijective, and by adding a point at $(0,0)$ we have, as above, a bijection with Dyck paths of length $\lfloor \frac{n}{2} \rfloor$. So $b_n = c(\lfloor \frac{n}{2} \rfloor)$. □

Corollary 4.5 $\#\text{Classes}(n; \{123, 132, 213, 321\}) = c(\lfloor \frac{n-1}{2} \rfloor) + c(\lfloor \frac{n}{2} \rfloor) + 1.$

4.6 $123 \leftrightarrow 231 \leftrightarrow 312 \leftrightarrow 321$

The equivalence class of the identity under $123 \leftrightarrow 231 \leftrightarrow 312 \leftrightarrow 321$ consists of the reversals of the permutations in the equivalence class of $D_n = n, n-1 \ldots 1$ under the transformations $321 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 123$, considered in the previous section. But we know that D_n was in the class of the identity under these transformations, because it was not isolated. This observation gives us immediately:

Proposition 4.10 *The only permutations not belonging to* $\mathcal{C}_n(123, 231, 312, 321)$ *are the alternating permutations in which the elements in odd positions form an increasing sequence, and the elements in even positions form an increasing sequence.*

The number of permutations in $\mathcal{C}_n(123, 231, 312, 321)$ *is* $c_n(123, 231, 312, 321) = n! - c(\lfloor \frac{n-1}{2} \rfloor) - c(\lfloor \frac{n}{2} \rfloor)$ *, where* $c(n)$ *is the* n*th Catalan number.*

Moreover, $\#\text{Classes}(n; \{123, 231, 312, 321\}) = c(\lfloor \frac{n-1}{2} \rfloor) + c(\lfloor \frac{n}{2} \rfloor) + 1.$

5 Replacement sets of size 5

5.1 $123 \leftrightarrow 132 \leftrightarrow 213 \leftrightarrow 231 \leftrightarrow 312$

Proposition 5.1 $\#\text{Classes}(n; \{123, 132, 213, 231, 312\}) = 2$*, and the two equivalence classes are* $\mathcal{C}_n(123, 132, 213, 231, 312)$ *and* $\{D_n\}$ *where* $D_n = n\,(n-1) \ldots 1.$

Proof: Let σ be a permutation not in $\mathcal{C}_n(123, 132, 213, 231, 312)$: by section 4.3, $\sigma_1 \ldots \sigma_{n-1}$ is decreasing and $\sigma_n \neq 2$. And by section 4.1, applying reverse complements, $\sigma_n = 1$, and so $\sigma = D_n$. Conversely, it is clear that $D_n \notin \mathcal{C}_n(123, 132, 213, 231, 312)$, because D_n contains only the pattern 321 and so is isolated. □

5.2 Remaining cases

Proposition 5.2 *If* $n \geq 4$ *there is only a single equivalence class for the sets* $\{123, 132, 213, 231, 321\}$, $\{123, 132, 213, 312, 321\}$, $\{123, 132, 231, 312, 321\}$ *or* $\{123, 213, 231, 312, 321\}$.

Proof: We will prove the first of the four statements, as the four proofs are straightforward and all very similar. Suppose, conversely, that we can find a permutation σ which is not in $\mathcal{C}_n(123, 132, 213, 231, 321)$; then by section 4.4, applying reverse complements, σ is a wedge alternation $>$, so by section 4.5 σ is alternating and the sequences $(\sigma_{2k})_k$ and $(\sigma_{2k+1})_k$ are both decreasing, which is impossible. □

6 Summary table

Replacement set	Enumeration	Sloane	Proof	Characterisation
$123, 132$ or $123, 213$	$\lfloor n/2\rfloor!\lceil n/2\rceil!$	A010551	[LPRW10]	
$123, 231$	$\lfloor n/2\rfloor!\lceil n/2\rceil!$	A010551	2.1	$\forall k$ the rank of σ_k among $\sigma_1 \dots \sigma_k$ is of the same parity as k
$123, 312$	$\lfloor n/2\rfloor!\lceil n/2\rceil!$	A010551	2.1 (RC)	$\forall k$ the rank of σ_k among $\sigma_k \dots \sigma_n$ is odd
$123, 321$	$\binom{n-1}{\lfloor (n-1)/2\rfloor}$	A001405	[LPRW10]	
$123, 132, 213$	$[3, 7, 35, 135, 945]$		[CEH+01]	
$123, 132, 231$	$n!\binom{n-1}{\lfloor (n-1)/2\rfloor}/2^{n-1}$	A000246	3.2	left-to-right minima are in odd position
$123, 132, 312$	$n!/2$	A001710	3.1 (RC)	1 is left of 2
$123, 132, 321$	$[3, 9, 54, 285, 2160]$		[LPRW10]	
$123, 213, 231$	$n!/2$	A001710	3.1	$n-1$ is left of n
$123, 213, 312$	$n!\binom{n-1}{\lfloor (n-1)/2\rfloor}/2^{n-1}$	A000246	3.2 (RC)	right-to-left maxima are in positions of same parity as n
$123, 213, 321$	$[3, 9, 54, 285, 2160]$		[LPRW10]	
$123, 231, 312$	$[3, 8, 45, 313, 2310]$		3.3	Non-isolated even permutations
$123, 231, 321$	$[3, 9, 54, 285, 2160]$		[LPRW10]	
$123, 312, 321$	$[3, 9, 54, 285, 2160]$		[LPRW10]	
$123, 132, 213, 231$	$n! - (n-1)!$		4.1	$\sigma_1 \neq n$
$123, 132, 213, 312$	$n! - (n-1)!$		4.1 (RC)	$\sigma_n \neq 1$
$123, 132, 213, 321$	$n! - c(\lfloor \frac{n-1}{2}\rfloor) - c(\lfloor \frac{n}{2}\rfloor)$		4.5	σ is not alternating or the sequence $(\sigma_{2k})_k$ or $(\sigma_{2k+1})_k$ is not decreasing
$123, 132, 231, 312$	$n! - (n-1)$		4.3	$\sigma_1 \dots \sigma_{n-1}$ is not decreasing or $\sigma_n = 2$
$123, 132, 231, 321$	$(n-1)!\lceil \frac{n}{2}\rceil$		4.2	1 is in odd position
$123, 132, 312, 321$	$n! - 2$		4.4	σ is not a wedge alternation $<$
$123, 213, 231, 312$	$n! - (n-1)$		4.3 (RC)	$\sigma_2 \dots \sigma_n$ is not decreasing or $\sigma_1 = n-1$
$123, 213, 231, 321$	$n! - 2$		4.4 (RC)	σ is not a wedge alternation $>$
$123, 213, 312, 321$	$(n-1)!\lceil \frac{n}{2}\rceil$		4.2 (RC)	n and its position have the same parity
$123, 231, 312, 321$	$n! - c(\lfloor \frac{n-1}{2}\rfloor) - c(\lfloor \frac{n}{2}\rfloor)$		4.6	σ is not alternating or the sequence $(\sigma_{2k})_k$ or $(\sigma_{2k+1})_k$ is not increasing
$123, 132, 213, 231, 312$	$n! - 1$		5.1	All except $n(n-1)\dots 1$
Other cases	$n!$		5.2	All

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The brick polytope of a sorting network[†]

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Abstract. The associahedron is a polytope whose graph is the graph of flips on triangulations of a convex polygon. Pseudotriangulations and multitriangulations generalize triangulations in two different ways, which have been unified by Pilaud and Pocchiola in their study of pseudoline arrangements with contacts supported by a given network. In this paper, we construct the "brick polytope" of a network, obtained as the convex hull of the "brick vectors" associated to each pseudoline arrangement supported by the network. We characterize its vertices, describe its faces, and decompose it as a Minkowski sum of simpler polytopes. Our brick polytopes include Hohlweg and Lange's many realizations of the associahedron, which arise as brick polytopes of certain well-chosen networks.

Résumé. L'associaèdre est un polytope dont le graphe est le graphe des flips sur les triangulations d'un polygone convexe. Les pseudotriangulations et les multitriangulations généralisent les triangulations dans deux directions différentes, qui ont été unifiées par Pilaud et Pocchiola au travers de leur étude des arrangements de pseudodroites avec contacts couvrant un support donné. Nous construisons ici le "polytope de briques" d'un support, obtenu comme l'enveloppe convexe des "vecteurs de briques" associés à chaque arrangement de pseudodroites couvrant ce support. Nous caractérisons les sommets de ce polytope, décrivons ses faces et le décomposons en somme de Minkowski de polytopes élémentaires. Notre construction contient toutes les réalisations de l'associaèdre d'Hohlweg et Lange, qui apparaissent comme polytopes de briques de certains supports bien choisis.

Keywords: associahedron, sorting networks, pseudoline arrangements with contacts

1 Introduction

This paper focusses on polytopes realizing *flip graphs* on certain geometric and combinatorial structures. The motivating example is the *associahedron* whose vertices correspond to triangulations of a convex polygon and whose edges correspond to flips between them — see Figure 5. The associahedron appears under various motivations ranging from geometric combinatorics to algebra, and several different constructions have been proposed by Lee (1989), Gel′fand et al. (1994), Billera et al. (1990), Loday (2004), and Hohlweg and Lange (2007). See the book of De Loera et al. (2010) about triangulations of point sets.

In their study of the graph of flips on *pseudoline arrangements with contacts* supported by a given network, Pilaud and Pocchiola (2010) unified two different generalizations of triangulations of convex polygons: *pseudotriangulations* of planar point sets (see Rote et al. (2008)) and *multitriangulations* of convex polygons (see Pilaud and Santos (2009) or Pilaud (2010)). Our paper is based on this framework, and we recall its definitions and basic properties in Section 2.

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In this paper, we define and study the *brick polytope* of a network $\mathcal{N}$, obtained as the convex hull of vectors associated to each pseudoline arrangement supported by $\mathcal{N}$. Our main result is the characterization of the pseudoline arrangements which give rise to the vertices of the brick polytope of $\mathcal{N}$, from which we derive a combinatorial description of its faces. We furthermore provide a natural decomposition of the brick polytope into a Minkowski sum of simpler polytopes. These results are presented in Section 3.

We observe in Section 4 that our brick polytopes coincide (up to translation) with the associahedra of Hohlweg and Lange (2007) for certain well-chosen networks. We thus provide a different point of view and complete the combinatorial description of their associahedra. As a supplementary motivation of our construction, let us also mentioned that permutahedra and cyclohedra naturally show up in our setting.

2 The brick polytope of a sorting network

2.1 Sorting networks, pseudoline arrangements, and flips

A *network* $\mathcal{N}$ is a set of n horizontal lines (called *levels*), together with m vertical segments (called *commutators*) joining two consecutive horizontal lines, such that no two commutators have a common endpoint — see Figure 1 (left). The *bricks* of $\mathcal{N}$ are its $m - n + 1$ bounded cells.

A *pseudoline* is an abscissa monotone path on $\mathcal{N}$ which starts at a level ℓ and ends at the level $n+1-\ell$. A *contact* between two pseudolines is a commutator whose endpoints are contained one in each pseudoline, and a *crossing* between two pseudolines is a commutator traversed by both pseudolines. A *pseudoline arrangement* (with contacts) is a set of n pseudolines supported by $\mathcal{N}$ such that any two of them have precisely one crossing, some (perhaps zero) contacts, and no other intersection — see Figure 1 (middle and right). Observe that a pseudoline arrangement supported by $\mathcal{N}$ is completely determined by its $\binom{n}{2}$ crossings, or equivalently by its $m - \binom{n}{2}$ contacts. We say that a network is *sorting* when it supports some pseudoline arrangements. We denote by $\mathrm{Arr}(\mathcal{N})$ the set of pseudoline arrangements supported by $\mathcal{N}$.

There is a natural *flip* operation which transforms a pseudoline arrangement supported by $\mathcal{N}$ into another one by exchanging the position of a contact. More precisely, if V is the set of contacts of a pseudoline arrangement Λ supported by $\mathcal{N}$, and if $v \in V$ is a contact between two pseudolines of Λ which cross at w, then $V \triangle \{v, w\}$ is the set of contacts of another pseudoline arrangement supported by $\mathcal{N}$ — see Figure 1 (middle and right). The *graph of flips* $G(\mathcal{N})$ is the graph whose nodes are the pseudoline arrangements supported by $\mathcal{N}$ and whose edges are the flips between them. This graph was introduced and studied by Pilaud and Pocchiola (2010), whose first statement is the following result:

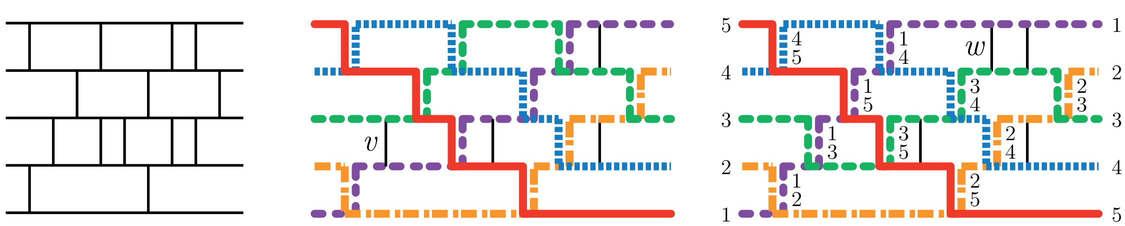

Fig. 1: A sorting network $\mathcal{N}$ (left) and two pseudoline arrangements supported by $\mathcal{N}$ and related by a flip (right). The rightmost pseudoline arrangement is the greedy pseudoline arrangement $\Gamma(\mathcal{N})$, whose flips are all increasing. It is obtained by sorting the permutation $(5, 4, 3, 2, 1)$ according to the network $\mathcal{N}$.

Theorem 2.1 (Pilaud and Pocchiola (2010)) *The graph of flips $G(\mathcal{N})$ of a sorting network $\mathcal{N}$ with n levels and m commutators is $\left(m - \binom{n}{2}\right)$-regular and connected.*

Regularity of the graph of flips is obvious since every contact induces a flip. For the connectivity, define a flip to be *increasing* if the added contact lies on the left of the removed contact. The oriented graph of increasing flips is clearly acyclic and is proved to have a unique source by Pilaud and Pocchiola (2010) (and thus, to be connected). This source is called the *greedy pseudoline arrangement* supported by $\mathcal{N}$ and is denoted by $\Gamma(\mathcal{N})$. It is characterized by the property that any of its contacts is located to the right of its corresponding crossing. It can be computed by sorting the permutation $(n, n-1, \ldots, 2, 1)$ according to the sorting network $\mathcal{N}$ — see Figure 1 (right). We will use this pseudoline arrangement later. We refer to Pilaud and Pocchiola (2010) for further details and applications of the greedy pseudoline arrangement.

Theorem 2.1 is a motivation to consider the simplicial complex $\Delta(\mathcal{N})$ whose maximal simplices are the sets of contacts of pseudoline arrangements supported by $\mathcal{N}$. As a particular subword complex of Knutson and Miller (2004), this complex $\Delta(\mathcal{N})$ is known to form a combinatorial sphere, but it remains open to know whether it is the boundary complex of a $\left(m - \binom{n}{2}\right)$-dimensional simplicial polytope. In this article, we construct a polytope whose graph is a subgraph of $G(\mathcal{N})$. For certain networks (see Definition 3.4), this subgraph is even the whole graph $G(\mathcal{N})$, and we obtain a polytope whose boundary complex is $\Delta(\mathcal{N})$.

2.2 The brick polytope

Definition 2.2 *Let $\mathcal{N}$ be a sorting network with n levels. The* brick vector *of a pseudoline arrangement Λ supported by $\mathcal{N}$ is the vector $\omega(\Lambda) \in \mathbb{R}^n$ whose ith coordinate is the number of bricks of $\mathcal{N}$ located below the ith pseudoline of Λ (the one which starts at level i and finishes at level $n+1-i$). The* brick polytope *$\Omega(\mathcal{N}) \subset \mathbb{R}^n$ of the sorting network $\mathcal{N}$ is the convex hull of the brick vectors of all pseudoline arrangements supported by $\mathcal{N}$.*

This article describes the combinatorial properties of the brick polytope $\Omega(\mathcal{N})$ in terms of the properties of the supporting network $\mathcal{N}$. In Section 3, we provide a characterization of the pseudoline arrangements supported by $\mathcal{N}$ whose brick vectors are vertices of $\Omega(\mathcal{N})$, from which we derive a description of the faces of $\Omega(\mathcal{N})$. We also provide a natural decomposition of $\Omega(\mathcal{N})$ into a Minkowski sum of simpler polytopes.

We start by observing that the brick polytope is not full dimensional. Define the *depth* of a brick of $\mathcal{N}$ to be the number of levels located above it, and let $D(\mathcal{N})$ be the sum of the depths of all the bricks of $\mathcal{N}$. Since any pseudoline arrangement supported by $\mathcal{N}$ covers each brick as many times as its depth, all brick vectors are contained in the following hyperplane:

Lemma 2.3 *The brick polytope $\Omega(\mathcal{N})$ is contained in the hyperplane of equation $\sum_{i=1}^{n} x_i = D(\mathcal{N})$.*

Example 2.4 (2-levels networks) *Consider the network $\mathcal{X}_m$ formed by two levels related by m commutators. Choosing the ith commutator of $\mathcal{X}_m$ as unique crossing, we obtain a pseudoline arrangement whose brick vector is $(m-i, i-1)$. The graph of flips is the complete graph on m vertices while the brick polytope $\Omega(\mathcal{X}_m)$ is a segment contained in the hyperplane of equation $x + y = m - 1$.*

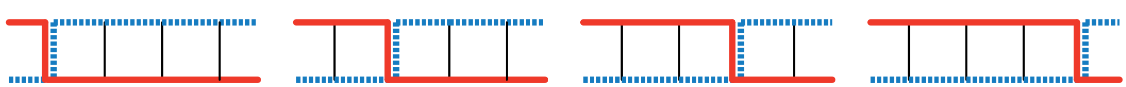

Fig. 2: The four pseudoline arrangements supported by the network $\mathcal{X}_4$ with two levels and four commutators.

3 Combinatorial description of the brick polytope

In this section, we provide a combinatorial description of the face structure of the brick polytope $\Omega(\mathcal{N})$. For this purpose, we study the cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector of a given pseudoline arrangement supported by $\mathcal{N}$. Our main tool is the incidence configuration of the contact graph of a pseudoline arrangement, which we define next.

3.1 The contact graph of a pseudoline arrangement

Let $\mathcal{N}$ be a sorting network with n levels and m commutators, supporting a pseudoline arrangement Λ.

Definition 3.1 *The* contact graph *of Λ is the directed multigraph $\Lambda^{\#}$ with a node for each pseudoline of Λ and an arc for each contact of Λ oriented from the pseudoline passing above the contact to the pseudoline passing below it.*

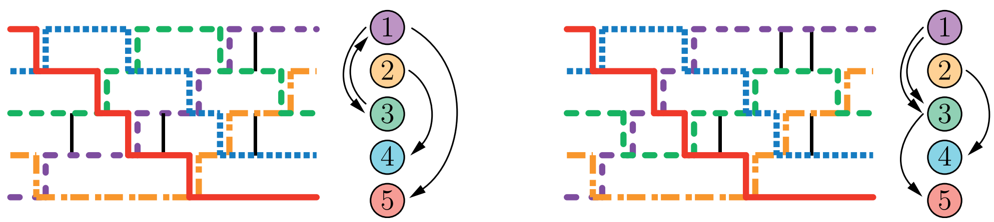

Fig. 3: The contact graphs of the pseudoline arrangements of Figure 1. Connected components are preserved by flip.

Let Λ and Λ' denote two pseudoline arrangements with support $\mathcal{N}$ and related by a flip involving their ith and jth pseudolines — see Figure 3. Then the directed multigraphs obtained by merging the vertices i and j in the contact graphs $\Lambda^{\#}$ and $\Lambda'^{\#}$ coincide. In particular, a flip preserves the connected components of the contact graph. Since the flip graph $G(\mathcal{N})$ is connected (Theorem 2.1), we obtain:

Lemma 3.2 *The contact graphs of all pseudoline arrangements supported by $\mathcal{N}$ have the same connected components.*

We say that a network is *irreducible* if the contact graphs of the pseudoline arrangements it supports are connected. It is easy to see that any sorting network can be decomposed into irreducible sorting networks:

Proposition 3.3 *If a sorting network $\mathcal{N}$ supports pseudoline arrangements whose contact graphs have p connected components, then it can be decomposed into p irreducible sorting networks $\mathcal{N}_1, \dots, \mathcal{N}_p$ such that $\Delta(\mathcal{N})$ is the join $\Delta(\mathcal{N}_1) * \cdots * \Delta(\mathcal{N}_p)$ and $\Omega(\mathcal{N})$ is a translate of the product $\Omega(\mathcal{N}_1) \times \cdots \times \Omega(\mathcal{N}_p)$.*

This proposition allow us to restrict our attention to irreducible sorting networks. Under this assumption, the following sorting networks have the fewest commutators:

Definition 3.4 *An irreducible sorting network $\mathcal{N}$ is* minimal *if the following equivalent conditions hold:*

- *$\mathcal{N}$ has n levels and $m = \binom{n}{2} + n - 1$ commutators.*
- *The contact graph of a pseudoline arrangement supported by $\mathcal{N}$ is a tree.*
- *The contact graphs of all pseudoline arrangements supported by $\mathcal{N}$ are trees.*

Minimal networks naturally show up in Section 3.3 since their brick polytopes are of particular interest.

3.2 The incidence cone of a directed multigraph

Let G be a directed multigraph on n vertices, whose underlying undirected graph is connected. We denote by $(e_1, \dots, e_n)$ the canonical basis of $\mathbb{R}^n$ and $\mathbb{1} := \sum e_i$.

Definition 3.5 *The* incidence configuration *of G is the set $I(G) := \{e_j - e_i \mid (i,j) \in G\} \subset \mathbb{R}^n$ of column vectors of its incidence matrix. The* incidence cone *of G is the cone $C(G) \subset \mathbb{R}^n$ generated by $I(G)$.*

Note that the incidence cone is contained in the linear subspace $\langle \mathbb{1} \mid x \rangle = 0$. There is a correspondence between the graph properties of G and the orientation properties of $I(G)$ — see Björner et al. (1999):

Remark 3.6 *The vectors of the incidence configuration $I(H)$ of a subgraph H of G*

1. *are independent if and only if H has no (non-necessarily oriented) cycle, that is, if H is a forest;*
2. *span the hyperplane $\langle \mathbb{1} \mid x \rangle = 0$ if and only if H is connected and spanning;*
3. *form a basis of the hyperplane $\langle \mathbb{1} \mid x \rangle = 0$ if and only if H is a spanning tree of G;*
4. *form a circuit if and only if H is a (non-necessarily oriented) cycle; the positive and negative parts of the circuit correspond to the subsets of edges oriented in one or the other direction along this cycle; in particular, $I(H)$ is a positive circuit if and only if H is an oriented cycle;*
5. *form a cocircuit if and only if H is a minimal (non-necessarily oriented) cut; the positive and negative parts of the cocircuit correspond to the edges in one or the other direction in this cut; in particular, $I(H)$ is a positive cocircuit if and only if H is an oriented cut.*

Remark 3.7 *The incidence configuration $I(H)$ of a subgraph H of G is the set of vectors of $I(G)$ contained in a k-face of $C(G)$ if and only if H has $n-k$ connected components and G/H is acyclic. Thus:*

1. *The incidence cone $C(G)$ has dimension $n-1$.*
2. *The incidence cone $C(G)$ is pointed if and only if G is an acyclic directed graph.*
3. *The facets of $C(G)$ correspond to the complements of the minimal directed cuts in G. Given a minimal directed cut, the characteristic vector of its sink is a normal vector of the corresponding facet.*

3.3 Vertex characterization and face description of the brick polytope

Let $\mathcal{N}$ be an irreducible sorting network supporting a pseudoline arrangement Λ. The contact graph $\Lambda^{\#}$ is our main tool to describe the cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$:

Theorem 3.8 *The cone of the brick polytope $\Omega(\mathcal{N})$ at the brick vector $\omega(\Lambda)$ is precisely the incidence cone $C(\Lambda^{\#})$ of the contact graph $\Lambda^{\#}$ of Λ.*

Proof: Assume that Λ' is obtained from Λ by flipping a contact from its ith pseudoline to its jth pseudoline. Then the difference $\omega(\Lambda') - \omega(\Lambda)$ is a positive multiple of $e_j - e_i$. This immediately implies that the incidence cone $C(\Lambda^{\#})$ is included in the cone of $\Omega(\mathcal{N})$ at $\omega(\Lambda)$.

Reciprocally, we shall prove that any facet F of the cone $C(\Lambda^{\#})$ is also a facet of the brick polytope $\Omega(\mathcal{N})$. According to Remark 3.7(3), there exists a minimal directed cut γ of $\Lambda^{\#}$ which partitions the vertices of $\Lambda^{\#}$ between its source U and its sink V such that $\mathbb{1}_V := \sum_{v \in V} e_v$ is a normal vector of F. We claim that for any pseudoline arrangement Λ' supported by $\mathcal{N}$, the scalar product $\langle \mathbb{1}_V \mid \omega(\Lambda') \rangle$ equals $\langle \mathbb{1}_V \mid \omega(\Lambda) \rangle$ when γ is a subset of the contacts of Λ', and is strictly bigger than $\langle \mathbb{1}_V \mid \omega(\Lambda) \rangle$ otherwise.

The set of all pseudoline arrangements supported by $\mathcal{N}$ whose set of contacts contains γ is connected by flips. Since a flip between two such pseudoline arrangements necessarily involves either two pseudolines of U or two pseudolines of V, the corresponding incidence vector is orthogonal to $\mathbb{1}_V$. Thus, the scalar product $\langle \mathbb{1}_V \,|\, \omega(\Lambda') \rangle$ is constant on all pseudoline arrangements whose set of contacts contains γ.

Reciprocally, since γ is a directed cut which separates U and V in the contact graph of Λ, the crossing between two pseudolines of Λ labeled by $u \in U$ and $v \in V$ is positioned as much as possible to the left of the network if $u < v$ and to the right of the network if $u > v$. This implies that the pseudolines labeled by V in any pseudoline arrangement Λ' supported by $\mathcal{N}$ cover each brick at least as many times as the pseudolines of Λ labeled by V. Now assume that γ is not a subset of the contacts of Λ', and consider any commutator of γ which is not a contact of Λ'. Then one of the brick immediately adjacent to this commutator is covered once more by the pseudolines of Λ' labeled by V than by the pseudolines of Λ labeled by V. This implies that $\langle \mathbb{1}_V \,|\, \omega(\Lambda') \rangle > \langle \mathbb{1}_V \,|\, \omega(\Lambda) \rangle$. □

This theorem together with Remark 3.7 have the following immediate applications:

Corollary 3.9 *The brick polytope of an irreducible sorting network with n levels has dimension $n-1$.*

Corollary 3.10 *The brick vector $\omega(\Lambda)$ is a vertex of the brick polytope $\Omega(\mathcal{N})$ if and only if $\Lambda^\#$ is acyclic.*

For example, since their contact graphs are sorted, the source and the sink of the oriented graph of increasing flips always appear as vertices of $\Omega(\mathcal{N})$. These two greedy pseudoline arrangements can be the only vertices of the brick polytope, as happens for 2-level networks.

In general, the application $\omega : \mathrm{Arr}(\mathcal{N}) \to \mathbb{R}^n$ is not injective on $\mathrm{Arr}(\mathcal{N})$ (see *e.g.* Example 3.16). However, the vertices of the brick polytope have precisely one preimage by ω:

Proposition 3.11 *The application $\omega : \mathrm{Arr}(\mathcal{N}) \to \mathbb{R}^n$ restricts to a bijection between the pseudoline arrangements supported by $\mathcal{N}$ whose contact graphs are acyclic and the vertices of the brick polytope $\Omega(\mathcal{N})$.*

Proof: According to Corollary 3.10, the application ω defines a surjection from the pseudoline arrangements supported by $\mathcal{N}$ whose contact graphs are acyclic to the vertices of the brick polytope $\Omega(\mathcal{N})$. To prove injectivity, we use an inductive argument based on the following claims: (i) the greedy pseudoline arrangement $\Gamma(\mathcal{N})$ is the unique preimage of $\omega(\Gamma(\mathcal{N}))$; and (ii) if a vertex of $\Omega(\mathcal{N})$ has a unique preimage by ω, then so do its neighbors in the graph of $\Omega(\mathcal{N})$.

To prove (i), consider a pseudoline arrangement Λ supported by $\mathcal{N}$ such that $\omega(\Lambda) = \omega(\Gamma(\mathcal{N}))$. According to Theorem 3.8, the contact graphs $\Lambda^\#$ and $\Gamma(\mathcal{N})^\#$ have the same incidence cone, which ensures that all arcs of $\Lambda^\#$ are sorted. In other words, all flips in Λ are increasing. Since this property characterizes the greedy pseudoline arrangement, we obtain that $\Lambda = \Gamma(\mathcal{N})$.

To prove (ii), consider two neighbors v, v' in the graph of $\Omega(\mathcal{N})$. Let $i, j \in [n]$ and $\alpha > 0$ be such that $v' - v = \alpha(e_j - e_i)$. Let Λ be a pseudoline arrangement supported by $\mathcal{N}$ such that $v = \omega(\Lambda)$. Let Λ' denote the pseudoline arrangement obtained from Λ by flipping the rightmost contact between its ith and jth pseudolines if $i < j$ and the leftmost one if $i > j$. Then $v' = \omega(\Lambda')$. In particular, if v has two distinct preimages by ω, then so does v'. This proves (ii). □

Corollary 3.12 *The graph of the brick polytope is a subgraph of $G(\mathcal{N})$ whose vertices are the pseudoline arrangements with acyclic contact graphs. However, note that the graph of the brick polytope is not always the subgraph of $G(\mathcal{N})$ induced by the pseudoline arrangements with acyclic contact graphs.*

Remark 3.7(3) and Theorem 3.8 also provides the normal vectors of the brick polytope:

Corollary 3.13 *The facet normal vectors of the brick polytope $\Omega(\mathcal{N})$ are precisely all normal vectors of the incidence cones of the contact graphs of the pseudoline arrangements supported by $\mathcal{N}$. Representative for them are given by the characteristic vectors of the sinks of the minimal directed cuts of these graphs.*

More generally, Theorem 3.8, Remark 3.7, and Proposition 3.11 provide a combinatorial description of the faces of the brick polytope. We need the following definition:

Definition 3.14 *A set γ of commutators of $\mathcal{N}$ is k-*admissible *if there exists a pseudoline arrangement $\Lambda \in \mathrm{Arr}(\mathcal{N})$ whose set of contacts contain γ and such that $\Lambda^{\#} \smallsetminus \gamma^{\#}$ has $n-k$ connected components and $\Lambda^{\#}/(\Lambda^{\#} \smallsetminus \gamma^{\#})$ is acyclic (where $\gamma^{\#}$ is the subgraph of $\Lambda^{\#}$ corresponding to the commutators of γ).*

Corollary 3.15 *Let Φ be the application which associates to a subset X of $\mathbb{R}^n$ the set of commutators of $\mathcal{N}$ which are contacts in all the pseudoline arrangements supported by $\mathcal{N}$ whose brick vectors lie in X. Let Ψ be the application which associates to a set γ of commutators of $\mathcal{N}$ the convex hull of the brick vectors of the pseudoline arrangements supported by $\mathcal{N}$ whose set of contacts contains γ. Then the applications Φ and Ψ define inverse bijections between the k-faces of $\Omega(\mathcal{N})$ and the k-admissible sets of commutators of $\mathcal{N}$.*

Example 3.16 (Duplicated networks) *Consider a network Π with n levels and $2\binom{n}{2}$ commutators obtained by duplicating each commutator of a sorting network $\bar{\Pi}$ with n levels and $\binom{n}{2}$ commutators. Any pseudoline arrangement Λ supported by Π has one contact and one crossing among each pair of adjacent commutators of Π, and the contact graph $\Lambda^{\#}$ is a tournament (an oriented complete graph). Thus, the vertices of the brick polytope $\Omega(\Pi)$ correspond to the permutations of $[n]$, representatives of its facet normal vectors are given by the vectors of $\{0,1\}^n$ except 0^n and 1^n, and its k-faces are in one-to-one correspondence with ordered $(n-k)$-partitions of $[n]$. The brick polytope $\Omega(\Pi)$ is a permutahedron. It is in fact a translate of the usual permutahedron $P_n := \mathrm{conv}\left\{(\sigma(1),\dots,\sigma(n))^T \mid \sigma \in \mathfrak{S}_n\right\}$.*

To finish this section, we apply our results to minimal sorting networks (*i.e.* such that the contact graphs of the pseudoline arrangements they support are trees — see Definition 3.4).

Theorem 3.17 *For any minimal irreducible sorting network $\mathcal{N}$, the simplicial complex $\Delta(\mathcal{N})$ is isomorphic to the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N})$. In particular, the graph of $\Omega(\mathcal{N})$ is isomorphic to the flip graph $G(\mathcal{N})$.*

Proof: Since any orientation on a tree is acyclic, any set γ of commutators of $\mathcal{N}$ such that $\mathcal{N} \smallsetminus \gamma$ is sorting is admissible. Thus, the boundary complex of $\Omega(\mathcal{N})$ is isomorphic to $\Delta(\mathcal{N})$ via Corollary 3.15. □

3.4 Brick polytopes and Minkowski decompositions

Let $\mathcal{N}$ be a sorting network with n levels and let b be a brick of $\mathcal{N}$. For any pseudoline arrangement Λ supported by $\mathcal{N}$, we denote by $\omega(\Lambda, b) \in \mathbb{R}^n$ the characteristic vector of the pseudolines of Λ passing above b. We define the polytope $\Omega(\mathcal{N}, b) := \mathrm{conv}\{\omega(\Lambda, b) \mid \Lambda \in \mathrm{Arr}(\mathcal{N})\} \subset \mathbb{R}^n$. Note that the vertex set of $\Omega(\mathcal{N}, b)$ is contained in the vertex set of an hypersimplex, since the number of pseudolines above b always equals the depth of b. These polytopes decompose $\Omega(\mathcal{N})$ into a Minkowski sum:

Proposition 3.18 *The brick polytope $\Omega(\mathcal{N})$ is the Minkowski sum of the polytopes $\Omega(\mathcal{N}, b)$ associated to all the bricks b of $\mathcal{N}$.*

Proof: Since $\omega(\Lambda) = \sum \omega(\Lambda, b)$ for any pseudoline arrangement Λ supported by $\mathcal{N}$, the brick polytope $\Omega(\mathcal{N})$ is included in the Minkowski sum $\sum \Omega(\mathcal{N}, b)$. To prove equality, we thus only have to prove that any vertex of $\sum \Omega(\mathcal{N}, b)$ is indeed a vertex of $\Omega(\mathcal{N})$.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a linear function, and Λ, Λ' be two pseudoline arrangements related by a flip involving their ith and jth pseudolines. If a brick b of $\mathcal{N}$ is not located between the ith pseudolines of Λ and Λ', then $f(\omega(\Lambda, b)) = f(\omega(\Lambda', b))$. Otherwise, the variation $f(\omega(\Lambda, b)) - f(\omega(\Lambda', b))$ has the same sign as the variation $f(\omega(\Lambda)) - f(\omega(\Lambda'))$. Consequently, the pseudoline arrangement Λ_f supported by $\mathcal{N}$ which minimizes f on $\Omega(\mathcal{N})$, also minimizes f on $\Omega(\mathcal{N}, b)$ for each brick b of $\mathcal{N}$.

Now let v be any vertex of $\sum \Omega(\mathcal{N}, b)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ denote a linear function which is minimized by v on $\sum \Omega(\mathcal{N}, b)$. Then v is the sum of the vertices which minimize f in each summand $\Omega(\mathcal{N}, b)$. Consequently, we obtain that $v = \sum \omega(\Lambda_f, b) = \omega(\Lambda_f)$ is a vertex of $\Omega(\mathcal{N})$. $\square$

Example 3.19 (Duplicated networks, continued) *A duplicated network* Π *(see Example 3.16) has two types of bricks: those located between two adjacent commutators (which replace the commutators of* $\bar{\Pi}$*) and the other ones (which correspond to the bricks of* $\bar{\Pi}$*). For any brick* b *of the later type, the polytope* $\Omega(\Pi, b)$ *is still a single point. Now let* b *be a brick of* Π *located between two adjacent commutators. Let* i, j *be such that* b *replaces the crossing of the* i*th and* j*th pseudolines of the pseudoline arrangement supported by* $\bar{\Pi}$*. Then* $\Omega(\Pi, b)$ *is (a translate of) the segment* $[e_i, e_j]$*. Summing the contributions of all bricks, we obtain that the permutahedron* $\Omega(\Pi)$ *is the Minkowski sum of all segments* $[e_i, e_j]$ *for* $1 \le i < j \le n$*.*

Another Minkowski decomposition of our brick polytopes can be derived from the study of the well-behaved class of generalized permutahedra developed by Postnikov (2009) and Ardila et al. (2010):

Definition 3.20 (Postnikov (2009)) *A* generalized permutohedra *is a polytope with inequality description:*

$$\mathbb{Z}\big(\{z_I\}_{I\in[n]}\big) := \Big\{(x_1,\dots,x_n)^T \in \mathbb{R}^n \;\Big|\; \textstyle\sum_{i=1}^n x_i = z_{[n]} \text{ and } \sum_{i\in I} x_i \ge z_I \text{ for } I \subset [n]\Big\}$$

for some family $\{z_I\}_{I\subset[n]} \in \mathbb{R}^{2^{[n]}}$*.*

Since $\mathbb{Z}(\{z_I\}) + \mathbb{Z}(\{z'_I\}) = \mathbb{Z}(\{z_I + z'_I\})$, the class of generalized associahedra is closed by Minkowski sum and difference (a Minkowski difference $P - Q$ of two polytopes $P, Q \subset \mathbb{R}^n$ is defined only if there exists a polytope R such that $P = Q + R$). Consequently, for any $\{y_I\}_{I\subset[n]} \in \mathbb{R}^{2^{[n]}}$, the Minkowski sum and difference $\mathbb{Y}\big(\{y_I\}_{I\in[n]}\big) := \sum_{I\subset[n]} y_I \triangle_I$ of faces $\triangle_I := \operatorname{conv}\{e_i \mid i \in I\}$ of the standard simplex $\triangle_{[n]}$ is a generalized permutahedron. Reciprocally, it turns out that any generalized permutahedron $\mathbb{Z}(\{z_I\})$ can be decomposed as such a Minkowski sum and difference $\mathbb{Y}(\{y_I\})$, and that $\{y_I\}$ is derived from $\{z_I\}$ by a simple inversion formula when all the inequalities defining $\mathbb{Z}(\{z_I\})$ are tight:

Proposition 3.21 (Postnikov (2009), Ardila et al. (2010)) *Every generalized permutahedron can be written uniquely as a Minkowski sum and difference of faces of the standard simplex:* $\mathbb{Z}(\{z_I\}) = \mathbb{Y}(\{y_I\})$*, where* $y_I = \sum_{J\subset I} (-1)^{|I\smallsetminus J|} z_J$ *if all inequalities* $\sum_{i\in I} x_i \ge z_I$ *defining* $\mathbb{Z}(\{z_I\})$ *are tight.*

Example 3.22 *The classical permutahedron can be written as* $\Pi_n = \mathbb{Z}(\{\frac{|I|(|I|+1)}{2}\}) = \sum_{|I|=2} \triangle_I$*.*

The Minkowski decomposition of Proposition 3.21 is used by Postnikov (2009) to compute the volume of the generalized permutohedra.

According to Lemma 2.3 and Corollary 3.13, all our brick polytopes are generalized permutohedra. It raises the question to compute efficiently their Minkowski decomposition into dilates of faces of the standard simplex. This question was recently addressed by Lange (2011) for the special sorting networks whose brick polytopes are associahedra and which we present in the next section.

4 Hohlweg and Lange's associahedra, revisited

Based on the duality between triangulations of a convex polygon and pseudoline arrangements supported by the 1-kernel of a reduced alternating sorting network, we observe in this section that the brick polytopes of these particular networks specialize to the associahedra of Hohlweg and Lange (2007).

4.1 Duality

We call *reduced alternating sorting network* any network with n levels and $\binom{n}{2}$ commutators such that the commutators adjacent to each intermediate level are alternatively located above and below it. Such a network supports a unique pseudoline arrangement, whose first and last pseudolines both touch its top and its bottom level, and whose intermediate pseudolines all touch either its top or its bottom level.

To a word $x \in \{a,b\}^{n-2}$, we associate the following two dual objects — see Figure 4 (left):

- $\mathcal{P}_x$ denotes the n-gon obtained as the convex hull of $\{p_i \mid i \in [n]\}$ where $p_1 = (1,0)$, $p_n = (n,0)$ and p_{i+1} is the point of the circle of diameter $[p_1, p_n]$ with abscissa $i+1$ and located above $[p_1, p_n]$ if $x_i = a$ and below $[p_1, p_n]$ if $x_i = b$, for all $i \in [n-2]$.
- $\mathcal{N}_x$ denotes the reduced alternating sorting network such that the $(i+1)$th pseudoline touches its top level if $x_i = a$ and its bottom level if $x_i = b$, for all $i \in [n-2]$.

For any $1 \le i < j \le n$, we naturally both label by (i,j) the diagonal $[p_i, p_j]$ of $\mathcal{P}_x$ and the commutator of $\mathcal{N}_x$ where cross the ith and jth pseudolines of the unique pseudoline arrangement supported by $\mathcal{N}_x$.

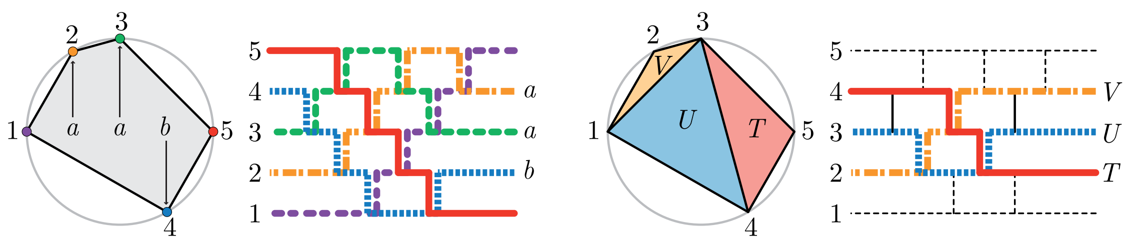

Fig. 4: The polygon $\mathcal{P}_{aab}$ and the network $\mathcal{N}_{aab}$ (left). A triangulation T of $\mathcal{P}_{aab}$ and its dual T^* on $\mathcal{N}_{aab}$ (right).

We call 1-*kernel* of a network $\mathcal{N}$ the network $\mathcal{N}^1$ obtained from $\mathcal{N}$ by erasing its first and last horizontal lines, as well as all commutators incident to them. For a word $x \in \{a,b\}^{n-2}$, the network $\mathcal{N}_x^1$ has $n-2$ levels and $\binom{n}{2} - n$ commutators. Since we erased the commutators between consecutive pseudolines on the top or bottom level of $\mathcal{N}_x$, the remaining commutators are labeled by the internal diagonals of $\mathcal{P}_x$.

Proposition 4.1 (Pilaud and Pocchiola (2010)) *Fix a word $x \in \{a,b\}^{n-2}$. The set of commutators of $\mathcal{N}_x$ labeled by the internal diagonals of a triangulation T of $\mathcal{P}_x$ is the set of contacts of a pseudoline arrangement T^* supported by $\mathcal{N}_x^1$. Reciprocally the internal diagonals of $\mathcal{P}_x$ which label the contacts of a pseudoline arrangement supported by $\mathcal{N}_x^1$ form a triangulation of $\mathcal{P}_x$.*

The dual pseudoline arrangement T^* of a triangulation T of $\mathcal{P}_x$ has one pseudoline Δ^* dual to each triangle Δ of T. A commutator is the crossing (resp. a contact) between two pseudolines Δ^* and Δ'^*

of T^* if it is labeled by the common bisector (resp. by a common edge) of the triangles Δ and Δ'. Consequently, this duality defines an isomorphism between the graph of flips on pseudoline arrangements supported by $\mathcal{N}_x^1$ and the graph of flips on triangulations of $\mathcal{P}_x$. Furthermore, we obtain the following interpretation of the contact graph of T^*:

Lemma 4.2 *The contact graph $(T^*)^\#$ of the dual pseudoline arrangement T^* of a triangulation T is precisely the dual tree of T, with some additional orientations on the edges.*

Remark 4.3 *This duality has been extended by Pilaud and Pocchiola (2010) in two different directions: on the one hand, to* pseudotriangulations *of point sets in general position (see Rote et al. (2008)), and on the other hand, to* multitriangulations *of convex polygon (see Pilaud and Santos (2009)). We refer to the work of Pilaud and Pocchiola (2010) or to Chapter 3 of Pilaud (2010) for precise definitions and details.*

4.2 Associahedra

Consider a reduced alternating sorting network $\mathcal{N}_x$ with n levels. According to Proposition 4.1 and Lemma 4.2, its 1-kernel $\mathcal{N}_x^1$ is a minimal network: the pseudoline arrangements it supports correspond to triangulations of $\mathcal{P}_x$ and their contact graphs are the dual trees of these triangulations. Thus, we obtain:

Proposition 4.4 *For any $x \in \{a,b\}^{n-2}$, the simplicial complex of crossing-free sets of internal diagonals of the convex n-gon $\mathcal{P}_x$ is (isomorphic to) the boundary complex of the polar of the brick polytope $\Omega(\mathcal{N}_x^1)$. Thus, the brick polytope $\Omega(\mathcal{N}_x^1)$ is a realization of the $(n-3)$-dimensional associahedron.*

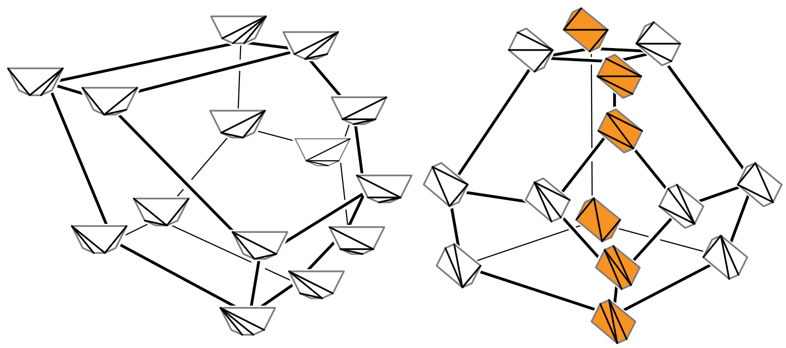

Fig. 5: The brick polytopes $\Omega(\mathcal{N}_{b^4}^1)$ (left) and $\Omega(\mathcal{N}_{a^2b^2}^1)$ (right) provide two different realizations of the 3-dimensional associahedron. The convex hull of the brick vectors of the centrally symmetric triangulations of $\mathcal{P}_{a^2b^2}$ (colored in the picture) is a realization of the 2-dimensional cyclohedron.

Up to translation, our brick polytopes are precisely the associahedra of Hohlweg and Lange (2007). In their presentation, the associahedron is obtained as the convex hull of vectors directly associated to each triangulation of the polygon $\mathcal{P}_x$.

Observe that the associahedron $\Omega(\mathcal{N}_x^1)$ does not depend on the first and last letters of x since we erase the first and last levels of $\mathcal{N}_x$. Furthermore, a network $\mathcal{N}_x$ and its reflection through the vertical (resp. horizontal) axis give rise to affinely equivalent associahedra. Affine equivalence between these associahedra is studied by Bergeron et al. (2009). Two non-affinely equivalent 3-dimensional associahedra are presented in Figure 5.

Example 4.5 *The brick polytope of the* 1*-kernel of the "bubble sort" network* $\mathcal{B}_n := \mathcal{N}_{b^{n-2}}$ *coincides (up to translation) with the* $(n-3)$*-dimensional associahedron of Loday (2004) — see Figure 5 (left).*

We now describe the normal vectors of the facets of our associahedra. For any $x \in \{a,b\}^{n-2}$, the facets of the brick polytope $\Omega(\mathcal{N}_x^1)$ are in bijection with the commutators of $\mathcal{N}_x^1$. The vertices of the facet corresponding to a commutator γ are the brick vectors of the pseudoline arrangements supported by $\mathcal{N}_x^1$ and with a contact at γ. We have already seen that a normal vector of this facet is given by the characteristic vector of the sink of the cut induced by γ in the contact graphs of the pseudoline arrangements supported by $\mathcal{N}_x^1$ and with a contact at γ. In the following lemma, we give an additional description of this characteristic vector:

Lemma 4.6 *Let* Λ *be a pseudoline arrangement supported by* $\mathcal{N}_x^1$ *and let* γ *be a contact of* Λ*. The arc corresponding to* γ *is a cut of the contact graph* $\Lambda^{\#}$ *which separates the pseudolines of* Λ *passing above* γ *from those passing below* γ*.*

Corollary 4.7 *For any* $x \in \{a,b\}^{n-2}$*, the brick polytope* $\Omega(\mathcal{N}_x^1)$ *has* $n-3$ *pairs of parallel facets. The diagonals of* $\mathcal{P}_x$ *corresponding to two parallel facets of* $\Omega(\mathcal{N}_x^1)$ *are crossing.*

Remark 4.8 *As discussed in Section 3.4, the associahedron* $\Omega(\mathcal{N}_x^1)$ *has two different Minkowski decompositions: as a positive Minkowski sum* $\sum_b \Omega(\mathcal{N}_x^1, b)$ *of the polytopes* $\Omega(\mathcal{N}_x^1, b)$ *associated to each brick* b *of* $\mathcal{N}_x^1$*, or as a Minkowski sum and difference* $\sum_{I \subset [n-2]} y_I \triangle_I$ *of faces* $\triangle_I$ *of the standard simplex* $\triangle_{[n-2]}$*.*

In Loday's associahedron (i.e. when $x = b^{n-2}$ *and* $\mathcal{N}_x := \mathcal{B}_n$*), these two decompositions coincide. Indeed, for any* $i, j \in [n]$ *with* $j \geq i+3$*, denote by* $b(i,j)$ *the brick of* $\mathcal{B}_n^1$ *located immediately below the contact between the* i*th and the* j*th pseudoline of the unique pseudoline arrangement supported by* $\mathcal{B}_n$*. Then the Minkowski summand* $\Omega(\mathcal{B}_n^1, b(i,j))$ *is the face* $\triangle_{\{i,\dots,j-2\}}$ *of the standard simplex (up to a translation of vector* $\mathbb{1}_{\{1,\dots,i-1\}\cup\{j-1,\dots,n-2\}}$*). This implies that* $\Omega(\mathcal{B}_n^1) = \sum_{1 \leq i < j \leq n-2} \triangle_{\{i,\dots,j\}}$*.*

For general x*, the Minkowski summands* $\Omega(\mathcal{N}_x^1, b)$ *are not always simplices. A recent work of Lange (2011) describes the coefficients* $\{y_I\}$ *in the Minkowski decomposition of any associahedra* $\Omega(\mathcal{N}_x^1)$*.*

Remark 4.9 *To close this section, we want to mention that we can similarly present Hohlweg & Lange's realizations of the cyclohedra. Namely, consider a antisymmetric word* $x \in \{a,b\}^{2n-2}$ *(i.e. which satisfies* $\{x_i, x_{2n-1-i}\} = \{a,b\}$ *for all* i*), such that the* $(2n)$*-gon* $\mathcal{P}_x$ *is centrally symmetric. Then the convex hull of the brick vectors of the dual pseudoline arrangements of all centrally symmetric triangulations of* $\mathcal{P}_x$ *is a realization of the* $(n-1)$*-dimensional cyclohedron. For example, the centrally symmetric triangulations of* $\mathcal{P}_{a^2b^2}$ *are colored in the right associahedron of Figure 5: the convex hull of the corresponding vertices is a realization of the* 2*-dimensional cyclohedron.*

5 Conclusion

We associated a brick vector to each pseudoline arrangement supported by a given network $\mathcal{N}$. The convex hull of these vectors, the brick polytope, has an interesting combinatorial structure: its graph is a subgraph of the flip graph $G(\mathcal{N})$ and it generalizes Hohlweg and Lange's associahedra.

This paper was originally motivated by the question of the polytopality of the simplicial complex $\Delta(\mathcal{N})$. We could expect our brick polytope to be a first step towards this question, in the sense that $\Omega(\mathcal{N})$ could be a projection of the polar of a realization of $\Delta(\mathcal{N})$. However, this is false in general. More details on this question will appear in a long version of this paper (see Pilaud and Santos (2010)).

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Cyclic sieving for two families of non-crossing graphs

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Abstract. We prove the cyclic sieving phenomenon for non-crossing forests and non-crossing graphs. More precisely, the cyclic group acts on these graphs naturally by rotation and we show that the orbit structure of this action is encoded by certain polynomials. Our results confirm two conjectures of Alan Guo.

Résumé. Nous prouvons le phénomène de crible cyclique pour les forêts et les graphes sans croisement. Plus précisément, le groupe cyclique agit sur ces graphes naturellement par rotation et nous montrons que la structure d'orbite de cette action est codée par certains polynômes. Nos résultats confirment deux conjectures de Alan Guo.

Keywords: cyclic sieving, non-crossing forests, non-crossing graphs

1 Introduction

The notion of cyclic sieving phenomenon was introduced by Reiner, Stanton, and White in [4], generalizing Stembridge's $q = -1$ phenomenon. It involves a finite set X, a cyclic group acting on X, and a polynomial $X(q) \in \mathbb{N}[q]$. The triple $(X, C, X(q))$ is said to exhibit the cyclic sieving phenomenon if for every $c \in C$ of order d,

$$\#\{x \in X : c(x) = x\} = X(w_d), \tag{1}$$

where w_d is a primitive d-th root of unity. In other words, the evaluations of the polynomial $X(q)$ at appropriate roots of unity carry all the numerical information about the C-orbit structure. In particular, $X(1)$ is the cardinality of X. For a survey of the literature on cyclic sieving, the reader is referred to [5].

A non-crossing graph of size n is a graph with vertex set $\{1, 2, \ldots, n\}$ arranged in a circle whose edges are straight line segments that do not cross. See Fig. 1 for illustration. From now on every graph will be non-crossing, and for simplicity, this word will often be omitted. The number of forests of size n and k components is

$$f_{n,k} = \frac{1}{2n-k}\binom{n}{k-1}\binom{3n-2k-1}{n-k}, \tag{2}$$

while the number of graphs of size n with k edges is

$$g_{n,k} = \frac{1}{n-1}\sum_{j=0}^{k}\binom{n-1}{k-j}\binom{n-1}{j+1}\binom{n-2+j}{n-2}. \tag{3}$$

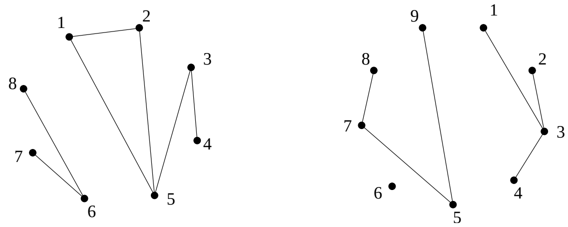

Fig. 1: A non-crossing graph of size 8 with 7 edges and a non-crossing forest of size 9 with 3 components.

Both (2) and (3) were derived in [2]. These formulas admit natural q-analogues:

$$X(q) = \frac{1}{[2n-k]_q} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} 3n-2k-1 \\ n-k \end{bmatrix}_q \tag{4}$$

$$Y(q) = \frac{1}{[n-1]_q} \sum_{j=0}^{n-2} \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_q \begin{bmatrix} n-1 \\ j+1 \end{bmatrix}_q \begin{bmatrix} n-2+j \\ n-2 \end{bmatrix}_q q^{j(j+n-k+2)}, \tag{5}$$

where we are using the usual notation:$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$, $[n]!_q = [n]_q[n-1]_q \cdots [1]_q$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]!_q}.$$

Let the cyclic group of order n act by rotation on graphs with n vertices. In this paper, we prove that forests and graphs, with this action of the cyclic group and the polynomials $X(q)$ and $Y(q)$, respectively, exhibit the cyclic sieving phenomenon. Namely,

Theorem 1.1. *Let X be the set of non-crossing forests on n vertices with k components. If $d|n$ and ω_d is a primitive d-th root of unity, then the number $s_d(n,k)$ of elements of X which are fixed under rotation by $\frac{2\pi}{d}$ is equal to $X(\omega_d)$.*

Theorem 1.2. *Let Y be the set of non-crossing graphs on n vertices with k edges. If $d|n$ and ω_d is a primitive d-th root of unity, then the number $s_d(n,k)$ of elements of Y which are fixed under rotation by $\frac{2\pi}{d}$ is equal to $Y(\omega_d)$.*

This proves the conjectures from [3], where Guo proves the cyclic sieving phenomenon for non-crossing connected graphs. It should be noted that the cyclic-sieving phenomenon for trees, namely the case $k = 1$ in Theorem 1.1, was first proved by Eu and Fu in [1]. The authors first prove that quadrangulations of a polygon exhibit the cyclic sieving phenomenon, where the cyclic action is cyclic rotation of the polygon, and then give a bijection between quadrangulations of a $2n$-gon and trees on n vertices, which preserves the cyclic sieving phenomenon.

In the following two sections we prove Theorem 1.1 and Theorem 1.2 by treating each case separately. We follow a similar line of proof as in the case of connected graphs. As the reader can note, there are structural similarities between these families of graphs that at places require similar arguments. Despite this, none of the results implies another one and all these graphs have to be treated separately. The proof in each case seems to work out “magically” due to the fact that the corresponding generating functions satisfy certain polynomial equations that help us simplify the expressions we obtain and apply Lagrange inversion. This note aims to highlight the similarities and the differences between the aforementioned graphs, with the hope that they would give an insight that would lead to a unifying proof.

As we have mentioned, in several places we will use Lagrange inversion to extract coefficients of certain generating functions.

Lagrange inversion. *Let $\phi(u) \in \mathbb{Q}[[u]]$ be a formal power series with $\phi(0) \neq 0$, and let $y(z) \in \mathbb{Q}[[u]]$ satisfy $y = z\phi(y)$. Then, for an arbitrary series ψ, the coefficient of z^n in $\psi(y)$ is given by*

$$[z^n]\psi(y(z)) = \frac{1}{n}[u^{n-1}]\phi(u)^n\psi'(u).$$

Lagrange inversion may be applied to bivariate generating functions by treating the second variable as a parameter.

2 Non-crossing forests

The authors in [2] computed the numbers $f_{n,k}$ using Lagrange inversion. In the process, they obtained some polynomial equations related to the generating function $F(z,w) = \sum f_{n,k} z^n w^k$ which will be useful in our proofs. We state them here.

Let $T(z)$ be the generating function for non-crossing trees with respect to size. Then T satisfies

$$T^3 - zT + z^2 = 0. \tag{6}$$

Each forest can be obtained from a tree by substituting each vertex by a pair (vertex, forest). The substitution yields

$$F = 1 + T(zF) \tag{7}$$

whereby we are counting the empty forests as well. One can use (6) to eliminate T, which yields

$$F^3 + (z^2 - z - 3)F^2 + (z+3)F - 1 = 0. \tag{8}$$

This equation admits a Lagrange form, upon setting $F = 1 + wy$,

$$y = z(1+wy)\left(\frac{1-\sqrt{1-4y}}{2y}\right), \tag{9}$$

from which the explicit formula (2) for $f_{n,k}$ follows.

We will also use the following property of the Catalan generating function $Cat(z) = \frac{1-\sqrt{1-4z}}{2z}$, which can also be obtained using Lagrange inversion:

$$[z^m]Cat(z)^n = \frac{n}{n+m}\binom{2m+n-1}{m}. \tag{10}$$

We will prove Theorem 1.1 by verifying the condition (1) stated in the introduction. For that, we first evaluate $X(q)$ at roots of unity. The following lemma is repeatedly used.

Lemma 2.1. *Let n, m_1, m_2, k, and d be positive integers. Let ω_d be a primitive d-th root of unity. Then*

(a) *The factor* $[m]_q$ *has a simple zero at* $q = w_d$ *if and only if* $d|m$, $d \neq 1$.

(b) *If* $m_1 \equiv m_2$ *mod* d, *then*

$$\lim_{q \to w_d} \frac{[m_1]_q}{[m_2]_q} = \begin{cases} \frac{m_1}{m_2} & \text{if } m_1 \equiv m_2 \equiv 0 \text{ mod } d, \\ 1 & \text{if } m_1 \equiv m_2 \not\equiv 0 \text{ mod } d. \end{cases}$$

(c) *(q -Lucas theorem) If* $n = ad + b$ *and* $k = rd + s$, *where* $0 \geq r, s \leq d - 1$, *then*

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q=\omega_d} = \binom{a}{r} \begin{bmatrix} b \\ s \end{bmatrix}_{q=\omega_d}.$$

Proposition 2.2. *Let* $d|n$ *and* $n' = \frac{n}{d}$. *Then*

$$X(\omega_d) = \begin{cases} \frac{n'-k'+1}{2n'-k'} \binom{n'}{k'-1} \binom{3n'-2k'-1}{n'-k'}, & \text{if } d \geq 2 \text{ and } k' = \frac{k}{d} \in \mathbb{N}, \\ \binom{n'}{k'} \binom{3n'-2k'-2}{n'-k'-1}, & \text{if } d = 2 \text{ and } k' = \frac{k+1}{2} \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: If $d|k$, then by part (c) of Lemma 2.1,

$$\begin{bmatrix} 3n - 2k - 1 \\ n - k \end{bmatrix}_{q=\omega_d} = \binom{3n' - 2k' - 1}{n' - k'}.$$

On the other hand, both $[2n - k]_{q=\omega_d} = 0$ and $\begin{bmatrix} n \\ k-1 \end{bmatrix}_{q=\omega_d} = 0$. By examining the factors in the numerator and denominator and using parts (a) and (b) of Lemma 2.1, one gets

$$\frac{1}{[2n - k]_{q=\omega_d}} \begin{bmatrix} n \\ k - 1 \end{bmatrix}_{q=\omega_d} = \frac{n' - k' + 1}{2n' - k'} \binom{n'}{k' - 1}.$$

For $d = 2$ and k odd, the result follows by applying the q-Lucas theorem to both q-binomial coefficients. Finally, if $d \geq 3$ and d does not divide k, $[2n - k]_{q=\omega_d} \neq 0$, the first q binomial coefficient vanishes for $k \not\equiv 1$ mod d, and the second one vanishes for $k \equiv 1$ mod d. □

2.1 The case $d = 2$ and k is odd

Every centrally symmetric forest on n vertices has exactly one diameter edge and is determined by the forest on one side of that diameter which necessarily has $k' + 1$ components. Denote by $\tilde{f}_{n,k}$ the number of forests on n vertices with k components such that the vertices 1 and n are connected by an edge. Since there are n' ways to choose the diameter edge, we have

$$s_2(2n', 2k' + 1) = n' \tilde{f}_{n'+1,k'+1}.$$

Proposition 2.3. *The number of non-crossing forests on n vertices with k components in which the vertices 1 and n are connected by an edge is given by*

$$\tilde{f}_{n,k} = \frac{1}{n-1}\binom{n-1}{k-1}\binom{3n-2k-3}{n-k-1}. \tag{11}$$

Consequently,

$$s_2(2n', 2k'+1) = \binom{n'}{k'}\binom{3n'-2k'-2}{n'-k'-1}.$$

Proof: Removing the edge $\{1, n\}$ from such a forest would produce a forest with $k+1$ components in which the vertices 1 and n are in different components. Such a forest can be obtained from two trees by substituting each vertex except n by a pair (vertex, forest). The substitution construction yields

$$\sum \tilde{f}_{n,k} z^n w^{k+1} = \frac{(wT(zF))^2}{F}.$$

One can eliminate T using (7) and then, since F satisfies (8), we get

$$\sum \tilde{f}_{n,k} z^n w^{k+1} = F^2 + (z^2w^3 - zw^2 - 2)F + (zw^2 + 1). \tag{12}$$

Therefore,

$$\tilde{f}_{n,k} = [z^n w^{k+1}]F^2 + [z^{n-2}w^{k-2}]F - [z^{n-1}w^{k-1}]F - 2[z^n w^{k+1}]F. \tag{13}$$

Only one of these coefficients needs to be computed:

$$[z^n w^{k+1}]F^2 = [z^n w^{k+1}](1+wy)^2 = 2[z^n w^k]y + [z^n w^{k-1}]y^2$$

and, using Lagrange inversion,

$$\begin{aligned}
[z^n w^{k-1}]y^2 &= \frac{1}{n}[u^{n-1}w^{k-1}](1+wu)^n Cat(u)^n 2u \\
&= \frac{2}{n}\binom{n}{k-1}[u^{n-k-1}]Cat(u)^n = \frac{2}{2n-k-1}\binom{n}{k-1}\binom{3n-2k-3}{n-k-1}.
\end{aligned}$$

Substituting this into (13) yields

$$\tilde{f}_{n,k} = \frac{2}{2n-k-1}\binom{n}{k-1}\binom{3n-2k-3}{n-k-1} + f_{n-2,k-2} - f_{n-1,k-1},$$

which simplifies to (11). $\square$

2.2 The case $d = 2$ and k is even

For each centrally symmetric forest of size $2n'$, there are well-defined vertices

$$1 \le v_1 < v_2 < \cdots < v_m \le n',$$

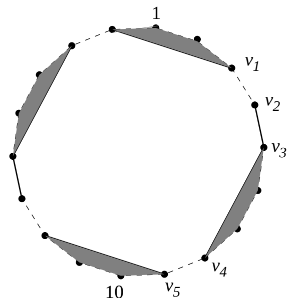

Fig. 2: The vertices $v_1, \ldots, v_m$ for a forest with $m = 5$.

such that for $1 \leq i \leq m$, the graph on the vertices between v_i and v_{i+1}, inclusive, is a forest with an edge between v_i and v_{i+1} (we define v_{m+1} to be the vertex $v_1 + n'$), or possibly empty in the case when $v_{i+1} - v_i = 1$, and v_i and v_j are not connected by an edge if $|j - i| \geq 2$. See Fig. 2 for illustration.

Recall that $\tilde{f}_{n,n-k}$ counts forests of size n and $n-k$ components, and therefore k edges, which contain the edge $\{1, n\}$. Define $f^*_{n,k}$ to be

$$f^*_{n,k} = \begin{cases} 1, & \text{if } n = 2, k = 0 \\ \tilde{f}_{n,n-k}, & \text{otherwise.} \end{cases}$$

Then, the number $a_{n,k}$ of centrally symmetric graphs of size $2n$ and k pairs of antipodal edges (where we count each diameter edge as a pair) that can be obtained by gluing together forests counted by $f^*_{n,k}$ is

$$a_{n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots+k_m=k} \sum_{1\leq v_1<v_2<\cdots<v_m\leq n} \prod f^*_{v_{i+1}-v_i+1,k_i}. \tag{14}$$

Note that not all the graphs counted by $a_{n,k}$ are forests. Namely, a cycle will be formed exactly when none of the forests we use in the gluing process is the forest with 2 vertices and no edges. The number of such graphs is

$$b_{n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots+k_m=k} \sum_{1\leq v_1<v_2<\cdots<v_m\leq n} \prod \tilde{f}_{v_{i+1}-v_i+1,v_{i+1}-v_i+1-k_i}. \tag{15}$$

Therefore,

$$s_2(2n', 2k') = a_{n',n'-k'} - b_{n',n'-k'} \tag{16}$$

and this difference indeed does not count forests with odd number of edges because they are counted by both $a_{n',n'-k'}$ and $b_{n',n'-k'}$. Let $A(z,w)$, $B(z,w)$, $\tilde{F}(z,w)$, and $F^*(z,w)$ denote the generating functions for $a_{n,k}$, $b_{n,k}$, $\tilde{f}_{n,k}$, and $f^*_{n,k}$, respectively. Then

$$F^*(z,w) = \tilde{F}(zw, \frac{1}{w}) + z^2. \tag{17}$$

As explained in [3, Lemma 4.2], (14) and (15) imply that

$$A(z,w) = z\frac{\frac{\partial}{\partial z}\left(\frac{F^*(z,w)}{z}\right)}{1-\frac{F^*(z,w)}{z}} \qquad \text{and} \qquad B(z,w) = z\frac{\frac{\partial}{\partial z}\left(\frac{\tilde{F}(zw,1/w)}{z}\right)}{1-\frac{\tilde{F}(zw,1/w)}{z}}.$$

Using (17), we get

$$A(zw,\frac{1}{w}) = z\frac{\frac{\partial}{\partial z}\left(\frac{\tilde{F}(z,w)+z^2w^2}{zw}\right)}{1-\frac{\tilde{F}(z,w)+z^2w^2}{zw}} \qquad \text{and} \qquad B(zw,\frac{1}{w}) = z\frac{\frac{\partial}{\partial z}\left(\frac{\tilde{F}(z,w)}{zw}\right)}{1-\frac{\tilde{F}(z,w)}{zw}}.$$

Solving these differential equations, we obtain

$$\begin{aligned}\sum \frac{a_{n,n-k}}{n}z^n w^k &= -\log(1-\frac{\tilde{F}(z,w)+z^2w^2}{zw}) \qquad \text{and}\\ \sum \frac{b_{n,n-k}}{n}z^n w^k &= -\log(1-\frac{\tilde{F}(z,w)}{zw}).\end{aligned}$$

From (12) we get

$$\tilde{F} = \frac{1}{w}F^2 + (z^2w^2 - zw - \frac{2}{w})F + (zw + \frac{1}{w}).$$

Using this and (8), we obtain

$$\sum \frac{a_{n,n-k}-b_{n,n-k}}{n}z^n w^k = \log(1-\frac{\tilde{F}}{zw}) - \log(1-\frac{\tilde{F}+z^2w^2}{zw}) = \log(F).$$

Finally, using (16) and Lagrange inversion, we have

$$\begin{aligned}s_2(2n',2k') = n'[z^{n'}w^{k'}]\log(F) &= \frac{n'}{2n'-k'}\binom{n'-1}{k'-1}\binom{3n'-2k'-1}{n'-k'} \\ &= \frac{n'-k'+1}{2n'-k'}\binom{n'}{k'-1}\binom{3n'-2k'-1}{n'-k'} = X(-1).\end{aligned} \tag{18}$$

2.3 *The case* $d \geq 3$

Since each edge in the forest in not longer than $\frac{n}{d}$, it lies in a free orbit under the action of rotation. Therefore, if d does not divide the number of edges $n-k$, which happens if and only if d does not divide k, then in fact there are no forests with k components that are fixed under rotation by $\frac{2\pi}{d}$. This agrees with the fact that in this case also $X(\omega_d) = 0$.

Consider now the case when $d|k$. Let $n' = \frac{n}{d}$ and $k' = \frac{k}{d}$. We have the following lemma.

Lemma 2.4. $s_d(n,k) = s_2(2n',2k')$.

Proof: We construct a bijection between forests on dn' vertices and dk' components that are fixed under rotation by $\frac{2\pi}{d}$ and forests counted by $s_2(2n',2k')$ in the following way. Construct a forest on $2n'$ vertices labeled $1,2,\ldots,2n'$ by drawing an edge from i to j for every such edge in the original graph. Moreover, for every edge from i to $(d-1)n'+j$, $1 \leq i, j \leq n'$ in the original graph draw an edge between i and $n'+j$. See Fig. 3 for illustration. Note that if F is a forest fixed by a rotation of $\frac{2\pi}{d}$, then all its edges are of length strictly less than d, where by length of an edge we mean the absolute difference of its endpoints. Moreover, it has a total of $d(n'-k')$ edges which form orbits of size d under this action. This implies that the graph obtained by the construction described above will be a non-crossing forest with $2n'$ vertices and $2(n'-k')$ edges, i.e., $2k'$ components. It is easy to see that this map is a bijection. $\square$

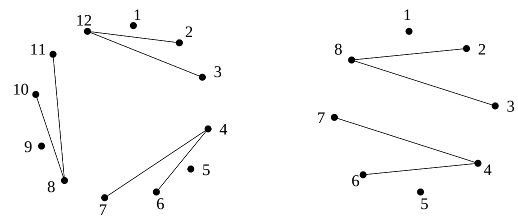

Fig. 3: Example of the bijection for $n = 12$, $k = 3$, and $d = 3$. The edges $\{2, 8\}$ and $\{3, 8\}$ in the image correspond to the edges $\{2, 12\}$ and $\{3, 12\}$ in the original forest.

Combining Lemma 2.4, (18), and Proposition 2.2 we get

$$s_d(n, k) = s_2(2n', 2k') = \frac{n'-k'+1}{2n'-k'}\binom{n'}{k'-1}\binom{3n'-2k'-1}{n'-k'} = X(\omega_d).$$

This completes the proof of Theorem 1.1.

3 Non-crossing graphs

Let $C(z, w)$ and $G(z, w)$ be the generating functions for connected graphs and graphs, respectively, where z marks vertices and w marks edges. Using a combinatorial argument, it can be shown [2] that

$$wC^3 + wC^2 - z(1+2w)C + z^2(1+w) = 0.$$

Each graph can then be obtained by replacing a vertex in a non-crossing graph by a (vertex, graph). Therefore,

$$G(z, w) = 1 + C(zG(z, w), w). \tag{19}$$

Eliminating C in (19) we arrive at

$$wG^2 + ((1+w)z^2 - (1+2w)z - 2w)G + w + z(1+2w) = 0. \tag{20}$$

This becomes amenable to Lagrange inversion upon the change of variables $G = 1+z+zy$ that transforms it into

$$y = z(1+w)\frac{1+y}{1-wy}. \tag{21}$$

Proposition 3.1. *Let $d|n$ and $n' = \frac{n}{d}$. Then*

$$Y(\omega_d) = \begin{cases} \sum_{j=0}^{k'} \binom{n'-1}{k'-j}\binom{n'-1}{j}\binom{n'+j-2}{n'-2} & \text{if } d = 2 \text{ and } k' = \frac{k}{2} \in \mathbb{N}, \\ \sum_{j=0}^{k'} \binom{n'-1}{k'-j}\binom{n'}{j+1}\binom{n'+j-1}{n'-1} & \text{if } d = 2 \text{ and } k' = \frac{k+1}{2} \in \mathbb{N}, \\ \sum_{j=0}^{k'} \binom{n'}{k'-j}\binom{n'-1}{j}\binom{n'+j-1}{n'-1} & \text{if } d \geq 3 \text{ and } k' = \frac{k}{d} \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: Set $j' = \lfloor \frac{j}{d} \rfloor$. In the case $d = 2$, using part (c) of Lemma 2.1, one gets $\begin{bmatrix} n-2+j \\ n-2 \end{bmatrix}_{q=\omega_d} = \binom{n'+j'-1}{n'-1}$,

$$\begin{bmatrix} n-1 \\ j+1 \end{bmatrix}_{q=\omega_d} = \begin{cases} \binom{n'-1}{j'} & \text{if } j \text{ is even,} \\ \binom{n'-1}{j'+1} & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{bmatrix} n-1 \\ k-j \end{bmatrix}_{q=\omega_d} = \begin{cases} \binom{n'-1}{k'-j'-1} & \text{if } k \text{ is even, } j \text{ is odd} \\ \binom{n'-1}{k'-j'} & \text{otherwise.} \end{cases}$$

Summing over all j' yields the first two parts of the proposition. The case $d \geq 3$ can be analyzed similarly, by evaluating each q-binomial coefficient separately. In both cases the summands of $Y(\omega_d)$ are nonzero only when $j \equiv 0, 1 \bmod d$. If $d|k$ the resulting sum is the third part of the proposition. Otherwise, the terms pairwise cancel and the sum is zero. □

3.1 The case $d = 2$ and k is odd

Similarly to the case of forests, every centrally symmetric graph with odd number of edges has exactly one diameter edge and is completely determined by the subgraph on one side of that edge. Let $\tilde{g}_{n,k}$ be the number of graphs with n vertices and k edges that contain the edge $\{1, n\}$. Then

$$s_2(n, k) = n'\tilde{g}_{n'+1,k'+1}.$$

Proposition 3.2. *The number of non-crossing graphs on n vertices with k edges in which the vertices 1 and n are connected by an edge is given by*

$$\tilde{g}_{n,k} = \frac{1}{n-1} \sum_{j=0}^{k-1} \binom{n-2}{k-j-1}\binom{n-1}{j+1}\binom{n+j-2}{n-2}. \tag{22}$$

Consequently,

$$s_2(2n', 2k'+1) = Y(-1).$$

Proof: Let $\tilde{G}(z, w)$ be the generating function for $\tilde{g}_{n,k}$ and let $\tilde{C}(z, w)$ be the generating function for connected graphs in which 1 and n are connected by a vertex. Each graph which has the edge $\{1, n\}$ can be obtained from a connected graph that has the edges $\{1, n\}$ by substituting each vertex with a pair (vertex, graph). Therefore,

$$\tilde{G} = \frac{\tilde{C}(zG, w)}{G}.$$

The author in [3] derives the following formula for $\tilde{C}$, although not explicitly stated in this form:

$$\tilde{C} = \frac{w}{w+1}(C^2 + C - z).$$

Using (19) we can eliminate C and we arrive at

$$\tilde{G} = \frac{w}{w+1}\,(G-1-z) = \frac{zwy}{w+1}. \tag{23}$$

From here using Lagrange Inversion, one obtains the formula (22). □

3.2 The case $d = 2$ and k is even

Let $d_{n,k}$ denote the number of centrally symmetric graphs with $2n$ vertices and k pairs of antipodal edges, where each diameter edge is again counted as a pair.

Lemma 3.3.

$$d_{n,k} = \sum_{j=0}^{k} \binom{n}{k-j}\binom{n-1}{j}\binom{n+j-1}{n-1}. \tag{24}$$

Proof: For each graph counted by $d_{n,k}$ there is a unique subset of vertices $1 \le v_1 < \cdots < v_m \le n$, such that the graph on the vertices between v_i and v_{i+1}, inclusive, has an edge between v_i and v_{i+1} (we define v_{m+1} to be the vertex $v_1 + n'$), or is possibly empty in the case when $v_{i+1} - v_i = 1$, and v_i and v_j are not connected by an edge if $|j-i| \ge 2$. Set

$$g^*_{n,k} = \begin{cases} 1, & \text{if } n = 2, k = 0, \\ \tilde{g}_{n,k}, & \text{otherwise.} \end{cases}$$

Then, from the argument above, we have

$$d_{n,k} = \sum_{m=1}^{n} \sum_{k_1+\cdots+k_m=k} \sum_{1\le v_1<v_2<\cdots<v_m\le n} \prod g^*_{v_{i+1}-v_i+1,k_i}, \tag{25}$$

and, consequently,

$$\sum d_{n,k} z^n w^k = z\frac{\frac{\partial}{\partial z}\left(\frac{\tilde{G}+z^2}{z}\right)}{1-\frac{\tilde{G}+z^2}{z}}.$$

Dividing both sides by z and integrating yields

$$\sum \frac{d_{n,k}}{n} z^n w^k = -\log\left(1-\frac{\tilde{G}+z^2}{z}\right) = \log(1+y).$$

The last equality follows from (21) and (23). Extracting the coefficients of $\log(1+y)$ yields (24). □

On the other hand, $d_{n,k} = s_2(2n, 2k-1) + s_2(2n, 2k)$. This can be used to compute $s_2(2n, 2k)$:

$$\begin{aligned} s_2(2n,2k) &= d_{n,k} - s_2(2n,2k-1) \\ &= \sum_{j=0}^{k} \binom{n}{k-j}\binom{n-1}{j}\binom{n+j-1}{n-1} - \sum_{j=0}^{k-1} \binom{n-1}{k-j-1}\binom{n}{j+1}\binom{n+j-1}{n-1} \\ &= \sum_{j=0}^{k} \binom{n-1}{k-j}\binom{n-1}{j}\binom{n+j-2}{n-2}. \end{aligned}$$

The last equality follows by comparing the coefficients in front of $a^{n-1}b^k$ on both sides of the identity

$$\frac{(1+a)^{n-1}(1+b)^n}{(1-ab)^n} - \frac{b(1+a)^n(1+b)^{n-1}}{(1-ab)^n} = \frac{(1+a)^{n-1}(1+b)^{n-1}}{(1-ab)^{n-1}}.$$

This completes the proof of $s_2(2n, 2k) = Y(-1)$.

3.3 The case $d \geq 3$

Consider first the case when d does not divide k. Again every edge lies in a free orbit under the action of rotation. Therefore, there are no graphs with k edges that are fixed under rotation by $\frac{2\pi}{d}$. This agrees with the fact that in this case also $Y(\omega_d) = 0$.

Suppose now that $d|k$. Note that the edges of each graph counted by $s_d(n, k)$ are no longer than d. Those graphs that have no edges of length d are in bijection with centrally symmetric graphs on $2n'$ vertices and $2k'$ edges via a map defined as in the proof of Lemma 2.4. On the other hand, the same map takes the graphs that have edges of length d (those edges form a regular n'-gon) to centrally symmetric graphs on $2n'$ vertices and $2k' - 1$ edges. So,

$$\begin{aligned} s_d(n,k) &= s_2(2n', 2k') + s_2(2n', 2k'-1) \\ &= \sum_{j=0}^{k'} \binom{n'-1}{k'-j}\binom{n'-1}{j}\binom{n'+j-2}{n'-2} + \sum_{j=0}^{k'-1} \binom{n'-1}{k'-j-1}\binom{n'}{j+1}\binom{n'+j-1}{n'-1} \\ &= \sum_{j=0}^{k'} \binom{n'}{k'-j}\binom{n'-1}{j}\binom{n'+j-1}{n'-1} = Y(\omega_d). \end{aligned}$$

This completes the proof of Theorem 1.2.

4 Final remarks

Recall that the definition of the cyclic sieving phenomenon asks for polynomials with nonnegative integer coefficients. For completeness, here we prove that our functions $X(q)$ and $Y(q)$ indeed have that property.

Proposition 4.1. $X(q), Y(q) \in \mathbb{N}[q]$.

Proof: To show that $X(q) \in \mathbb{Q}$, it suffices to show that $X(q)$ is a polynomial in q. Since all the roots of the numerator and the denominator of $X(q)$ are roots of unity, this follows from the fact that for each d-th primitive root of unity ω_d, the order of the zero at $q = \omega_d$ is no smaller in the numerator than in the denominator. Namely, consider the expansion

$$X(q) = \frac{\overbrace{[n]_q \cdots [n-k+2]_q}^{A}}{\underbrace{[k]_q \cdots [1]_q}_{C}} \frac{\overbrace{[3n-2k-1]_q \cdots [2n-2k+1]_q}^{B}}{\underbrace{[n-k]_q \cdots [1]_q}_{D}}.$$

The denominator has zero at $q = \omega_d$ with multiplicity $\lfloor \frac{k-1}{d} \rfloor + \lfloor \frac{n-k}{d} \rfloor$. A, and B have zero at $q = \omega_d$ with multiplicities at least $\lfloor \frac{k-1}{d} \rfloor$ and $\lfloor \frac{n-k-1}{d} \rfloor$, respectively. If d does not divide $(n-k)$ then $\lfloor \frac{n-k-1}{d} \rfloor =$

$\lfloor \frac{n-k}{d} \rfloor$ and we are done. If d divides $(n-k)$ but not n, then in fact B has zero of order $\lfloor \frac{n-k}{d} \rfloor$. In the last case, when d divides both $(n-k)$ and n, A has in fact zeroof order $\lceil \frac{k-1}{d} \rceil = \lfloor \frac{k-1}{d} \rfloor + 1$.

It is well known that the q-binomial coefficients are polynomials with symmetric unimodal nonnegative integer coefficients. Hence, so is the product

$$\begin{bmatrix} n \\ k-1 \end{bmatrix}_q \begin{bmatrix} 3n-2k-1 \\ n-k \end{bmatrix}_q .$$

The fact that $X(q) \in \mathbb{N}[q]$ now follows from [4, Proposition 10.1]. Similarly, one can show that each summand of $Y(q)$ is a polynomial with nonnegative integer coefficients. Namely, consider the expansion of the j-th term in the sum of $Y(q)$:

$$\frac{\overbrace{[n-1]_q \cdots [n-k+j]_q}^{A} \overbrace{[n-2-j]_q \cdots [j+2]_q}^{B}}{\underbrace{[k-j]_q \cdots [1]_q}_{C} \underbrace{[j]_q \cdots [1]_q}_{D} \underbrace{[n-j-2]_q \cdots [1]_q}_{E}} q^{j(j+n-k+2)}.$$

The denominator has zero at $q = \omega_d$ of order $\lfloor \frac{k-j}{d} \rfloor + \lfloor \frac{j}{d} \rfloor + \lfloor \frac{n-j-2}{d} \rfloor$, while the numerator has zero of order at least $\lfloor \frac{k-j}{d} \rfloor + \lfloor \frac{n-3}{d} \rfloor$. Note $\lfloor \frac{n-3}{d} \rfloor \geq \lfloor \frac{j}{d} \rfloor + \lfloor \frac{n-j-2}{d} \rfloor$ unless d divides both j and $(n-j-2)$. But in this case, B has in fact a zero of order $\lceil \frac{n-3}{d} \rceil = \lfloor \frac{j}{d} \rfloor + \lfloor \frac{n-j-2}{d} \rfloor$. This proves that each summand in $Y(q)$ is in $\mathbb{Q}(q)$. The fact that it is in $\mathbb{N}[q]$ follows the same way as for $X(q)$. $\square$

Given that the polynomials $X(q)$ and $Y(q)$ are natural q-analogues of the cardinalities of the corresponding sets, a natural question to ask is whether they are generating functions for some statistics defined on these sets. To the best of the author's knowledge, such statistics have not been discovered. As suggested by Sagan [5, Section 12.3], if found, they could lead to a purely combinatorial proof of the cyclic sieving phenomenon.

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Isotropical Linear Spaces and Valuated Delta-Matroids

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Abstract. The spinor variety is cut out by the quadratic Wick relations among the principal Pfaffians of an $n \times n$ skew-symmetric matrix. Its points correspond to n-dimensional isotropic subspaces of a $2n$-dimensional vector space. In this paper we tropicalize this picture, and we develop a combinatorial theory of tropical Wick vectors and tropical linear spaces that are tropically isotropic. We characterize tropical Wick vectors in terms of subdivisions of Delta-matroid polytopes, and we examine to what extent the Wick relations form a tropical basis. Our theory generalizes several results for tropical linear spaces and valuated matroids to the class of Coxeter matroids of type D.

Résumé. La variété spinorielle est decoupée par les relations quadratiques de Wick parmi les Pfaffiens principaux d'une matrice antisymétrique $n \times n$. Ses points correspondent aux sous-espaces isotropes à n dimensions d'un espace vectoriel de dimension $2n$. Dans cet article nous tropicalison cette description, et nous développons une théorie combinatoire de vecteurs tropicaux de Wick et d'espaces linéaires tropicaux qui sont tropicallement isotropes. Nous caractérisons des vecteurs tropicaux de Wick en termes de subdivisions des polytopes Delta-matroïde, et nous étudions dans quelle mesure les relations de Wick forment une base tropicale. Notre théorie généralise plusieurs résultats pour les espaces linéaires tropicaux et évaluait des matroïdes à la classe des matroïdes de Coxeter du type D.

Keywords: spinor variety, isotropic subspace, tropical linear space, valuated matroid, delta-matroid, matroid subdivision.

1 Introduction

Let n be a positive integer, and let V be a $2n$-dimensional vector space over an algebraically closed field K of characteristic 0. Fix a basis $e_1, e_2, \dots, e_n, e_{1^*}, e_{2^*}, \dots, e_{n^*}$ for V, and consider the symmetric bilinear form on V defined as

$$Q(x, y) = \sum_{i=1}^{n} x_i \, y_{i^*} + \sum_{i=1}^{n} x_{i^*} \, y_i,$$

for any $x, y \in V$ with coordinates $x = (x_1, \dots, x_n, x_{1^*}, \dots, x_{n^*})$ and $y = (y_1, \dots, y_n, y_{1^*}, \dots, y_{n^*})$. An n-dimensional subspace $U \subseteq V$ is called (totally) **isotropic** if for all $u, v \in U$ we have $Q(u, v) = 0$,

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or equivalently, for all $u \in U$ we have $Q(u, u) = 0$. Denote by $\mathcal{P}(n)$ the collection of subsets of the set $[n] := \{1, 2, \ldots, n\}$. The space of pure spinors $\mathrm{Spin}^{\pm}(n)$ is an algebraic set in projective space $\mathbb{P}^{\mathcal{P}(n)-1}$ that parametrizes totally isotropic subspaces of V. Its defining ideal is generated by very special quadratic equations, known as Wick relations. We will discuss these relations in Section 2. Since any linear subspace $W \subseteq K^n$ defines an isotropic subspace $U := W \times W^{\perp} \subseteq K^{2n}$, all Grassmannians $G(k, n)$ can be embedded naturally into the space of pure spinors, and in fact, Wick relations can be seen as a natural generalization of Plücker relations.

In [Spe08], Speyer studied tropical Plücker relations, tropical Plücker vectors (or valuated matroids [DW92]), and their relation with tropical linear spaces. In his study he showed that these objects have a beautiful combinatorial structure, which is closely related to matroid polytope decompositions. In this paper we will study the tropical variety and prevariety defined by all Wick relations, the combinatorics satisfied by the vectors in these spaces (valuated Δ-matroids [DW91]), and their connection with tropical linear spaces that are tropically isotropic (which we will call isotropical linear spaces). Much of our work can be seen as a generalization to type D of some of the results obtained by Speyer, or as a generalization of the theory of Δ-matroids to the "valuated" setup.

We will say that a vector $p \in \mathbb{T}^{\mathcal{P}(n)}$ with coordinates in the tropical semiring $\mathbb{T} := \mathbb{R} \cup \{\infty\}$ is a tropical Wick vector if it satisfies the tropical Wick relations. A central object for our study of tropical Wick vectors will be that of an even Δ-matroid [Bou87]. Even Δ-matroids are a natural generalization of classical matroids, and much of the theory of matroids can be extended to them. In particular, their associated polytopes are precisely those $0/1$ polytopes whose edges have the form $\pm e_i \pm e_j$, with $i \neq j$. In this sense, even Δ-matroids can be seen as Coxeter matroids of type D, while classical matroids correspond to Coxeter matroids of type A. We will present all the necessary background on even Δ-matroids in Section 3. Tropical Wick vectors will be valuated Δ-matroids: real functions on the set of bases of an even Δ-matroid satisfying certain "valuated exchange property" which is amenable to the greedy algorithm (see [DW91]).

In Section 4 we will be interested in determining for what values of n the Wick relations form a tropical basis, and we will provide an answer for all $n \neq 6$. We will prove in Section 5 that in fact tropical Wick vectors can be characterized in terms of even Δ-matroid polytope subdivisions. We give a complete list of all even Δ-matroids up to isomorphism on a ground set of at most 5 elements, together with their corresponding spaces of valuations, in the website `http://math.berkeley.edu/~felipe/delta/`. In Section 6 we will extend some of the theory of even Δ-matroids to the valuated setup. We will define duality, circuits, and cycles for a tropical Wick vector p, generalizing the corresponding definitions for even Δ-matroids. We will be mostly interested in studying the cocycle space of a tropical Wick vector, which can be seen as an analog in type D to the tropical linear space associated to a tropical Plücker vector, and we will give a parametric description of it in terms of cocircuits. We will then specialize our results to tropical Plücker vectors, unifying in this way several results for tropical linear spaces given by Murota and Tamura [MT01], Speyer [Spe08], and Ardila and Klivans [AK06]. In Section 7 we will define isotropical linear spaces and study their relation with tropical Wick vectors. We will give an effective characterization for determining when a tropical linear space is isotropical, and we will show that the correspondence between isotropic linear spaces and points in the pure spinor space is lost after tropicalizing. Nonetheless, we will prove that this correspondence still holds when we restrict our attention only to vectors having "small support".

2 Isotropic Linear Spaces and Spinor Varieties

Let n be a positive integer, and let V be a $2n$-dimensional vector space over an algebraically closed field K of characteristic 0, with a fixed basis $e_1, e_2, \ldots, e_n, e_{1^*}, e_{2^*}, \ldots, e_{n^*}$. Denote by $\mathcal{P}(n)$ the collection of subsets of the set $[n] := \{1, 2, \ldots, n\}$. In order to simplify the notation, if $S \in \mathcal{P}(n)$ and $a \in [n]$ we will write Sa, $S - a$, and $S\Delta a$ instead of $S \cup \{a\}$, $S \setminus \{a\}$, and $S\Delta\{a\}$, respectively. Given an n-dimensional isotropic subspace $U \subseteq V$, one can associate to it a vector $w \in \mathbb{P}^{\mathcal{P}(n)-1}$ of Wick coordinates as follows. Write U as the rowspace of some $n \times 2n$ matrix M with entries in K. If the first n columns of M are linearly independent, we can row reduce the matrix M and assume that it has the form $M = [I|A]$, where I is the identity matrix of size n and A is an $n \times n$ matrix. The fact that U is isotropic is equivalent to the property that the matrix A is skew-symmetric. The vector $w \in \mathbb{P}^{\mathcal{P}(n)-1}$ is then defined as

$$w_{[n]\setminus S} := \begin{cases} \mathrm{Pf}(A_S) & \text{if } |S| \text{ is even,} \\ 0 & \text{if } |S| \text{ is odd;} \end{cases}$$

where $S \in \mathcal{P}(n)$ and $\mathrm{Pf}(A_S)$ denotes the Pfaffian of the principal submatrix A_S of A whose rows and columns are indexed by the elements of S. If the first n columns of M are linearly dependent then we proceed in a similar way but working over a different affine chart of $\mathbb{P}^{\mathcal{P}(n)-1}$. In this case, we can first reorder the elements of our basis (and thus the columns of M) using a permutation of **2n** $:= \{1, 2, \ldots, n, 1^*, 2^*, \ldots, n^*\}$ consisting of transpositions of the form (j, j^*) for all j in some index set $J \subseteq [n]$, so that we get a new matrix that can be row-reduced to a matrix of the form $M' = [I|A]$ (with A skew-symmetric). We then compute the Wick coordinates as

$$w_{[n]\setminus S} := \begin{cases} (-1)^{\mathrm{sg}(S,J)} \cdot \mathrm{Pf}(A_{S\Delta J}) & \text{if } |S\Delta J| \text{ is even,} \\ 0 & \text{if } |S\Delta J| \text{ is odd;} \end{cases}$$

where $(-1)^{\mathrm{sg}(S,J)}$ is some sign depending on S and J that will not be important for us. The vector $w \in \mathbb{P}^{\mathcal{P}(n)-1}$ of Wick coordinates depends only on the subspace U, and the subspace U can be recovered from its vector w of Wick coordinates.

The **space of pure spinors** is the set $\mathrm{Spin}^{\pm}(n) \subseteq \mathbb{P}^{\mathcal{P}(n)-1}$ of Wick coordinates of all n-dimensional isotropic subspaces of V, and thus it is a parameter space for these subspaces. It is an algebraic set, and it decomposes into two isomorphic irreducible varieties as $\mathrm{Spin}^{\pm}(n) = \mathrm{Spin}^{+}(n) \sqcup \mathrm{Spin}^{-}(n)$, where $\mathrm{Spin}^{+}(n)$ consists of all Wick coordinates w whose **support** $\mathrm{supp}(w) := \{S \in \mathcal{P}(n) : w_S \neq 0\}$ is made of even-sized subsets, and $\mathrm{Spin}^{-}(n)$ consists of all Wick coordinates whose support is made of odd-sized subsets. The irreducible variety $\mathrm{Spin}^{+}(n)$ is called the **spinor variety**; it is the projective closure of the image of the map sending an $n \times n$ skew-symmetric matrix to its vector of Pfaffians. Its defining ideal consists of all polynomial relations among the Pfaffians of a skew-symmetric matrix, and it is generated by the following quadratic relations:

$$\sum_{i=1}^{s} (-1)^i \, w_{\tau_i \sigma_1 \sigma_2 \cdots \sigma_r} \cdot w_{\tau_1 \tau_2 \cdots \hat{\tau_i} \cdots \tau_s} + \sum_{j=1}^{r} (-1)^j \, w_{\sigma_1 \sigma_2 \cdots \hat{\sigma_j} \cdots \sigma_r} \cdot w_{\sigma_j \tau_1 \tau_2 \cdots \tau_s}, \tag{1}$$

where $\sigma, \tau \in \mathcal{P}(n)$ have odd cardinalities r, s, respectively, and the variables w_σ are understood to be alternating with respect to a reordering of the indices, e.g. $w_{2134} = -w_{1234}$ and $w_{1135} = 0$. The ideal defining the space of pure spinors is generated by all quadratic relations having the form (1), but now with $\sigma, \tau \in \mathcal{P}(n)$ having any cardinality. These relations are known as **Wick relations**. The shortest nontrivial Wick relations are obtained when $|\sigma\Delta\tau| = 4$; they have the form $w_{Sabcd} \cdot w_S - w_{Sab} \cdot w_{Scd} + w_{Sac} \cdot$

$w_{Sbd} - w_{Sad} \cdot w_{Sbc}$ and $w_{Sabc} \cdot w_{Sd} - w_{Sabd} \cdot w_{Sc} + w_{Sacd} \cdot w_{Sb} - w_{Sbcd} \cdot p_{Sa}$, where $S \subseteq [n]$ and $a, b, c, d \in [n] \setminus S$ are distinct. These relations will be of special importance for us; they will be called the **4-term Wick relations**.

3 Delta-Matroids

In this section we review some of the basic theory of even Δ-matroids.

3.1 Bases

Our first description of even Δ-matroids is the following.

Definition 3.1 *An* ***even*** $\boldsymbol{\Delta}$***-matroid*** *(or* ***even Delta-matroid****) is a pair $M = (E, \mathcal{B})$, where E is a finite set and $\mathcal{B}$ is a nonempty collection of subsets of E satisfying the following* ***symmetric exchange axiom****: for all $A, B \in \mathcal{B}$ and for all $a \in A\Delta B$, there exists $b \in A\Delta B$ such that $b \neq a$ and $A\Delta\{a, b\} \in \mathcal{B}$. Here Δ denotes symmetric difference: $X\Delta Y = (X \setminus Y) \cup (Y \setminus X)$. The set E is called the* ***ground set*** *of M, and $\mathcal{B}$ is called the collection of* ***bases*** *of M. We also say that M is an even Δ-matroid over the set E.*

Even Delta-matroids are a natural generalization of classical matroids; in fact, it is easy to see that matroids are precisely those even Δ-matroids whose bases have all the same cardinality (the reader not familiar with matroids can take this as a definition). The following proposition is easy to prove.

Proposition 3.2 *Let M be an even Δ-matroid. Then all the bases of M have the same parity.*

It should be mentioned that the bases of an even Δ-matroid can all have odd cardinality; unfortunately, the name used for even Δ-matroids might be a little misleading.

The notion of duality for matroids generalizes naturally to even Δ-matroids.

Definition 3.3 *Let $M = (E, \mathcal{B})$ be an even Δ-matroid. Directly from the definition it follows that the collection $\mathcal{B}^* := \{E \setminus B : B \in \mathcal{B}\}$ is also the collection of bases of an even Δ-matroid M^* over E. We will refer to M^* as the* ***dual*** *even Δ-matroid to M.*

3.2 Representability

Our interest in even Δ-matroids comes from the study of the possible supports of Wick vectors.

Proposition 3.4 *Let V be a $2n$-dimensional vector space over the field K. If $U \subseteq V$ is an n-dimensional isotropic subspace with Wick coordinates w, then the subsets in $\mathrm{supp}(w) := \{S \in \mathcal{P}(n) : w_S \neq 0\}$ form the collection of bases of an even Δ-matroid $M(U)$ over $[n]$. An even Δ-matroid arising in this way is said to be a* ***representable*** *even Δ-matroid (over the field K).*

This notion of representability generalizes the classical notion of representability for classical matroids. For matroids, some work has succeeded in studying this property over fields of very small characteristic, but there is no simple and useful characterization of representable matroids over a field of characteristic zero. The study of representability for even Δ-matroids shares the same difficulties, and there seems to be almost no research done in this direction so far.

3.3 Matroid Polytopes

Given any collection $\mathcal{B}$ of subsets of $[n]$ one can associate to it the polytope $\Gamma_{\mathcal{B}} := \text{convex}\{e_S : S \in \mathcal{B}\}$, where $e_S := \sum_{i \in S} e_i$ is the indicator vector of the subset S. The following theorem characterizes the polytopes associated to even Δ-matroids; it is a special case of a more general and fundamental theorem characterizing the associated polytopes of Coxeter matroids (see [BGW03]).

Theorem 3.5 *If $\mathcal{B} \subseteq \mathcal{P}(n)$ is nonempty then $\mathcal{B}$ is the collection of bases of an even Δ-matroid if and only if all the edges of the polytope $\Gamma_{\mathcal{B}}$ have the form $\pm e_i \pm e_j$, where $i, j \in [n]$ are distinct.*

These results let us think of even Δ-matroids as "matroids of type D" (while classical matroids are "matroids of type A").

3.4 Circuits and Symmetric Matroids

We will now describe a notion of circuits for even Δ-matroids. We will present here only the basic properties needed for the rest of the paper; a much more detailed description can be found in [BGW03].

Consider the sets $[n] := \{1, 2, \ldots, n\}$ and $[n]^* := \{1^*, 2^*, \ldots, n^*\}$. Define the map $* : [n] \to [n]^*$ by $i \mapsto i^*$ and the map $* : [n]^* \to [n]$ by $i^* \mapsto i$. We can think of $*$ as an involution of the set **2n** $:= [n] \cup [n]^*$, where for any $j \in$ **2n** we have $j^{**} = j$. If $J \subseteq$ **2n** we define $J^* := \{j^* : j \in J\}$. We say that the set J is **admissible** if $J \cap J^* = \emptyset$, and that it is a **transversal** if it is an admissible subset of size n. For any $S \subseteq [n]$, we define its **extension** $\bar{S} \subseteq$ **2n** to be the transversal given by $\bar{S} := S \cup ([n] \setminus S)^*$, and for any transversal J we will define its **restriction** to be the set $J \cap [n]$. Extending and restricting are clearly bijections (inverse to each other) between the set $\mathcal{P}(n)$ and the set of transversals $\mathcal{V}(n)$ of **2n**.

Given an even Δ-matroid $M = ([n], \mathcal{B})$, the **symmetric matroid** associated to M is the collection $\bar{\mathcal{B}}$ of transversals defined as $\bar{\mathcal{B}} := \{\bar{B} : B \in \mathcal{B}\}$. There is of course no substantial difference between an even Δ-matroid and its associated symmetric matroid; however, working with symmetric matroids will allow us to simplify the forthcoming definitions.

Definition 3.6 *Let $M = ([n], \mathcal{B})$ be an even Δ-matroid over $[n]$. A subset $S \subseteq$ **2n** is called **independent** in M if it is contained in some transversal $\bar{B} \in \bar{\mathcal{B}}$, and it is called **dependent** in M if it is not independent. A subset $C \subseteq$ **2n** is called a **circuit** of M if C is a minimal dependent subset which is admissible. A **cocircuit** of M is a circuit of the dual even Δ-matroid M^*. The set of circuits of M will be denoted by $\mathcal{C}(M)$, and the set of cocircuits by $\mathcal{C}^*(M)$. An admissible union of circuits of M is called a **cycle** of M. A **cocycle** of M is a cycle of the dual even Δ-matroid M^*.*

This definition of circuits for even Δ-matroids generalizes the concept of circuits for matroids. In fact, if $M = ([n], \mathcal{B})$ is a matroid, $\mathcal{C}$ is its collection of (classical) circuits and $\mathcal{K}$ is its collection of (classical) cocircuits, then the collection of circuits of M, when considered as an even Δ-matroid, is $\{C : C \in \mathcal{C}\} \cup \{K^* : K \in \mathcal{K}\}$.

Example 3.7 *Take $n = 3$, and let U be the isotropic subspace of $\mathbb{C}^6$ defined as the rowspace of the matrix*

$$
\begin{array}{cc}
 & \begin{array}{cccccc} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{1^*} & \mathbf{2^*} & \mathbf{3^*} \end{array} \\
M = & \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -2 & 0 \end{pmatrix}.
\end{array}
$$

The even Δ-matroid M represented by U has bases $\mathcal{B} = \{123, 1, 2, 3\}$, corresponding to the support of its vector of Wick coordinates. Its associated polytope is the tetrahedron with vertices $(1,1,1)$, $(1,0,0)$,

$(0,1,0)$*, and* $(0,0,1)$*; whose edges are indeed of the form* $\pm e_i \pm e_j$*. The circuits of* M *are the admissible subsets* $1^*23, 12^*3, 123^*, 1^*2^*3^*$*. The dual even* Δ*-matroid* M^* *has bases* $\mathcal{B}^* = \{\emptyset, 12, 13, 23\}$*. The cocircuits of* M *are the admissible subsets* $123, 12^*3^*, 1^*23^*, 1^*2^*3$*.*

4 Tropical Wick Relations

We now turn to the study of the tropical prevariety and tropical variety defined by the Wick relations. Due to space constraints, we will assume the reader is familiar with the basic notions of tropical geometry.

Definition 4.1 *A vector* $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ *is called a* ***tropical Wick vector*** *if it satisfies the tropical Wick relations, that is, for all* $S, T \in \mathcal{P}(n)$ *the minimum*

$$\min_{i \in S \Delta T} (p_{S\Delta i} + p_{T\Delta i}) \tag{2}$$

is achieved at least twice (or it is equal to ∞*). The* ***Δ-Dressian*** $\Delta\mathrm{Dr}(n) \subseteq \mathbb{T}^{\mathcal{P}(n)}$ *is the space of all tropical Wick vectors in* $\mathbb{T}^{\mathcal{P}(n)}$*, i.e., the tropical prevariety defined by the Wick relations.*

Tropical Wick vectors have also been studied in the literature under the name of **valuated Δ-matroids** (see [DW91]). The following definition will be central to our study, and it is the reason why working over $\mathbb{R} \cup \infty$ and not just $\mathbb{R}$ is fundamental for us.

Definition 4.2 *The* ***support*** *of a vector* $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ *is* $\mathrm{supp}(p) := \{S \subseteq [n] : p_S \neq \infty\}$*.*

We will later see (Theorem 5.1) that the support of any tropical Wick vector consists of subsets whose cardinalities have all the same parity, so the Δ-Dressian decomposes as the disjoint union of two tropical prevarieties: the **even Δ-Dressian** $\Delta\mathrm{Dr}^+(n) \subseteq \mathbb{T}^{\mathcal{P}(n)}$ (consisting of all tropical Wick vectors whose support has only subsets of even cardinality) and the **odd Δ-Dressian** $\Delta\mathrm{Dr}^-(n) \subseteq \mathbb{T}^{\mathcal{P}(n)}$ (defined analogously).

One of the main advantages of allowing our vectors to have ∞ entries is that tropical Wick vectors can be seen as a generalization of tropical Plücker vectors (or valuated matroids), as explained below.

Definition 4.3 *A tropical Wick vector* $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ *is called a* ***tropical Plücker vector*** *(or a* ***valuated matroid****) if all the subsets in* $\mathrm{supp}(p)$ *have the same cardinality* r_p*, called the* ***rank*** *of* p*. The name is justified by noting that in this case, the tropical Wick relations become just the tropical Plücker relations: For all* $S, T \in \mathcal{P}(n)$ *such that* $|S| = r_p - 1$ *and* $|T| = r_p + 1$*, the minimum*

$$\min_{i \in T \setminus S} (p_{Si} + p_{T-i}) \tag{3}$$

is achieved at least twice (or it is equal to ∞*). The space of tropical Plücker vectors of rank* k *is called the* ***Dressian*** $\mathrm{Dr}(k,n)$*; it is the tropical prevariety defined by the Plücker relations of rank* k*.*

Tropical Plücker vectors play a central role in the combinatorial study of tropical linear spaces done by Speyer (see [Spe08]). In his paper he only deals with tropical Plücker vectors whose support is the collection of all subsets of $[n]$ of some fixed size k; we will later see that our definition is the "correct" generalization to more general supports.

Definition 4.4 *The* ***tropical pure spinor space*** $\mathrm{TSpin}^{\pm}(n) \subseteq \mathbb{T}^{\mathcal{P}(n)}$ *is the tropicalization of the space of pure spinors, i.e., it is the tropical variety defined by all polynomials in the ideal generated by the Wick relations. A tropical Wick vector in the tropical pure spinor space is said to be* ***realizable****. The decomposition of the* Δ*-Dressian into its even an odd parts induces a decomposition of the tropical pure* *spinor space as the disjoint union of two "isomorphic" tropical varieties* $\mathrm{TSpin}^+(n)$ *and* $\mathrm{TSpin}^-(n)$*, namely, the tropicalization of the spinor varieties* $\mathrm{Spin}^+(n)$ *and* $\mathrm{Spin}^-(n)$ *described in Section 2. The tropicalization* $\mathrm{TSpin}^+(n) \subseteq \mathbb{T}^{\mathcal{P}(n)}$ *of the even part* $\mathrm{Spin}^+(n)$ *will be called the* ***tropical spinor variety****.*

By definition, we have that the tropical pure spinor space $\mathrm{TSpin}^\pm(n)$ is contained in the Δ-Dressian $\Delta\mathrm{Dr}(n)$. A first step in studying representability of tropical Wick vectors (i.e. valuated Δ-matroids) is to determine when these two spaces are the same, or equivalently, when the Wick relations form a tropical basis. Our main result in this section answers this question for almost all values of n.

Theorem 4.5 *If* $n \leq 5$ *then the tropical pure spinor space* $\mathrm{TSpin}^\pm(n)$ *is equal to the* Δ*-Dressian* $\Delta\mathrm{Dr}(n)$*, i.e., the Wick relations form a tropical basis for the ideal they generate. If* $n \geq 7$ *then* $\mathrm{TSpin}^\pm(n)$ *is strictly smaller than* $\Delta\mathrm{Dr}(n)$*; in fact, there is a vector in the even* Δ*-Dressian* $\Delta\mathrm{Dr}^+(n)$ *whose support consists of all even-sized subsets of* $[n]$ *which is not in the tropical spinor variety* $\mathrm{TSpin}^+(n)$*.*

To show that the tropical pure spinor space and the Δ-Dressian agree when $n \leq 5$ we used of Anders Jensen's software Gfan [Jen]. The results of our computations can be found at the website `http://math.berkeley.edu/~felipe/delta/`. It is still unclear what happens when $n = 6$. In this case, the spinor variety is described by 76 nontrivial Wick relations (60 of which are 4-term Wick relations) on 32 variables, and a Gfan computation requires a long time to finish. We state the following conjecture.

Conjecture 4.6 *The tropical pure spinor space* $\mathrm{TSpin}^\pm(6)$ *is equal to the* Δ*-Dressian* $\Delta\mathrm{Dr}(6)$*.*

Our equality between $\mathrm{TSpin}^\pm(n)$ and $\Delta\mathrm{Dr}(n)$ for $n \leq 5$ implies the following corollary about representability of even Δ-matroids.

Corollary 4.7 *Let* M *be an even* Δ*-matroid on a ground set of at most* 5 *elements. Then* M *is a representable even* Δ*-matroid over any algebraically closed field of characteristic* 0*.*

5 Tropical Wick Vectors and Delta-Matroid Subdivisions

In this section we provide a description of tropical Wick vectors in terms of polytopal subdivisions. We start with a useful local characterization, which was basically proved by Murota in [Mur06].

Theorem 5.1 *Suppose* $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ *has nonempty support. Then* p *is a tropical Wick vector if and only if supp(*p*) is the collection of bases of an even* Δ*-matroid over* $[n]$ *and the vector* p *satisfies the 4-term tropical Wick relations: For all* $S \in \mathcal{P}(n)$ *and all* $a, b, c, d \in [n] \setminus S$ *distinct, the minima* $\min(p_{Sabcd} + p_S, p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc})$ *and* $\min(p_{Sabc} + p_{Sd}, p_{Sabd} + p_{Sc}, p_{Sacd} + p_{Sb}, p_{Sbcd} + p_{Sa})$ *are achieved at least twice (or are equal to* ∞*).*

Corollary 5.2 *Suppose* $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ *has nonempty support. Then* p *is a tropical Plücker vector if and only if supp(*p*) is the collection of bases of matroid over* $[n]$ *(of rank* r_p*) and the vector* p *satisfies the 3-term tropical Plücker relations: For all* $S \in \mathcal{P}(n)$ *such that* $|S| = r_p - 2$ *and all* $a, b, c, d \in [n] \setminus S$ *distinct, the minimum* $\min(p_{Sab} + p_{Scd}, p_{Sac} + p_{Sbd}, p_{Sad} + p_{Sbc})$ *is achieved at least twice (or it is equal to* ∞*).*

Corollary 5.2 shows that our notion of tropical Plücker vector is indeed a generalization of the one given by Speyer in [Spe08] to the case where $\mathrm{supp}(p)$ is not necessarily the collection of bases of a uniform matroid.

Definition 5.3 *Given a vector $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$, denote by $\Gamma_p \subseteq \mathbb{R}^n$ its* ***associated polytope*** $\Gamma_p := \mathrm{convex}\{e_S : S \in \mathrm{supp}(p)\}$. *The vector p induces naturally a regular subdivision $\mathcal{D}_p$ of Γ_p in the following way. Consider the vector p as a height function on the vertices of Γ_p, so "lift" vertex e_S of Γ_p to height p_S to obtain the* ***lifted polytope*** $\Gamma'_p = \mathrm{convex}\{(e_S, p_S) : S \in \mathrm{supp}(p)\} \subseteq \mathbb{R}^{n+1}$. *The* ***lower faces*** *of Γ'_p are the faces of Γ'_p minimizing a linear form $(v, 1) \in \mathbb{R}^{n+1}$; their projection back to $\mathbb{R}^n$ form the polytopal subdivision $\mathcal{D}_p$ of Γ_p, called the* ***regular subdivision induced by*** p.

We now come to the main result of this section. It describes tropical Wick vectors as the height vectors that induce "nice" polytopal subdivisions. After finishing this paper, it was pointed out to the author that an equivalent formulation of this result had already been proved by Murota in [Mur97], under the language of maximizers of an even Δ-matroid.

Theorem 5.4 *Let $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$. Then p is a tropical Wick vector if and only if the regular subdivision $\mathcal{D}_p$ induced by p is an even Δ-matroid subdivision, i.e., it is a subdivision of an even Δ-matroid polytope into even Δ-matroid polytopes.*

If we restrict Theorem 5.4 to the case where all subsets in $\mathrm{supp}(p)$ have the same cardinality, we get the following corollary. It generalizes the results of Speyer in [Spe08] for subdivisions of a hypersimplex.

Corollary 5.5 *Let $p \in \mathbb{T}^{\mathcal{P}(n)}$. Then p is a tropical Plücker vector if and only if the regular subdivision $\mathcal{D}_p$ induced by p is a matroid subdivision, i.e., it is a subdivision of a matroid polytope into matroid polytopes.*

6 The Cocycle Space

In this section we define the notion of circuits, cocircuits and duality for tropical Wick vectors, and study the space of vectors which are "tropically orthogonal" to all circuits. The admissible part of this space will be called the cocycle space, for which we give a parametric representation. Most of our results can be seen as a generalization of results for matroids and even Δ-matroids to the "valuated" setup. For this purpose it is useful to keep in mind that for any even Δ-matroid $M = ([n], \mathcal{B})$, by Theorem 5.4 there is a natural tropical Wick vector associated to it, namely, the vector $p_M \in \mathbb{T}^{\mathcal{P}(n)}$ defined as

$$(p_M)_I := \begin{cases} 0 & \text{if } I \in \mathcal{B}, \\ \infty & \text{otherwise.} \end{cases}$$

Definition 6.1 *Suppose $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ is a tropical Wick vector. It follows easily from the definition that the vector $p^* = (p^*_S) \in \mathbb{T}^{\mathcal{P}(n)}$ defined as $p^*_S := p_{[n]\setminus S}$ is also a tropical Wick vector, called the* ***dual tropical Wick vector*** *to p. Note that the even Δ-matroid associated to p^* is the dual even Δ-matroid to the one associated to p.*

Definition 6.2 *Recall that a subset $J \subseteq$ **2n** is said to be admissible if $J \cap J^* = \emptyset$. An admissible subset of **2n** of size n is called a transversal; the set of all transversals of **2n** is denoted by $\mathcal{V}(n)$. For any subset $S \in \mathcal{P}(n)$ we defined its extension to be the transversal $\bar{S} := S \cup ([n] \setminus S)^* \subseteq$ **2n**. There is of course a bijection $S \mapsto \bar{S}$ between $\mathcal{P}(n)$ and $\mathcal{V}(n)$.*

Now, let $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ be a tropical Wick vector. It will be convenient for us to work with the natural ***extension*** *$\bar{p} \in \mathbb{T}^{\mathcal{V}(n)}$ of p defined as $\bar{p}_{\bar{S}} := p_S$. For any $T \in \mathcal{P}(n)$ we define the vector $c_T \in \mathbb{T}^{\mathbf{2n}}$ (also denoted $c_{\bar{T}}$) as*

$$(c_T)_i = (c_{\bar{T}})_i := \begin{cases} \bar{p}_{\bar{T}\Delta\{i,i^*\}} & \textit{if } i \in \bar{T}, \\ \infty & \textit{otherwise.} \end{cases}$$

It can be checked that if $\mathrm{supp}(c_T) \neq \emptyset$ *then* $\mathrm{supp}(c_T)$ *is one of the circuits of the even* Δ*-matroid* M_p *whose collection of bases is* $\mathrm{supp}(p)$*. We will say that the vector* $c \in \mathbb{T}^{\boldsymbol{2n}}$ *is a* ***circuit*** *of the tropical Wick vector* p *if* $\mathrm{supp}(c) \neq \emptyset$ *and there is some* $T \in \mathcal{P}(n)$ *and some* $\lambda \in \mathbb{R}$ *such that* $c = \lambda \odot c_T$ *(or in classical notation,* $c = c_T + \lambda \cdot \boldsymbol{1}$*, where* ***1*** *denotes the vector in* $\mathbb{T}^{\boldsymbol{2n}}$ *whose coordinates are all equal to* 1*). It is not hard to see that* $\mathcal{C}(M_p) = \{\mathrm{supp}(c) : c$ *is a circuit of* $p\}$*, so this notion of circuits indeed generalizes the notion of circuits for even* Δ*-matroids to the "valuated" setup. The collection of circuits of* p *will be denoted by* $\mathcal{C}(p) \subseteq \mathbb{T}^{\boldsymbol{2n}}$*. A* ***cocircuit*** *of the tropical Wick vector* p *is just a circuit of the dual vector* p^**, i.e., a vector of the form* $\lambda \odot c_T^*$*, where* $c_T^* \in \mathbb{T}^{\boldsymbol{2n}}$ *(also denoted* $c_{\bar{T}}^*$*) is the vector*

$$(c_T^*)_i = (c_{\bar{T}}^*)_i := \begin{cases} \bar{p}_{\bar{T}\Delta\{i,i^*\}} & \text{if } i \notin \bar{T}, \\ \infty & \text{otherwise.} \end{cases}$$

We now define the concept of "tropical orthogonality", which is just the tropicalization of the usual notion of orthogonality in terms of the dot product.

Definition 6.3 *Two vectors* $x, y \in \mathbb{T}^N$ *are said to be* ***tropically orthogonal****, denoted by* $x \top y$*, if the minimum* $\min(x_1 + y_1, x_2 + y_2, \ldots, x_N + y_N)$ *is achieved at least twice (or it is equal to* ∞*). If* $X \subseteq \mathbb{T}^N$ *then its* ***tropically orthogonal set*** *is* $X^\top := \{y \in \mathbb{T}^N : y \top x$ *for all* $x \in X\}$*.*

We now turn to the study of the space of vectors which are tropically orthogonal to all circuits. Our motivation for this will be clear later, when we deal with tropical linear spaces.

Definition 6.4 *A vector* $x \in \mathbb{T}^{\boldsymbol{2n}}$ *is said to be* ***admissible*** *if* $\mathrm{supp}(x)$ *is an admissible subset of* ***2n****. Let* $p \in \mathbb{T}^{\mathcal{P}(n)}$ *be a tropical Wick vector. If* $x \in \mathcal{C}(p)^\top$ *is admissible then* x *will be called a* ***cocycle*** *of* p*. The set of all cocycles of* p *will be called the* ***cocycle space*** *of* p*, and will be denoted by* $\mathcal{Q}(p) \subseteq \mathbb{T}^{\boldsymbol{2n}}$*.*

We will now give a parametric description for the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\mathbf{2n}}$ of a tropical Wick vector $p \in \mathbb{T}^{\mathcal{P}(n)}$. For this purpose we first introduce the concept of tropical convexity.

Definition 6.5 *A set* $X \subseteq \mathbb{T}^N$ *is called* ***tropically convex*** *if it is closed under tropical linear combinations, i.e., for any* $x_1, \ldots, x_r \in X$ *and any* $\lambda_1, \ldots, \lambda_r \in \mathbb{T}$ *we have that* $\lambda_1 \odot x_1 \oplus \cdots \oplus \lambda_r \odot x_r \in X$*. For any* $a_1, \ldots, a_r \in \mathbb{T}^N$*, their* ***tropical convex hull*** *is defined to be* $\mathrm{tconvex}(a_1, \ldots, a_r) := \{\lambda_1 \odot a_1 \oplus \cdots \oplus \lambda_r \odot a_r : \lambda_1, \ldots, \lambda_r \in \mathbb{T}\}$*; it is the smallest tropically convex set containing the vectors* $a_1, \ldots, a_r$*. A set of the form* $\mathrm{tconvex}(a_1, \ldots, a_r)$ *is usually called a* ***tropical polytope****.*

Theorem 6.6 *Let* $p \in \mathbb{T}^{\mathcal{P}(n)}$ *be a tropical Wick vector. Then the cocycle space* $\mathcal{Q}(p) \subseteq \mathbb{T}^{\boldsymbol{2n}}$ *of* p *is the set of admissible vectors in the tropical convex hull of the cocircuits of* p*.*

Theorem 6.6 implies that if p is a tropical Wick vector and M is its associated even Δ-matroid then the set of supports of all cocycles of p is precisely the set of cocycles of M (see Definition 3.6), so our definition of cocycles for tropical Wick vectors extends the usual definition of cocycles for even Δ-matroids to the valuated setup. Theorem 6.6 implies the following corollary.

Corollary 6.7 *Let* $p \in \mathbb{T}^{\mathcal{P}(n)}$ *be a tropical Wick vector. Then* $\mathcal{Q}(p^*) \subseteq \mathbb{T}^{\boldsymbol{2n}}$ *is the set of admissible vectors in* $\mathcal{Q}(p)^\top$*.*

6.1 Tropical Linear Spaces

We will now specialize some of the results presented above to tropical Plücker vectors (i.e. valuated matroids). In this way we will unify several results for tropical linear spaces given by Murota and Tamura in [MT01], Speyer in [Spe08], and Ardila and Klivans in [AK06]. Unless otherwise stated, all matroidal terminology in this section will refer to the classical matroidal notions and not to the Δ-matroidal notions discussed above.

Definition 6.8 *Let $p = (p_S) \in \mathbb{T}^{\mathcal{P}(n)}$ be a tropical Plücker vector of rank r_p. For $T \in \mathcal{P}(n)$ of size $r_p + 1$, we define the vector $d_T \in \mathbb{T}^n$ as*

$$(d_T)_i := \begin{cases} p_{T-i} & \text{if } i \in T, \\ \infty & \text{otherwise}. \end{cases}$$

*If $\operatorname{supp}(d_T) \neq \emptyset$ then $\operatorname{supp}(d_T)$ is one of the circuits of the matroid M_p whose collection of bases is $\operatorname{supp}(p)$. We will say that the vector $d \in \mathbb{T}^n$ is a **Plücker circuit** of p if $\operatorname{supp}(d) \neq \emptyset$ and there is some $T \in \mathcal{P}(n)$ of size $r_p + 1$ and some $\lambda \in \mathbb{R}$ such that $d = \lambda \odot d_T$ (or in classical notation, $d = d_T + \lambda \cdot \boldsymbol{1}$, where $\boldsymbol{1}$ denotes the vector in $\mathbb{T}^n$ whose coordinates are all equal to 1). It is not hard to see that $\mathcal{C}(M_p) = \{\operatorname{supp}(d) : d \text{ is a Plücker circuit of } p\}$, so this notion of Plücker circuits generalizes the notion of circuits for matroids to the "valuated" setup. The collection of Plücker circuits of p will be denoted by $\mathcal{PC}(p)$. A **Plücker cocircuit** of p is just a Plücker circuit of the dual vector p^*, i.e., a vector of the form $\lambda \odot d_T^*$ where $T \in \mathcal{P}(n)$ has size $n - r_p - 1$ and $d_T^* \in \mathbb{T}^n$ denotes the vector*

$$(d_T^*)_i := \begin{cases} p_{T\cup i} & \text{if } i \notin T, \\ \infty & \text{otherwise}. \end{cases}$$

The reason we are using the name "Plücker circuits" is just so that they are not confused with the circuits of p in the Δ-matroidal sense; a more appropriate name (but not very practical for the purposes of this paper) would be "circuits in type A" (while the Δ-matroidal circuits are "circuits in type D").

The following definition was introduced by Speyer in [Spe08].

Definition 6.9 *Let $p \in \mathbb{T}^{\mathcal{P}(n)}$ be a tropical Plücker vector. The space $L_p := \mathcal{PC}(p)^\top \subseteq \mathbb{T}^n$ is called the **tropical linear space** associated to p.*

The tropical linear space L_p should be thought of as the space of cocyles of p "in type A" (while $\mathcal{Q}(p)$ is the space of cocycles of p "in type D"). Tropical linear spaces have a very special geometric importance, for more information the reader is invited to consult [Spe08]. The following proposition will allow us to apply the "type D" results that we got in previous sections to the study of tropical linear spaces.

Proposition 6.10 *Let $p \in \mathbb{T}^{\mathcal{P}(n)}$ be a tropical Plücker vector, and let $L_p \subseteq \mathbb{T}^n$ be its associated linear space. Then, under the natural identification $\mathbb{T}^{\boldsymbol{2n}} \cong \mathbb{T}^n \times \mathbb{T}^n$, we have $\mathcal{C}(p)^\top = L_p \times L_{p^*}$.*

The following theorem provides a parametric description of any tropical linear space. It was first proved by Murota and Tamura in [MT01]. In the case of realizable tropical linear spaces it also appears in work of Yu and Yuster [YY07]. It follows easily from the results in the previous section.

Theorem 6.11 *Suppose $p \in \mathbb{T}^{\mathcal{P}(n)}$ is a tropical Plücker vector. Then the tropical linear space $L_p \subseteq \mathbb{T}^n$ is the tropical convex hull of the Plücker cocircuits of p.*

It is instructive to see what Theorem 6.11 is saying when applied to tropical Plücker vectors with only zero and infinity entries (what is sometimes called the "constant coefficient case" in tropical geometry). In this case, since the complements of unions of cocircuits of the associated matroid M are exactly the flats of M, we get precisely the description of the tropical linear space in terms of the flats of M that was given by Ardila and Klivans in [AK06].

Another useful application of our results to the study of tropical linear spaces is the following. It was also proved by Murota and Tamura in [MT01].

Theorem 6.12 *If $p \in \mathbb{T}^{\mathcal{P}(n)}$ is a tropical Plücker vector then $L_{p^*} = L_p^\top$. In particular, for any tropical linear space L, we have $(L^\top)^\top = L$.*

7 Isotropical Linear Spaces

Definition 7.1 *Let $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$ be an n-dimensional tropical linear space. We say that L is (totally)* ***isotropic*** *if for any two $x, y \in L$ we have that the minimum $\min(\, x_1+y_{1^*}\,,\, \dots\,,\, x_n+y_{n^*}\,,\, x_{1^*}+y_1\,,\, \dots\,,\, x_{n^*}+y_n\,)$ is achieved at least twice (or it is equal to ∞). In this case, we also say that L is an* ***isotropical linear space****. Note that if $K = \mathbb{C}\{\{t\}\}$ and $V = K^{\boldsymbol{2n}}$, the tropicalization of any n-dimensional isotropic subspace U of V (see Section 2) is an isotropical linear space $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$. In this case we say that L is* ***isotropically realizable*** *by U.*

We mentioned in Section 2 that if U is an isotropic linear subspace then its vector of Wick coordinates w carries all the information of U. One might expect something similar to hold tropically, that is, that the valuation of the Wick vector w still carries all the information of the tropicalization of U. This is not true, as the next example shows.

Example 7.2 *We present two n-dimensional isotropic linear subspaces of $\mathbb{C}\{\{t\}\}^{\boldsymbol{2n}}$ whose corresponding tropicalizations are distinct tropical linear spaces, but whose Wick coordinates have the same valuation. Take $n = 4$. Let U_1 be the 4-dimensional isotropic linear subspace of $\mathbb{C}\{\{t\}\}^{\boldsymbol{8}}$ defined as the rowspace of the matrix*

$$\begin{array}{cc} & \begin{array}{cccccccc} \boldsymbol{1} & \boldsymbol{2} & \boldsymbol{3} & \boldsymbol{4} & \boldsymbol{1}^* & \boldsymbol{2}^* & \boldsymbol{3}^* & \boldsymbol{4}^* \end{array} \\ M_1 = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -2 & -2 & -1 & 0 \end{pmatrix}, \end{array}$$

and U_2 be the 4-dimensional isotropic linear subspace of $\mathbb{C}\{\{t\}\}^{\boldsymbol{8}}$ defined as the rowspace of

$$\begin{array}{cc} & \begin{array}{cccccccc} \boldsymbol{1} & \boldsymbol{2} & \boldsymbol{3} & \boldsymbol{4} & \boldsymbol{1}^* & \boldsymbol{2}^* & \boldsymbol{3}^* & \boldsymbol{4}^* \end{array} \\ M_2 = & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 0 & -1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -4 & -2 & -1 & 0 \end{pmatrix}. \end{array}$$

Their corresponding tropical linear spaces L_1 and L_2 are distinct since, for example, the Plücker coordinate indexed by the subset 343^*4^* is nonzero for U_1 but zero for U_2. However, the Wick coordinates of U_1 and U_2 are all nonzero scalars (the ones indexed by even subsets), and thus their valuations give rise to the same tropical Wick vector.

It is important to have an effective way for deciding if a tropical linear space is isotropical or not. For this purpose, if $v \in \mathbb{T}^{\mathbf{2n}}$, we call its **reflection** to be the vector $v^r \in \mathbb{T}^{\mathbf{2n}}$ defined as $v_i^r := v_{i^*}$. If $X \subseteq \mathbb{T}^{\mathbf{2n}}$ then its reflection is the set $X^r := \{x^r : x \in X\}$. The following theorem gives us a simple criterion for identifying isotropical linear spaces.

Theorem 7.3 *Let $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$ be a tropical linear space with associated tropical Plücker vector p (whose coordinates are indexed by subsets of $\boldsymbol{2n}$). Then the following are equivalent:*

1. *L is an n-dimensional isotropical linear space.*
2. *$L^\top = L^r$.*
3. *$p_{\boldsymbol{2n} \setminus T} = p_{T^*}$ for all $T \subseteq \boldsymbol{2n}$ of size n.*

If L is an isotropical linear space which is isotropically realizable by U then we have seen that the valuation p of the Wick vector w associated to U does not determine L. Nonetheless, the following theorem shows that p does determine the admissible part of L.

Theorem 7.4 *Let $L \subseteq \mathbb{T}^{\boldsymbol{2n}}$ be an n-dimensional isotropical linear space which is isotropically realizable by the isotropic subspace $U \subseteq \mathbb{C}\{\{t\}\}^{\boldsymbol{2n}}$. Let $p \in \mathbb{T}^{\mathcal{P}(n)}$ be the tropical Wick vector obtained as the valuation of the Wick vector w associated to U. Then the set of admissible vectors in L is the cocycle space $\mathcal{Q}(p) \subseteq \mathbb{T}^{\boldsymbol{2n}}$.*

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Submaximal factorizations of a Coxeter element in complex reflection groups

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Abstract. When W is a finite reflection group, the noncrossing partition lattice $NC(W)$ of type W is a very rich combinatorial object, extending the notion of noncrossing partitions of an n-gon. A formula (for which the only known proofs are case-by-case) expresses the number of multichains of a given length in $NC(W)$ as a generalized Fuß-Catalan number, depending on the invariant degrees of W. We describe how to understand some specifications of this formula in a case-free way, using an interpretation of the chains of $NC(W)$ as fibers of a "Lyashko-Looijenga covering". This covering is constructed from the geometry of the discriminant hypersurface of W. We deduce new enumeration formulas for certain factorizations of a Coxeter element of W.

Résumé. Lorsque W est un groupe de réflexion fini, le treillis $NC(W)$ des partitions non-croisées de type W est un objet combinatoire très riche, qui généralise la notion de partitions non-croisées d'un n-gone. Une formule (seulement prouvée au cas par cas à l'heure actuelle) exprime le nombre de chaînes de longueur donnée dans $NC(W)$ sous la forme d'un nombre de Fuß-Catalan généralisé, qui dépend des degrés invariants de W. Nous décrivons une stratégie visant à comprendre certaines spécifications de cette formule de manière uniforme, en utilisant une interprétation des chaînes de $NC(W)$ comme fibres d'un "revêtement de Lyashko-Looijenga". Ce revêtement est construit à partir de la géométrie de l'hypersurface du discriminant de W. Nous en déduisons de nouvelles formules de comptage pour certaines factorisations d'un élément de Coxeter de W.

Keywords: complex reflection groups, generalized noncrossing partition lattice, generalized Fuss-Catalan numbers, factorizations of a Coxeter element

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1 Introduction

Let W be a well-generated irreducible complex reflection group[(i)]. We define a partial order $\preccurlyeq$ on W, related to the reflection length of the elements in W (see Def. 3). The noncrossing partition lattice of type W, denoted $\mathrm{NC}(W)$, is a particular interval for this order. It is an algebraic generalization of the lattice of noncrossing partitions of an n-gon, and it has many important combinatorial properties. One of the most amazing ones is the following:

Proposition 1 ("Chapoton's formula") *Let W be an irreducible, well-generated complex reflection group, of rank n. Then, for any $p \in \mathbb{N}$, the number of multichains $w_1 \preccurlyeq \ldots \preccurlyeq w_p$ in the poset $\mathrm{NC}(W)$ is equal to*

$$\mathrm{Cat}^{(N)}(W) = \prod_{i=1}^{n} \frac{d_i + ph}{d_i},$$

(i) The precise definitions will be given in Sect 2.

where $d_1 \leq \cdots \leq d_n = h$ are the invariant degrees of W (defined in Sec. 2.1).

The numbers $\mathrm{Cat}^{(p)}(W)$ are called *Fuß-Catalan numbers of type W*. They also appear in other combinatorial objects related to the group W, *e.g.* in the context of cluster algebras of finite type.

In the real case, this formula was first observed by Chapoton in [Cha05, Prop. 9]. The proof is case-by-case (using the classification of finite Coxeter groups), and it mainly uses results by Athanasiadis and Reiner [Rei97, AR04]. The remaining complex cases are checked by Bessis in [Bes07] using results of [BC06]. There is still no case-free proof of this formula, even for the simplest case where $p = 1$ which gives the cardinality of $\mathrm{NC}(W)$ as the generalized Catalan number $\prod_{i=1}^{n}(d_i + h)/d_i$.

This very simple formula naturally motivates the search for a uniform proof that could shed light on the mysterious relation between the combinatorics of $\mathrm{NC}(W)$ and the invariant theory of W.

Multichains in $\mathrm{NC}(W)$ are directly related to certain *block factorizations* of a Coxeter element c of W (see Def. 4). In fact, Chapoton's formula can be reformulated in terms of these factorizations.

In [Bes07], Bessis discovered that some instances of the formula (namely, the number of maximal factorizations of c) can be explained through the geometry of a map, called the *Lyashko-Looijenga covering* LL, constructed from the geometry of the discriminant of W.

The object of this paper is to explain how, by studying this map in more detail, we can obtain further enumerative results, giving formulas for the number of submaximal factorizations of c.

Theorem 2 (*cf.* Thm. 8 and Cor. 11) *Let c be a Coxeter element of W and Λ a conjugacy class of elements of reflection length 2 in $\mathrm{NC}(W)$. Then:*

(a) the number of block factorizations of c, constituting of $n-2$ reflections and one element of length 2 and of conjugacy class Λ, is

$$|\,\mathrm{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\; h^{n-1}}{|W|} \deg D_\Lambda \ ,$$

where D_Λ is a homogeneous polynomial attached to Λ, determined by the geometry of the discriminant hypersurface of W (see Sec. 4);

(b) the total number of block factorizations of c in $n-1$ factors (or submaximal factorizations) is

$$|\,\mathrm{FACT}_{n-1}(c)| = \frac{(n-1)!\; h^{n-1}}{|W|} \left(\frac{(n-1)(n-2)}{2} h + \sum_{i=1}^{n-1} d_i \right).$$

While the first point is new, the second one is not. In fact, it is a direct consequence of Chapoton's formula. The main interest here is that the proof is geometric and almost[(ii)] case-free. The structure of the proof is as follows:

1. we use new geometric properties of the morphism LL to prove formula (a);
2. we find a uniform way to compute $\sum_\Lambda \deg D_\Lambda$, using an algebraic study of the Jacobian of LL;
3. we deduce formula (b) since $|\,\mathrm{FACT}_{n-1}(c)| = \sum_\Lambda |\,\mathrm{FACT}_{n-1}^{\Lambda}(c)|$.

[(ii)] We have to rely on some structural properties of LL, proved in [Bes07] case-by-case.

Thus, even if the method used here does not seem directly generalizable to factorizations with fewer blocks, it is a new interesting avenue towards a geometric case-free explanation of Chapoton's formula.

Outline. In Section 2 we give some background and notations about complex reflection groups and the noncrossing partition lattice, and we define the block factorizations of a Coxeter element. Section 3 is devoted to the construction and properties of the Lyashko-Looijenga covering of type W, and in particular its relation with factorizations. In Section 4 we give the precise formulas for submaximal factorizations, and explain the steps of the proof.

All the details of the constructions and proofs can be found in [Rip10c] and [Rip10b] (see also [Rip10a]).

2 The noncrossing partition lattice and block factorizations of a Coxeter element

2.1 Complex reflection groups

First we recall some notations and definitions about complex reflection groups.

For V a finite dimensional complex vector space, we call a *reflection*(iii) of $\mathrm{GL}(V)$ an automorphism r of V of *finite order* and such that the invariant space $\mathrm{Ker}(r-1)$ is a *hyperplane* of V. We call a *complex reflection group* a finite subgroup of $\mathrm{GL}(V)$ generated by reflections.

A simple way to construct such a group is to take a finite real reflection group (or, equivalently, a finite Coxeter group together with its natural geometric realization) and to complexify it. There are of course many other examples that cannot be seen in a real space. A complete classification of irreducible complex reflection groups has been given by Shephard-Todd in [ST54] : it consists in an infinite series with three parameters and 34 exceptional groups of small ranks. For more details on these groups, we refer to the book [LT09].

We denote by W a subgroup of $\mathrm{GL}(V)$ which is a complex reflection group. Note that for real reflection groups the results of this paper are already interesting (and, most of them, new).

From now on we suppose that W is irreducible of rank n(iv). If $(v_1, \dots, v_n)$ denotes a basis of V, W acts naturally on the polynomial algebra $\mathbb{C}[V] = \mathbb{C}[v_1, \dots, v_n]$. Chevalley-Shephard-Todd's theorem implies that the invariant algebra $\mathbb{C}[V]^W$ is again a polynomial algebra, and it can be generated by n algebraically independent homogeneous polynomials $f_1, \dots, f_n$ (called the *fundamental invariants*). The degrees $d_1, \dots, d_n$ of these invariants do not depend on the choices for the f_i's (if we require $d_1 \leq \cdots \leq d_n$) and they are called the *invariant degrees* of W. As for finite Coxeter groups, we will denote by h the highest degree d_n (Coxeter number).

We will also require that W is a *well-generated* (irreducible) complex reflection group, *i.e.* , it can be generated by n reflections(v). Then there exist in W so-called *Coxeter elements*, which generalize the usual notion of a Coxeter element in finite Coxeter groups : a Coxeter element c of W is a $e^{2i\pi/h}$-regular element (in the sense of Springer's regularity), *i.e.* , it is such that there exists $v \in V$ outside the reflecting hyperplanes such that $cv = e^{2i\pi/h}v$.

(iii) This is called *pseudo-reflection* by certain authors.
(iv) That is, the linear action on V is irreducible, and the dimension of V is n.
(v) This is, of course, always verified in the real case.

2.2 The noncrossing partition lattice of type W

Denote by $\mathcal{R}$ the set of all reflections in W. For w in W, let $\ell(w)$ denote the minimal length of w as a word on the alphabet $\mathcal{R}$. This is called the reflection length or absolute length(vi).

Definition 3 *We denote by $\preccurlyeq$ the absolute order on W, that is:*

$$u \preccurlyeq v \qquad \text{if and only if} \qquad \ell(u) + \ell(u^{-1}v) = \ell(v).$$

If c is a Coxeter element of W, the noncrossing partition lattice *of (W, c) is:*

$$\mathrm{NC}(W, c) = \{w \in W \mid w \preccurlyeq c\}.$$

It is easy to see that the structure of $\mathrm{NC}(W, c)$ does not depend(vii) on the choice of the Coxeter element c; thus we will just write $\mathrm{NC}(W)$ for short. In the prototypal case of type A, where $W = \mathfrak{S}_n$, $\mathcal{R}$ is the set of all transpositions and c is an n-cycle; then $\mathrm{NC}(W)$ is isomorphic to the poset of noncrossing partitions of an n-gon, as introduced by Kreweras in [Kre72]. In general, the noncrossing partition lattice of type W has a very rich combinatorial structure, we refer to [Arm09, Ch. 1] or [Rip10a, Chap. 0].

2.3 Multichains in $\mathrm{NC}(W)$ and block factorizations of a Coxeter element

As described in the introduction, Chapoton's formula expresses the number of multichains in $\mathrm{NC}(W)$. Here we prefer to work with block factorizations of a Coxeter element, which are directly related to multichains.

Definition 4 *For a Coxeter element c, $(w_1, \dots, w_p)$ is a* block factorization(viii) *of c if:*

- $\forall i, w_i \in W - \{1\}$*;*
- $w_1 \dots w_p = c$*;*
- $\ell(w_1) + \dots + \ell(w_p) = \ell(c)$.

We denote by $\text{FACT}(c)$ (resp. $\text{FACT}_p(c)$), the set of block factorizations of c (resp. factorizations in p factors).

Note that the length of c equals the rank of W (denoted by n), so any block factorization of c determines a composition of the integer n, by taking the distribution of the lengths of the factors. The set $\text{FACT}_n(c)$ is also called the set of reduced decompositions of c into reflections.

If $(w_1, \dots, w_p)$ is a factorization of c, then we get a (strict) chain in $\mathrm{NC}(W)$:

$$w_1 \prec w_1 w_2 \prec \dots \prec w_1 \dots w_p = c.$$

Strict chains are related to multichains by straightforward formulas, so that we can pass from enumeration of multichains in $\mathrm{NC}(W)$ to enumeration of block factorizations of c, and vice versa (see [Rip10a, App. B] or [Sta97, Ch. 3.11] for example).

In the following section, we describe a geometric construction of these block factorizations, and how they are related to the fibers of a topological covering.

(vi) In contrast with the weak length ℓ_S, relative to the set S of Coxeter generators, which exists only in the real case.
(vii) Because all the Coxeter elements are conjugated, and the reflection length is invariant under conjugation.
(viii) We will often simply write *factorization* in the following.

3 Lyashko-Looijenga covering and factorizations of a Coxeter element

3.1 Discriminant of a well-generated reflection group and Lyashko-Looijenga covering

Let W be a well-generated, irreducible complex reflection group W, with invariant polynomials $f_1, \dots, f_n$, homogeneous of degrees $d_1 \leq \dots \leq d_n = h$. Note that the quotient space V/W is then isomorphic to $\mathbb{C}^n$:

$$\begin{array}{ccc} V/W & \xrightarrow{\sim} & \mathbb{C}^n \\ \bar{v} & \mapsto & (f_1(v), \dots, f_n(v)) \end{array} .$$

We recall here the construction of the Lyashko-Looijenga map of type W (see [Bes07, Sec. 5] or [Rip10c, Sec. 3] for details).

We denote by Δ_W the discriminant of W. It is a polynomial in $\mathbb{C}[V]^W = \mathbb{C}[f_1, \dots, f_n]$, and it is an equation of the image of the union of the reflecting hyperplanes in the quotient space V/W. It is known (see *e.g.* [Bes07, Thm. 2.4]) that when W is well-generated, the fundamental invariants $f_1, \dots, f_n$ can be chosen such that the discriminant of W is a monic polynomial of degree n in f_n of the form:

$$\Delta_W = f_n^n + a_2 f_n^{n-2} + \dots + a_n ,$$

with $a_i \in \mathbb{C}[f_1, \dots, f_{n-1}]$ (see *e.g.* [Bes07, Thm. 2.4]). Let us denote by $\mathcal{H}$ the discriminant hypersurface:

$$\mathcal{H} := \{\bar{v} \in V/W \mid \Delta_W(\bar{v}) = 0\}.$$

and define $Y := \operatorname{Spec} \mathbb{C}[f_1, \dots, f_{n-1}] \simeq \mathbb{C}^{n-1}$, so that $V/W \simeq Y \times \mathbb{C}$.

The monic property given above implies that if we fix $f_1, \dots, f_{n-1}$, then Δ_W always has n roots (counting multiplicities) in f_n. Or, geometrically, that the intersection of the hypersurface $\mathcal{H}$ with the complex line $\{(y, f_n) \mid f_n \in \mathbb{C}\}$ (for a fixed $y \in Y$) generically has cardinality n. The definition of the Lyashko-Looijenga map comes from these observations.

Definition 5 *We denote by E_n be the set of centered configurations of n points in $\mathbb{C}$, that is*

$$E_n := H_0/\mathfrak{S}_n \text{ , where } H_0 = \big\{(x_1, \dots, x_n) \in \mathbb{C}^n \mid \sum_i x_i = 0\big\}.$$

The Lyashko-Looijenga map of type W is:

$$\begin{array}{ccc} Y & \xrightarrow{\mathrm{LL}} & E_n \\ y = (f_1, \dots, f_{n-1}) & \mapsto & \textit{multiset of roots of } \Delta_W(f_1, \dots, f_n) \textit{ in the variable } f_n. \end{array}$$

Remark 6 *We can also see* LL *as an algebraic morphism. Indeed, the natural coordinates for E_n as an algebraic variety are the $n-1$ elementary symmetric polynomials $e_2(x_1 \dots, x_n), \dots, e_n(x_1, \dots, x_n)$. Thus, the algebraic version of the map* LL *is simply the morphism*

$$\begin{array}{cccc} \widehat{\mathrm{LL}} : & \mathbb{C}^{n-1} & \to & \mathbb{C}^{n-1} \\ & (f_1, \dots, f_{n-1}) & \mapsto & (a_2(f_1, \dots, f_{n-1}) \text{ , } \dots \text{ , } a_n(f_1, \dots, f_{n-1})). \end{array}$$

We denote by E_n^{reg} the set of configurations in E_n with n distinct points, and we define the bifurcation locus of LL, namely $\mathcal{K} := \mathrm{LL}^{-1}(E_n - E_n^{\mathrm{reg}})$. Equivalently, we have

$$\mathcal{K} := \{y \in Y \mid D_{\mathrm{LL}}(y) = 0\},$$

where D_{LL} is called the LL-discriminant and is defined as:

$$D_{\mathrm{LL}} := \mathrm{Disc}(\Delta_W(y, f_n); f_n) \in \mathbb{C}[f_1, \dots, f_{n-1}].$$

The first important property is the following (from [Bes07, Thm. 5.3]):

$$\text{The restriction of } \mathrm{LL} : Y - \mathcal{K} \twoheadrightarrow E_n^{\mathrm{reg}} \text{ is a topological covering of degree } \frac{n!\, h^n}{|W|} \tag{P0}$$

3.2 Geometric construction of factorizations

Discriminant stratification

Before explaining the construction of factorizations from the discriminant hypersurface, we recall some useful properties of the geometric stratification associated to the parabolic subgroups of W.

The space V, together with the hyperplane arrangement $\mathcal{A}$, admits a natural stratification by the *flats*, namely, the elements of the intersection lattice $\mathcal{L} := \left\{\bigcap_{H \in \mathcal{B}} H \mid \mathcal{B} \subseteq \mathcal{A}\right\}$.

As the W-action on V maps a flat to a flat, this stratification gives rise to a quotient stratification $\bar{\mathcal{L}}$ of $W\backslash V$:

$$\bar{\mathcal{L}} = W\backslash\mathcal{L} = (p(L))_{L \in \mathcal{L}} = (W \cdot L)_{L \in \mathcal{L}} \ ,$$

where p is the projection $V \twoheadrightarrow W\backslash V$. For each stratum Λ in $\bar{\mathcal{L}}$, we denote by Λ^0 the complement in Λ of the union of the strata strictly included in Λ. The $(\Lambda^0)_{\Lambda \in \bar{\mathcal{L}}}$ form an open stratification of $W\backslash V$, called the *discriminant stratification*.

There is a natural bijection between the set of flats in V and the set of parabolic subgroups of W (Steinberg's theorem). This leads to a bijection between the stratification $\bar{\mathcal{L}}$ and the set of conjugacy classes of parabolic subgroups. Moreover, $\bar{\mathcal{L}}$ is in bijection with the set of conjugacy classes of *parabolic Coxeter elements* (which are Coxeter elements of parabolic subgroups), and with the set of conjugacy classes of elements of $\mathrm{NC}(W)$. Through these bijections, the codimension of a stratum Λ corresponds to the rank of the associated parabolic subgroup and to the length of the parabolic Coxeter element. We refer to [Rip10c, Sect.6] for details and proofs.

Geometric factorizations and compatibilities

In [Rip10c] we established a way to exhibit factorizations from the geometry of the discriminant hypersurface $\mathcal{H}$. The starting point is the construction of a map

$$\begin{array}{rccc} \rho : & \mathcal{H} & \to & W \\ & (y, x) & \mapsto & c_{y,x} \ , \end{array}$$

by the following steps (note that (y, x) lies in $\mathcal{H}$ if and only if the multiset $\mathrm{LL}(y)$ contains x).

1. Consider a small loop in $\mathbb{C}^n - \mathcal{H}$, which always stays in the fiber $\{(y, t), t \in \mathbb{C}\}$, and turns once around x (but not around any other x' in $\mathrm{LL}(y)$).

2. This loop determines an element $b_{y,x}$ of $\pi_1(\mathbb{C}^n - \mathcal{H}) = \pi_1(V^{\mathrm{reg}}/W) = B(W)$, *i.e.* , the braid group of W.

3. Send $b_{y,x}$ to $c_{y,x}$ *via* a fixed surjection $B(W) \twoheadrightarrow W$.

The details can be found in [Rip10c, Sect. 4]. The construction is adapted from that of Bessis in [Bes07, Sect. 6]. The map ρ has the following fundamental properties.

(P1) If $(x_1, \ldots, x_p)$ is the ordered support of $\mathrm{LL}(y)$ (for the lexicographical order on $\mathbb{C} \simeq \mathbb{R}^2$), then the p-tuple $(c_{y,x_1}, \ldots, c_{y,x_p})$ lies in $\mathrm{FACT}_p(c)$.

(P2) For all $x \in \mathrm{LL}(y)$, $c_{y,x}$ is a parabolic Coxeter element; its length is equal to the multiplicity of x in $\mathrm{LL}(y)$, and its conjugacy class corresponds (*via* the bijection mentioned above) to the unique stratum Λ in $\bar{\mathcal{L}}$ such that $(y, x) \in \Lambda^0$.

According to property (P1), we call the tuple $(c_{y,x_1}, \ldots, c_{y,x_p})$ (where $(x_1, \ldots, x_p)$ is the ordered support of $\mathrm{LL}(y)$) the *factorization of* c *associated to* y, and we denote it by $\underline{\mathrm{facto}}(y)$.

Any block factorization determines a composition of n. To any configuration of E_n we can also associate a composition of n, formed by the multiplicities of its elements in the lexicographical order. Then property (P2) implies that for any y in Y, the compositions associated to $\mathrm{LL}(y)$ and $\underline{\mathrm{facto}}(y)$ are the same. The third fundamental property (see [Rip10c, Thm. 5.1] or [Bes07, Thm. 7.9]) is the following.

(P3) The map $\mathrm{LL} \times \underline{\mathrm{facto}} : Y \to E_n \times \mathrm{FACT}(c)$ is injective, and its image is the entire set of compatible pairs (*i.e.* , pairs with same associated composition).

In other words, for each $y \in Y$, the fiber $\mathrm{LL}^{-1}(\mathrm{LL}(y))$ is in bijection (via $\underline{\mathrm{facto}}$) with the set of factorizations whose associated composition of n is the same as that associated to $\underline{\mathrm{facto}}(y)$.

4 Combinatorics of the submaximal factorizations

Property (P3) is particularly helpful to compute algebraically certain classes of factorizations. For example, if y lies in $Y - \mathcal{K}$, then $\underline{\mathrm{facto}}(y)$ is in $\mathrm{FACT}_n(c)$ (in other words, it is a reduced decomposition of c), *i.e.* the associated composition is $(1, 1, \ldots, 1)$. Thus, from (P3), the set of reduced decomposition of c is in bijection with any generic fiber of LL (the fiber of any point in E_n^{reg}), so it has cardinality $n!h^n/|W|$, because of property (P0). Note that this number has been computed algebraically, using the fact that the algebraic morphism $\widehat{\mathrm{LL}}$ is "weighted-homogeneous".

The natural question is: can we go further, and obtain enumeration of more complicated factorizations, using the property (P3) and a geometric study of the morphism $\widehat{\mathrm{LL}}$? This section gives a positive answer for the case of submaximal factorizations.

4.1 Restriction of LL and submaximal factorizations of a given type

A *submaximal factorization* of a Coxeter element c is a block factorization of c whose underlying partition of the associated composition is $\alpha = 2^1 1^{n-2} \vdash n$. In other words, these are factorizations of c in $n-1$ blocks ($(n-2)$ reflections and one factor of length 2), and as such, they are a natural first generalization of the set of reduced decompositions $\mathrm{FACT}_n(c)$.

Obviously we have to study the restriction of LL to the bifurcation locus $\mathcal{K}$. In fact, it is easier to first study finer restrictions, because $\mathcal{K}$ is not irreducible. Before stating the properties that we obtain, we need some notations.

Definition 7 *We call $\bar{\mathcal{L}}_2$ the set of strata of $\bar{\mathcal{L}}$ of codimension 2. Its elements correspond (via the bijection described earlier) to the conjugacy classes of parabolic Coxeter elements of length 2.*

We define φ to be the projection:

$$\begin{array}{rcl} V/W \simeq Y \times \mathbb{C} & \xrightarrow{\varphi} & Y \\ (y,x) & \mapsto & y. \end{array}$$

Let Λ be a stratum of $\bar{\mathcal{L}}_2$. We define:

- $\mathrm{FACT}_{n-1}^{\Lambda}(c)$*: the set of submaximal factorizations* of type Λ, i.e. *factorizations whose unique factor of length 2 has a conjugacy class corresponding to Λ.*
- $E_\alpha := E_n - E_n^{\mathrm{reg}}$.
- E_α^0*: the subset of E_α consisting of configurations whose partition of multiplicities is $\alpha = 2^1 1^{n-2} \vdash n$.*
- $\varphi(\Lambda)^0 := \varphi(\Lambda) \cap \mathrm{LL}^{-1}(E_\alpha^0)$.
- $\mathcal{K}^0 = \mathrm{LL}^{-1}(E_\alpha^0) = \bigcup_{\Lambda \in \bar{\mathcal{L}}_2} \varphi(\Lambda)^0$.
- LL_Λ*: the restriction of* LL: $\varphi(\Lambda) \xrightarrow{\mathrm{LL}_\Lambda} E_\alpha$.

Then, from [Rip10c], we have the following properties.

(i) The irreducible components of $\mathcal{K}$ are the $\varphi(\Lambda)$, for $\Lambda \in \bar{\mathcal{L}}_2$.

(ii) The connected components of $\mathcal{K}^0$ are the $\varphi(\Lambda)^0$, for $\Lambda \in \bar{\mathcal{L}}_2$.

(iii) The restriction of $\mathrm{LL} : \mathcal{K}^0 \twoheadrightarrow E_\alpha^0$ is a (possibly not connected) unramified covering [Rip10c, Thm. 5.2].

(iv) The image, by the map <u>facto</u>, of $\varphi(\Lambda)^0$ is exactly $\mathrm{FACT}_{n-1}^{\Lambda}(c)$.

For each Λ, let us denote by D_Λ an irreducible polynomial in $f_1, \ldots, f_{n-1}$ such that $\varphi(\Lambda) = \{D_\Lambda = 0\}$. From (i) we obtain a decomposition of the polynomial D_{LL} (the equation of $\mathcal{K}$, see Sec. 3.1):

$$D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda} \text{ , for some } r_\Lambda \geq 1.$$

Using the algebraic property of the restriction LL_Λ defined above, we can then obtain the following results.

Theorem 8 *Let Λ be a strata of $\bar{\mathcal{L}}_2$. Then:*

(a) LL_Λ *is a finite quasi-homogeneous morphism of degree* $\frac{(n-2)!\, h^{n-1}}{|W|} \deg D_\Lambda$*;*

(b) the number of submaximal factorizations of c *of type* Λ *is equal to*

$$|\,\mathrm{FACT}_{n-1}^{\Lambda}(c)| = \frac{(n-1)!\; h^{n-1}}{|W|} \deg D_\Lambda \ .$$

Proof: (outline) We use the fact that the map LL_Λ defined above can be viewed as an algebraic morphism $\widehat{\mathrm{LL}}_\Lambda$, corresponding to the extension

$$\mathbb{C}[a_2, \ldots, a_n]/(D) \subseteq \mathbb{C}[f_1, \ldots, f_{n-1}]/(D_\Lambda) \ .$$

Then we compute Hilbert series to obtain the degree of this extension.

(b) is a direct consequence of (a) together with Property (P3). The multiplicative factor $(n-1)$ is because there are $(n-1)$ compositions of n whose underlying partition is $2^1 1^{n-2}$.

□

Remark 9 *In [KM10], motivated by the enumerative theory of the generalized non-crossing partitions, Krattenthaler and Müller defined and computed the* decomposition numbers *of a Coxeter element, for all irreducible* real *reflection groups. In our terminology, these are the numbers of block factorizations according to the Coxeter type of the factors. Note that the Coxeter type of a parabolic Coxeter element is the type of its associated parabolic subgroup, in the sense of the classification of finite Coxeter groups. So the conjugacy class for a parabolic elements is a finer characteristic than the Coxeter type*[(ix)]*.*

Nevertheless, when W *is real*[(x)]*, most of the results obtained from formula (b) in Thm. 8 are very specific cases of the computations in [KM10]. But the method of proof is completely different, geometric instead of combinatorial. Note that another possible way to tackle this problem is to use a recursion, to obtain data for the group from the data for its parabolic subgroups. A recursion formula (for factorizations where the rank of each factor is dictated) is indeed given by Reading in [Rea08], but the proof is very specific to the real case.*

For W non-real, formula (b) implies new combinatorial results on the factorizations of a Coxeter element. The numerical data for all irreducible well-generated complex reflection groups are gathered in the Table 2 of [Rip10b]. In particular, we obtain (geometrically) general formulas for the submaximal factorizations of a given type in $G(e, e, n)$.

4.2 *Jacobian and discriminant of* LL

Now we would like to compute the total number of submaximal factorizations. For this, we need to work out the sum of the formulas of Thm. 8(b). The problem is that we don't know explicitly the degrees of the polynomials D_Λ.

We recall that $D_{\mathrm{LL}} = \prod_{\Lambda \in \bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda}$. We use the following theorem to get through it in a uniform way.

(ix) Take for example D_4, where there are three conjugacy classes of parabolic elements of type $A_1 \times A_1$.

(x) The computation of all decomposition numbers for complex groups, by combinatorial means, is also a work in progress (Krattenthaler, personal communication).

Theorem 10 *Define J_{LL} to be the Jacobian of the morphism $\widehat{\mathrm{LL}}$:*

$$J_{\mathrm{LL}} = \mathrm{Jac}((a_2,\dots,a_n)/(f_1,\dots,f_{n-1})) = \det\left(\frac{\partial a_i}{\partial f_j}\right)_{\substack{2\le i\le n\\ 1\le j\le n-1}}.$$

Then (up to a multiplicative constant):

$$J_{\mathrm{LL}} = \prod_{\Lambda\in\bar{\mathcal{L}}_2} D_\Lambda^{r_\Lambda-1}.$$

The proof involves a detailed study of the polynomial ring extension associated to $\widehat{\mathrm{LL}}$. We prove that the polynomials D_Λ correspond to the ramified ideals of this extension, and that the integers r_Λ are their ramification indices. We also interpret them combinatorially, using the covering properties of LL. Then we deduce that the extension is “well-ramified”, as defined in [Rip10a, Ch. 2], which implies the expected factorization of the Jacobian. We refer to [Rip10b, Thm. 3.4] for details.

4.3 Enumeration of submaximal factorizations

We deduce easily the computation of the total number of submaximal factorizations.

Corollary 11 *Let W be an irreducible, well-generated complex reflection group, with invariant degrees $d_1\le\cdots\le d_n = h$. Then, the number of submaximal factorizations of a Coxeter element c is equal to:*

$$|\,\mathrm{FACT}_{n-1}(c)| = \frac{(n-1)!\,h^{n-1}}{|W|}\left(\frac{(n-1)(n-2)}{2}h + \sum_{i=1}^{n-1} d_i\right).$$

Proof: Using Thm. 8(b) and Thm. 10, we obtain:

$$\begin{aligned}
|\,\mathrm{FACT}_{n-1}(c)| &= \sum_{\Lambda\in\bar{\mathcal{L}}_2} |\,\mathrm{FACT}^{\Lambda}_{n-1}(c)|\\
&= \frac{(n-1)!\,h^{n-1}}{|W|}\sum_{\Lambda\in\bar{\mathcal{L}}_2}\deg D_\Lambda\\
&= \frac{(n-1)!\,h^{n-1}}{|W|}\left(\deg D_{\mathrm{LL}} - \deg J_{\mathrm{LL}}\right),
\end{aligned}$$

As D_{LL} is a classical discriminant with respect to the variable f_n of degree h, we have $\deg D_{\mathrm{LL}} = n(n-1)h$. Moreover: $\deg J_{\mathrm{LL}} = \sum_{i=2}^{n}\deg(a_i) - \sum_{j=1}^{n-1}\deg(f_j) = \sum_{i=2}^{n} ih - \sum_{j=1}^{n-1} d_j$. A quick computation allows to conclude. □

The formula in the above theorem is actually included in Chapoton’s formula: indeed, there exist easy combinatorial tricks allowing to pass from the numbers of multichains to the numbers of strict chains, which are roughly the numbers of block factorizations (see [Rip10a, App. B] for details.

However, the proof we obtained here is more satisfactory (and more enlightening) than the one using Chapoton’s formula. Indeed, if we recapitulate the ingredients of the proof, we only made use of the formula for the number of reduced decompositions —necessary to prove the first properties of LL in

[Bes07]—, the remaining being the geometric properties of LL, for which we never used the classification. In other words, we travelled from the numerology of $\mathrm{FACT}_n(c)$ to that of $\mathrm{FACT}_{n-1}(c)$, without adding any case-by-case analysis to the setting of [Bes07].

Yet, the method used here to compute the number of submaximal factorizations is difficult to generalize to factorizations with fewer blocks. A more promising approach would be to avoid computing explicitely these factorizations, and to try to understand globally Chapoton's formula as some ramification formula for the morphism LL. A reformulation of the formula gives indeed:

$$\forall p \in \mathbb{N}, \sum_{k=1}^{n} \binom{p+1}{k} |\,\mathrm{FACT}_k(c)| = \prod_{i=1}^{n} \frac{d_i + ph}{d_i} \ ,$$

where the $|\,\mathrm{FACT}_k(c)|$ are closely related to the cardinalities of the fibers of LL.

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Local extrema in random permutations and the structure of longest alternating subsequences

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Abstract. Let $\mathbf{as}_n$ denote the length of a longest alternating subsequence in a uniformly random permutation of order n. Stanley studied the distribution of $\mathbf{as}_n$ using algebraic methods, and showed in particular that $\mathbb{E}(\mathbf{as}_n) = (4n+1)/6$ and $\mathrm{Var}(\mathbf{as}_n) = (32n - 13)/180$. From Stanley's result it can be shown that after rescaling, $\mathbf{as}_n$ converges in the limit to the Gaussian distribution. In this extended abstract we present a new approach to the study of $\mathbf{as}_n$ by relating it to the sequence of local extrema of a random permutation, which is shown to form a "canonical" longest alternating subsequence. Using this connection we reprove the abovementioned results in a more probabilistic and transparent way. We also study the distribution of the values of the local minima and maxima, and prove that in the limit the joint distribution of successive minimum-maximum pairs converges to the two-dimensional distribution whose density function is given by $f(s,t) = 3(1-s)te^{t-s}$.

Résumé. Pour une permutation aléatoire d'ordre n, on désigne par $\mathbf{as}_n$ la longueur maximale d'une de ses sous-suites alternantes. Stanley a étudié la distribution de $\mathbf{as}_n$ en utilisant des méthodes algébriques, et il a démontré en particulier que $\mathbb{E}(\mathbf{as}_n) = (4n + 1)/6$ et $\mathrm{Var}(\mathbf{as}_n) = (32n - 13)/180$. A partir du résultat de Stanley on peut montrer qu'après changement d'échelle, $\mathbf{as}_n$ converge vers la distribution normale. Nous présentons ici une approche nouvelle pour l'étude de $\mathbf{as}_n$, en la reliant à la suite des extrema locaux d'une permutation aléatoire, dont nous montrons qu'elle constitue une sous-suite alternante maximale "canonique". En utilisant cette relation, nous prouvons à nouveau les résultats mentionnés ci-dessus d'une façon plus probabiliste et transparente. En plus, nous prouvons un résultat asymptotique sur la distribution limite des paires formées d'un minimum et d'un maximum locaux consécutifs.

Keywords: longest alternating subsequences, permutation statistics, random permutation

1 Introduction

Let $x_1, \ldots, x_n$ be a sequence of distinct real numbers. A subsequence $x_{i_1}, \ldots, x_{i_k}$, where $1 \le i_1 < \ldots < i_k \le n$, is called an *alternating subsequence* if it satisfies

$$x_{i_1} > x_{i_2} < x_{i_3} > \ldots x_k.$$

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Let $\mathrm{as}_{\max}(x_1, \ldots, x_n)$ be the maximal length of an alternating subsequence of $x_1, \ldots, x_n$. Note that there may be more than one alternating subsequence of this maximal length. For example, the sequence $4, 2, 1, 3$ has three longest alternating subsequences, namely $(4, 1, 3)$, $(4, 2, 3)$ and $(2, 1, 3)$.

In this extended abstract we are concerned with the random variable

$$\mathbf{as}_n = \mathrm{as}_{\max}(\sigma(1), \sigma(2), \ldots, \sigma(n)),$$

where σ is a uniformly random permutation in the symmetric group S_n. Equivalently, for our purposes it will be convenient to realize $\mathbf{as}_n$ as

$$\mathbf{as}_n = \mathrm{as}_{\max}(X_1, X_2, \ldots, X_n), \tag{1}$$

where $X_1, X_2, \ldots, X_n$ is a sequence of independent and identically distributed random variables with the uniform distribution $U[0, 1]$. The fact that this realization gives a random variable with the same distribution is a consequence of the well-known fact that the order structure of $(X_1, \ldots, X_n)$ is a uniformly random element of S_n.

Stanley (2008) studied $\mathbf{as}_n$, and proved, among other results, the following exact formulas for the expectation and variance of $\mathbf{as}_n$:

$$\mathbb{E}(\mathbf{as}_n) = \frac{2}{3}n + \frac{1}{6}, \qquad (n \geq 2), \tag{2}$$

$$\mathrm{Var}(\mathbf{as}_n) = \frac{8}{45}n - \frac{13}{180}, \qquad (n \geq 4). \tag{3}$$

In particular, for large values of n, $\mathbf{as}_n$ takes values which are with high probability concentrated around its mean value of approximately $2n/3$, with a standard deviation of order $\sqrt{n}$. Stanley also noted that the distribution of $\mathbf{as}_n$ converges in the limit as $n \to \infty$ to the Gaussian distribution. This was proved by Widom (2006) using a generating function identity derived by Stanley. Stanley also pointed out (see also Stanley (2010)) that the limiting Gaussian distribution follows using the same generating function identity from general results of Pemantle and Wilson (2002), and sketches a different proof based on unpublished results of Wilf (1998), which also rely on properties of specific generating functions related to $\mathbf{as}_n$.

In this extended abstract, we present a new approach to the study of the distribution of $\mathbf{as}_n$. The main new idea is that one can construct a specific longest alternating subsequence of a given sequence $x_1, \ldots, x_n$ in a simple way using the "local extrema" of the sequence. In the probabilistic setting, the length $\mathbf{as}_n$ is then equal (modulo some boundary corrections) to the number of local extrema of the sequence of random variables $X_1, \ldots, X_n$. This number can be represented explicitly as a sum of Bernoulli random variables with simple correlations, which leads immediately to a new and more transparent derivation of the relations (2), (3) and the limiting Gaussian distribution. Furthermore, with this approach it is natural to try to understand the structure of this distinguished longest alternating subsequence formed from the local extrema; we will derive explicit formulas for the limiting densities of the local minima and maxima, and for the limiting two-dimensional density of "minimum-maximum pairs", which consist of a local minimum and the local maximum that follows it.

2 Local extrema and the canonical alternating subsequence

Let us start with the combinatorial description of the sequence of local extrema and its relation to longest alternating subsequences. Let $x_1, \ldots, x_n$ be a sequence of distinct numbers. For $2 \leq k \leq n - 1$, we say that x_k is a **local minimum** if $x_{k-1} > x_k < x_{k+1}$. We say that x_k is a **local maximum** if $x_{k-1} < x_k > x_{k+1}$, and we say that it is a **local extremum** if it is a local minimum or maximum. Define the **canonical alternating subsequence** to be the sequence of local extrema of $x_1, \ldots, x_n$, together with the last element x_n, and together with the first element x_1 if it satisfies $x_1 > x_2$.

Lemma 1 *The canonical alternating subsequence is in fact an alternating subsequence, and its length is* $\mathrm{as}_{max}(x_1, \ldots, x_n)$.

Proof: Because local minima and maxima appear in alternation, the sequence of local extrema meets the definition of an alternating subsequence, except possibly the requirement that the subsequence starts with a descent rather than an ascent. To make sure this requirement is also met, we add x_1 to the subsequence if the first local extremum is a minimum, which happens exactly if $x_1 > x_2$. It is also easy to check that adding x_n does not damage the alternating property in any case, so the canonical subsequence is alternating.

Denote the canonical alternating subsequence by $x_{j_1}, \ldots, x_{j_m}$, and denote $A = \mathrm{as}_{\mathrm{max}}(x_1, \ldots, x_n)$. By definition we have that $m \leq A$. Conversely, to show that $m \geq A$, observe that the sequence of elements between successive local extrema is necessarily monotone. Therefore any alternating subsequence can contain at most one index from each of $[1, j_1), [j_1, j_2), \ldots, [j_{m-1}, j_m)$ and $[j_m, n] = \{n\}$ and therefore has length $\leq m$. $\square$

3 A new probabilistic representation of $\mathbf{as}_n$

As a corollary to the last result, we get the following convenient representation for the random variable $\mathbf{as}_n$ as defined in (1).

Corollary 2 *Define events $A_1, \ldots, A_{n-1}$ depending on the random variables $X_1, \ldots, X_n$ by*

$$A_1 = \{X_1 > X_2\},$$
$$A_k = \{X_{k-1} < X_k > X_{k+1}\} \cup \{X_{k-1} > X_k < X_{k+1}\}, \qquad (2 \leq k \leq n-1).$$

Let I_A denote the indicator random variable of an event A. Then we have

$$\mathbf{as}_n = 1 + \sum_{k=1}^{n-1} I_{A_k}. \tag{4}$$

Proof: For $2 \leq k \leq n-1$, A_k represents the event that X_k is a local extremum in the sequence $X_1, \ldots, X_n$. So, the right-hand side of (4) exactly counts the number of terms in the canonical alternating subsequence. $\square$

4 Computation of the mean and variance

Next, we compute the expectations of the indicator random variables I_{A_k}, $1 \leq k \leq n-1$, and their correlations.

Lemma 3 *The means of the indicator random variables I_{A_k}, $1 \le k \le n-1$, are given by*

$$\mathbb{E}(I_{A_k}) = \mathbb{P}(A_k) = \begin{cases} 1/2 & k=1, \\ 2/3 & 2 \le k \le n-1. \end{cases}$$

The covariances $\mathrm{Cov}(I_{A_j}, I_{A_k}) = \mathbb{E}(I_{A_j} I_{A_k}) - \mathbb{E}(I_{A_j})\mathbb{E}(I_{A_k})$, $1 \le j, k \le n-1$, *are given in the following covariance matrix:*

$$\Big(\mathrm{Cov}(I_{A_j}, I_{A_k})\Big)_{j,k=1}^{n-1} = \begin{pmatrix}
\frac{1}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} & 0 & 0 & 0 & 0 & \ldots & 0 \\
0 & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} & 0 & 0 & 0 & \ldots & 0 \\
0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} & 0 & 0 & \cdots & 0 \\
0 & 0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} & 0 & \cdots & 0 \\
0 & 0 & 0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} & \cdots & 0 \\
\vdots & \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & \cdots & & 0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} & \frac{1}{180} \\
0 & 0 & \cdots & & & 0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9} & \frac{-1}{36} \\
0 & 0 & \cdots & & & & 0 & \frac{1}{180} & \frac{-1}{36} & \frac{2}{9}
\end{pmatrix}.$$

Proof: The computation of the means is trivial and is omitted. For the covariances, note that the events A_j and A_k are independent (and therefore the corresponding covariance is 0) if $|j-k| > 2$, since each A_j is a function of X_{j-1}, X_j and X_{j+1} only; this leaves the diagonal strip $|j-k| \le 2$ containing potentially non-zero terms, which are computed as follows. By symmetry assume that $j \le k$. On the main diagonal $j = k$, it is easy to compute the variances

$$\mathrm{Cov}(I_{A_k}, I_{A_k}) = \mathrm{Var}(I_{A_k}) = \mathbb{P}(A_k)(1 - \mathbb{P}(A_k)) = \begin{cases} 1/4 & k=1, \\ 2/9 & 2 \le k \le n-1. \end{cases}$$

Next, if $j = 1$, we have for $k = 2$ that

$$\mathrm{Cov}(I_{A_1}, I_{A_2}) = \mathbb{P}(A_1 \cap A_2) - \mathbb{P}(A_1)\mathbb{P}(A_2) = \mathbb{P}(X_1 > X_2 < X_3) - \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3} - \frac{1}{3} = 0,$$

and similarly for $k = 3$,

$$\begin{aligned} \mathrm{Cov}(I_{A_1}, I_{A_3}) &= \mathbb{P}(A_1 \cap A_3) - \frac{1}{2} \cdot \frac{2}{3} = \mathbb{P}(X_1 > X_2 > X_3 < X_4) + \mathbb{P}(X_1 > X_2 < X_3 > X_4) - \frac{1}{3} \\ &= \frac{1}{8} + \frac{5}{24} - \frac{1}{3} = 0. \end{aligned}$$

(The number $5/24$ comes from the fact that there are 5 alternating permutations of order 4.) Next, if $k = j + 1 \geq 3$, we have

$$\begin{aligned}\mathrm{Cov}(I_{A_j}, I_{A_{j+1}}) &= \mathbb{P}(A_2 \cap A_3) - \frac{2}{3} \cdot \frac{2}{3} \\ &= \mathbb{P}(X_1 > X_2 < X_3 > X_4) + \mathbb{P}(X_1 < X_2 > X_3 < X_4) - \frac{4}{9} = \frac{10}{24} - \frac{4}{9} = \frac{-1}{36}.\end{aligned}$$

Finally, for $k = j + 2 \geq 4$ we have

$$\begin{aligned}\mathrm{Cov}(I_{A_j}, I_{A_{j+2}}) &= \mathbb{P}(A_2 \cap A_4) - \frac{2}{3} \cdot \frac{2}{3} \\ &= \mathbb{P}(X_1 < X_2 > X_3 < X_4 > X_5) + \mathbb{P}(X_1 > X_2 < X_3 > X_4 < X_5) \\ &\quad + \mathbb{P}(X_1 < X_2 > X_3 > X_4 < X_5) + \mathbb{P}(X_1 > X_2 < X_3 < X_4 > X_5) - \frac{4}{9} \\ &= 2 \cdot \frac{16}{120} + 2 \cdot \frac{11}{120} - \frac{4}{9} = \frac{1}{180},\end{aligned}$$

where the fraction $16/120$ comes from the fact that there are 16 alternating permutations of order 5, and the fraction $11/120$ comes from the fact that there are 11 permutations σ of order 5 satisfying the order relations

$$\sigma(1) < \sigma(2) > \sigma(3) > \sigma(4) < \sigma(5)$$

(see also Lemma 6 below for an alternative way to compute this number). □

Using Lemma 3 and the representation (4), we can now easily compute the mean and variance of $\mathbf{as}_n$ to recover the relations (2), (3):

$$\begin{aligned}\mathbb{E}(\mathbf{as}_n) &= \mathbb{E}\left(1 + \sum_{k=1}^{n-1} I_{A_k}\right) = 1 + \frac{1}{2} + (n-2)\frac{2}{3} = \frac{2}{3}n + \frac{1}{6}, \\ \mathrm{Var}(\mathbf{as}_n) &= \mathrm{Var}\left(1 + \sum_{k=1}^{n-1} I_{A_k}\right) = \sum_{j,k=1}^{n} \mathrm{Cov}(I_{A_j}, I_{A_k}) \\ &= \frac{1}{4} + (n-2)\frac{2}{9} + 2(n-3)\frac{-1}{36} + 2(n-4)\frac{1}{180} = \frac{8}{45}n - \frac{13}{180},\end{aligned}$$

where in the computation of the mean we assume that $n \geq 2$, and for the variance computation we assume that $n \geq 4$. Note that in principle, higher moments of $\mathbf{as}_n$ can also be computed in the same way, although the computations require higher-order correlations (e.g., of the form $\mathbb{E}(I_{A_j} I_{A_k} I_{A_\ell})$) and therefore become more tedious.

5 The limiting Gaussian distribution

We now show how the representation (4) can be used to deduce the limiting Gaussian distribution of $\mathbf{as}_n$.

Proposition 4 *For all $t \in \mathbb{R}$ we have*

$$\mathbb{P}\left(\frac{\mathbf{as}_n - \mathbb{E}(\mathbf{as}_n)}{(\mathrm{Var}(\mathbf{as}_n))^{1/2}} \leq t\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-x^2/2}\, dx \qquad \textit{as } n \to \infty.$$

Proof: First, note that in (4) we can ignore the constant 1 and the first summand I_{A_1} in the sum of indicators, since their contribution is negligible compared to the scale of $\sqrt{n}$. Thus, if we consider the modified random variable $\mathbf{as}'_n = \sum_{k=2}^{n-1} I_{A_k}$, it will be enough to prove that $\mathbf{as}'_n$ converges after scaling to the standard Gaussian distribution $N(0,1)$. This sum is a sum of a sequence of identically distributed random variables. The sequence of random variables is *not* independent, but it is "3-dependent", meaning that any two contiguous blocks of random variables that are separated by a gap of length 3 or more are independent of each other. It is a well-known fact that for bounded and identically distributed random variables (and even under much weaker conditions), one can replace the condition of independence by "m-dependence" (where m is the size of the gap needed to ensure independence of the two blocks), and this is enough to ensure that the standard central limit theorem from probability theory holds. Formally, our sequence satistfies the assumptions of Theorem 1 of Hoeffding and Robbins (1948) (whose proof is based on a simple reduction to the standard central limit theorem for independent sequences), which gives the desired conclusion on convergence to the Gaussian distribution. $\square$

We emphasize that, while the proofs of the convergence to Gaussian distribution given by Stanley (2008) and Widom (2006) also rely on fairly straightforward techniques, namely asymptotic analysis of generating functions, they provide little intuition regarding why one should expect to see a Gaussian limit. The proof above makes the situation much clearer, since it is based on the observation that $\mathbf{as}_n$ is a sum of many weakly independent components.

6 The distribution of local extrema

Having demonstrated the relevance of the sequence of local extrema to understanding the longest alternating subsequence statistic $\mathbf{as}_n$, it now makes sense to try to understand aspects of the behavior of the extrema, other than just how many of them there are. In particular, it is natural to look at the distribution of the extrema *values* — i.e., how small or large can we expect them to be? Of course, the extrema consist of local minima and local maxima appearing in alternation, so it makes sense to answer this question for the minima and maxima separately. The precise result is as follows.

Theorem 5 *Let N_{min} be the (random) number of local minima in the sequence of random variables $X_1, \ldots, X_n$, and let $m_1, m_2, \ldots, m_{N_{min}}$ denote the values of the local minima in the sequence, i.e., $m_j = X_{k_j}$ where k_j is the position where the j-th local minimum appears. For all $0 \le a < b \le 1$ we have the convergence in probability*

$$\frac{1}{N_{min}} \#\Big\{ 1 \le j \le N_{min} \ : \ m_j \in [a,b] \Big\} \xrightarrow[n\to\infty]{P} \int_a^b 3(1-t)^2\,dt = (1-a)^3 - (1-b)^3. \tag{5}$$

In other words, the local minima in the limit are distributed in $[0,1]$ according to the limiting probability density function $3(1-t)^2$. Similarly (and symmetrically), if we denote by N_{max} the number of local maxima*, and denote the values of the local maxima by $M_1, \ldots, M_{N_{max}}$, then for all $0 \le a < b \le 1$ we have the convergence in probability*

$$\frac{1}{N_{max}} \#\Big\{ 1 \le j \le N_{max} \ : \ M_j \in [a,b] \Big\} \xrightarrow[n\to\infty]{P} \int_a^b 3t^2\,dt = b^3 - a^3. \tag{6}$$

That is, the local maxima are in the limit distributed according to the density $3t^2$.

Proof: First, note that N_{min} and N_{max} differ by at most 1, and their sum is equal to $\mathbf{as}_n - 1$ or $\mathbf{as}_n - 2$ (depending on whether the term I_{A_1} in (4) is 0 or 1), so, by the results on the mean and variance of $\mathbf{as}_n$, we see that both N_{min} and N_{max} are equal to $n/3 + o(n)$ with asymptotically high probability as $n \to \infty$. Second, note that it is enough to prove (5) (which by symmetry implies (6)) in the case $b = 1$. Next, observe that we can write the number of local minima falling in the interval $[a, 1]$ as a sum of indicator random variables, namely

$$\#\Big\{1 \leq j \leq N_{\text{min}} \;:\; m_j \in [a, 1]\Big\} = \sum_{k=2}^{n-1} 1_{\{X_{k-1} > X_k < X_{k+1},\; X_k \geq a)\}} \tag{7}$$

It is trivial to compute the mean of each of these (identically distributed) indicators. It is equal to

$$\mathbb{E}\left(1_{\{X_{k-1} > X_k < X_{k+1},\; X_k \geq a)\}}\right) = \mathbb{P}\big(X_1, X_2, X_3 \geq a, X_2 = \min(X_1, X_2, X_3)\big) = \frac{1}{3}(1-a)^3.$$

It is also easy to see that the variance of the sum of the indicators is $O(n)$, since, as before, each of the indicators is correlated with only the two adjacent ones on each side. Therefore we get (for example using Chebyshev's inequality) that as $n \to \infty$ the left-hand side of (7) is with high probability equal to $\frac{1}{3}(n-2)(1-a)^3 + o(n)$ (in fact, the error is of order $O(n^{1/2})$, and one can use the standard results from probability theory mentioned above to get a limiting Gaussian convergence for this random variable as well). Combining this with the previous observation about the asymptotic behavior of N_{min} gives (5). $\square$

7 The joint distribution of a minimum-maximum pair

Our final result concerns a formula for the limiting *joint distribution* of a local minimum and the local maximum that follows it. We start with a lemma.

Lemma 6 *Let $d \geq 4$, and let A be an open subset of $\{(s,t) : 0 \leq s \leq t \leq 1\}$. We have*

$$\mathbb{P}\Big(X_1 > X_2 < X_3 < \ldots < X_{d-1} > X_d, (X_2, X_{d-1}) \in A\Big) = \frac{1}{(d-4)!} \iint_A (1-s)t(t-s)^{d-4}\, ds\, dt. \tag{8}$$

Proof: The probability on the left-hand side can be expressed in an obvious way as a d-dimensional multiple integral of the constant function 1 on the subset of $[0,1]^d$ consisting of vectors $(x_1, \ldots, x_d)$ satisfying $x_1 > x_2 < \ldots < x_{d-1} > x_d$ and $(x_2, x_{d-1}) \in A$. Choosing x_2 and x_{d-1} as the outer variables of integration, we can compute this integral as the iterated integral

$$\iint_A \left(\int_{x_2}^1 dx_1 \int_0^{x_{d-1}} dx_d \iint \ldots \int_{\{x_2 \leq x_3 \leq \ldots \leq x_{d-2} \leq x_{d-1}\}} dx_3 \ldots dx_{d-2} \right) dx_2\, dx_{d-1}$$
$$= \iint_A (1-x_2) x_{d-1} \frac{(x_{d-1} - x_2)^{d-4}}{(d-4)!}\, dx_2\, dx_{d-1}.$$

This is equal to the right-hand side of (8). $\square$

The result on the joint distribution of a minimum-maximum pair is as follows.

Theorem 7 *Let A be an open subset of $\{(s,t) : 0 \le s \le t \le 1\}$. Denote by $N_{\textit{min-max}}$ the number of* minimum-maximum pairs, *which are defined as pairs (i,j) of positions where $i < j$, X_i is a local minimum of the sequence $X_1,\ldots,X_n$, X_j is a local maximum, and X_i and X_j are not separated by another local extremum. Denote the values of these minimum-maximum pairs (X_i, X_j) (arranged in order of their appearance) by $(m_1,\mu_1),(m_2,\mu_2),\ldots,(m_{N_{\textit{min-max}}},\mu_{N_{\textit{min-max}}})$ (in the notation of Theorem 5, $\mu_j = M_j$ or M_{j+1} depending on whether the first local maximum appears after the first local minimum or before it). Then we have the convergence in probability*

$$\frac{1}{N_{\textit{min-max}}}\#\Big\{1 \le k \le N_{\textit{min}} \ : \ (m_j,\mu_j) \in A\Big\} \xrightarrow[n\to\infty]{P} \iint_A 3(1-s)t\,e^{t-s}\,ds\,dt. \tag{9}$$

That is, the joint distribution of a local minimum-maximum pair is represented in the limit by the density function $f(s,t) = 3(1-s)te^{t-s}$, $(0 < s < t < 1)$.

Proof: It is easy to see that $N_{\text{min-max}}$ differs from N_{min} by at most 1, so as before, we know that it is with high probability approximately equal to $n/3 + o(n)$. Denote

$$T_n = \#\Big\{1 \le k \le N_{\text{min-max}} \ : \ (m_j,\mu_j) \in A\Big\}.$$

The key observation is that we can decompose T_n based on the size of the gap between the position of the local minimum and the subsequent maximum. This leads to a representation

$$T_n = Y_{n,1} + Y_{n,2} + \ldots + Y_{n,n-3},$$

where we define random variables $Y_{n,1},\ldots,Y_{n,n-3}$ by

$$\begin{aligned}
Y_{n,1} &= \#\Big\{2 \le k \le n-2 \ : \ X_{k-1} > X_k < X_{k+1} > X_{k+2},\ (X_k, X_{k+1}) \in A\Big\},\\
Y_{n,2} &= \#\Big\{2 \le k \le n-3 \ : \ X_{k-1} > X_k < X_{k+1} < X_{k+2} > X_{k+3},\ (X_k, X_{k+2}) \in A\Big\},\\
&\ \vdots\\
Y_{n,j} &= \#\Big\{2 \le k \le n-j-1 \ : \ X_{k-1} > X_k < X_{k+1} < \ldots < X_{k+j} > X_{k+j+1},\ (X_k, X_{k+j}) \in A\Big\},\\
&\ \vdots\\
Y_{n,n-3} &= \#\Big\{2 \le k \le 2 \ : \ X_1 > X_2 < X_3 < \ldots < X_{n-1} > X_n,\ (X_2, X_{n-1}) \in A\Big\}.
\end{aligned}$$

Now observe that each $Y_{n,j}$ can in turn be represented as a sum of $n-j-2$ indicator random variables of events of the form

$$B_{n,j,k} = \Big\{X_{k-1} > X_k < X_{k+1} < \ldots < X_{k+j} > X_{k+j+1},\ (X_k, X_{k+j}) \in A\Big\}. \tag{10}$$

For a fixed j, all of these events have the same expectation, given by the right-hand side of (8) with $d = j+3$. It follows that the expectation of T_n is given by

$$\mathbb{E}(T_n) = \sum_{j=1}^{n-3}(n-j-2)\frac{1}{(j-1)!}\iint_A (1-s)t(t-s)^{j-1}\,ds\,dt. \tag{11}$$

Because of the fast decay of the coefficients $1/(j-1)!$, it is an easy exercise to sum this series asymptotically (for example by truncating it around $j \approx \log n$; see below for a related estimate), to obtain that

$$\mathbb{E}(T_n) = n \iint_A (1-s)te^{t-s}\, ds\, dt + O\left(n^{-10}\right) \qquad \text{as } n \to \infty.$$

So, we get that at least the mean of the variable $\frac{1}{N_{\text{min-max}}} T_n$ approaches the expression on the right-hand side of (9) in the limit. It remains to show that this random variable is concentrated around its mean. To see this, set $p_n = \lfloor \log n \rfloor$ (where $\lfloor x \rfloor$ denotes the integer part of x), and define

$$T'_n = \sum_{j=1}^{p_n} Y_{n,j}.$$

Observe that $T_n = T'_n$ with high probability, since, by Markov's inequality,

$$\begin{aligned} \mathbb{P}(T_n \neq T'_n) &= \mathbb{P}\left(\sum_{j=p_n+1}^{n-3} Y_{n,j} > 0\right) \leq \mathbb{E}\left(\sum_{j=p_n+1}^{n-3} Y_{n,j}\right) \\ &\leq n \sum_{j=p_n+1}^{\infty} \frac{1}{(j-1)!} = O(n^{-10}). \end{aligned} \tag{12}$$

But we know that the variance of T'_n is given by

$$\operatorname{Var}(T'_n) = \sum_{i,j=1}^{p_n} \operatorname{Cov}(Y_{n,i}, Y_{n,j}).$$

We claim that these covariances satisfy $\operatorname{Cov}(Y_{n,i}, Y_{n,j}) \leq 10n \log n$; if true, this implies that $\operatorname{Var}(T'_n) \leq 10n(\log n)^3$, so that T'_n has standard deviation of order at most $n^{1/2}(\log n)^{3/2}$ and is therefore concentrated around its mean. To see where the covariance bound comes from, consider that each $Y_{n,j}$ is represented by a sum of indicator random variables $I_{B_{n,j,k}}$, where $B_{n,j,k}$ is defined in (10) above. The number of indicators is at most n, and furthermore the only nonzero correlations between $I_{B_{n,j,k}}$ and $I_{B_{n,i,k'}}$ can appear when $|k-k'| \leq \log n$, since otherwise the indicators are functions of independent blocks of random variables from the sequence $X_1, \ldots, X_n$. This easily implies the stated covariance bound. To summarize, we have shown concentration of T'_n (and therefore also of T_n, by (12)) around its mean, which is also very close to the mean of T_n. Combining these facts with (11) and dividing by $n/3$, the approximate value of $N_{\text{min-max}}$ with high probability, gives the result. □

8 Concluding remarks

It is interesting to contrast, as Stanley (2008) did, the results for longest alternating subsequences of random permutations with the well-developed theory of longest *increasing* subsequences (see Aldous and Diaconis (1999), Stanley (2007)). In all honesty, it must be admitted that the latter subject leads to a richer and more interesting theory... Still, the study of longest alternating subseqences is not without its own rewards, and provides a nice example of the interaction of algebraic-combinatorial and probabilistic ideas. In particular, the connection made in this extended abstract between this permutation statistic and the study of the sequence of local extrema of permutations raises interesting new questions. If one starts with a doubly-infinite sequence $\ldots, X_{-1}, X_0, X_1, X_2, \ldots$ of i.i.d. $U[0,1]$ random variables and considers the local extrema (so that the restriction to any finite block says something about longest alternating subsequences in that block), the sequence of local extrema can be thought of as an interesting "stationary point process" on $\mathbb{Z}$. This is analogous to the sequence of descents in random sequences, which was studied by many authors. Most recently, Borodin et al. (2010) proved that the sequence of descents is a determinantal point process. One natural question that seems worthy of further study is whether this property is shared by the sequence of local extrema.

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Maximal 0*-*1*-fillings of moon polyominoes with restricted chain lengths and rc-graphs*

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Abstract. We show that maximal 0-1-fillings of moon polynomials, with restricted chain lengths, can be identified with certain rc-graphs, also known as pipe dreams. In particular, this exhibits a connection between maximal 0-1-fillings of Ferrers shapes and Schubert polynomials. Moreover, it entails a bijective proof showing that the number of maximal fillings of a stack polyomino S with no north-east chains longer than k depends only on k and the multiset of column heights of S.

Our main contribution is a slightly stronger theorem, which in turn leads us to conjecture that the poset of rc-graphs with covering relation given by generalised chute moves is in fact a lattice.

Résumé. Nous démontrons que les remplissages maximaux avec 0 et 1 des polyominos L-convexes, avec longueurs de chaînes restreintes, peuvent être identifiés avec certains *rc-graphes*, également connus sous le nom de *pipe dreams*. En particulier, ceci montre un lien entre ces remplissages d'un diagramme de Ferrers et les polynômes de Schubert. On en déduit en outre une preuve bijective du fait que le nombre de remplissages maximaux d'un *stack polyomino* S, avec longueurs de chaînes bornées par un entier k, dépend seulement de k et du multi-ensemble des tailles des colonnes de S.

Notre contribution principale est un énoncé un peu plus fort, qui nous mène à conjecturer que l'ensemble ordonné (poset) des *rc-graphes* est en fait un treillis.

Keywords: multitriangulations, rc-graphs, Edelman-Greene insertion, Schubert polynomials

1 Introduction

1.1 Triangulations, multitriangulations and 0*-*1*-fillings*

The systematic study of 0-1-fillings of polyominoes with restricted chain lengths likely originates in an article by Jakob Jonsson [5]. At first, he was interested in a generalisation of triangulations, where the objects under consideration are maximal sets of diagonals of the n-gon, such that at most k diagonals are allowed to cross mutually. Thus, in the case $k = 1$ one recovers ordinary triangulations. He realised these objects as fillings of the staircase shaped polyomino with row-lengths $n-1, n-2, \dots, 1$ with zeros and ones. The condition that at most k diagonals cross mutually then translates into the condition that

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the longest north-east chain in the filling has length k, see Definition 2.3. Instead of studying fillings of the staircase shape only, he went on to consider more general shapes which he called *stack* and *moon polyominoes*, see Definition 2.2 and Figure 1.

For stack polyominoes he was able to prove that the number of maximal fillings depends only on k and the multiset of heights of the columns, not on the particular shape of the polyomino. He conjectured that this statement holds more generally for moon polyominoes, which was eventually proved by the author [13] using a technique introduced by Christian Krattenthaler [8] based on Sergey Fomin's growth diagrams for the Robinson-Schensted-Knuth correspondence. However, the proof given there is not fully bijective: what one would hope for is a correspondence between fillings of any two moon polyominoes that differ only by a permutation of the columns. This article is a step towards this goal.

1.2 RC-graphs and the subword complex

RC-graphs (for 'reduced word compatible sequence graphs', see [1], also known as 'pipe dreams' see [7]) were introduced by Sergey Fomin and Anatol Kirillov [3] to prove various properties of Schubert polynomials. Namely, for a given permutation w, the Schubert polynomial $\mathfrak{S}_w$ can be regarded as the generating function of rc-graphs, see the remark after Definition 2.5

A different point of view is to consider them as facets of a certain simplicial complex. Let w_0 be the long permutation $n \cdots 21$, and consider its reduced factorisation

$$Q = s_{n-1} \cdots s_2 s_1 \; s_{n-1} \cdots s_3 s_2 \; \cdots\cdots \; s_{n-1} s_{n-2} \; s_{n-1}.$$

Then the subword complex associated to Q and w introduced by Allen Knutson and Ezra Miller [7, 6] has as facets those subwords of Q that are reduced factorisations of w. Subword complexes enjoy beautiful topological properties, which are transferred by the main theorem of this article to the simplicial complex of 0-1-fillings, as observed by Christian Stump [17], see also the article by Luis Serrano and Christian Stump [15].

The intimate connection between maximal fillings and rc-graphs demonstrated by the main theorem of this article, Theorem 3.2, *should* not have come as a surprise. Indeed, Sergey Fomin and Anatol Kirillov [4] established a connection between reduced words and reverse plane partitions already thirteen years ago, which is not much less than the case of Ferrers shapes in Theorem 4.3. They even pointed towards the possibility of a bijective proof using the Edelman-Greene correspondence.

More recently, the connection between Schubert polynomials and triangulations was noticed by Alexander Woo [18]. Vincent Pilaud and Michel Pocchiola [11] discovered rc-graphs (under the name 'beam arrangements') more generally for multitriangulations, however, they were unaware of the theory of Schubert polynomials. In particular, Theorem 3.18 of Vincent Pilaud's thesis [10] (see also Theorem 21 of [11]) is a variant of our Theorem 3.2 for multitriangulations.

Finally, Christian Stump and the author of the present article became aware of an article by Vincent Pilaud and Francisco Santos [12] that describes the structure of multitriangulations in terms of so-called k-stars (introduced by Harold Coxeter). We then decided to translate this concept to the language of fillings, and discovered pipe dreams yet again.

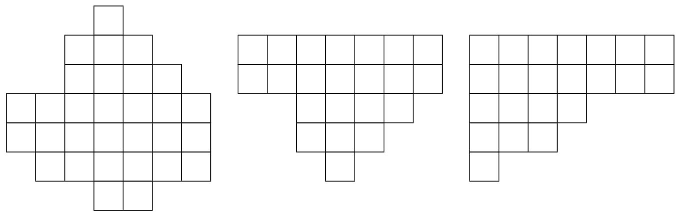


Fig. 1: a moon-polyomino, a stack-polyomino and a Ferrers diagram

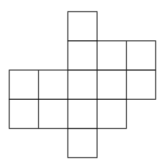
2 Definitions

2.1 Polyominoes

Definition 2.1 *A* polyomino *is a finite subset of the quarter plane* $\mathbb{N}^2$*, where we regard an element of* $\mathbb{N}^2$ *as a cell. A* column *of a polyomino is the set of cells along a vertical line, a* row *is the set of cells along a horizontal line. We are using 'English' (or matrix) conventions for the indexing of the rows and columns of polyominoes: the top row and the left-most column have index* 1*.*

The polyomino is convex*, if for any two cells in a column (rsp. row), the elements of* $\mathbb{N}^2$ *in between are also cells of the polyomino. It is* intersection-free*, if any two columns are* comparable*, i.e., the set of row coordinates of cells in one column is contained in the set of row coordinates of cells in the other. Equivalently, it is intersection-free, if any two rows are comparable.*

For example, the polyomino



is convex, but not intersection-free, since the first and the last columns are incomparable.

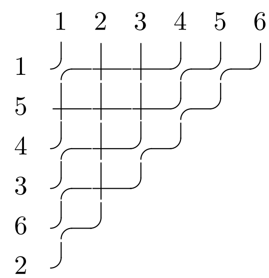
Definition 2.2 *A* moon polyomino *(or L-convex polyomino) is a convex, intersection-free polyomino. Equivalently we can require that any two cells of the polyomino can be connected by a path consisting of neighbouring cells in the polyomino, that changes direction at most once. A* stack polyomino *is a moon-polyomino where all columns start at the same level. A* Ferrers diagram *is a stack-polyomino with weakly decreasing row widths* $\lambda_1, \lambda_2, \ldots, \lambda_n$*, reading rows from top to bottom.*

Because a moon-polyomino is intersection free, the set of rows of maximal length in a moon polyomino must be consecutive. We call the set of rows including these and the rows above the top half *of the polyomino. Similarly, the set of columns of maximal length, and all columns to the right of these, is the* right half *of the polyomino. The intersection of the top and the right half is the* top right quarter *of* M*.*

2.2 Fillings and Chains

Definition 2.3 *A* 0-1-filling *of a polyomino is an assignment of the numbers* 0 *and* 1 *to the cells of the polyomino. Cells containing* 0 *are also called* empty.

A north-east chain *is a sequence of non-zero entries in a filling such that the smallest rectangle containing all its elements is completely contained in the moon polyomino and such that for any two of its elements one is strictly to the right and strictly above the other.*



	1	2	3	4	5	6
1	•	+	+	•	•	•
5	+	+	+	•	•	
4	•	+	•	•		
3	•	+	•			
6	•	•				
2	•					

	1	2	3	4	5	6
1	•	2	3	•	•	•
5	2	3	4	•	•	
4	•	4	•	•		
3	•	5	•			
6	•	•				
2	•					

Fig. 2: a reduced pipe dream associated to the reduced factorisation $s_3s_2s_4s_3s_2s_4s_5$ of $1, 5, 4, 3, 6, 2$.

As it turns out, it is more convenient to draw dots instead of ones and leave cells filled with zeros empty. Two examples of (rather special) fillings of a moon polyomino are depicted in Figure 4.

Definition 2.4 *$\mathcal{F}^{ne}_{01}(M, k)$ is the set of* 0-1-*fillings of the moon polyomino M whose longest north-east chain has length k and that are maximal, i.e., assigning an empty cell a* 1 *would create a north-east chain of length $k + 1$. For a vector* $\mathbf{r}$ *of integers $\mathcal{F}^{ne}_{01}(M, k, \mathbf{r})$ is the subset of $\mathcal{F}^{ne}_{01}(M, k)$ consisting of those fillings that have exactly $\mathbf{r}_i$ zero entries in row i.*

For any filling in $\mathcal{F}^{ne}_{01}(M, k)$, and an empty cell ϵ, there must be a chain C such that replacing the 0 *with* 1 *in ϵ, and adding ϵ to C, would make C into a $(k + 1)$-chain. In this situation, we say that C is a* maximal chain for ϵ.

Note that extending the first k rows and columns of a Ferrers diagram does not affect the set $\mathcal{F}^{ne}_{01}$, which is why we choose to fix the number of zero entries instead of entries equal to 1, which might seem more natural at first glance.

For the staircase shape $\lambda_0 = (n - 1, \dots, 1)$, the set $\mathcal{F}^{ne}_{01}(\lambda_0, k)$ has a particularly beautiful interpretation, namely as the set of k-triangulations of the n-gon. More precisely, label the vertices of the n-gon clockwise from 1 to n, and identify a cell of the shape in row i and column j with the pair $(n - i + 1, j)$ of vertices. Thus, the entries in the filling equal to 1 define a set of diagonals of the n-gon. It is not hard to check that a north-east chain of length k in the filling corresponds to a set of k mutually crossing diagonals in the n-gon. This correspondence was Jakob Jonsson's [5] starting point to prove (in a quite non-bijective fashion) that there are as many k-triangulations of the n-gon as fans of k non-intersecting Dyck paths with $n - 2k$ up steps each. For this case Luis Serrano and Christian Stump [15] provided the first completely bijective proof.

2.3 Pipe dreams

In this section we collect some results around rc-graphs. All of these can be found in [1] together with precise references.

Definition 2.5 *A* pipe dream *for a permutation w is a filling of a the quarter plane $\mathbb{N}^2$, regarding each element of $\mathbb{N}^2$ as a cell,* elbow joints ⌟⌜ *and a finite number of* crosses ┼*, such that a pipe entering from above in column i exits to the left from row $w^{-1}(i)$. A pipe dream is* reduced *if each pair of pipes* *crosses at most once, it is then also called* rc-graph. $\mathcal{RC}(w)$ *is the set of reduced pipe dreams for* w*, and* $\mathcal{RC}(w, \mathbf{r})$ *is the subset of* $\mathcal{RC}(w)$ *having precisely* $\mathbf{r}_i$ *crosses in row* i*.*

Usually it will be more convenient to draw dots instead of elbow joints and sometimes to omit crosses. We will do so without further notice.

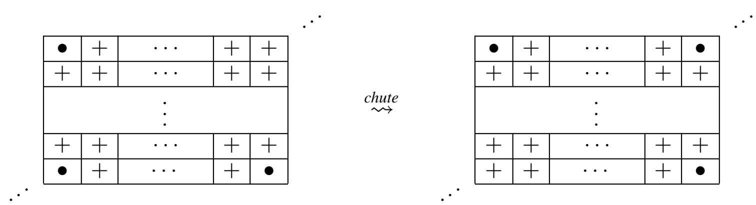
Every pipe dream in $\mathcal{RC}(w)$ is associated to a reduced factorisation of w as follows: replace each cross appearing in row i and column j of the pipe dream with the elementary transposition $(i + j - 1, i + j)$. Then the reduced factorisation of w is given by the sequence of transpositions obtained by reading each row of the pipe dream from right to left, and the rows from top to bottom. Figure 2 shows an example.

Using reduced pipe dreams, it is possible to define the Schubert polynomial $\mathfrak{S}_w$ for the permutation w in a very concrete way. For a reduced pipe dream $D \in \mathcal{RC}(w)$, define $x^D = \prod_{(i,j)\in D} x_i$, where the product runs over all crosses in the pipe dream. Then the Schubert polynomial is just the generating function for pipe dreams:

$$\mathfrak{S}_w = \sum_{D\in\mathcal{RC}(w)} x^D$$

The following operation on pipe dreams, in a slightly less general form, was introduced by Nantel Bergeron and Sara Billey [1]. It will be the main tool in the proof of Theorem 3.2.

Definition 2.6 *Let* $D \in \mathcal{RC}(w)$ *be a pipe dream. Then a* chute move *is a modification of* D *of the following form:*

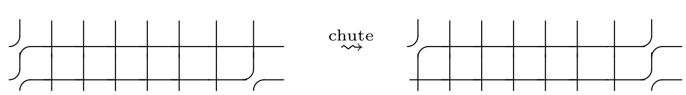


More formally, a chutable rectangle *is a rectangular region* r *inside a pipe dream* D *with at least two columns and two rows such that all but the following three locations of* r *are crosses: the north-west, south-west, and south-east corners. Applying a* chute move *to* D *is accomplished by placing a '+' in the south-west corner of a chutable rectangle* r *and removing the '+' from the north-east corner of* r*. We call the inverse operation* inverse chute move*.*

The following lemma was given by Nantel Bergeron and Sara Billey [1, Lemma 3.5] for two rowed chute moves, the proof is valid for our generalised chute moves without modification:

Lemma 2.7 *The set* $\mathcal{RC}(w)$ *of reduced pipe dreams for* w *is closed under chute moves.*

Proof: The pictorial description of chute moves immediately implies that the permutation associated to the pipe dream remains unchanged. For example, here is the picture associated with a two rowed chute move:



□

It follows that chute moves define a partial order on $\mathcal{RC}(w)$, where D is covered by E if there is a chute move transforming E into D. Nantel Bergeron and Sara Billey restricted their attention to two rowed chute moves. For this case, their main theorem states that the poset defined by chute moves has a unique maximal element, namely

$$D_{top}(w) = \left\{(c,j) : c \leq \#\{i : i < w_j^{-1}, w_i > j\}\right\}.$$

It is easy to see that considering general chute moves, the poset has also a unique minimal element, namely

$$D_{bot}(w) = \{(i,c) : c \leq \#\{j : j > i, w_j < w_i\}\}.$$

In the next section we will show a statement similar in spirit to the main theorem of Nantel Bergeron and Sara Billey for the more general chute moves defined above.

After generating and analysing some of these posets using `Sage` [14], see Figure 3 for an example, we became convinced that they should have much more structure:

Conjecture 2.8 *The poset of reduced pipe dreams defined by (general) chute moves is in fact a lattice.*

There is another natural way to transform one reduced pipe dream into another, originating in the concept of flipping a diagonal of a triangulation. Namely, consider an elbow joint in the pipe dream. Since any pair of pipes crosses at most once, there is at most one location where the pipes originating from the given elbow joint cross. If there is such a crossing, replace the elbow joint by a cross and the cross by an elbow joint. Clearly, the result is again a reduced pipe dream, associated to the same permutation.

It is believed (see Vincent Pilaud and Michel Pocchiola [11], Question 51) that the simplicial complex of multitriangulations can be realised as a polytope, in this case the graph of flips would be the graph of the polytope. Note that graph of chute moves is a subgraph of the graph of flips. Is Conjecture 2.8 related to the question of polytopality?

3 Maximal Fillings of Moon Polyominoes and Pipe Dreams

Consider a filling in $\mathcal{F}_{01}^{ne}(M,k)$. Replacing zeros with crosses, and all cells containing ones as well as all cells not in M with elbow joints we clearly obtain a pipe dream. We will see in this section that it is in fact reduced.

Even without that knowledge we can speak of chute moves applied to fillings in $\mathcal{F}_{01}^{ne}(M,k)$. However, a priori it is not clear under which conditions the result of such a move is again a filling in $\mathcal{F}_{01}^{ne}(M,k)$. In particular we have to deal with the fact that under this identification all cells outside M are also filled with *elbow joints* corresponding to *ones*. Of course, to determine the set of north-east chains we have to consider the original filling and the boundary of M and disregard elbow joints outside.

Similar to the article of Nantel Bergeron and Sara Billey we will also consider two special fillings $D_{bot}(M,k)$ and $D_{top}(M,k)$. These will turn out to be the minimal and the maximal element in the poset having elements $\mathcal{F}_{01}^{ne}(M,k)$, where one filling is smaller than another if it can be obtained by applying chute moves to the latter.

Definition 3.1 *Let M be a moon polyomino and $k \geq 0$. Then $D_{top}(M,k) \in \mathcal{F}_{01}^{ne}(M,k)$ is obtained by putting ones into all cells that can be covered by any rectangle of size at most $k \times k$ which is completely contained in the moon polyomino and that touches the boundary of M with its lower-left corner.*

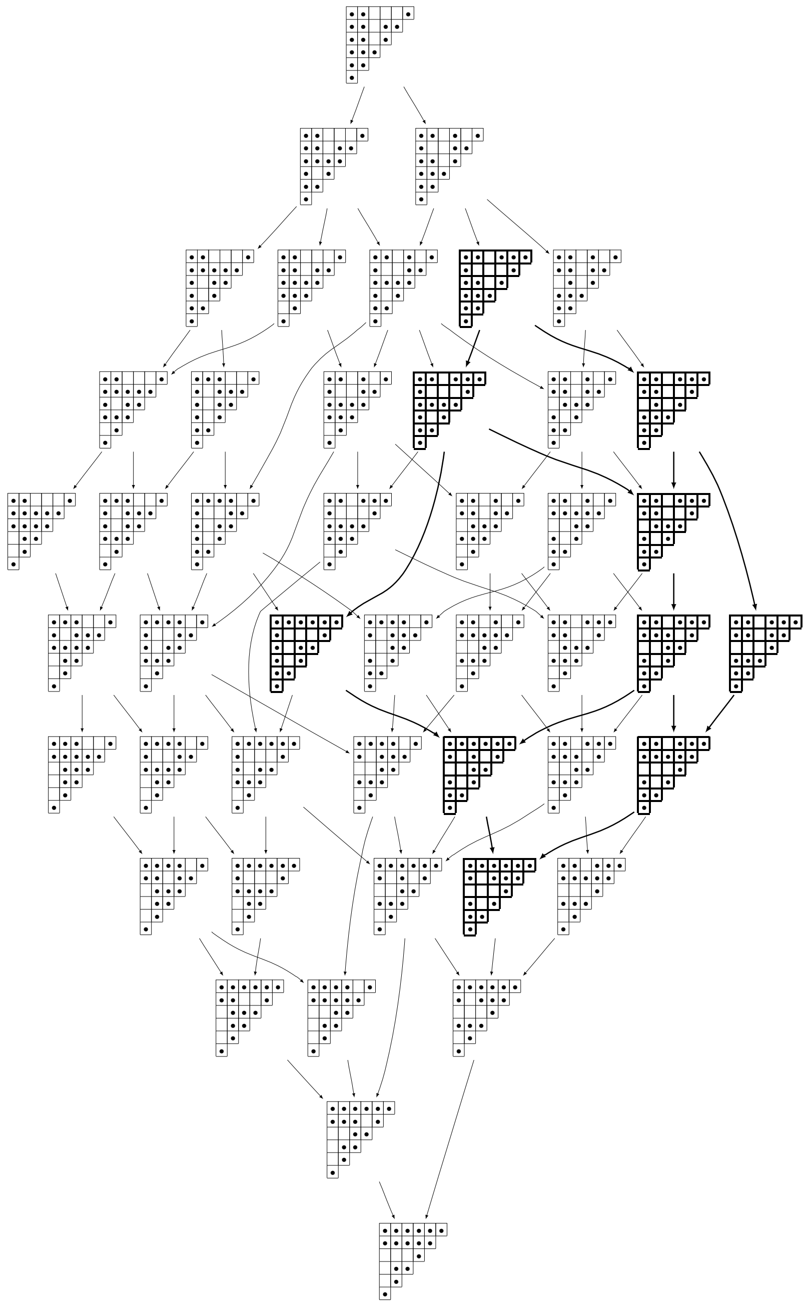
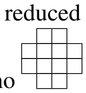


Fig. 3: the poset of reduced pipe dreams for the permutation $1, 2, 6, 4, 5, 3$. The interval of 0-1-fillings with $k = 1$ of the moon polyomino  is shown in bold.

Similarly, $D_{bot}(M,k) \in \mathcal{F}_{01}^{ne}(M,k)$ is obtained by putting ones into all cells that can be covered by any rectangle of size at most $k \times k$ which is completely contained in the moon polyomino and that touches the boundary of M with its upper-right corner.

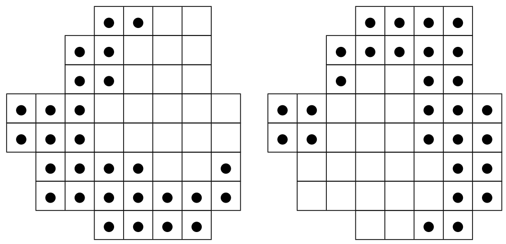


Fig. 4: The special fillings $D_{top}(M,k)$ and $D_{bot}(M,k)$ for $k = 2$ of a moon polyomino.

We can now state the main theorem of this article:

Theorem 3.2 *Let M be a moon polyomino and $k \geq 0$. The set $\mathcal{F}_{01}^{ne}(M,k,\mathbf{r})$ can be identified with the set of reduced pipe dreams $\mathcal{RC}\big(w(M,k),\mathbf{r}\big)$ having all crosses inside of M for some permutation depending only on M and k: replace zeros with crosses and all cells containing ones as well as all cells not in M with elbow joints.*

More precisely, the set $\mathcal{F}_{01}^{ne}(M,k)$ is an interval in the poset of reduced pipe dreams $\mathcal{RC}\big(w(M,k)\big)$ with minimal element $D_{bot}(M,k)$ and maximal element $D_{top}(M,k)$.

As already remarked in the introduction various versions of this theorem were independently proved by various authors by various methods. The most general version is due to Luis Serrano and Christian Stump [15, Theorem 2.6] who use properties of subword complexes and obtain additionally many properties of the simplicial complex of 0-1-fillings.

The advantage of our approach using chute moves is the demonstration of the property that $\mathcal{F}_{01}^{ne}(M,k)$ is in fact an interval in the bigger poset of reduced pipe dreams. In particular, if Conjecture 2.8 turns out to be true $\mathcal{F}_{01}^{ne}(M,k)$ is also a lattice. An illustration is given in Figure 3.

Let us first state a very basic property of chute moves as applied to fillings:

Lemma 3.3 *Let M be a moon polyomino. Chute moves and their inverses applied to a filling in $\mathcal{F}_{01}^{ne}(M,k)$ produce another filling in $\mathcal{F}_{01}^{ne}(M,k)$ whenever all zero entries remain in M.*

Proof: We only have to check that chain lengths are preserved which is not hard. □

Most of what remains of this section is devoted to prove that there is precisely one filling in $\mathcal{F}_{01}^{ne}(M,k)$ that does not admit a chute move such that the result is again in $\mathcal{F}_{01}^{ne}(M,k)$, namely $D_{bot}(M,k)$, and precisely one filling that does not admit an inverse chute move with the same property, namely $D_{top}(M,k)$.

Although the strategy itself is actually very simple the details turn out to be quite delicate. Because of space constraints we omit the proofs in this extended abstract. However, we try to give an impression of the overall structure of the proof and state some of the auxiliary lemmas. Let us fix k, a moon polyomino M and a maximal filling $D \in \mathcal{F}_{01}^{ne}(M,k)$ different from $D_{bot}(M,k)$. We will then explicitly locate a chutable rectangle. Note that maximality of the filling will play a crucial role throughout. The first lemma is used to show that certain cells of the polyomino must be empty because otherwise the filling would contain a chain of length $k+1$:

Lemma 3.4 (Chain induction) *Consider a maximal filling of a moon polyomino. Let ϵ be an empty cell such that all cells below ϵ in the same column are empty too, except possibly those that are below the lowest cell of the column left of ϵ. Assume that for* each *of these cells δ there is a maximal chain for δ strictly north-east of δ. Then there is a maximal chain for ϵ strictly north-east of ϵ.*

Note that for the conclusion of Lemma 3.4 to hold we really have to assume that *all* cells below ϵ are empty: in the maximal filling for $k=1$

$\bullet$	ϵ	$\bullet$
	δ	$\bullet$
$\bullet$	$\bullet$	

there is a maximal chain for δ north-east of δ, but no maximal chain for ϵ north-east of ϵ. The following example demonstrates that it is equally necessary that the filling is maximal:

$\bullet$	
ϵ	$\bullet$
	$\bullet$

The next lemma parallels the main Lemma 3.6 in the article by Nantel Bergeron and Sara Billey [1]:

Lemma 3.5 *Consider a maximal filling of a moon polyomino. Suppose that there is a cell γ containing a 1 with an empty cell ϵ in the neighbouring cell to its right such that there are at least as many cells above γ as above ϵ. Then the filling contains a chutable rectangle.*

Finally, the main statement follows from a careful analysis of fillings different from $D_{bot}(M,k)$, repeatedly applying the previous lemmas to exclude obstructions to the existence of a chutable rectangle:

Theorem 3.6 *Any maximal filling other than $D_{bot}(M,k)$ admits a chute move such that the result is again a filling of M. Any maximal filling other than $D_{top}(M,k)$ admits an inverse chute move such that the result is again a filling of M.*

Proof of Theorem 3.2: All pipe dreams in $\mathcal{RC}(w)$ contained in M are maximal 0-1 fillings of M since they can be generated by applying sequences of chute moves to $D_{top}(M,k)$.

Since we can apply chute moves to any maximal 0-1-filling of M except $D_{bot}(M,k)$ all such fillings arise in this fashion. (We have to remark here that in case the pipe dream associated to some filling would not be reduced, applying chute moves eventually exhibits that the filling was not maximal.) Together with Lemma 3.3 this implies that all fillings $F_{01}^{ne}(M,k)$ have the same associated permutation.

Note that, as a by-product, this procedure implies that all maximal 0-1-fillings of M have the same number of entries equal to zero, *i.e.*, the simplicial complex of fillings is pure. $\square$

4 Applying the Edelman-Greene correspondence

Using the identification described in the previous section, we can apply a correspondence due to Paul Edelman and Curtis Greene [2], that associates pairs of tableaux to reduced factorisations of permutations. This in turn will yield the desired bijective proof of Jakob Jonsson's result at least for stack polyominoes.

The main result of this section was obtained for Ferrers shapes earlier by Luis Serrano and Christian Stump [15] using the same proof strategy. For stack polyominoes the description of the P-tableau is different, thus we believe it is useful to repeat the arguments here.

The following theorem is a collection of results from Paul Edelman and Curtis Greene [2], Richard Stanley [16], Alain Lascoux and Marcel-Paul Schützenberger [9], and describes properties of the *Edelman-Greene* correspondence:

Theorem 4.1 *There is a bijection between pairs of words reduced factorisations of a permutation w and pairs (P, Q) of Young tableaux of the same shape, such that P is column strict with reading word reduced equivalent to w, and Q is standard. Moreover, if w is vexillary, i.e., 2143-avoiding, the tableau P is the same for all reduced factorisations of w.*

It turns out that the permutations associated to moon polyominoes are indeed vexillary:

Proposition 4.2 *For any moon-polyomino M and any k, the permutation $w(M, k)$ is vexillary.*

There are vexillary permutations which do not correspond to moon polyominoes. For example, the only two reduced pipe dreams for the permutation $4, 2, 5, 1, 3$ are as follows:

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 4 & + & + & + & \bullet & \bullet \\ 2 & + & \bullet & + & \bullet & \\ 5 & + & \bullet & \bullet & & \\ 1 & \bullet & \bullet & & & \\ 3 & \bullet & & & & \end{array} \quad \text{and} \quad \begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 4 & + & + & + & \bullet & \bullet \\ 2 & + & \bullet & \bullet & \bullet & \\ 5 & + & + & \bullet & & \\ 1 & \bullet & \bullet & & & \\ 3 & \bullet & & & & \end{array}$$

Proof: It is sufficient to prove the claim for $k = 0$, since the empty cells in the filling $D_{top}(M, k)$ for any k again form a moon polyomino. Thus, suppose that the permutation associated to M is not vexillary. Then we have indices $i < j < k < \ell$ such that $w(j) < w(i) < w(\ell) < w(k)$. It follows that the pipes entering in columns i and j from above cross, and so do the two pipes entering in columns k and ℓ, and thus correspond to cells of the moon polyomino. Since any two cells in the moon polyomino can be connected by a path of neighbouring cells changing direction at most once, there is a third cell where either the pipes entering from i and ℓ or from j and k cross, which is impossible. □

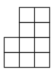
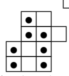
Theorem 4.3 (for Ferrers shapes, Luis Serrano and Christian Stump [15]) *For a stack polyomino S, consider the set $\mathcal{F}_{01}^{ne}(S, k, \mathbf{r})$. Let μ_i be the number of cells the i^{th} row of S is indented to the right, and suppose that $\mu_1 = \cdots = \mu_k = \mu_{k+1} = 0$.*

Let u be the word $1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \ldots$ and let v be the reduced factorisation of w associated to a given pipe dream. Then the Edelman-Greene correspondence applied to the pair of words (u, v) induces a bijection between $\mathcal{F}_{01}^{ne}(S, k, \mathbf{r})$ and the set of pairs (P, Q) of Young tableaux satisfying the following conditions:

- *the common shape of P and Q is the multiset of column heights of the empty cells in $D_{top}(S, k)$,*
- *the first row of P equals $(k + 1, k + 2 + \mu_{k+2}, k + 3 + \mu_{k+3}, \ldots)$, and the entries in columns are consecutive,*
- *Q has type $\{1^{\mathbf{r}_1}, 2^{\mathbf{r}_2}, \ldots\}$, and entries in column i are at most $i + k$.*

Thus, the common shape of P and Q encodes the row lengths of S, the entries of the first row of P encode the left border of S, and the entries of Q encode the filling.

In particular, this theorem implies an explicit bijection between the sets $\mathcal{F}^{ne}_{01}(S_1, k, \mathbf{r})$ and $\mathcal{F}^{ne}_{01}(S_2, k, \mathbf{r})$, given that the multisets of column heights of S_1 and S_2 coincide.

Curiously, the most natural generalisation of the above theorem to moon polyominoes is not true. Namely, one may be tempted to replace the condition on Q by requiring that the entries of Q are between Q_{top} and Q_{bot} component-wise. However, this fails already for $k = 1$ and the reverse Ferrers shape  with $P = \begin{array}{|c|c|c|}\hline 3&4&5\\\hline 5\\\cline{1-1}\end{array}$, $Q_{top} = \begin{array}{|c|c|c|}\hline 1&2&3\\\hline 3\\\cline{1-1}\end{array}$ and $Q_{bot} = \begin{array}{|c|c|c|}\hline 2&3&4\\\hline 4\\\cline{1-1}\end{array}$. In this case the tableau $Q = \begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ has preimage 

One might hope to prove Conjecture 2.8 by applying the Edelman-Greene correspondence, and checking that the poset is a lattice on the tableaux. However, at least for the natural component-wise order on tableaux, the correspondence is not order preserving, not even for the case of Ferrers shapes.

Proof: In view of Proposition 4.2, to obtain the tableau P it is enough to insert the reduced word given by the filling $D_{top}(S, k)$ using the Edelman-Greene correspondence, which is not hard for stack polyominoes.

It remains to prove that the entries in column i of Q are at most $i + k$ precisely if (u, v) comes from a filling in $\mathcal{F}^{ne}_{01}(S, k)$. To this end, observe that the shape of the first i columns of P equals the shape of the tableau obtained after inserting the pair of words $((u_1, u_2, \dots, u_\ell), (v_1, v_2, \dots, v_\ell))$, where ℓ is such that $u_\ell \le k + i$ and $u_{\ell+1} > k + i$.

Namely, this is the case if and only if the first $i+k+\mu_{i+k+1}$ positions of the permutation corresponding to $(v_1, v_2, \dots, v_\ell)$ coincide with those of the permutation w corresponding to v itself, as can be seen by considering $D_{top}(w)$, whose empty cells form again a stack polyomino.

This in turn is equivalent to all letters v_m being at least $k + i + 1 + \mu_{k+i+1}$ for $m > \ell$, *i.e.*, whenever the corresponding empty cell of the filling occurs in a row below the $(i + k)^{\text{th}}$ of S, and thus, when it is inside S. □

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I would like to acknowledge that Christian Stump provided a preliminary version of [17]. Luis Serrano and Christian Stump informed me privately that they were able to prove that all k-fillings of Ferrers shapes yield the same permutation w, however, their ideas would not work for stack polyominoes. I was thus motivated to attempt the more general case.

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Asymptotics of several-partition Hurwitz numbers

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Abstract. We derive in this paper the asymptotics of several-partition Hurwitz numbers, relying on a theorem of Kazarian for the one-partition case and on an induction carried on by Zvonkine. Essentially, the asymptotics for several partitions is the same as the one-partition asymptotics obtained by concatenating the partitions.

Résumé. Dans cet article, nous donnons l'asymptotique générale des nombres de Hurwitz à plusieurs partitions, s'appuyant sur un théorème de Kazarian pour le cas d'une partition et s'inspirant d'une récurrence menée par Zvonkine. En substance, l'asymptotique pour plusieurs partitions est la même que celle à une partition obtenue en concaténant les partitions.

Keywords: Hurwitz numbers, asymptotics, many partitions, transitive factorisations

1 Introduction

In the end of the XIX[th] century, Hurwitz computed the number of ways to factorise in the symmetric group $\mathfrak{S}_n$ a permutation of given cyclic type λ into a product of a minimal number of transpositions which generate a transitive subgroup. If one denotes by $h_n^0(\lambda)$ that number divided par $n!$, Hurwitz proved that

$$\frac{h_n^0(\lambda)}{(n+p-2)!} = \frac{1}{|\mathrm{Aut}\lambda|}\left(\prod_{i=1}^{p}\frac{d_i^{d_i}}{d_i!}\right)n^{p-3} \qquad \text{(write } \lambda = (d_1, ..., d_p) \text{).}$$

A fruitful generalisation of Hurwitz's original question (see [5] and [6]) is to seek such factorisation numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ with prescribed number of transpositions (the minimal case corresponds to $g = 0$), by remplacing the single permutation σ by a product of an arbitrary number of permutations of given types $\overrightarrow{\lambda} = (\lambda_1, ..., \lambda_k)$ (see Section 2.2 for reminders on partitions), and by adding a transitiveness condition – without the latter, such "disconnected" Hurwitz numbers would be given by Frobenius's formula. Section 2.3 recalls the definitions of the numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ and of their corresponding generating fonction $H^g\left(\overrightarrow{\lambda}\right)$. Section 2.2 defines convenient renormalisations $\mathbb{h}_n^g\left(\overrightarrow{\lambda}\right)$ and $\mathbb{H}_n^g\left(\overrightarrow{\lambda}\right)$.

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In genera 0 and 1, some considerations from algebraic geometry (more precisely the ELSV formula, see [3]) yield explicit formulæ for $h_n^0(\lambda)$ and $h_n^1(\lambda)$, whence closed formulæ for series $H^0(\lambda)$ and $H^1(\lambda)$ (see Section 2.4). Moreover, Kazarian [4] used the integration frame of the ELSV formula to give an explicit formula for the series H^g when $g \geq 2$ (see Section 2.5). However, very little is known on the numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ when the number of partitions is strictly greater than 1. In this paper, we will only be concerned with determining the asymptotics of $h_n^g\left(\overrightarrow{\lambda}\right)$ when n grows to ∞.

Zvonkine introduced in [7] an algebra of power series $\mathcal{A} := \mathbb{Q}[Y, Z]$ which has the following properties (see [8] and [9] for details):

1. the asymptotics of the leading coefficient of a series lying in $\mathcal{A}$ is determined by the leading coefficient in Z (see Claim 1 in Section 2.1);
2. all series $H^g(\lambda_1, ..., \lambda_k)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$ (see [7]).

Zvonkine proved the latter by induction on the number k of partitions, relying when $k = 1$ on already-known formulæ for spherical and toric genera (see Section 2.4) and on Kazarian's formula for higher genera (see Section 2.5). However, Zvonkine did not make explicit the formula he used; since we want to precisely compute the Z-degree of $H^g\left(\overrightarrow{\lambda}\right)$, we will carry out the explicitation of this reduction formula (see Theorem 10 in Section 3.1). Then, we will be able to prove our main result (Theorem 8) by induction (see Section 3.2), hence the sought asymptotics of all numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ (see Corollary 9). Both Theorem 8 and Corollary 9 are stated in Section 2.6.

The constants in the obtained asymptotics involves some rationnal-valued intersection numbers whose generating function, up to some renormalisation, satisfy Painlevé's equation I ($\frac{d^2f}{dq^2} + f(q)^2 = q$, see [5]) and can hence be recursively computed (see last lines of [1]). Section 2.6 recalls such a recursion formula, allowing one to retrieve the map asymptotics constants t_g defined in [2].

For sake of conciseness, we will use throughout the paper the genus-notation

$$g' := g - 1.$$

2 Hurwitz numbers and the algebra $\mathcal{A}$

2.1 *The algebra* $\mathcal{A} = \mathbb{Q}\left[\sum_{n\geq 1} \frac{n^{n-1}}{n!} q^n, \sum_{n\geq 1} \frac{n^n}{n!} q^n\right]$

Let us define an algebra $\mathcal{A} := \mathbb{Q}[Y, Z]$ where $Y := \sum_{n\geq 1} \frac{n^{n-1}}{n!} q^n$ and $Z := \sum_{n\geq 1} \frac{n^n}{n!} q^n = DY$ with $D : \sum_{n\geq 0} a_n q^n \mapsto \sum_{n\geq 1} n a_n q^n$. Define also a pseudo-inverse $D^{-1} : \sum_{n\geq 0} a_n q^n \mapsto \sum_{n\geq 1} \frac{a_n}{n} q^n$. The combinatorial identity $Y = qe^Y$ allows one to linearise the product $YZ = Z - Y$, whence the description $\mathcal{A} = \mathbb{Q}[Y] + \mathbb{Q}[Z]$. The latter entitles one to assign to every series in $\mathcal{A}^Z := \mathcal{A} \backslash \mathbb{Q}[Y]$ a polynomial in Z (up to the constant coefficient) that completly describes the asymptotics of the corresponding series thanks to the following claim:

Claim 1 (asymptotics in the algebra $\mathcal{A}$). *One has for any positive integers i and k*

$$\begin{array}{c} \textit{leading coefficient in } \frac{Y^i}{i} \sim C_{-1} \frac{e^n}{n} \sqrt{n}^{-1} \\ \textit{leading coefficient in } Z^k \sim C_k \frac{e^n}{n} \sqrt{n}^k \end{array} \quad \textit{where} \quad \begin{array}{c} \frac{1}{C_{-1}} = \sqrt{2\pi} \\ \frac{1}{C_k} = \Gamma\left(\frac{k}{2}\right) 2^{\frac{k}{2}} \end{array} .$$

Two series S and T in $\mathcal{A}^Z$ have therefore the same asymptotics if and only if their Z-leading terms are equal, which we will denote by a Z-equality $S \overset{Z}{=} T$. For instance, one has $P(Z)\,Q(Y) \overset{Z}{=} P(Z) \iff Q(1) \neq 0$ for any polynomials P and Q and the equality $DP(Z) \overset{Z}{=} Z^3 P'(Z)$ if $P \neq 0$.

2.2 Reminders on partitions

Recall that a *partition* of an integer a is any finite non-increasing sequence $\lambda = (d_1 \geq d_2 \geq \cdots \geq d_p)$ of positive integers (the *parts* of λ) summing up to a. Define the *length* $l(\lambda) := p$, the *size* $|\lambda| := a$, the *multiplicity* $m_k(\lambda) := \mathrm{Card}\{i;\ d_i = k\}$ of any integer k, the *ramification* $r(\lambda) := |\lambda| - l(\lambda)$, the *number of symmetries* $|\mathrm{Aut}\lambda| = \prod_{k\geq 1} m_k(\lambda)!$, the *reduction* $\mathring{\lambda} := \lambda \backslash 1^{m_1(\lambda)}$, the ($n$-th) *completion* $\overline{\lambda} := \lambda \sqcup 1^{n-m_1(\lambda)}$ for any integer $n \geq m_1(\lambda)$. A partition λ is called *reduced* if $m_1(\lambda) = 0$. The *concatenation* $\lambda \sqcup \mu$ of two partitions λ and μ is the partition whose parts are those of λ union those of μ. The length, size and ramification are morphisms from the concatenation to the addition and can therefore be extended to a tuple of partitions by concatenating the latter. At last, it will be convenient to use the following notations and renormalisations (see Definition 2 to define $h_n^g\left(\overrightarrow{\lambda}\right)$ and $H^g\left(\overrightarrow{\lambda}\right)$):

$$\boxed{\lambda} := \frac{1}{|\mathrm{Aut}\lambda|} \frac{d_1^{d_1} \cdots d_p^{d_p}}{d_1! \cdots d_p!}, \quad \boxed{\overrightarrow{\lambda}} := \boxed{\lambda_1}\boxed{\lambda_2} \cdots \boxed{\lambda_k}, \quad \boxed{h}_n^g\left(\overrightarrow{\lambda}\right) := \frac{h_n^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}, \quad \boxed{H}^g\left(\overrightarrow{\lambda}\right) := \frac{H^g\left(\overrightarrow{\lambda}\right)}{\boxed{\overrightarrow{\lambda}}}.$$

Let $n \geq 1$ an integer and σ a permutation in $\mathfrak{S}_n$. Its *support* is the complement $S\sigma = \mathrm{Supp}\sigma$ in $[1, n]$ of all σ-fixed points and its *type* is the partition $\mathrm{type}(\sigma)$ whose parts are the lengths of the cycles of σ (including fixed cycles). For instance, the type of the disjoint product of two permutations is the concatenation of their types and the cardinality of the support of a permutation equals the size of the reduction of its type. Recall that conjugacy classes in $\mathfrak{S}_n$ are indexed by n-sized partitions.

2.3 Constellations and Hurwitz numbers

Let n and k be positive integers. Define a *k-constellation* of degree n to be a k-tuple $\overrightarrow{\sigma} \in \mathfrak{S}_n^k$ such that $\sigma_1 \cdots \sigma_k = \mathrm{Id}$ and that the subgroup $\langle \sigma_1, ..., \sigma_k \rangle$ acts transitively on $[1, n]$. The *type* of a k-constellation $\overrightarrow{\sigma}$ is the k-tuple of the types of the σ_i. Its *ramification* r is the sum of those of the σ_i's. Its *genus* $g \geq 0$ is defined by the *Riemann-Hurwitz formula* $r = 2n + 2g'$. (Recall that $g' = g - 1$).

Definition 2 (Hurwitz numbers $h_n^g\left(\overrightarrow{\lambda}\right)$ and Hurwitz series $H^g\left(\overrightarrow{\lambda}\right)$). *Let g and n be two non-negative integers and λ_1, ..., λ_k be partitions of non-negative integers.*

Define $T = T_n = T_n^g(\lambda_1, ..., \lambda_k) := 2n + 2g' - r$ where $r := \sum r(\lambda_i)$.

Define $h_n^g(\lambda_1, ..., \lambda_k)$ by $\frac{1}{n!}$ times the number of pairs (C, F) where C is a constellation $(\overrightarrow{\sigma}, \overrightarrow{\tau}) \in \mathfrak{S}_n^k \times \mathfrak{S}_n^T$ of type $\left\{ \begin{array}{c} \forall i,\ \mathrm{type}(\sigma_i) = \overline{\lambda_i} \\ \forall j,\ \mathrm{type}(\tau_j) = \overline{2} \end{array} \right.$ *and where $F \subset [1, n]^k$ satisfies* $\left\{ \forall i, \begin{array}{c} F_i \subset \mathrm{Fix}\sigma_i \\ |F_i| = m_1(\lambda_i) \end{array} \right.$.

Define Hurwitz series by the following generating functions:

$$H^g\left(\overrightarrow{\lambda}\right) := \sum_{n\geq 1} \frac{h_n^g\left(\overrightarrow{\lambda}\right)}{T_n!} q^n \qquad\qquad H^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) := \sum_{n\geq 1} \frac{h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right)}{T_n!} q^n.$$

By choosing first the constellation then the fixed parts, one obtains the relation

$$h_n^g\left(\lambda_1, ..., \lambda_k\right) = h_n^g\left(\overline{\lambda_1}, ..., \overline{\lambda_k}\right) \times \prod_{i=1,...,k} \binom{n - \left(|\lambda_i| - m_1\left(\lambda_i\right)\right)}{m_1\left(\lambda_i\right)}.$$

2.4 Hurwitz series in genera 0 *and* 1

In spherical or toric genus, one has closed formulæ stemming from the ELSV formula for one-partition Hurwitz numbers. When one sees these relations in the series $H^g(\lambda)$, one obtains the following claim, which is a reformulation of unpublished results already known by Kazarian in [4]. We will need to define $e_k(\lambda) := \sum_{i_1<i_2<\cdots<i_k} d_{i_1} d_{i_2} \cdots d_{i_k}$ for any partition $\lambda = (d_1, ..., d_p)$ and any integer $k \in [0, p]$.

Claim 3 (Hurwitz series in genera 0 and 1). *Set a partition λ of an integer $a \geq 0$ in $p \geq 0$ parts. Then, one has the identities* $\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right)$ *and*

$$24\mathbb{H}^1(\lambda) = D^{p-1}\left(Y^{a-1}Z^2\right) + (a-1)\,D^{p-1}\left(Y^{a-1}Z\right) - \sum_{x=2}^{p} (x-2)!\,e_x(\lambda)\,D^{p-x}\left(Y^{a-x}Z^x\right).$$

Examples.

$$\mathbb{H}^0_{p=2} = \tfrac{Y^a}{a} \qquad \mathbb{H}^0_{p=1} = \tfrac{1}{a}\left(\tfrac{Y^a}{a} - \tfrac{Y^{a+1}}{a+1}\right) \qquad \mathbb{H}^0_{p=0} = Y - \tfrac{3}{2}\left(\tfrac{Y^2}{2}\right) + \tfrac{1}{2}\left(\tfrac{Y^3}{3}\right)$$

$$24\mathbb{H}^1(\emptyset) = D^{-1}Z^2 \qquad 24\mathbb{H}^1((1)) = Z^2 \qquad 24\mathbb{H}^1((2)) = Z^2$$

for any $d \geq 0$: $\quad 24\mathbb{H}^1((d+1)) = Y^a Z\,(Z+d) = Z^2 - Y^2 - 2Y^3 - 3Y^4 - \cdots - (d-1)\,Y^d.$

Corollary 4 (asymptotics of one-partition Hurwitz numbers in genera 0 and 1). *For any partition λ and any genus $g \in \{0, 1\}$, one has the following asymptotics*

$$\frac{\mathbb{h}^g_n(\lambda)}{T_n!} \sim c_g e^n n^{\frac{5}{2}g'+p-1} \text{ where } (c_0, c_1) := \left(\frac{1}{\sqrt{2\pi}}, \frac{1}{48}\right).$$

Proof of Corollary 4. In null genus, one has the relation $\mathbb{H}^0(\lambda) = D^{p-3}\left(Y^{a-1}Z\right) \stackrel{Z}{\equiv} D^{p-3}Z$. Its leading coefficient is therefore equivalent to n^{p-3} times $C_1 \frac{e^n}{n}\sqrt{n}^1$ thanks to Claim 1. In toric genus, the first term $D^{p-1}\left(Y^{a-1}Z^2\right) \stackrel{Z}{\equiv} D^{p-1}Z^2$ has degree $2 + 2(p-1) = 2p$ whereas the following terms $D^{p-x}\left(Y^{a-x}Z^x\right) \stackrel{Z}{\equiv} D^{p-x}Z^x$ for $x \geq 1$ have Z-degrees $x + 2(p-x) < 2p$. One has therefore $24\mathbb{H}^1(\lambda) \stackrel{Z}{\equiv} D^{p-1}Z^2$, whose leading coefficient is equivalent to n^{p-1} times $C_2 \frac{e^n}{n}\sqrt{n}^2$. □

2.5 Kazarian's formulæ, and the asymptotics of one-partition Hurwitz numbers

Considering integration theory on the moduli space of complex curves with some marked points led Kazarian to the following formula (see [9]). The latter involves some rational-valued intersection numbers $\langle \tau_0^u \tau_2^v \rangle$ defined (in [10]) for some integers $u, v \geq 0$. We will simply write $\langle \tau_2^v \rangle$ for $\langle \tau_0^0 \tau_2^v \rangle$.

Theorem 5 (Kazarian's formula). *Let $\mu = (d_1, ..., d_p)$ be a partition of an integer $a \geq 0$ and $g \geq 0$ be a genus. Then, whenever $n + 2g' > 0$, one has $\boxed{H}^g(\mu) = Y^a (Z+1)^{2g'+p} P(Z)$ where $P(Z)$ is a polynomial of leading term $\frac{\left\langle \tau_0^p \tau_2^{3g'+p} \right\rangle}{(3g'+p)!} Z^{2g'+p}$.*

Corollary 6 (Kazarian's Z-formula). *For any partition λ and genus $g \geq 0$ such that $p + 2g' > 0$, one has the Z-equality $\boxed{H}^g(\lambda) \overset{Z}{=} \frac{\left\langle \tau_0^p \tau_2^{3g'+p} \right\rangle}{(3g'+p)!} Z^{5g'+2p}$.*

Combining Corollary 6 with Claim 1 immediately yields the asymptotics of all $h_n^g(\lambda)$'s when $g \geq 2$:

$$\frac{\boxed{h}_n^g(\lambda)}{T_n!} \sim \frac{\left\langle \tau_2^{3g'+p} \tau_0^p \right\rangle}{(3g'+p)!} C_{5g'+2p} \frac{e^n}{n} \sqrt{n}^{5g'+2p} = Cst_g(\lambda) \times e^n n^{\frac{5}{2}g'+p-1}.$$

A little more work (use the string and dilaton equations in [10]) on the numbers $\langle \tau_2^u \tau_0^v \rangle$ can show that the constant $Cst_g(\lambda) := \frac{\left\langle \tau_2^{3g'+p} \tau_0^p \right\rangle}{(3g'+p)!} C_{5g'+2p}$ is actually λ-free, as we already know in genus 0 and 1 thanks to Corollary 4, hence the following.

Theorem 7 (asymptotics of one-partition Hurwitz numbers in any genus). *For any partition λ and any genus $g \geq 0$, one has the asymptotics*

$$\frac{h_n^g(\lambda)}{T_n!} \overset{n\infty}{\sim} c_g e^n n^{\frac{5}{2}g'+p-1} \textit{ where } \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{48} \end{pmatrix} \textit{ and } c_{g\geq 2} := \frac{1}{\Gamma\left(\frac{5}{2}g'\right) 2^{\frac{5}{2}g'}} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!}.$$

2.6 The main theorem and the general asymptotics of Hurwitz numbers

We can now state our main result, proven in section 3.2, which reduces the understanding of the asymptotics of several-partition Hurwitz numbers to that of single-partition Hurwitz numbers. Recall from [7] that all series $DH^g(\lambda_1, ..., \lambda_k)$ lie in the algebra $\mathcal{A}$ (it is a consequence of the conjunction of Claim 3, Theorem 5 and Theorem 10).

Theorem 8. *For any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following Z-equality in the algebra $\mathcal{A}^Z$:*

$$D^3 \boxed{H}^g(\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{3+m_1(\lambda_i)+\cdots+m_1(\lambda_k)} \boxed{H}^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right).$$

Corollary 9 (general asymptotics of Hurwitz numbers). *For any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following asymptotics for some constant c_g:*

$$\frac{\mathbb{h}_n^g\left(\lambda_1, ..., \lambda_k\right)}{T_n!} \overset{n\infty}{\sim} c_g \frac{e^n}{n} n^{\frac{5}{2}g'} n^{l(\lambda_1)+\cdots+l(\lambda_k)} \textit{ with } \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} := \begin{pmatrix} \frac{1}{\sqrt{2\pi}} \\ \frac{1}{48} \end{pmatrix} \textit{ and } c_{g\geq 2} := \frac{1}{\Gamma\left(\frac{5}{2}g'\right) 2^{\frac{5}{2}g'}} \frac{\left\langle \tau_2^{3g'} \right\rangle}{(3g')!}.$$

Proof of Corollary 9. If one sets $m_1 := \sum m_1\left(\lambda_i\right)$ and $p := \sum l\left(\lambda_i\right)$, Theorem 8 states the Z-equality $D^3 \mathbb{H}^g\left(\lambda_1, ..., \lambda_k\right) \overset{Z}{=} D^{3+m_1} \mathbb{H}^g\left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)$, whence the asymptotics

$$\frac{\mathbb{h}_n^g\left(\lambda_1, ..., \lambda_k\right)}{T_n!} \sim n^{m_1} \frac{\mathbb{h}_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)}{T_n^g\left(\mathring{\lambda}_1 \sqcup \cdots \sqcup \mathring{\lambda}_k\right)!} \overset{\text{Theorem 7}}{\sim} n^{m_1} c_g e^n n^{\frac{5}{2}g' + (p-m_1)-1} = c_g e^n n^{\frac{5}{2}g'+p-1}. \square$$

Remark. The constants $\left\langle \tau_2^{3g'} \right\rangle$ can recursively be computed thanks to Witten's conjecture (see [10] and [6]): if one sets $\alpha_0 := \frac{1}{12}$ and $\frac{\alpha_k}{5k(5k+2)} := \frac{\left\langle \tau_2^{3k} \right\rangle}{(3k)!}$ for any $k \geq 1$, then one will obtain the recursion

$$\forall k \geq 1,\ \alpha_k = \frac{25k^2-1}{12}\alpha_{k-1} + \frac{1}{2}\sum_{a,b\geq 0}^{a+b=k-1} \alpha_p \alpha_q.$$

The latter being very similar to that in [2] which defines the map asymptotics constants t_g, it is then easy to derive the identity $c_g = \sqrt{2}^{g-3} t_g$ for any integer $g \geq 0$.

3 Reduction formulæ

Zvonkine proved in [7] that all series $H^g\left(\lambda_1, ..., \lambda_k\right)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$ by induction on the number k of partitions, the case $k = 1$ being an immediate corollary from Theorem 5. We explicit the (unexplicited) induction formula used by Zvonkine so as to control the leading coefficients in Z of the series $H^g\left(\lambda_1, ..., \lambda_k\right)$ and derive their asymptotics.

3.1 The reduction formula

Let us first carry out an analysis of what becomes a constellation after merging its first two permutations. We reproduce mostly what is explained in [7] but retain some more information.

Let $(\sigma, \rho, \sigma_3, \sigma_4, ..., \sigma_k)$ be a constellation and denote $\pi := \sigma\rho$. One gets $k-1$ permutations $\pi, \sigma_3, ..., \sigma_k$ whose product is the identity, but one generally loses the transitivity condition. Denote Ω^1, ..., Ω^N the orbits of our new group $\langle \pi, \sigma_3, ..., \sigma_k \rangle$ and set σ_i^j for the permutation σ_i induced on Ω^j. One thus obtains for any j a constellation $\left(\pi^j, \sigma_3^j, ..., \sigma_k^j\right)$ on the orbit Ω^j.

Notice that one always has $N \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$. This is trivial when $S\sigma \cup S\rho$ is empty (since one then has $\sigma = \mathrm{Id}$ and $N = 1$) and let us explain why, when $S\sigma \cup S\rho$ is non-empty, every orbit must intersect it (hence $N \leq |\lambda| + |\mu|$): if the group $\langle \sigma\rho, \sigma_3, ..., \sigma_k \rangle$ stabilised an orbit disjoint from $S\sigma \cup S\rho$, then so would the group $\langle \sigma, \rho, \sigma_3, ..., \sigma_k \rangle$ since σ and ρ acts trivially out of $S\sigma \cup S\rho$, but the latter group is by assumption transitive, so the mentionned orbit must equal all $[1, n]$, consequently intersecting $S\sigma \cup S\rho$.

The genera g^j's satisfy the Riemann-Hurwitz relation $2n^j + 2g^{j\prime} = r\left(\pi^j\right) + \sum_{i=3}^k r\left(\sigma_i^j\right)$. By summing up these relations and recalling that of our first constellation, one gets $\sum g^{j\prime} = g' - \frac{r(\lambda)+r(\mu)-r(\pi)}{2}$.

Furthermore, since $S\pi \subset S\sigma \cup S\rho$, one can consider the type ν of $\pi_{|S\sigma\cup S\rho}$ as a partition of an integer smaller than $|S\sigma \cup S\rho| \leq |\lambda| + |\mu|$. Let us be more precise and set ν^j for the type of the permutation $\pi_{|S\sigma\cup S\rho}$ induced of Ω^j: the ν^j's are all non-empty (unless $S\sigma \cup S\rho = \emptyset$, namely unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$) and their sizes always sum up to that of $|\nu|$. Then $\pi_{|S\sigma\cup S\rho}$ has $m_1\left(\nu^j\right)$ fixed points in Ω^j and the knowledge of these fixed points for all j's allows one to rebuild $S\sigma \cup S\rho$ (add for any j these $m_1\left(\nu^j\right)$ points to the support of π^j).

Finally, when one wants to factorise back $\pi = \sigma\rho$, one has to choose σ and ρ satisfying the three following conditions: σ and ρ have respective types $\overline{\lambda}$ and $\overline{\mu}$; the union $S\sigma \cup S\rho$ equals $S\pi$ union the preceedingly-chosen point; the group $\langle \sigma, \rho, \sigma_3, ..., \sigma_k, \tau_1, ..., \tau_T \rangle$ acts transitively. A conjugation argument shows that the number $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ of such choices depends only on the partitions $\lambda, \mu, \overrightarrow{\nu}$; it is besides not hard to show that the transitiveness condition amounts to a junction condition on the orbits Ω^j by the cycles of σ or ρ (see Definition 11).

By collecting constellations according to the datas above (N, $\overrightarrow{g}$, $\overrightarrow{\nu}$), one can explicit the formula used by Zvonkine in [7] to prove that all series $H^g\left(\overrightarrow{\lambda}\right)$ but $H^1(\emptyset)$ lay in the algebra $\mathcal{A}$. (Consider the above analysis as a sketch of proof). The reduction formula thus obtained relies on a family $\left(f_{\lambda,\mu}^{\overrightarrow{\nu}}\right)$ of non-negative integers that we will define just after stating the reduction formula.

Theorem 10 (reduction formula). *Let $g \geq 0$ be a genus and $\overrightarrow{\lambda} = (\lambda, \mu, \lambda_3, \lambda_4, ..., \lambda_k)$ be k partitions where $k \geq 2$ is an integer. One then has the following formula*

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} f_{\lambda,\mu}^{\overrightarrow{\nu}} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_j H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$$

where one sums over: integers $N \geq 1$ smaller or equal to $\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1$; the N-tuples $(\overrightarrow{\nu}, \overrightarrow{g})$ such that $2g' = r(\lambda) + r(\mu) - r(\overrightarrow{\nu}) + \sum 2g^{j\prime}$ (all ν^j's being non-empty unless $\mathring{\lambda} = \mathring{\mu} = \emptyset$); for any $i = 3, ..., k$ partitions[(i)] *$\left(\lambda_i^1, ..., \lambda_i^N\right)$ whose concatenation is λ_i.*

Definition 11 (the numbers $f_{\lambda,\mu}^{\overrightarrow{\nu}}$). *Let N be a positive integer and set $N+2$ partitions $\lambda, \mu, \overrightarrow{\nu}$. For any j, consider Ω^j a $\left|\nu^j\right|$-sized set and π^j a ν^j -typed permutation in $\mathfrak{S}_{\Omega^j}$. Define $f_{\lambda,\mu}^{\overrightarrow{\nu}}$ to be the number of factorisations in $\mathfrak{S}_{\bigsqcup \Omega^j}$ of the permutation $\prod \pi^j$ in a product $\sigma\rho$ satisfying the three conditions:*

1. *the types of σ and ρ are respectively $\overline{\lambda}$ and $\overline{\mu}$;*
2. *the supports of σ and ρ cover all $\bigsqcup \Omega^j$, namely $Fix\sigma \cap Fix\rho = \emptyset$;*

[(i)] when $k = 2$, one sums (not over nothing but) over the empty list

3. *(junction condition) for any $j \neq j'$, there is a finite sequence $j = j_0, ..., j_L = j'$ such that, for any $p = 1, ..., L$, there is a cycle of σ or ρ which intersects both orbits $\Omega^{j_{p-1}}$ and Ω^{j_p}.*

Remarks. The first condition shows that $f^{\overrightarrow{\nu}}_{\lambda,\mu} = f^{\overrightarrow{\nu}}_{\mathring{\lambda},\mathring{\mu}}$ while the second condition yields the implication $f^{\overrightarrow{\nu}}_{\lambda,\mu} > 0 \implies |\overrightarrow{\nu}| \leq \left|\mathring{\lambda}\right| + |\mathring{\mu}|$, which ensures that the sum in Theorem 10 is *finite*.

When ν is made with the only one partition $\lambda \sqcup \mu$, the above inequality implies that λ and μ are reduced and supports $S\sigma$ and $S\rho$ are disjoint. Then, choosing a factorisation amounts to choosing for any $k \geq 2$ which k-lengthed cycles of π will appear in σ. Therefore, one has $f^{\mathring{\lambda}\sqcup\mathring{\mu}}_{\lambda,\mu} = \prod_{k\geq 2} \binom{m_k(\lambda) + m_k(\mu)}{m_k(\lambda)}$, which can be rewritten in a more convient way (for future application) as $\frac{f^{\mathring{\lambda}\sqcup\mathring{\mu}}_{\lambda,\mu}}{m_1(\lambda)!\, m_1(\mu)!} \frac{\boxed{\mathring{\lambda} \sqcup \mathring{\mu}}}{\boxed{\lambda}\boxed{\mu}} = 1.$

3.2 Proof of Theorem 8

We restate Theorem 8: *for any partitions $\lambda_1, ..., \lambda_k$ and any genus $g \geq 0$, one has the following Z-equality in the algebra $\mathcal{A}^Z$ for M large enough:*

$$D^M \mathbb{H}^g (\lambda_1, ..., \lambda_k) \overset{Z}{=} D^{M+m_1(\lambda_i)+\cdots+m_1(\lambda_k)} \mathbb{H}^g \left(\mathring{\lambda}_1 \sqcup \mathring{\lambda}_2 \sqcup \cdots \sqcup \mathring{\lambda}_k\right).$$

For the wondering reader, the exponent M is a trick to get rid of the exceptional cases.

As a first example, the information about the Z-degree in Kazarian's formulæ can be stated without the condition $p + 2g' > 0$ by the simple equality $\deg_Z D^3 H^g(\lambda) = 5g + 2p + 1$.

Let us prove the following generalisation *for any $M \geq 0$: if the series $D^M H^g(\lambda)$ lies in $\mathcal{A}^Z$, then it has degree* $\deg_Z D^M H^g(\lambda) = 2M + 5g' + 2p$. Indeed, setting $S := H^g(\lambda)$, one can write on the one hand $D^3\left(D^M S\right) = 2 \cdot 3 + \deg_Z D^M S$ and on the other hand $D^M\left(D^3 S\right) = 2M + 5g' + 2p + 6$; equalling both members leads to the conclusion. □

Let us now prove *for any $S \in \mathcal{A}$, $S \in \mathcal{A}^Z \iff \forall M \geq 0,\ \deg_Z D^M S \geq 2M$*. The arrow $\implies$ stems from $DP(Z) \overset{Z}{=} Z^3 P'(Z)$. Conversely, if S is a polynomial $P(Y)$, then $DS = P'(Y)Z$ has Z-degree ≤ 1 and hence $D^M S = D^{M-1} DS$ has degree $\leq 1 + 2(M-1) < 2M$. □

Finally, let us prove the following corollary of Theorem 8.

Corollary 12 (which series H^g lie in $\mathcal{A}^Z$). *For any **non-empty** partitions $\lambda_1, ..., \lambda_k, \lambda, \mu$:*

1. *$H^g(\lambda_1, ..., \lambda_k)$ always lies in $\mathcal{A}^Z$ when $k \geq 3$.*
2. *$H^g(\lambda, \mu)$ does **not** lie in $\mathcal{A}^Z$ if and only if $g = 0$ and if both λ and μ have one part.*
3. *$H^g(\lambda)$ does **not** lie in $\mathcal{A}^Z$ if and only if $\binom{g}{l(\lambda)} \in \left\{\binom{0}{0}, \binom{0}{1}, \binom{0}{2}, \binom{1}{0}\right\}$.*

Proof. Take the Z-degree in the given Z-equality and use Corollary 6:

$$\deg D^M \mathbb{H}^g (\lambda_1, ..., \lambda_k) = 2M + 2\sum m_1(\lambda_i) + \left(5g' + 2\sum l\left(\mathring{\lambda}_i\right)\right) = 2M + 5g' + 2\sum l(\lambda_i).$$

Since all lengths are ≥ 1, the above degree is $\geq 2M$ when $k \geq 3$. When $k = 2$, the above degree is $< 2M$ if and only if $g = 0$ and $l(\lambda_i) = 1$ for $i = 1, 2$. When $k = 1$, one retrieves the already-known exceptional cases of Kazarian's formulæ. $\square$

We now proceed with the proof of Theorem 8, by induction on the number k of partitions. The case $k = 1$ is immediate from Corollary 6. Because of the number of exceptional cases, the case $k = 2$ will be the longest to deal with, the case $k = 3$ much similar and much easier, and greater k's will be almost straightforward. We start with $k \geq 4$ to get used to the idea, then $k = 3$ and finally $k = 2$, the induction hypothesis allowing one to use the corresponding parts of Corollary 12.

To derive the wanted Z-equality from Theorem 10, one has to analyse the contribution in Z of each product $\prod H^{g^j}$; one will eventually prove the following Z-equality:

$$H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) \stackrel{Z}{=} f_{\lambda,\mu}^{\mathring{\lambda}\sqcup\mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right)$$

(notice it already stands as a plain equality when $\mathring{\lambda} = \mathring{\mu} = \emptyset$, the reason for which we will leave that case aside below). It is then easy to derive the Z-equality of Theorem 8: remove the bars on top of λ and μ by multiplying by the binomials $\binom{D-(|\lambda|-m_1(\lambda))}{m_1(\lambda)}\binom{D-(|\lambda|-m_1(\lambda))}{m_1(\lambda)}$; since D strictly increases $\deg_Z$, one can multiply instead by $\frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\,m_1(\mu)!}$ and still get a Z-equality $H^g(\lambda, \mu, \lambda_3, ..., \lambda_k) \stackrel{Z}{=} \frac{D^{m_1(\lambda)+m_1(\mu)}}{m_1(\lambda)!\,m_1(\mu)!} f_{\lambda,\mu}^{\mathring{\lambda}\sqcup\mathring{\mu}} H^g\left(\mathring{\lambda} \sqcup \mathring{\mu}, \lambda_3, ..., \lambda_k\right)$; to get from H to $\boxed{H}$, divide both sides by $\boxed{\lambda}\,\boxed{\mu}\,\boxed{\lambda_3}\cdots\boxed{\lambda_k}$; to conclude, use the identity $\frac{f_{\lambda,\mu}^{\mathring{\lambda}\sqcup\mathring{\mu}}}{m_1(\lambda)!\,m_1(\mu)!}\frac{\boxed{\mathring{\lambda}\sqcup\mathring{\mu}}}{\boxed{\lambda}\,\boxed{\mu}} = 1$ and the induction hypothesis.

Case $k \geq 4$. One has $H^g\left(\overline{\lambda}, \overline{\mu}, \lambda_3, ..., \lambda_k\right) = \sum_{\overrightarrow{\nu}, \overrightarrow{g}} f_{\lambda,\mu}^{\overrightarrow{\nu}} \sum_{\overrightarrow{\lambda_3}, ..., \overrightarrow{\lambda_k}} \prod_j H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ by Theorem 10 where every factor $H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ lies in $\mathcal{A}^Z$ by Corollary 12 for $k-1$ partitions (recall all v^j's are non-empty since we left aside the case $\mathring{\lambda} = \mathring{\mu} = \emptyset$). The product $\prod H^{g^j}\left(\nu^j, \lambda_3^j, ..., \lambda_k^j\right)$ has therefore Z-degree

$$\begin{aligned}
&\sum_j 5g^{j\prime} + 2\left(l\left(\nu^j\right) + \sum_{i\geq 3} l\left(\lambda_i^j\right)\right) \\
&\quad = 5g' - 5\frac{r(\lambda) + r(\mu)}{2} + \frac{5}{2}\left(|\overrightarrow{\nu}| - l(\overrightarrow{v})\right) + 2l(\overrightarrow{\nu}) + 2\sum_{i\geq 3} l(\lambda_i) \\
&\quad = 2\sum_{i\geq 3} l(\lambda_i) + 5g' - \frac{5}{2}\left(r(\lambda) + r(\mu)\right) + \frac{5|\nu| - l(\nu)}{2}.
\end{aligned}$$

Everything is constant except $\frac{5|\nu|-l(\nu)}{2}$. Lemma 13 then shows that the above quantity is maximal if and only if $\nu = \mathring{\lambda} \sqcup \mathring{\mu}$; since this implies $N = 1$ and $\overrightarrow{g} = (g)$, one gets the announced Z-reduction formula.

Case $k=3$. We go along the same idea. Fix a genus $g\geq 0$ and three partitions λ,μ,ξ. Let $p:=l(\nu)+l(\xi)$ and p^j defined alike for all j. Theorem 10 then implies for any integer $M\geq 0$

$$D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{\xi},\overrightarrow{M}} f_{\lambda,\mu}^{\overrightarrow{\nu}}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$$

where the sum over $\overrightarrow{M}$ is taken over the N-tuples of non-negative integers M^j's which sum up to M. By the induction hypothesis for $k=2$, the term $D^M H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\xi\right)$ lies in $\mathcal{A}^Z$ for M large enough. Fix such an M. We then show that all other terms have Z-degree smaller than the latter.

By Corollary 12 for two partitions, a factor $D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\left(g^j,p^j,M^j\right)=(0,2,0)$; multiplying by such an element will decrease[(ii)] the Z -degree. As for the other factors, the D-trick combined with Corollary 12 for two partitions shows that their Z-degree is $5g^{j\prime}+2p^j+2M^j$. The product $\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\leq\sum_Z 5g^{j\prime}+2p^j+2M^j$ where the index Z means that $D^{M^j}H^{g^j}\left(\nu^j\right)$ lies in $\mathcal{A}^Z$.

Set $e:=\#\left\{j;\ \left(g^j,p^j,M^j\right)=(0,2,0)\right\}$ for the number of exceptional factors with no Z. The three previous Z-sums can be linked to the same sums without restriction:

$$\sum_Z g^{j\prime}=e+g'-\frac{r(\lambda)+r(\mu)-r\left(\overrightarrow{\nu}\right)}{2},\ \sum_Z p^j=l\left(\overrightarrow{\nu}\right)+l(\xi)-2e,\ \sum_Z M^j=M.$$

One can thus derive the majoration

$$\deg_Z\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)\leq 2M+5g'-\frac{5}{2}\left(r(\lambda)+r(\mu)\right)+l(\xi)+\frac{5|\nu|-l(\nu)}{2}+e.$$

Like when $k\geq 4$, everything is constant except $\frac{5|\nu|-l(\nu)}{2}+e$; since there is at least one $M^j\geq 1$ (thanks to the trick of applying D), one has $e\leq N-1\leq 3(N-1)$ and Lemma 13 still holds: the maximal-Z-degreed term $\prod_j D^{M^j}H^{g^j}\left(\nu^j,\xi^j\right)$ in the sum $D^M H^g\left(\overline{\lambda},\overline{\mu},\xi\right)$ is precisely $D^M H^g\left(\mathring{\lambda}\sqcup\mathring{\mu},\xi\right)$.

Case $k=2$. The proof goes as above. Fix $g\geq 0$ any genus and λ,μ two partitions. For any $M\geq 0$, Theorem 10 implies that $D^M H^g\left(\overline{\lambda},\overline{\mu}\right)=\sum_{\overrightarrow{\nu},\overrightarrow{g},\overrightarrow{M}} f_{\lambda,\mu}^{\overrightarrow{\nu}}\binom{M}{\overrightarrow{M}}\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$. By Corollary 12 for one partition (namely Corollary 6), a factor $D^{M^j}H^{g^j}\left(\nu^j\right)$ will belong to $\mathbb{Q}[Y]$ if and only if $\left(g^j,p^j,M^j\right)\in\{(0,1,1),(0,1,0),(0,2,0)\}$. For the other factors, we have already stated that their degrees were $5g^{j\prime}+2p^j+2M^j$. The product $\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)$ has therefore Z-degree $\leq\sum_Z 5g^{j\prime}+2p^j+2M^j$. After linking the Z-sums to the (no Z)-sums, one obtains the majoration

$$\begin{aligned}\deg_Z\prod_j D^{M^j}H^{g^j}\left(\nu^j\right)\quad\leq\quad & 2M+5g'-\frac{5}{2}\left(r(\lambda)+r(\mu)\right)+\frac{5|\nu|-l(\nu)}{2}\\ & +\#\left\{j;\ \begin{array}{c}g^j=0\\ p^j=1\\ M^j=1\end{array}\right\}+3\#\left\{j;\ \begin{array}{c}g^j=0\\ p^j=1\\ M^j=0\end{array}\right\}+\left\{j;\ \begin{array}{c}g^j=0\\ p^j=2\\ M^j=0\end{array}\right\}.\end{aligned}$$

(ii) strictly if and only if its (Y-)coefficients sum up to zero

The three sets whose cardinalities are involved being mutually disjoint, the corresponding sum is $\leq N$ and one can even remplace N by $N-1$ if there is at least one $M^j \geq 2$, which can be realised by choosing $M \geq 2\left(\left|\mathring{\lambda}\right| + |\mathring{\mu}| + 1\right) \geq 2N$. Therefore, one can still apply Lemma 13 and conclude, which finishes the proof of Theorem 8. □

Lemma 13. *Let λ, μ be two partitions and σ, ρ two permutation in $\mathfrak{S}_\infty$ of type $\left(\overline{\lambda}, \overline{\mu}\right)$. Denote ν the partition of $\sigma\rho$ induced on $S\sigma \cup S\rho$. Cluster the cycles of ν into N orbits such that the junction condition of Definition 11 is satisfied. Then the quantity $\frac{5|\nu|-l(\nu)}{2} + 3(N-1)$ is maximal if and only if σ and ρ have disjoint supports. (And, in that case, one has $N = 1$.)*

Proof. Call a cycle of σ or ρ to be *interlaced* if it encounters another cycle of σ or ρ (and two such cycles will be called *interlaced with* each other). Set c for the number of interlaced cycles and c' for the number of cycles (included fixed cycles) of the product $\sigma\rho$ induced on the interlaced cycles (of σ and ρ).

A crucial remark is the following: for the junction condition to be satisfied, every cycle of ν must lie in the same orbit as an interlaced cycle, whence the inequality $N \leq c'$.

If one sets $k := |S\sigma \cap S\rho|$ for the number of contact points of the supports, one can write

$$\begin{cases} |\nu| = \left|\mathring{\lambda}\right| + |\mathring{\mu}| - k \\ l(\nu) = l\left(\mathring{\lambda}\right) + l(\mathring{\mu}) - c + c' \end{cases}, \quad \text{hence the quantity to be upper-bounded:} \quad Q := \frac{5(-k) - (c' - c)}{2} + 3(N-1).$$

When $S\sigma \cap S\rho = \emptyset$, all variables $c, c', k, N-1$ equal 0 and so does Q. One has therefore to show $Q < 0$, namely $-2Q \geq 1$, for any other ν than $\mathring{\lambda} \sqcup \mathring{\mu}$. By the crucial remark, it suffices to show the same inequality $5k + c' - c - 2(3N-3) \geq 1$ with some N's been replaced by the same number of c''s: so as to kill the c' in the inequality, we remplace one N out of six, which lead us to wonder if the inequality $5(k+1-N) \geq c$ holds. We are going to show by induction on $|S\sigma| + |S\rho|$ the stronger inequality

$$2(k - N + 1) \geq c.$$

When $\sigma = \rho = \mathrm{Id}$, then all three quantitites $c, k, N-1$ equal 0, whence the above inequality.

Suppose now $\left|\mathring{\lambda}\right| + |\mathring{\mu}| > 0$. Because of the assumption $\nu \neq \mathring{\lambda} \sqcup \mathring{\mu}$, one has $k \geq 1$: take one contact point x in $S\sigma \cap S\rho$, set $y := \sigma(x)$ and $\tau := (x, y)$ the transposition exchanging these points. Finally, write $\sigma = \tau\sigma_*$ where $\sigma_* := \tau\sigma$ fixes x and therefore satifies $|\sigma^*| < |\sigma|$. Thus, one obtains the cycle decomposition of $\sigma\rho$ by multiplying that of $\sigma_*\rho$ by the transposition τ on the left (and conversely). Denote by a $*$-subscript the quantities c_*, k_*, N_* associated to the product of σ_* and ρ; notice that N_* is not well-defined and can be chosen arbitrarily as long as the junction condition is satisfied. For such an N_*, one has the induction hypothesis $c_* \leq 2(k_* - N_* + 1)$. What we want is to dispose of the $*$'s.

Since x is fixed by σ_*, it disappears from the contact points, hence $k_* < k$. Besides, σ_* loses at most one interlaced cycle (it can only be the σ-orbit of x) and ρ loses at most two interlaced cycles (those maybe interlaced with τ), hence $c_* \geq c - 3$. But the case $c_* = c - 3$ implies x's σ-orbit to be a transposition interlaced with two ρ-cycles, each of which not being interlaced with another σ-cycle; since σ and ρ play symmetric roles (set $y := \rho(x)$ instead of $\sigma(x)$), one can avoid this case and hence assume $c_* \geq c - 2$.

Let us look at what happens to the cycles of $\sigma\rho$ when composing (on the left) by τ. If a ($\sigma\rho$-)cycle γ is split in two cycles, cluster them in the same orbit as γ's orbit (hence $N_* = N$). If two cycles are joined, either both cycles were in the same orbit (then, do not change the orbits, hence $N_* = N$) or they were in distinct orbits (then, merge these orbits and do not change the others, hence $N_* = N - 1$). Whenever $N_* = N$, one can conclude by writing

$$c \leq c_* + 2 \leq 2\left(k_* - N_* + 1\right) + 2 \leq 2\left((k-1) - N + 1\right) + 2 = 2\left(k - N + 1\right).$$

We can consequently assume $N_* = N - 1$ and hence τ *joining* two $\sigma\rho$-cycles, which goes the same as saying x and y not to lie in the same $\sigma\rho$-orbit. But that implies both σ- and ρ-orbits of x to remain interlaced for σ_* and ρ (if not, iterate $\sigma\rho$ in a not-interlaced orbit to join x and y), hence $c_* = c$ and the induction hypothesis yields

$$c = c_* \leq 2\left(k_* - N_* + 1\right) \leq 2\left((k-1) - (N-1) + 1\right) = 2\left(k - N + 1\right). \square$$

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Demazure crystals and the energy function

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Abstract. There is a close connection between Demazure crystals and tensor products of Kirillov–Reshetikhin crystals. For example, certain Demazure crystals are isomorphic as classical crystals to tensor products of Kirillov–Reshetikhin crystals via a canonically chosen isomorphism. Here we show that this isomorphism intertwines the natural affine grading on Demazure crystals with a combinatorially defined energy function. As a consequence, we obtain a formula of the Demazure character in terms of the energy function, which has applications to nonsymmetric Macdonald polynomials and q-deformed Whittaker functions.

Résumé. Les cristaux de Demazure et les produits tensoriels de cristaux Kirillov–Reshetikhin sont étroitement liés. Par exemple, certains cristaux de Demazure sont isomorphes, en tant que cristaux classiques, à des produits tensoriels de cristaux Kirillov–Reshetikhin via un isomorphisme que l'on peut choisir canoniquement. Ici, nous montrons que cet isomorphisme entremêle la graduation affine naturelle des cristaux de Demazure avec une fonction énergie définie combinatoirement. Comme conséquence, nous obtenons une formule pour le caractère de Demazure exprimée au moyen de la fonction énergie, avec des applications aux polynômes de Macdonald non symétriques et aux fonctions de Whittaker q-déformées.

Keywords: Demazure crystals, affine crystals, nonsymmetric Macdonald polynomials, Whittaker functions

1 Introduction

Kashiwara's theory of crystal bases [20] provides a remarkable combinatorial tool for studying highest weight representations of symmetrizable Kac–Moody algebras and their quantizations. Here we consider finite-dimensional representations of the quantized universal enveloping algebra $U_q'(\mathfrak{g})$ corresponding to the derived algebra $\mathfrak{g}'$ of an affine Kac–Moody algebra. These representations do not extend to representations of $U_q(\mathfrak{g})$, but one can nonetheless define the notion of a crystal basis. In this setting crystal bases do not always exist, but there is an important class of finite-dimensional modules for $U_q'(\mathfrak{g})$ that are known to admit crystal bases. These are tensor products of the Kirillov–Reshetikhin modules $W^{r,s}$ [24] (denoted $W(s\omega_r)$ in that paper), where r is a node in the classical Dynkin diagram and s is a positive integer.

The modules $W^{r,s}$ were first conjectured to admit crystal bases $B^{r,s}$ in [14, Conjecture 2.1], and moreover it was conjectured that these crystals are perfect whenever s is a multiple of a particular constant c_r (perfectness is a technical condition which allows one to use the finite crystal to construct highest

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weight crystals, see [17]). This conjecture has now been proven in all non-exceptional cases (see [32, 33] for a proof that the crystals exist, and [7, Theorem 1.2] for a proof that they are perfect). We call $B^{r,s}$ a Kirillov–Reshetikhin (KR) crystal.

The perfectness of KR crystals ensures that they are related to highest weight affine crystals via the construction in [17]. In [21], Kashiwara proposed that this relationship is connected to the theory of Demazure crystals [19, 29], by conjecturing that perfect KR crystals are isomorphic as classical crystals to certain Demazure crystals (which are subcrystals of affine highest weight crystals). This was proven in most cases in [4, 5]. More relations between Demazure crystals and tensor products of perfect KR crystals were investigated in [25, 26, 27, 8].

There is a natural grading $\deg$ on a highest weight affine crystal $B(\Lambda)$, where $\deg(b)$ records the number of f_0 in a string of f_i's that act on the highest weight element to give b. Due to the ideas discussed above, it seems natural that this grading should transfer to a grading on a tensor product of KR crystals. Gradings on tensor products of KR crystals have in fact been studied, and are usually referred to as "energy functions." The idea dates to the earliest works on perfect crystals [17, 18], and was expanded in [34] following conjectural definitions in [13]. A function D, which we will refer to as the D-function, is defined as a sum involving local energy functions for each pair of factors in the tensor product and an 'intrinsic energy' of each factor. It has been suggested that there is a simple global characterization of the D function related to the affine grading on a corresponding highest weight crystal (see [35, Section 2.5], [13, Proof of Proposition 3.9]). Here we will formulate this precisely, and provide a proof.

1.1 Results

In the present work, we restrict to non-exceptional type (i.e. all affine Kac–Moody algebras except $A_2^{(2)}$, $G_2^{(1)}$, $F_4^{(1)}$, $E_6^{(1)}$, $E_7^{(1)}$, $E_8^{(1)}$, $E_6^{(2)}$ and $D_4^{(3)}$), where KR crystals are known to exist. We consider a tensor product B of perfect KR crystals, all of the same level. We define the intrinsic energy function E^{int} on B by letting $E^{\text{int}}(b)$ record the minimal number of f_0 in a path from a certain fixed $u \in B$ to b. One purpose of this note is to show that E^{int} agrees with the D-function up to a shift (i.e. addition of a global constant).

Our main tool is an enhancement of the relationship between KR crystals and Demazure crystals due to Fourier, Shimozono, and the first author. In [8, Theorem 4.4] it was shown that, under certain assumptions, there is a unique bijection from the Demazure crystal to the KR crystal respecting the classical crystal structure and such that all zero edges in the Demazure crystal are taken to zero edges in the KR crystal (although the KR crystal has more zero arrows). In most cases the assumptions from [8] follow from [6], and we deal with the remaining cases separately in Section 5, thereby firmly establishing this relationship between KR crystals and Demazure crystals in all non-exceptional types. We show in Theorem 6.2 that the resulting map intertwines the basic grading on the Demazure crystal with the D-function on the KR crystal, up to a shift. This in turn allows us to prove that E^{int} agrees with D up to a shift, and in fact the above map intertwines the basic grading with E^{int} exactly.

In the long version of this note [38], we also consider the more general setting when B is a tensor product of KR crystals which are not assumed to be perfect or of the same level. The D function is still well-defined, and we give a precise relationship between D and the affine grading on a related direct sum of highest weight modules. However, we no longer give an interpretation in terms of Demazure modules.

1.2 Applications

Our results express the characters of certain Demazure modules in terms of the intrinsic energy on a related tensor product of KR crystals (see Corollary 7.1). This has potential applications whenever these Demazure characters appear.

For untwisted simply-laced root systems, Ion [15], generalizing results of Sanderson [36] in type A, showed that the specializations $E_\lambda(q,0)$ of nonsymmetric Macdonald polynomials at $t=0$ coincide with specializations of Demazure characters of level one affine integrable modules. If λ is anti-dominant, then $E_\lambda(q,0)$ is actually a symmetric Macdonald polynomial $P_\lambda(q,0)$. In this case, the relevant Demazure module is associated to a tensor product B of level one KR crystals as above, so our results imply that $P_\lambda(q,0)$ is the character of B, where the powers of q are given by $-D$. Hence the coefficients in the expansion of $P_\lambda(q,0)$ in terms of the irreducible characters are the one-dimensional configuration sums defined in terms of the intrinsic energy in [13].

There is also a relation between Demazure characters and q-deformed Whittaker functions for $\mathfrak{gl}_n$ [9, Theorem 3.2]. Hence our results allow one to study Whittaker functions via KR crystals.

For more details, including proofs, see the long version of this paper [38].

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2 Kac–Moody algebras and Crystals

Let $\mathfrak{g}$ be a Kac–Moody algebra. Let $\Gamma=(I,E)$ be its Dynkin diagram, where I is the set of vertices and E the set of edges. Let Δ denote the root system associated to $\mathfrak{g}$, and let P denote the weight lattice of $\mathfrak{g}$ and $P^\vee$ the coweight lattice. We denote by $\{\alpha_i \mid i\in I\}$ the set of simple roots and $\{\alpha_i^\vee \mid i\in I\}$ the set of simple coroots, with $Q=\bigoplus_{i\in I}\mathbb{Z}\alpha_i$ the root lattice and $Q^\vee=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$ the coroot lattice.

Let $U_q(\mathfrak{g})$ be the corresponding quantum enveloping algebra, defined over $\mathbb{Q}(q)$. Let $\{E_i,F_i\}_{i\in I}$ be the standard elements in $U_q(\mathfrak{g})$ corresponding to the Chevalley generators of the derived algebra $\mathfrak{g}'$. We recall the triangular decomposition $U_q(\mathfrak{g})\cong U_q(\mathfrak{g})^{<0}\otimes U_q(\mathfrak{g})^0\otimes U_q(\mathfrak{g})^{>0}$, where $U_q(\mathfrak{g})^{<0}$ is the subalgebra generated by the F_i, $U_q(\mathfrak{g})^{>0}$ is the subalgebra generated by the E_i, and $U_q(\mathfrak{g})^0$ is the abelian group algebra generated by the usual elements K_w for $w\in P^\vee$, and the isomorphism is as vector spaces. Let $U_q'(\mathfrak{g})$ be the subalgebra generated by E_i,F_i and $K_i:=K_{H_i}$ for $i\in I$.

We are particularly interested in the case when $\mathfrak{g}$ is of affine type. We will use the following conventions: W, P and Λ_i denote the affine Weyl group, the affine weight lattice, and the affine fundamental weight corresponding to $i\in I$, respectively, while $\overline{W},\overline{P}$ and ω_i denote the weight lattice, Weyl group and fundamental weights corresponding to the finite type Dynkin diagram $I\backslash\{0\}$.

2.1 Crystals for $U_q(\mathfrak{g})$

We refer the reader to [12] for more details. For us, a crystal is a nonempty set B along with operators $e_i: B\to B\cup\{0\}$ and $f_i: B\to B\cup\{0\}$ for $i\in I$, which satisfy some conditions. The set B records certain combinatorial data associated to a representation V of a symmetrizable Kac–Moody algebra $\mathfrak{g}$, and the operators e_i and f_i correspond to the Chevalley generators E_i and F_i of $\mathfrak{g}$. Often the definition of a crystal includes three functions $\mathrm{wt}, \varphi, \varepsilon : B \to P$, where P is the weight lattice. In the case of crystals of integrable modules, these functions can be recovered (up to a global shift in a null direction if the Cartan matrix is not invertible) from the e_i and f_i. Explicitly, the weight of the highest weight element in the crystal $B(\lambda)$ of an irreducible highest weight module is λ, and each operator f_i has weight $-\alpha_i$.

An important theorem of Kashiwara states that every integrable $U_q(\mathfrak{g})$-highest weight module $V(\lambda)$ has a crystal basis. We denote the resulting $U_q(\mathfrak{g})$ crystal by $B(\lambda)$.

2.2 $U_q'(\mathfrak{g})$ crystals

In the case when the Cartan matrix is not invertible, one can define an extended notion of $U_q'(\mathfrak{g})$ crystals that includes some cases which do not lift to $U_q(\mathfrak{g})$ crystals. Such a crystal is still a set B along with operators $e_i, f_i : B \to B \cup \{0\}$ related to a $U_q'(\mathfrak{g})$ modules (see e.g. [22]). We consider only crystals coming from integrable modules, so we can define

$$\varepsilon_i(b) := \max\{m \mid e_i^m(b) \neq 0\}, \qquad \varphi_i(b) := \max\{m \mid f_i^m(b) \neq 0\},$$

$$\varphi(b) := \sum_{i \in I} \varphi_i(b)\Lambda_i, \quad \varepsilon(b) := \sum_{i \in I} \varepsilon_i(b)\Lambda_i, \quad \text{and} \quad \mathrm{wt}(b) := \varphi(b) - \varepsilon(b).$$

Then $\mathrm{wt}(b)$ corresponds to the classical weight grading of the corresponding module. Notice that $\mathrm{wt}(b)$ is always in the space $P' := \mathrm{span}\{\Lambda_i \mid i \in I\}$. If the Cartan matrix of $\mathfrak{g}$ is not invertible, P' is a proper sublattice of P.

Remark 2.1 *The simple roots α_i are not in general in the span of the fundamental weights, so in this case the weight of the operator f_i is not $-\alpha_i$. It is rather the projection of $-\alpha_i$ onto the space of the fundamental weights in the direction which sends the null root to* 0.

The tensor product rule for $U_q'(\mathfrak{g})$ or $U_q(\mathfrak{g})$ modules leads to a tensor product rule for the corresponding crystals. If A and B are two crystals, the tensor product $A \otimes B$ is the crystal whose underlying set is $\{a \otimes b \mid a \in A, b \in B\}$ with operators f_i defined by:

$$f_i(a \otimes b) = \begin{cases} f_i(a) \otimes b & \text{if } \varepsilon_i(a) \geq \varphi_i(b), \\ a \otimes f_i(b) & \text{otherwise,} \end{cases} \tag{2.1}$$

and e_i defined by the rule $e_i(b) = b'$ if and only if $f_i(b') = b$.

2.3 Extended affine Weyl group

Fix $\mathfrak{g}$ of affine type. Write the null root as $\delta = \sum_{i \in I} a_i \alpha_i$. Following [13], for each $i \in I \backslash \{0\}$, define $c_i = \max(1, a_i/a_i^\vee)$. It turns out that $c_i = 1$ in all cases except (1) $c_i = 2$ for $\mathfrak{g} = B_n^{(1)}$ and $i = n$, $\mathfrak{g} = C_n^{(1)}$ and $1 \leq i \leq n-1$, $\mathfrak{g} = F_4^{(1)}$ and $i = 3, 4$, and (2) $c_2 = 3$ for $\mathfrak{g} = G_2^{(1)}$. Here we use Kac's indexing of affine Dynkin diagrams from [16, Table Fin, Aff1 and Aff2]. Consider the sublattices of $\overline{P}$ given by

$$M = \bigoplus_{i \in I \backslash \{0\}} \mathbb{Z} c_i \alpha_i = \mathbb{Z}\overline{W} \cdot \theta / a_0 \quad \text{and} \quad \widetilde{M} = \bigoplus_{i \in I \backslash \{0\}} \mathbb{Z} c_i \omega_i.$$

The finite type Weyl group $\overline{W}$ acts on $\overline{P}$ by linearizing the rules $s_i\lambda = \lambda - \langle \alpha_i^\vee, \lambda\rangle \alpha_i$. Clearly $M \subset \widetilde{M}$ and the action of $\overline{W}$ on $\overline{P}$ restricts to actions on M and $\widetilde{M}$. Let $T(\widetilde{M})$ (resp. $T(M)$) be the subgroup of $T(\overline{P})$ generated by the translations t_λ by $\lambda \in \widetilde{M}$ (resp. $\lambda \in M$).

There is an isomorphism [16, Prop. 6.5] $W \cong \overline{W} \ltimes T(M)$ as subgroups of Aut(P), where W is the affine Weyl group. Under this isomorphism we have $s_0 = t_{\theta/a_0} s_\theta$, where θ is the highest root of $I\backslash\{0\}$. Define the extended affine Weyl group to be the subgroup of Aut(P) given by $\widetilde{W} = \overline{W} \ltimes T(\widetilde{M})$.

Define $\theta^\vee \in \mathfrak{h}^*$ so that $s_\theta(\lambda) = \lambda - \langle \theta^\vee, \lambda\rangle\theta$. Let $C \subset \overline{P} \otimes_{\mathbb{Z}} \mathbb{R}$ be the fundamental chamber, that is the set of elements λ such that $\langle \alpha_i^\vee, \lambda\rangle \geq 0$ for all $i \in I\backslash\{0\}$, and $\langle \theta^\vee, \lambda\rangle \leq 1/a_0$. Any automorphism τ of the affine Dynkin diagram Γ induces a symmetry of $\overline{P} \otimes_{\mathbb{Z}} \mathbb{R}$ preserving C, which we also denote by τ. Any element of $\widetilde{W}$ can be written uniquely as $v\tau$ for for $v \in W$ and diagram automorphism τ. Not all τ show up in such expressions, and the set of τ that do is subgroup of $Aut(\Gamma)$ which we denote by Σ.

Remark 2.2 *When $\mathfrak{g}$ is of untwisted type, $M \cong Q^\vee$, $\widetilde{M} \cong P^\vee$, with the isomorphism ν given by $c_i\omega_i = \nu(\omega_i^\vee)$, and $c_i\alpha_i = \nu(\alpha_i^\vee)$ for $i \in I\backslash\{0\}$.*

2.4 Demazure modules and crystals

In this section $\mathfrak{g}$ is an arbitrary symmetrizable Kac–Moody algebra. Let λ be a dominant integral weight for $\mathfrak{g}$. Define $W^\lambda := \{w \in W \mid w\lambda = \lambda\}$. Fix $\mu \in W\lambda$, and recall that the μ weight space in $V(\lambda)$ is one-dimensional. Let u_μ be a non-zero element of the μ weight space in $V(\lambda)$. Write $\mu = w\lambda$ where w is the shortest element in the coset wW^λ.

Define the Demazure module

$$V_w(\lambda) := U_q(\mathfrak{g})^{>0} \cdot u_{w(\lambda)}.$$

It was conjectured by Littelmann [30] and proven by Kahshiwara [19] that $V_w(\lambda)$ has a crystal base $B_w(\lambda)$. Define the set

$$f_w(b) := \{\, f_{i_N}^{m_N} \cdots f_{i_1}^{m_1}(b) \mid m_k \in \mathbb{Z}_{\geq 0}\}, \tag{2.2}$$

where $w = s_{i_N} \cdots s_{i_1}$ is any fixed reduced decomposition of w. By [19, Proposition 3.2.3], as sets, $B_w(\lambda) = f_w(u_\lambda)$.

2.5 Non-exceptional finite type crystals

The standard crystals of type $X_n = A_n, B_n, C_n, D_n$ can be realized as in Figure 1. We call the set of symbols that show up in this realization the type X_n alphabet. Impose a partial order $\prec$ on this alphabet by saying $x \prec y$ iff x is to the left of y in Figure 1 (so in type D_n, n and $\overline{n}$ are incomparable).

Definition 2.3 *Fix $\mathfrak{g}$ of type X_n, for $X = A, B, C, D$. Fix a dominant integral weight γ for $\mathfrak{g} = X_n$. Write $\gamma = m_1\omega_1 + m_2\omega_2 + \cdots + m_{n-1}\omega_{n-1} + m_n\omega_n$. Define a generalized partition $\Lambda(\gamma)$ associated to γ, which is defined case by case as follows:*

- *If $X = A, C$, $\Lambda(\gamma)$ has m_i columns of each height i for each $1 \leq i \leq n$;*
- *If $X = B$, $\Lambda(\gamma)$ has m_i columns of height i for $1 \leq i \leq n-1$, and $m_n/2$ columns of height n;*
- *If $X = D$, $\Lambda(\gamma)$ has m_i columns of each height i for each $1 \leq i \leq n-2$, $\min(m_{n-1}, m_n)$ columns of height $n-1$, and $|m_n - m_{n-1}|/2$ columns of height n. Color columns of height n using color 1 if $m_n > m_{n-1}$ and color 2 if $m_n < m_{n-1}$.*

A_n	$\boxed{1} \xrightarrow{1} \boxed{2} \xrightarrow{2} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{n+1}$
B_n	$\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{0} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}}$
C_n	$\boxed{1} \xrightarrow{1} \cdots \xrightarrow{n-1} \boxed{n} \xrightarrow{n} \boxed{\overline{n}} \xrightarrow{n-1} \cdots \xrightarrow{1} \boxed{\overline{1}}$
D_n	$\begin{tikzcd}[row sep=small] & & & \boxed{n} \arrow[dr, "n"] & & & \\ \boxed{1} \arrow[r, "1"] & \cdots \arrow[r, "n-2"] & \boxed{n-1} \arrow[ur, "n-1"] \arrow[dr, "n"'] & & \boxed{\overline{n-1}} \arrow[r, "n-1"] & \cdots \arrow[r, "1"] & \boxed{\overline{1}} \\ & & & \boxed{\overline{n}} \arrow[ur, "n-1"'] & & & \end{tikzcd}$

Fig. 1: Standard crystals $B(\omega_1)$

In cases where the above formulas involve a fractional number x of columns at some height, we denote this by putting $\lfloor x \rfloor$ columns in addition to a single column of half width. Notice that this can only happen for columns of height n, and at worst we get a single column of width $1/2$.

In [23], the highest weight crystals $B(\gamma)$ of types A_n, B_n, C_n, D_n were constructed in terms of tableaux, now known as Kashiwara–Nakashima (KN) tableaux, of shape $\Lambda(\gamma)$ and containing the symbols from Figure 1 of the relevant type.

3 Kirillov–Reshetikhin modules and their crystals

Let $\mathfrak{g}$ be an affine Kac–Moody algebra with index set I. The Kirillov–Reshetikhin modules were first introduced for the Yangian of $\mathfrak{g}'$ in [24], and developed for $U'_q(\mathfrak{g})$ in [3]. One can characterize the KR module $W^{r,s}$ for $U'_q(\mathfrak{g})$, where $r \in I \setminus \{0\}$ and $s \geq 1$, as the irreducible representations of $U'_q(\mathfrak{g})$ whose Drinfeld polynomials are given by $P_i(u) = (1 - q_i^{1-s}u)(1 - q_i^{3-s}u) \cdots (1 - q_i^{s-1}u)$ if $i = r$ and 1 otherwise. Here $q_i = q^{(\alpha_i|\alpha_i)/2}$.

Theorem 3.1 *[33, 7] In all non-exceptional types, $W^{r,s}$ has a crystal base $B^{r,s}$. Furthermore, if s is a multiple of c_r, then the resulting crystals are perfect, where $c_r = 2$ for type $B_n^{(1)}$ and $r = n$, and for type $C_n^{(1)}$ and $r < n$, and $c_r = 1$ in all other non-exceptional cases.* □

Set

$$\diamond = \begin{cases} \emptyset & \text{for type } A_n^{(1)} \text{ and } 1 \leq r \leq n \\ & \text{for types } C_n^{(1)}, D_{n+1}^{(2)} \text{ and } r = n \\ & \text{for type } D_n^{(1)} \text{ and } r = n-1, n \\ \text{vertical domino} & \text{for type } D_n^{(1)} \text{ and } 1 \leq r \leq n-2 \\ & \text{for types } B_n^{(1)}, A_{2n-1}^{(2)} \text{ and } 1 \leq r \leq n \\ \text{horizontal domino} & \text{for types } C_n^{(1)}, D_{n+1}^{(2)} \text{ and } 1 \leq r < n \\ \text{box} & \text{for type } A_{2n}^{(2)} \text{ and } 1 \leq r \leq n. \end{cases} \tag{3.1}$$

As shown by Chari [1] in the untwisted case and more recently established in the twisted case, every $B^{r,s}$ decomposes as a classical crystal as

$$B^{r,s} \cong \bigoplus_{\lambda} B(\lambda), \tag{3.2}$$

where the sum is over those λ which can be obtained from $s\omega_r$ by removing some number of $\diamond$, each occurring with multiplicity 1.

By [28, Proposition 3.8], any tensor product $B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_N,s_N}$ of KR-crystals is connected. We refer to such a B as a *composite KR-crystal*. As in [17], if the factors are all perfect KR crystals of the same level ℓ then $B = B^{r_1,\ell c_{r_1}} \otimes \cdots \otimes B^{r_N,\ell c_{r_N}}$ is also perfect of level ℓ, and we refer to such a crystal as a *composite KR-crystal of level ℓ*.

Explicit combinatorial models for KR crystals $B^{r,s}$ in non-exceptional types were constructed in [6] in terms of KN tableaux. Using these models, we obtain the following crucial lemma. The proof requires case by case analysis, and makes heavy use of [37, Lemma 5.1], which leads to a description of the action of e_0 on X_{n-2} highest weight elements, where X_n is the underlying classical type.

Lemma 3.2 *Let $B^{r,s}$ be a KR crystal of non-exceptional type. Fix $b \in B^{r,s}$, and assume that b (resp. $e_0(b)$) lies in the classical component $B(\gamma)$ (resp. $B(\gamma')$) of* (3.2)*. If $\varepsilon_0(b) \leq \lceil s/c_r \rceil$ then $\diamond = \emptyset$, and otherwise:*

(i) $\Lambda(\gamma')$ is either equal to $\Lambda(\gamma)$, or else is obtained from $\Lambda(\gamma)$ by adding or removing a single $\diamond$.

(ii) If $\varepsilon_0(b) > \lceil s/c_r \rceil$, then $\Lambda(\gamma')$ is obtained from $\Lambda(\gamma)$ by removing a $\diamond$.

4 Energy functions

We define two energy functions on tensor products of KR crystals. The function E^{int} is given by a fairly natural "global" condition on tensor products of level-ℓ KR crystals. The function D is defined by summing up combinatorially defined "local" contributions, and makes sense for general tensor products of KR crystals. In Theorem 6.2 below we establish that these two functions agree up to a shift, as was suggested in [35, Section 2.5].

4.1 The function E^{int}

The following is essentially the definition of a ground state path from [17].

Definition 4.1 *Let $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ be a composite level ℓ KR crystal. Define $u_B = u_B^N \otimes \cdots \otimes u_B^1$ to be the unique element of B such that $\varepsilon(u_B^1) = \ell\Lambda_0$ and, for each $1 \leq j < N$, $\varepsilon(u_B^{j+1}) = \varphi(u_B^j)$. This is well-defined by the definition of a perfect crystal. The element u_B is called the ground state path of B.*

Definition 4.2 *Let B be a composite KR crystal of level ℓ and consider u_B as in Definition 4.1. Define the intrinsic energy $E^{\mathit{int}}(b)$ for $b \in B$ to be the minimal number of f_0 in a string $f_{i_N} \cdots f_{i_1}$ such that $f_{i_N} \cdots f_{i_1}(u_B) = b$.*

4.2 The D function

Definition 4.3 *The D-function on $B^{r,s}$ is the function defined as follows:*

(i) $D_{B^{r,s}} : B^{r,s} \to \mathbb{Z}$ is constant on all classical components.

(ii) On the component $B(\lambda)$, $D_{B^{r,s}}$ records the maximum number of $\diamond$ that can be removed from $\Lambda(\lambda)$ such that the result is still a (generalized) partition, where $\diamond$ is as in (3.1).

In those cases when $\diamond = \emptyset$, this is interpreted as saying that $D_{B^{r,s}}$ is the constant function 0.

Let B_1, B_2 be two affine crystals with generators v_1 and v_2, respectively, such that $B_1 \otimes B_2$ is connected and $v_1 \otimes v_2$ lies in a one-dimensional weight space. By [28, Proposition 3.8], this holds for any two KR crystals. The *combinatorial R-matrix* [17, Section 4] is the unique crystal isomorphism $\sigma : B_2 \otimes B_1 \to B_1 \otimes B_2$. By weight considerations, this must satisfy $\sigma(v_2 \otimes v_1) = v_1 \otimes v_2$.

As in [17], [34, Theorem 2.4], there is a function $H = H_{B_2,B_1} : B_2 \otimes B_1 \to \mathbb{Z}$, unique up to global additive constant, such that, for all $b_2 \in B_2$ and $b_1 \in B_1$,

$$H(e_i(b_2 \otimes b_1)) = H(b_2 \otimes b_1) + \begin{cases} -1 & \text{if } i = 0 \text{ and LL}, \\ 1 & \text{if } i = 0 \text{ and RR}, \\ 0 & \text{otherwise}. \end{cases} \tag{4.1}$$

Here LL (resp. RR) indicates that e_0 acts on the left (resp. right) tensor factor in both $b_2 \otimes b_1$ and $\sigma(b_2 \otimes b_1)$. When B_1 and B_2 are KR crystals, we normalize H_{B_2,B_1} by requiring $H_{B_2,B_1}(u_{B_2} \otimes u_{B_1}) = 0$, where u_{B_1} and u_{B_2} are as in Definition 4.1.

Definition 4.4 *For $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$, set $D_j := D_{B^{r_j,s_j}} \sigma_1 \sigma_2 \cdots \sigma_{j-1}$ and set $H_{j,i} := H_i \sigma_{i+1} \sigma_{i+2} \cdots \sigma_{j-1}$, where σ_j and H_j act on the j-th and $(j+1)$-st tensor factors and $D_{B^{r_j,s_j}}$ is the D-function for B^{r_j,s_j} as given in Definition 4.3 acting on the rightmost factor. The D-function $D_B : B \to \mathbb{Z}$ is defined as*

$$D_B := \sum_{N \geq j > i \geq 1} H_{j,i} + \sum_{j=1}^{N} D_j. \tag{4.2}$$

5 Perfect KR crystals and Demazure crystals

We now state a precise relationship between KR crystals and Demazure crystals (see Theorem 5.1). This was proven by Fourier, Schilling, and Shimozono [8] under some additional assumptions on the KR crystals, most of which follow from the later results [7] showing that the relevant KR crystals are perfect. In types $A_{2n}^{(2)}$ and for the exceptional nodes in type $D_n^{(1)}$ the assumptions from [8] need to be proven separately or slightly modified, which we do in the long version of this paper. Thus we establish:

Theorem 5.1 *Let $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ be a level-ℓ composite KR crystal of non-exceptional type. Define $\lambda = -(c_{r_1} \omega_{r_1^*} + \cdots + c_{r_N} \omega_{r_N^*})$, where r^* is defined by $\omega_{r^*} = -w_0(\omega_r)$ with w_0 the longest element of $\overline{W}$, and write $t_\lambda \in T(\widetilde{M}) \subset \widetilde{W}$ as $t_\lambda = v\tau$ for $v \in W, \tau \in \Sigma$. Then there is a unique isomorphism of affine crystals*

$$j : B(\ell\Lambda_{\tau(0)}) \to B \otimes B(\ell\Lambda_0).$$

This satisfies $j(u_{\ell\Lambda_{\tau(0)}}) = u_B \otimes u_{\ell\Lambda_0}$, where u_B is the distinguished element from Definition 4.1, and

$$j\left(B_v(\ell\Lambda_{\tau(0)})\right) = B \otimes u_{\ell\Lambda_0}, \tag{5.1}$$

where $B_v(\ell\Lambda_{\tau(0)})$ is the Demazure as defined in Section 2.4.

6 The affine grading via the energy function

In this section we give precise statements of our main results. We show that for $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ a composite level-ℓ KR crystal the map j from Theorem 5.1 intertwines the D function from Section 4.2 with the affine degree up to a shift. This allows us to show that j intertwines E^{int} with $\deg$ exactly, and in particular E^{int} agrees with D up to a shift.

Definition 6.1 *For $t_\lambda = v\tau \in \widetilde{W}$, let $\deg : B_v(\ell\Lambda_{\tau(0)}) \to \mathbb{Z}_{\geq 0}$ be the affine degree map, defined by $\deg(u_{\ell\Lambda_{\tau(0)}}) = 0$, and each f_i has degree $\delta_{i,0}$.*

Theorem 6.2 *With the same assumptions and notation as in Theorem 5.1, let $\tilde{j} : B_v(\ell\Lambda_{\tau(0)}) \to B$ be the restriction of the map j to $B_v(\ell\Lambda_{\tau(0)})$, where $B \otimes u_{\ell\Lambda_0}$ is identified with just B. Then for all $b \in B_v(\ell\Lambda_{\tau(0)})$ we have $\deg(b) = D(\tilde{j}(b)) - D(u_B) = E^{int}(\tilde{j}(b))$.*

These results are proven using the following lemma, which in turn follows from Lemma 3.2.

Lemma 6.3 *Let $B = B^{r_N,s_N} \otimes \cdots \otimes B^{r_1,s_1}$ be a tensor product of KR crystals and fix an integer ℓ such that $\ell \geq \lceil s_k/c_k \rceil$ for all $1 \leq k \leq N$. If $e_0(b) \neq 0$ then $D(e_0(b)) \geq D(b) - 1$, and if $\varepsilon_0(b) > \ell$ then this is an equality.*

Notice that Lemma 6.3 holds in greater generality than Theorem 6.2. In the long version of this paper [38, Section 8], we give a relationship between the D function and the affine grading on a related sum of highest weight crystals in this more general setting. However, we no longer give a connection with Demazure modules.

7 Applications

We now discuss how the relation between the affine grading in the Demazure crystal and the energy function can be used to derive a formula for the Demazure character using the energy function, as well as showing how they are related to nonsymmetric Macdonald polynomials and Whittaker functions.

7.1 Demazure characters

By definition the Demazure character is $\mathrm{ch}V_w(\lambda) = \sum_\mu \dim(V_w(\lambda)_\mu)e^\mu$, where $V_w(\lambda)_\mu$ is the μ weight space of the Demazure module $V_w(\lambda)$. This can be expressed in terms of the Demazure crystal as

$$\mathrm{ch}V_w(\lambda) = \sum_{b \in B_w(\lambda)} e^{\mathrm{wt}\,(b)}. \tag{7.1}$$

It follows immediately from Theorem 6.2 that:

Corollary 7.1 *Let $B = B^{r_N,\ell c_{r_N}} \otimes \cdots \otimes B^{r_1,\ell c_{r_1}}$ be a $U_q'(\mathfrak{g})$-composite level-ℓ KR crystal, $\lambda = -(c_{r_1}\omega_{r_1^*} + \cdots + c_{r_N}\omega_{r_N^*})$, and $t_\lambda = v\tau$ as in Theorem 5.1. Then*

$$\mathrm{ch}V_v(\ell\Lambda_{\tau(0)}) = e^{\ell\Lambda_0} \sum_{b \in B} e^{\mathrm{wt}\,(b) - \delta E^{int}(b)} = e^{\ell\Lambda_0} \sum_{b \in B} e^{\mathrm{wt}\,(b) - \delta(D(b) - D(u_B))}. \tag{7.2}$$

7.2 Nonsymmetric Macdonald polynomials

Fix $\mathfrak{g}$ of affine type. Let $\widetilde{P} \subset P$ be the sublattice of level 0 weights. Recall that $\widetilde{P}$ is naturally contained in $\overline{P} + \mathbb{Z}\delta$, where δ is the null root (and this containment is equality except in type $A_{2n}^{(2)}$). Let t be the collection of indeterminates t_α for each root α such that $t_\alpha = t_{\alpha'}$ if α and α' have the same length. Consider the following elements of the group algebra $\mathbb{Q}(q,t)\overline{P}$:

$$\Delta := \prod_{\alpha \in R_+^{\mathrm{aff}}} \frac{1-e^\alpha}{1-t_\alpha e^\alpha}\Big|_{e^\delta = q}, \quad \text{and} \quad \Delta_1 := \Delta/([e^0]\Delta),$$

where $[e^0]$ means the coefficient of e^0 and R_+^{aff} is the set of positive affine real roots. Cherednik's inner product [2] on $\mathbb{Q}(q,t)\overline{P}$ is $\langle f, g\rangle_{q,t} = [e^0](f\overline{g}\Delta_1)$, where $\bar{\cdot}$ is the involution $\overline{q} = q^{-1}$, $\overline{t} = t^{-1}$, $\overline{e^\lambda} = e^{-\lambda}$.

The nonsymmetric Macdonald polynomials $E_\lambda(q,t) \in \mathbb{Q}(q,t)\overline{P}$ for $\lambda \in \overline{P}$ were introduced by Opdam [31] in the differential setting and Cherednik [2] in general (although here we follow conventions of Haglund, Haiman, Loehr [10, 11]). They are uniquely characterized by two conditions: Triangularity: $E_\lambda \in x^\lambda + \mathbb{Q}(q,t)\{x^\mu \mid \mu < \lambda\}$ and orthogonality: $\langle E_\lambda, E_\mu\rangle_{q,t} = 0$ for $\lambda \neq \mu$. Here $<$ is the Bruhat order on $\overline{P}$ identified with the set of minimal coset representatives in $\widetilde{W}/\overline{W}$.

Extending Sanderson's work [36] for type A, Ion [15] showed that, for all simply laced untwisted affine root systems, we have the following: Write $t_\lambda \in \widetilde{W}$ as $t_\lambda = w\tau$, where $w \in W, \tau \in \Sigma$. Then $E_\lambda(q,0) = q^c\,\mathrm{ch}(V_w(\Lambda_{\tau(0)}))|_{e^\delta = q,\, e^{\Lambda_0}=1}$, where c is a specific exponent described in [15, 36]. When λ is anti-dominant, $E_\lambda(q,t)$ is actually the symmetric Macdonald polynomial, so we denote it by $P_\lambda(q,t)$. In types $A_n^{(1)}$ and $D_n^{(1)}$, Corollary 7.1 allows us write $P_\lambda(q,0)$ in terms of the energy function on a tensor product B of KR crystals, and we can show that $c = -D(u_B)$. Thus in these cases

$$P_\lambda(q,0) = \sum_{b \in B} q^{-D(b)} e^{\mathrm{wt}\,(b)}. \tag{7.3}$$

There is a similar expression for other $E_\lambda(q,0)$ in type $A_n^{(1)}$ and $D_n^{(1)}$ where the sum is over some $B' \subset B$.

Example 7.2 *The Macdonald polynomial of type $A_2^{(1)}$ indexed by $(0,0,2)$ is given by*

$$P_{(0,0,2)}(q,0) = x_1^2 + (q+1)x_1x_2 + x_2^2 + (q+1)x_1x_3 + (q+1)x_2x_3 + x_3^2.$$

The corresponding KR crystal is $B = B^{1,1} \otimes B^{1,1}$. Drawing only arrows which also exist in the corresponding Demazure crsytal $B_{s_2s_1s_0s_2}(\Lambda_2)$, this is

$$\begin{tikzcd}
2\otimes 1 \arrow[r, "2"] & 3\otimes 1 \arrow[r, "0"] \arrow[dr, "1"'] & 1\otimes 1 \arrow[r, "1"] & 1\otimes 2 \arrow[r, "1"] \arrow[dr, "2"'] & 2\otimes 2 \arrow[r, "2"] & 2\otimes 3 \arrow[r, "2"] & 3\otimes 3 \\
& & 3\otimes 2 & & 1\otimes 3 \arrow[ur, "1"'] & &
\end{tikzcd}$$

$D(2\otimes 1) = D(3\otimes 1) = D(3\otimes 2) = -1$ *and the rest of the terms have $D = 0$, confirming* (7.3).

7.3 Whittaker functions

Gerasimov, Lebedev, Oblezin [9, Theorem 3.2] showed that q-deformed $\mathfrak{gl}_n$-Whittaker functions are Macdonald polynomials specialized at $t = 0$. As above this also gives a link to Demazure characters, and hence by the results in Section 6 to KR crystals graded by their energy functions. It would be interesting to generalize this to other types. The q-deformed $\mathfrak{gl}_n$-Whittaker functions are simultaneous eigenfunctions of a q-deformed Toda chain, which might serve as a starting point for this generalization.

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The equivariant topology of stable Kneser graphs

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Abstract. Schrijver introduced the stable Kneser graph $SG_{n,k}$, $n \geq 1$, $k \geq 0$. This graph is a vertex critical graph with chromatic number $k+2$, its vertices are certain subsets of a set of cardinality $m = 2n + k$. Björner and de Longueville have shown that its box complex is homotopy equivalent to a sphere, $\mathrm{Hom}(K_2, SG_{n,k}) \simeq \mathbb{S}^k$. The dihedral group D_{2m} acts canonically on $SG_{n,k}$. We study the D_{2m} action on $\mathrm{Hom}(K_2, SG_{n,k})$ and define a corresponding orthogonal action on $\mathbb{R}^{k+1} \supset \mathbb{S}^k$. We establish a close equivariant relationship between the graphs $SG_{n,k}$ and Borsuk graphs of the k-sphere and use this together with calculations in the $\mathbb{Z}_2$-cohomology ring of D_{2m} to tell which stable Kneser graphs are test graphs in the sense of Babson and Kozlov.

The graphs $SG_{2s,4}$ are test graphs, i.e. for every graph H and $r \geq 0$ such that $\mathrm{Hom}(SG_{2s,4}, H)$ is $(r-1)$-connected, the chromatic number $\chi(H)$ is at least $r+6$. On the other hand, if $k \notin \{0, 1, 2, 4, 8\}$ and $n \geq N(k)$ then $SG_{n,k}$ is not a homotopy test graph, i.e. there are a graph G and an $r \geq 1$ such that $\mathrm{Hom}(SG_{n,k}, G)$ is $(r-1)$-connected and $\chi(G) < r + k + 2$. The latter result also depends on a new necessary criterion for being a test graph, which involves the automorphism group of the graph.

Résumé. Schrijver a défini le graphe de Kneser stable $SG_{n,k}$, avec $n \geq 1$ et $k \geq 0$. Le graphe $SG_{n,k}$ est un graphe critique (par rapport aux sommets) de nombre chromatique $k+2$, dont les sommets correspondent à certains sous-ensembles d'un ensemble de cardinalité $m = 2n + k$. Björner et de Longueville ont démontré que son complexe de boîtes et la sphère sont homotopiquement équivalents, c'est-à-dire $\mathrm{Hom}(K_2, SG_{n,k}) \simeq \mathbb{S}^k$. Le groupe diédral D_{2m} agit sur $SG_{n,k}$ canoniquement. Nous étudions l'action de D_{2m} sur $\mathrm{Hom}(K_2, SG_{n,k})$ et nous définissons une action orthogonale correspondante sur $\mathbb{R}^{k+1} \supset \mathbb{S}^k$. Par ailleurs, nous fournissons une relation équivariante étroite entre les graphes $SG_{n,k}$ et les graphes de Borsuk de la sphère de dimension k. Utilisant cette relation et certains calculs dans l'anneau de cohomologie de D_{2m} sur $\mathbb{Z}_2$, nous décrivons quels graphes de Kneser stables sont des graphes de tests selon la notion de Babson et Kozlov.

Les graphes $SG_{2s,4}$ sont des graphes de tests, c'est-à-dire que pour tout H et $r \geq 0$ tels que $\mathrm{Hom}(SG_{2s,4}, H)$ est $(r-1)$-connexe, le nombre chromatique $\chi(H)$ est au moins $r+6$. D'autre part, si $k \notin \{0, 1, 2, 4, 8\}$ et $n \geq N(k)$, alors $SG_{n,k}$ n'est pas un graphe de tests d'homologie: il existe un graphe G et un entier $r \geq 1$ tels que $\mathrm{Hom}(SG_{n,k}, G)$ est $(r-1)$-connexe et $\chi(G) < r + k + 2$. Ce dernier résultat dépend d'un nouveau critère nécessaire pour être un graphe de tests, qui implique le groupe d'automorphismes du graphe.

Keywords: stable Kneser graph, Hom complex, graph homomorphism, test graph, alternating oriented matroid

1 Introduction

Background

The subject of topological obstructions to graph colourings was started when Lovász determined the chromatic numbers of Kneser graphs in Lovász (1978).

1.1 Definition. Let $n \geq 1$, $k \geq 0$. The Kneser graph $KG_{n,k}$ is a graph with vertices the n-element subsets of a fixed set of cardinality $2n + k$, say

$$V(KG_{n,k}) = \{S \subset \mathbb{Z}_{2n+k} \colon |S| = n\}\,.$$

Two such sets are neighbours in $KG_{n,k}$ if and only if they are disjoint,

$$E(KG_{n,k}) = \{(S, T) \in V(KG_{n,k}) \colon S \cap T = \emptyset\}\,.$$

It is easy to see that $KG_{n,k}$ admits a $(k + 2)$-colouring, $\chi(KG_{n,k}) \leq k + 2$. Lovász assigned to each graph G a simplicial complex, its *neighbourhood complex* $\mathcal{N}(G)$ and proved the following two theorems.

1.2 Theorem. *If G is a graph and $r \geq 0$ such that $\mathcal{N}(G)$ is $(r-1)$-connected, then $\chi(G) \geq r + 2$.*

1.3 Theorem. *The complex $\mathcal{N}(KG_{n,k})$ is $(k-1)$-connected.*

These establish $\chi(KG_{n,k}) = k + 2$ as conjectured by Kneser.

The proof of Theorem 1.2 uses the Borsuk–Ulam theorem. This led Bárány to a simpler proof of $\chi(KG_{n,k}) \geq k+2$, which does not use any graph complexes but applies the Borsuk–Ulam theorem more directly Bárány (1978). This proof uses the existence of certain generic configurations of vectors in $\mathbb{R}^{k+1}$. Using a specific configuration of this kind, Schrijver found an induced subgraph of $KG_{n,k}$, the graph $SG_{n,k}$, with the property that already $\chi(SG_{n,k}) = k + 2$ Schrijver (1978).

1.4 Definition. The *stable Kneser graph* $SG_{n,k}$ is the induced subgraph of $KG_{n,k}$ on the vertex set

$$V(SG_{n,k}) = \{S \in V(KG_{n,k}) \colon \{i, i+1\} \not\subset S \text{ for all } i \in \mathbb{Z}_{2n+k}\}\,.$$

The vertices of $SG_{n,k}$ are called *stable subsets* of $\mathbb{Z}_{2n+k}$.

Schrijver also proves that the graph $SG_{n,k}$ is vertex critical in the sense that it becomes $(k + 1)$-colourable if an arbitrary vertex is removed.

A more systematic treatment of topological obstructions to the existence of graph colourings was suggested by Lovász and started by Babson and Kozlov (2006). For graphs G and H, they define a cell complex $\mathrm{Hom}(H, G)$. The vertices of $\mathrm{Hom}(H, G)$ are the graph homomorphisms from H to G. They introduce the concept of a test graph.

1.5 Definition. A graph T is a *homotopy test graph* if for all loopless graphs G and $r \geq 0$ such that $\mathrm{Hom}(T, G)$ is $(r - 1)$-connected the inequality $\chi(G) \geq \chi(T) + r$ holds.

Since the complex $\mathrm{Hom}(K_2, G)$ is homotopy equivalent to $\mathcal{N}(G)$, Theorem 1.2 states that K_2 is a homotopy test graph. In Babson and Kozlov (2006) this result is extended to all complete graphs and in Babson and Kozlov (2007) to odd cycles. These proofs show a graph T to be a test graph by studying the spaces $\mathrm{Hom}(T, K_n)$ and C_2-actions on them induced by a C_2-action on T. Here C_2 denotes the cyclic group of order 2. Indeed, for a graph T with an action of a group Γ one can define the property of being a

Γ-*test graph* (Definition 3.1), which implies being a homotopy test graphs, and the homotopy test graphs mentioned above are shown to be C_2-test graphs.

In Schultz (2009) easier proofs, in particular for odd cycles, are obtained by instead studying the complex $\mathrm{Hom}(K_2, T)$ together with two involutions, one induced by the non-trivial involution of K_2 and another by an involution on T. This also yielded the somewhat isolated result that $KG_{2s,2}$ is a C_2-test graph. All known test graphs at that point were Kneser graphs or stable Kneser graphs, since $KG_{1,k} = SG_{1,k}$ is a complete graph on $k+2$ vertices and $SG_{n,1}$ is a cycle of length $2n+1$

In Dochtermann and Schultz (2010) it was shown that test graphs T can be obtained by constructing graphs T with prescribed topology of $\mathrm{Hom}(K_2, T)$.

Overview and results

Since all "naturally occurring" graphs which have so far been identified as test graphs are stable Kneser graphs (the result for $KG_{2s,4}$ can be derived from one for $SG_{2s,4}$), it is natural to ask if more or even all stable Kneser graphs are test graphs. Our main goal in this work is to decide which of them are test graphs. We can answer this question to a large extent. We find new test graphs, but it turns out that most stable Kneser graphs are not test graphs. We point out some of the results that we obtain on the way.

It has been known that stable Kneser graphs are related to spheres, for example Björner and de Longueville (2003) have shown the homotopy equivalence $\mathcal{N}(SG_{n,k}) \simeq \mathbb{S}^k$. We describe more aspects of this relationship and also make it equivariant. Braun (2010) has shown that for $n > 1$ the automorphism group of $SG_{n,k}$ is the dihedral group D_{2m} with $2m$ elements, $m = 2n+k$. In Section 2 we define a $(k+1)$-dimensional orthogonal representation of D_{2m} and an explicit map from its unit sphere to $\mathrm{Hom}(K_2, SG_{n,k})$,

$$\mathbb{S}(W_{n,k}) \to_{C_2 \times D_{2m}} \mathrm{Hom}(K_2, SG_{n,k}). \tag{1}$$

This map is equivariant with respect to the action of the dihedral group, and also with respect to the 2-element group C_2 acting as the antipodal map on the sphere and via K_2 on $\mathrm{Hom}(K_2, SG_{n,k})$. This map is a D_{2m}-homotopy equivalence. Its construction involves the alternating oriented matroid. It uses a connection between the approaches of Bárány (1978) and Schrijver (1978) on one hand and Lovász (1978) on the other, which has not been made explicit previously.

A construction which is in some sense dual to (1) is best formulated using the notion of a Borsuk graph. We write $C_2 = \{e, \tau\}$.

1.6 Definition. Let (X, d) be a metric space with an isometric C_2-action and $\varepsilon > 0$. We define the *ε-Borsuk graph of X*, $B_\varepsilon(X)$, as follows. The vertex set of $B_\varepsilon(X)$ is the set of all points of X and $x \sim y \iff d(x, \tau y) < \varepsilon$.

1.7 Theorem (Proposition 5.3)**.** *Let $k \geq 0$ and $\varepsilon > 0$. Then for large enough n and $m = 2n+k$ there is an equivariant graph homomorphism*

$$SG_{n,k} \to_{D_{2m}} B_\varepsilon(\mathbb{S}(W_{n,k})).$$

The first construction will help us to show that certain stable Kneser graphs are test graphs, the second that certain stable Kneser graphs are not. These constructions are also dual in the way that the first uses properties of covectors of the alternating oriented matroid and the second uses properties of its vectors.

The D_{2m}-representation $W_{n,k}$ gives rise to a $(k+1)$-dimensional vector bundle $\xi_{n,k}$ over the classifying space BD_{2m}, Definition 4.2. This in turn defines Stiefel-Whitney classes $w_i(\xi_{n,k}) \in H^i(D_{2m}; \mathbb{Z}_2)$ and also classes $\overline{w}_i(\xi_{n,k}) \in H^i(D_{2m};\mathbb{Z}_2)$ by $\sum_{i\geq 0} w_i \cdot \sum_{i\geq 0} \overline{w}_i = 1$. An approach relying on the map (1) and similar to that in Schultz (2009), but with different topological tools, yields:

1.8 Proposition (Section 4)**.** *Let $n, k \geq 1$, $m = 2n + k$. If $\overline{w}_r(\xi_{n,k}) \neq 0$ for all $r \geq 1$, then $SG_{n,k}$ is a D_{2m}-test graph and hence a homotopy test graph.*

Calculations in the cohomology ring $H^*(D_{2m};\mathbb{Z}_2)$ identify the cases in which this criterion is applicable.

1.9 Theorem (see Corollary 4.4)**.** *Let $n, k \geq 0$. If $k \in \{0, 1, 2\}$ or if $k = 4$ and n is even, then $SG_{n,k}$ is a homotopy test graph.*

Here, mainly the case $k = 4$, n even, is new, and for it, it would not have been sufficent to restrict the calculations to a 2-element subgroup of D_{2m}. Consequently, this result could not have been obtained by a direct application of the test graph criteria given in Schultz (2009).

So far, all proofs that certain graphs are test graphs used actions of a group on this graph. We show that this is not merely because it was the only known technique.

1.10 Theorem (see Theorem 3.3)**.** *Let T be a finite, connected, vertex critical graph. Then T is a homotopy test graph if and only if T is an $\operatorname{Aut}(T)$-test graph.*

This also gives us a tool to prove that graphs are *not* test graphs. For example we note the curious consequence that a connected vertex critical graph without non-trivial automorphisms cannot be a test graph.

Using Theorem 1.7, some algebraic topology of vector bundles and a few constructions in the category of graphs, we obtain the following criterion.

1.11 Proposition (Proposition 6.3)**.** *Let $k \geq 1$. Then there is an $N > 0$ such that for all $n \geq N$ the following holds: If there is an $r > 0$ such that $\overline{w}_r(\xi_{n,k}) = 0$ and $r = 1$ or $r \equiv 0 \pmod 2$, then $SG_{n,k}$ is not a D_{2m}-test graph and hence not a homotopy test graph.*

Again we do some calculations and arrive at the following result.

1.12 Theorem. *Let $k \geq 0$, $k \notin \{0, 1, 2, 4, 8\}$. Then there is an $N > 0$ such that for no $n \geq N$ the graph $SG_{n,k}$ is a homotopy test graph.*

Also, there is an $N > 0$ such that for no odd $n \geq N$ the graph $SG_{n,8}$ is a homotopy test graph.

This text is an extended abstract of Schultz (2010), and we will concentrate here on the constructions that we deem interesting and omit almost all calculations.

Definitions and notation

We will use the same terminology and notation as in Dochtermann and Schultz (2010) and refer the reader there for details.

Our graphs are undirected and without multiple edges. Even though we are mostly interested in graphs without loops, allowing loops is important for some constructions. We call a graph without loops loopless, and one in which every vertex is looped we call reflexive. A *graph homomorphism* $f\colon G \to H$ is a function $f\colon V(G) \to V(H)$ between the vertex sets, which preserves the adjacency relation, $(f(u), f(v)) \in E(H)$ for all $(u, v) \in E(G)$ (for which we also write $u \sim v$). We denote the category of graphs and graph homomorphisms by $\mathcal{G}$.

The graph $\mathbf{1}$ consisting of one vertex and a loop is a final object in the category of graphs. Any two graphs G and H have a categorical product $G \times H$. For every graph G, the functor $\bullet \times G$ has a right adjoint $[G, \bullet]$, i.e. there is a natural equivalence $\mathcal{G}(Z \times G, H) \cong \mathcal{G}(Z, [G, H])$. The graph $[G, H]$ is also written H^G and called an exponential graph. The graph homomorphisms from G to H correspond to the looped vertices of $[G, H]$ in accordance with $\mathcal{G}(G, H) \cong \mathcal{G}(\mathbf{1} \times G, H) \cong \mathcal{G}(\mathbf{1}, [G, H])$.

For graphs G and H we define a poset

$$\operatorname{Hom}(G, H) = \left\{ f \in (\mathcal{P}(V(H)) \setminus \{\emptyset\})^{V(G)} : f(u) \times f(v) \subset E(H) \text{ f. a. } (u, v) \in E(G) \right\}$$

with $f \leq g$ if and only if $f(u) \subset g(u)$ for all $u \in V(G)$. Hom is a functor from $\mathcal{G}^{\mathrm{opp}} \times \mathcal{G}$ to the category of posets and order preserving maps. $\operatorname{Hom}(G, H)$ is the face poset of a cell complex first described in Babson and Kozlov (2006). The special case $\operatorname{Hom}(\mathbf{1}, H)$ is the poset of cliques of looped vertices of H. The atoms of $\operatorname{Hom}(G, H)$ correspond to the graph homomorphisms from G to H. More is true: There is a natural homotopy equivalence $|\operatorname{Hom}(G, H)| \simeq |\operatorname{Hom}(\mathbf{1}, [G, H])|$ induced by a poset map which preserves atoms and with a homotopy inverse of the same kind. Also $|\operatorname{Hom}(\mathbf{1}, \bullet)|$ preserves products up to such an equivalence, see Dochtermann (2009b). More formal properties of Hom can be derived from the above facts, in particular the existence of a map

$$\operatorname{Hom}(G, H) \times \operatorname{Hom}(H, Z) \to \operatorname{Hom}(G, Z) \tag{2}$$

which on atoms corresponds to composition of graph homomorphisms and has all the expected properties. Of course such a map is also easy to write down explicitly, it was first used in Schultz (2009).

Another construction that we will use takes a poset P and assigns to it a reflexive graph P^1. The vertices of P^1 are the atoms of P, and two atoms are adjacent in P^1, if and only if they have a common upper bound in P. This construction played in important role in Dochtermann and Schultz (2010) and we will use several results from there. Usually P is the face poset of a cell complex X, in which case we also write X^1 instead of P^1 and call it the looped 1-skeleton of X.

2 The Bárány-Schrijver construction

Stable Kneser graphs and the alternating oriented matroid

Our starting point is the realization that parts of the construction which was used by Bárány (1978) to prove $\chi(KG_{n,k}) = k + 2$ *without* using graph complexes and later refined by Schrijver (1978) to prove $\chi(SG_{n,k}) = k + 2$ can also be used to tell us something about graph complexes, namely the existence of an equivariant map $\mathbb{S}^k \to_{C_2} \operatorname{Hom}(K_2, SG_{n,k})$. This was also implicitly used by Ziegler (2002) in a combinatorial proof of $\chi(SG_{n,k}) = k+2$, which built open a combinatorial proof of Kneser's conjecture by Matoušek (2004).

Schrijver uses vectors $v_0, \dots, v_{2n+k-1}$ on the moment curve to define a covering of $\mathbb{S}^k$ by the system of sets $(\{x \in \mathbb{S}^k : (-1)^j \langle x, v_j \rangle > 0 \text{ f.a. } j \in S\})_{S \in V(SG_{n,k})}$ and use the Borsuk-Ulam theorem to conclude that $\chi(SG_{n,k}) \geq k + 2$.

We define the alternating oriented matroid $C^{m,k+1}$, $m > k \geq 0$, to be the oriented matroid associated to the vector configuration $v_0, \dots, v_{m-1} \in \mathbb{R}^{k+1}$ with $v_j = (1, t_j, \dots, t_j^k)$ for some real numbers $t_0 < t_1 < \dots < t_{m-1}$ (Björner et al., 1999, 9.4). The set of non-zero covectors is

$$\mathcal{L}(C^{m,k+1}) = \left\{ (\operatorname{sign}\langle x, v_0 \rangle, \dots, \operatorname{sign}\langle x, v_{m-1} \rangle) : x \in \mathbb{R}^{k+1} \setminus \{0\} \right\} \subset \{-1, 0, +1\}^m .$$

We regard $\mathcal{L}(C^{m,k+1})$ as a poset with the partial order induced by the partial order on $\{-1, 0, +1\}$ given by $s \leq s' \iff s = 0 \vee s = s'$. The elements of $\mathcal{L}(C^{m,k+1})$ are exactly the sign vectors with at most

k sign changes. By this we mean those sign vectors obtained as $(\operatorname{sign} p(0), \dots, \operatorname{sign} p(m-1))$ with p a polynomial of degree k. The minimal elements, called cocircuits, are those with exactly k zeros (and hence a sign change at every zero, to be interpreted as above).

2.1 Proposition and Definition. *Let $n \geq 1$, $k \geq 0$ and $m = 2n+k$. For $s = (s_0, \dots, s_{m-1}) \in \mathcal{L}(C^{m,k+1})$ let the sets $S_0(s), S_1(s) \subset \{0, \dots, m-1\}$ be defined by*

$$S_l(s) = \left\{ j \colon (-1)^j s_j = (-1)^l \right\}.$$

Then

$$\begin{aligned} \mathcal{L}(C^{m,k+1}) &\to \operatorname{Hom}(K_2, SG_{n,k}) \\ s &\mapsto (l \mapsto \{T \in V(SG_{n,k}) \colon T \subset S_l(s)\}), \end{aligned}$$

with $V(K_2) = \{0,1\}$, is a well-defined order preserving map.

Proof: Denote the map by g. For $s \in \mathcal{L}(C^{m,k+1})$ obviously $S_0(s) \cap S_1(s) = \emptyset$, so $(T_0, T_1) \in E(SG_{n,k})$ for all $T_l \in g(s)_l$. We only have to check that $g(s)_l \neq \emptyset$, and since g is order preserving by construction, we can assume that s is a cocircuit. But then s contains exactly k zeros. Let $i_0 < i_1 < \cdots < i_{2n-1}$ be the indices at which s is non-zero. That s has sign changes at exactly the zeros means that $s_{i_{j+1}} = (-1)^{i_{j+1}-i_j-1} s_{i_j}$, i.e. $(-1)^{i_{j+1}} s_{i_{j+1}} = -(-1)^{i_j} s_{i_j}$. Therefore $S_0(s)$ and $S_1(s)$ are interleaved n-sets and $S_0(s), S_1(s) \in V(SG_{n,k})$. □

2.2 Proposition and Definition. *Let $n \geq 1$, $k \geq 0$ and $m = 2n+k$ and (v_j) the vector configuration above or any other vector configuration realizing $C^{m,k+1}$.*

The covector poset $\mathcal{L}(C^{m,k+1})$ is the face poset of a cell decomposition of $\mathbb{S}^k$, where the open cell corresponding to $s \in \mathcal{L}(C^{m,k+1})$ is $\left\{ x \in \mathbb{S}^k \colon \operatorname{sign}\langle x, v_j \rangle = s_j \right\}$. Therefore the poset map of Definition 2.1 induces a continuous map

$$f \colon \mathbb{S}^k \to |\operatorname{Hom}(K_2, SG_{n,k})|.$$

If we write the group with 2 elements as $C_2 = \{e, \tau\}$ and have C_2 operate via the antipodal map on $\mathbb{S}^k$ and via the isomorphism $\operatorname{Aut}(K_2) \cong C_2$ on $\operatorname{Hom}(K_2, SG_{n,k})$, then the map f satisfies $f(\tau \cdot x) = \tau \cdot f(x)$ for all $x \in \mathbb{S}^k$. □

The action of the dihedral group

The map of Definition 2.2 turns out to be useful in studying $\operatorname{Hom}(K_2, SG_{n,k})$. Since we are interested in the action of the automorphism group of $SG_{n,k}$ on $\operatorname{Hom}(K_2, SG_{n,k})$, we will choose different vectors in the construction to obtain an equivariant map with respect to an orthogonal action of $\operatorname{Aut}(SG_{n,k})$ on $\mathbb{S}^k$. We do this by replacing the moment curve with the trigonometric moment curve.

2.3 Definition. For $m \geq 2$ let $D_{2m} = \left\langle \sigma, \rho \mid \rho^2 = \sigma^m = (\sigma\rho)^2 = 1 \right\rangle$ denote the dihedral group with $2m$ elements.

For $m = 2n+k$ we define a right D_{2m} action on $KG_{n,k}$ by

$$S \cdot \sigma = \{j+1 \colon j \in S\}, \qquad S \cdot \rho = \{-j \colon j \in S\},$$

where all arithmetic is modulo m. The subgraph $SG_{n,k}$ is invariant under this action. We also set $C_2 = \left\langle \tau \mid \tau^2 = 1 \right\rangle$ and have this group act nontrivially on K_2 from the right. Since Hom is contravariant in the first and covariant in the second parameter, this defines a left C_2-action and a right D_{2m}-action on $\operatorname{Hom}(K_2, SG_{n,k})$, and these commute.

For $n > 1$ the homomorphism $D_{2m} \to \operatorname{Aut}(SG_{n,k})$ given by this action has been shown to be an isomorphism by Braun (2010).

Our goal is to choose vectors v_i in Definition 2.2 in such a way that that map becomes D_{2m}-equivariant with respect to an easy to define D_{2m}-action on $\mathbb{S}^k$. We achieve this by essentially replacing the moment curve by the trigonometric moment curve.

2.4 Definition. Let $n \geq 1$, $k \geq 0$, $m = 2n + k$, We define orthogonal actions on $\mathbb{R}^{k+1}$ and vectors in $\mathbb{R}^{k+1}$. In the following $R_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$. First we set $\tau \cdot x = -x$.

For $k = 2r$ we set

$$x \cdot \sigma = -\operatorname{diag}(1, R_{2\pi/m}, R_{4\pi/m}, \ldots, R_{k\pi/m}) \cdot x, \qquad x \cdot \rho = \operatorname{diag}(1, 1, -1, \ldots, 1, -1) \cdot x,$$
$$v_j = (-1)^j (1, 1, 0, \ldots, 1, 0) \cdot \sigma^j.$$

For $k = 2r + 1$ we set

$$x \cdot \sigma = -\operatorname{diag}(R_{\pi/m}, R_{3\pi/m}, \ldots, R_{k\pi/m}) \cdot x, \qquad x \cdot \rho = \operatorname{diag}(1, -1, \ldots, 1, -1) \cdot x,$$
$$v_j = (-1)^j (1, 0, \ldots, 1, 0) \cdot \sigma^j.$$

We denote $\mathbb{R}^{k+1}$ equipped with the orthogonal right action of D_{2m} defined above by $W_{n,k}$. The unit sphere in $W_{n,k}$ is denoted by $\mathbb{S}(W_{n,k})$.

It is now not too hard to check that the system $(v_j)_{0 \leq j < m}$ realizes $C^{m,k+1}$ and that using it in Definition 2.1 makes the resulting map D_{2m}-equivariant.

2.5 Theorem. *Let $n \geq 1$, $k \geq 0$, and $m = 2n + k$. There is a continuous map $f \colon \mathbb{S}(W_{n,k}) \to |\operatorname{Hom}(K_2, SG_{n,k})|$ which is equivariant with respect to the actions of C_2 and D_{2m} defined above.* □

3 Homotopy test graphs and Γ-test graphs

In addition to Definition 1.5 of a homotopy test graph we also define a Γ-test graph for a group Γ. We denote by $E_r\Gamma$ any r-dimensional, $(r-1)$-connected CW-space with a free Γ-action, and by $E\Gamma$ any contractible CW-space with a free Γ-action. We write $B_r\Gamma = E_r\Gamma/\Gamma$ and $B\Gamma = E\Gamma/\Gamma$ for the coresponding orbit spaces.

3.1 Definition. Let T be a graph with a right action of a finite group Γ. We call T a Γ-*test graph*, if for all loopless graphs G and integers $r \geq 0$ such that there exists an equivariant map $E_r\Gamma \to_\Gamma |\operatorname{Hom}(T, G)|$ the inequality $\chi(G) \geq \chi(T) + r$ holds.

3.2 Proposition. *Let T be a graph with a right action of a finite group Γ. If T is a Γ-test graph, then T is a homotopy test graph.*

Proof: If $|\operatorname{Hom}(T, G)|$ is $(r-1)$-connected, then there is an equivariant map $E_r\Gamma \to_\Gamma |\operatorname{Hom}(T, G)|$, since $E_r\Gamma$ is an r-dimensional free Γ-space. □

We have announced a partial converse of this result in Theorem 1.10. It follows from the following theorem by setting $\Gamma = \operatorname{Aut}(T)$, $s = \chi(T) + r$ and considering the equivalence of (ii) and (iii).

In (i), X^1 denotes the looped 1-skeleton of X and $T \times_\Gamma X^1$ the orbit graph of the diagonal action of Γ on $T \times X^1$. This construction is explored in Dochtermann and Schultz (2010).

3.3 Theorem. *Let T be a finite, connected graph equipped with a right action of a finite group Γ. Let $r \geq 0$, $s \geq 1$. Then each of the following statements implies the next.*

(i) For every Γ-invariant triangulation X of $E_r\Gamma$ the inequality $\chi(T \times_\Gamma X^1) \geq s$ holds.

(ii) For all graphs G such that there is a Γ-equivariant map $E_r\Gamma \to |\mathrm{Hom}(T, G)|$ the inequality $\chi(G) \geq s$ holds.

(iii) For all graphs G such that $|\mathrm{Hom}(T, G)|$ is $(r-1)$-connected the inequality $\chi(G) \geq s$ holds.

If T is vertex critical and $\Gamma = \mathrm{Aut}(T)$, then (iii) *implies* (i) *and all of the statements are equivalent.*

Proof: The implication (i) $\Longrightarrow$ (ii) is a standard application of the techniques developped in Dochtermann and Schultz (2010) and earlier papers, the implication (ii) $\Longrightarrow$ (iii) is immediate as in Proposition 3.2.

To prove the implication (iii) $\Longrightarrow$ (i) we assume that a triangulation X of $E_r\Gamma$ is given. If we obtain Y from X by repeated barycentric subdivision, then Y is also Γ-invariant, and there is an equivariant graph homomorphism $Y^1 \to_\Gamma X^1$. If the subdivision Y is fine enough, then

$$|\mathrm{Hom}(T, T \times_\Gamma Y^1)| \simeq |\mathrm{Hom}(T, T)| \times_\Gamma |\mathrm{Hom}(T, Y^1)|$$

by Sec. 5.2 of Dochtermann and Schultz (2010). Since we assumed T to be vertex critical, the only endomorphisms of T are the automorphisms. It also follows that $\mathrm{Hom}(T, T)$ is 0-dimensional, $\mathrm{Hom}(T, T) \cong \mathrm{Aut}(T)$. We assumed $\Gamma = \mathrm{Aut}(T)$. Therefore

$$|\mathrm{Hom}(T, T)| \times_\Gamma |\mathrm{Hom}(T, Y^1)| \approx \mathrm{Aut}(T) \times_{\mathrm{Aut}(T)} |\mathrm{Hom}(T, Y^1)| \approx |\mathrm{Hom}(T, Y^1)|.$$

But by Thm 3.1 of Dochtermann (2009a), since T is connected, and again if Y is a fine enough subdivision,

$$|\mathrm{Hom}(T, Y^1)| \simeq |Y| \approx E_r\Gamma.$$

Therefore $|\mathrm{Hom}(T, T \times_\Gamma Y^1)|$ is $(r-1)$-connected. Hence $\chi(T \times_\Gamma X^1) \geq \chi(T \times_\Gamma Y^1) \geq s$. □

4 Stable Kneser graphs which are test graphs

The approach that we use to show that for certain n, k the graph $SG_{n,k}$ is a test graph is similar to that used for odd cycles ($k = 1$) in Schultz (2009).

We assume the existence of an equivariant map $E_r D_{2m} \to_{D_{2m}} \mathrm{Hom}(SG_{n,k}, G)$. In Theorem 2.5 we have constructed a map $\mathbb{S}(W_{n,k}) \to_{D_{2m}} \mathrm{Hom}(K_2, SG_{n,k})$. It is known that $\mathrm{Hom}(K_2, K_{k+r+1}) \simeq_{C_2} \mathbb{S}^{k+r-1}$. So if $\chi(G) < k + r + 2$, then we obtain a map

$$\begin{aligned} \mathbb{S}(W_{n,k}) \times_{D_{2m}} E_r D_{2m} &\to_{C_2} \mathrm{Hom}(K_2, SG_{n,k}) \times_{D_{2m}} \mathrm{Hom}(SG_{n,k}, G) \\ &\to_{C_2} \mathrm{Hom}(K_2, G) \to_{C_2} \mathrm{Hom}(K_2, K_{k+r+1}) \simeq_{C_2} \mathbb{S}^{k+r-1}, \end{aligned}$$

the second arrow being the map (2). Now $\mathbb{S}(W_{n,k}) \times_{D_{2m}} E_r D_{2m}$ is the total space of the sphere bundle associated to the euclidean vector bundle $W_{n,k} \times_{D_{2m}} E_r D_{2m} \to B_r D_{2m}$, and vector bundle theory yields obstructions to the existence of an equivariant map as above.

4.1 Theorem. *Let Γ be a finite group and $\mathbb{R}^{k+1}$ be equipped with an orthogonal right Γ-action. Let ξ be the $(k+1)$-dimensional bundle $\mathbb{R}^{k+1} \times_\Gamma E\Gamma \to B\Gamma$. Consider $\mathbb{S}^k \times_\Gamma E_r$ as a (free) C_2-space via the antipodal action on $\mathbb{S}^k$. If there is an equivariant map*

$$\mathbb{S}^k \times_\Gamma E_r\Gamma \to_{C_2} \mathbb{S}^{r+k-1}$$

for some $r \geq 0$, then it follows that $\overline{w}_r(\xi) = 0 \in H^r(B\Gamma; \mathbb{Z}_2) = H^r(\Gamma; \mathbb{Z}_2)$.

Here $\overline{w}_r(\xi)$ denotes the r-th dual Stiefel-Whitney class of the bundle ξ. These classes and the Stiefel-Whitney classes are related by $1 = w(\xi)\overline{w}(\xi) = \sum_{r\geq 0} w_r(\xi) \cdot \sum_{r \geq 0} \overline{w}_r(\xi)$. We see that it is important to calculate the Stiefel-Whitney classes of the following bundles.

4.2 Definition. Let $n \geq 1$, $k \geq 0$, $m = 2n + k$. We denote the $(k+1)$-dimensional vector bundle $W_{n.k} \times_{D_{2m}} ED_{2m} \to BD_{2m}$ by $\xi_{n,k}$.

Theorem 4.1 and the discussion preceding it already prove Proposition 1.8.

With some knowledge of the $\mathbb{Z}_2$-cohomology of the dihedral groups, the classes $w_r(\xi_{n,k})$ can more or less directly be read off from the action described in Definition 2.4. The results are the following.

4.3 Proposition. *Let $k = 2r + 1$, $r \geq 0$, $n > 0$, and $m = 2n + k$. Then*

$$w(\xi_{n,k}) = (1 + \alpha)^{r+1} \in H^*(D_{2m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha], \qquad |\alpha| = 1.$$

Here $\mathbb{Z}_2[\alpha]$ refers to the free graded $\mathbb{Z}_2$-algebra with one generator α, which is of degree 1 ($|\alpha| = 1$).

Let $k = 2r$, $r \geq 1$, $n > 0$, $n \equiv r + 1 \pmod 2$, and $m = 2n + k$. Then

$$w(\xi_{n,k}) = (1 + \alpha)(1 + \beta)^{\lceil r/2 \rceil}((1 + \alpha)(1 + \alpha + \beta))^{\lfloor r/2 \rfloor} \in H^*(D_{2m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[\alpha, \beta], \qquad |\alpha| = |\beta| = 1.$$

Let $k = 2r$, $r \geq 1$, $n > 0$, $n \equiv r \pmod 2$, $m = 2n + k$. Then

$$w(\xi_{n,k}) = (1 + y)(1 + x + y + u)^{\lceil r/2 \rceil}(1 + x + y)^{\lfloor r/2 \rfloor} \in H^*(D_{2m}; \mathbb{Z}_2) \cong \mathbb{Z}_2[x, y, u]/(xy), \qquad |x| = |y| = 1, |u| = 2.$$

From these and Proposition 1.8 we obtain Theorem 1.9 by calculation. We show the most interesting example.

4.4 Corollary. *Let $s \geq 1$, then $SG_{2s,4}$ is a D_{2m}-test graph.*

Proof: We want to show that $\overline{w}_r(\xi_{n,k}) \neq 0$ for all r. If j is the homomorphism from the ring $H^*(D_{2m}; \mathbb{Z}_2)$ to its quotient by the ideal $(x - y)$, then

$$j(w(\xi_{2s,4})) = (1 + x)(1 + 2x + u)(1 + 2x) = (1 + x)(1 + u),$$
$$j(\overline{w}(\xi_{2s,4})) = (1 + x)^{-1}(1 + u)^{-1} = (1 + x)\sum_{i \geq 0} u^i,$$

i.e. $j(\overline{w}_{2\ell})(\xi_{2s,4})) = u^\ell \neq 0$, $j(\overline{w}_{2\ell+1}(\xi_{2s,4})) = xu^\ell \neq 0$. □

4.5 Remark. Since the homomorphism j is actually induced by the inclusion of a cyclic subgroup $\Gamma \subset D_{2m}$, the graph $SG_{2s,4}$ is a Γ-test graph for that Γ.

5 Stable Kneser graphs and Borsuk graphs

We investigate the relationship between Borsuk graphs (Definition 1.6) of spheres and stable Kneser graphs. In Section 2 we have constructed a map $\mathbb{S}(W_{n,k}) \to_{C_2\times D_{2m}} |\mathrm{Hom}(K_2, SG_{n,k})|$ via a poset map

$$\mathcal{L}(C^{m,k+1}) \to_{C_2\times D_{2m}} \mathrm{Hom}(K_2, SG_{n,k})$$

which sends atoms to atoms and therefore gives rise to a graph homomorphism

$$\mathcal{L}(C^{m,k+1})^1 \to_{C_2\times D_{2m}} \mathrm{Hom}(K_2, SG_{n,k})^1 \cong_{C_2\times D_{2m}} [K_2, SG_{n,k}],$$

which in turn is adjoint to a graph homomorphism

$$K_2 \times_{C_2} \mathcal{L}(C^{m,k+1})^1 \to_{D_{2m}} SG_{n,k}.$$

Regarding $\mathcal{L}(C^{m,k+1})$ as the face poset of a cellular decomposition of $\mathbb{S}^k$, the graph $K_2 \times_{C_2} \mathcal{L}(C^{m,k+1})^1$ can be described as the graph whose vertices are the vertices of that cell complex and in which the neighbours of a vertex are those vertices which share a face with its antipodal vertex. This could be called the Borsuk graph of this C_2-cell complex and is a discrete analog of the ε-Borsuk graph defined in Definition 1.6. We will now construct a graph homomorphism in the other direction and using $B_\varepsilon(\mathbb{S}(W_{n,k}))$.

5.1 Lemma. *Let $m = 2n + k$, $S \in V(SG_{n,k})$ and $(v_i)_{0\le i<m}$ a system of vectors in $\mathbb{R}^{k+1}$ realizing the alternating oriented matroid $C^{m,k+1}$. Then $\sum_{i\in S}(-1)^i v_i \neq 0$.*

Proof: Assume that $S \subset \{0, \ldots, m-1\}$ and $\sum_{i\in S}(-1)^i v_i = 0$. Since every minimal linear dependency of the vectors v_i is of cardinality $k + 2$ and has coefficients with alternating signs (see the proof of Prop. 9.4.1 in Björner et al. (1999)) there are $j_0 < j_1 < \cdots < j_{k+1}$ with $\{j_s\} \subset S$ and $(-1)^{j_{s+1}} = (-1)^{j_s+1}$ for $0 \le s \le k$. If S is a stable set, i.e. one which does not contain consecutive elements, then this implies $m \ge 2|S| + k + 1$ and hence $|S| < n$. □

This justifies the following definition.

5.2 Definition. For $S \in V(SG_{n,k})$ let

$$v(S) := \frac{\sum_{i\in S}(-1)^i v_i}{\|\sum_{i\in S}(-1)^i v_i\|},$$

where $v_i \in \mathbb{R}^{k+1}$ is as in the proof of Theorem 2.5.

5.3 Proposition. *Let $k \ge 0$ and $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that for all $n \ge N$ the function $v\colon V(SG_{n,k}) \to \mathbb{S}^k$ is a D_{2m}-equivariant graph homomorphism*

$$SG_{n,k} \to_{D_{2m}} B_\varepsilon(\mathbb{S}(W_{n,k})).$$

Proof (sketch): Equivariance follows from the choice of the vectors v_i. For the homomorphism property, we have to show that there is an N such that $\|v(S) + v(T)\| < \varepsilon$ for all $(S, T) \in E(SG_{n,k})$ with $n \ge N$. Now for fixed k and large n the sets S and $T\sigma$ must have a small symmetric difference, if S and T are disjoint and both stable. Now, again for large n, $(-1)^i v_i$ is close to $-(-1)^{i+1}v_{i+1}$. This shows that $\sum_{i\in S}(-1)^i v_i + \sum_{i\in T}(-1)^i v_i$ will indeed be small. It remains to show that neither of the two summands can have small norm. This requires a quantitative version of Lemma 5.1 for this particular choice of (v_i), which can be obtained using some analysis. □

6 Stable Kneser graphs which are not test graphs

We want to prove Proposition 1.11, which states that the vanishing of a topological obstruction lets us indeed prove that certain stable Kneser graphs are not test graphs. We first state a purely topological result and then turn the existence of the vector bundle guaranteed by it into the existence of a graph with lower chromatic number than expected.

6.1 Proposition. *Let ξ be a vector bundle over an r-dimensional finite simplicial complex X and $\overline{w}_r(\xi) = 0$. If $r = 1$ or $r \equiv 0 \pmod 2$, then there is an $(r-1)$-dimensional vector bundle η over X such that $\xi \oplus \eta$ is trivial.* □

6.2 Proposition. *Let $k \geq 0$, $r > 0$. Then there is an $\varepsilon > 0$ such that the following holds.*

Let Γ be a finite group which acts from the right on $\mathbb{R}^{k+1}$ by orthogonal maps. Let X be a finite simplicial complex with a free Γ-action and let ξ be the vector bundle $\mathbb{R}^{k+1} \times_\Gamma |X| \to |X|/\Gamma$. If there exists an $(r-1)$-dimensional vector bundle η over $|X|/\Gamma$ such that $\xi \oplus \eta$ is trivial, then there is a Γ-invariant subdivision Y of X such that

$$\chi(B_\varepsilon(\mathbb{S}^k) \times_\Gamma Y^1) < k + 2 + r.$$

Proof: We choose a covering $(A_i)_{i=0,\dots,k+r}$ of $\mathbb{S}^{k+r-1}$ by closed subsets such that no A_i contains a pair of antipodal points. Let $D := \min_i \operatorname{dist}(A_i, -A_i) > 0$, $0 < \varepsilon < D$ and $\varepsilon' := D - \varepsilon$.

Since Γ acts by orthogonal maps, the bundle ξ is a Euclidean vector bundle. The bundle η can be made into a Euclidean vector bundle, and the $r + l$ linear independent sections of $\xi \oplus \eta$ which define the trivialization can be made orthogonal using Gram-Schmidt. Therefore there is a continuous map $E(\xi) = \mathbb{R}^{k+1} \times_\Gamma |X| \to \mathbb{R}^{k+r}$ such that the restriction to each fibre of ξ is a linear isometry. Denoting the space of linear isometries from $\mathbb{R}^{k+1}$ to $\mathbb{R}^{k+r}$ by $\operatorname{Iso}(\mathbb{R}^{k+1}, \mathbb{R}^{k+r})$ and viewing it as a Γ-space via the action on $\mathbb{R}^{k+1}$, this is equivalent to the existence of an equivariant continuous map

$$f \colon |X| \to_\Gamma \operatorname{Iso}(\mathbb{R}^{k+1}, \mathbb{R}^{k+r}).$$

We let Y be a subdivision of X such that for all pairs y, y' of neighbouring vertices of Y we have $\|f(y) - f(y')\| < \varepsilon'$, where $\|\bullet\|$ denotes the operator norm. We define

$$\begin{aligned} c \colon \mathbb{S}^k \times V(Y) &\to \{0, \dots, k+r\}, \\ (v, y) &\mapsto \min\{i \colon f(y)(v)\}. \end{aligned}$$

Now if $v, v' \in \mathbb{S}^k$, $\|v + v'\| < \varepsilon$, and $y, y' \in V(Y)$, $\{y, y'\} \in Y$, then

$$\operatorname{dist}(A_{c(v,y)}, -A_{c(v',y')}) \leq \|f(y)(v) - (-f(y')(v'))\| \leq \|f(y) - f(y')\| + \|v - (-v')\| < \varepsilon' + \varepsilon = D$$

and hence $c(v, y) \neq c(v', y')$. This shows that the function c is a graph homomorphism $B_\varepsilon(\mathbb{S}^k) \times Y^1 \to K_{k+r+1}$. Since for $\gamma \in \Gamma$ we have $f(\gamma y)(v) = (\gamma f)(v) = f(v\gamma)$, we have $c(v, \gamma y) = c(v\gamma, y)$, and $[(v, y)] \mapsto c(v, y)$ defines a $(k+1+r)$-colouring of $B_\varepsilon(\mathbb{S}^k) \times_\Gamma Y^1$. □

Putting everything together yields the following more explicit version of Proposition 1.11.

6.3 Proposition. *Let $k, r \geq 1$ and $r = 1$ or $r \equiv 0 \pmod 2$. Then there is an $N \geq 2$ such that for all $n \geq N$ with $\overline{w}_r(\xi_{n,k}) = 0$ there is a graph G such that $\operatorname{Hom}(SG_{n,k}, G)$ is $(r-1)$-connected and $\chi(G) < k + 2 + r$.*

Proof: Given k and r we choose $\varepsilon > 0$ as in Proposition 6.2 and N as in Proposition 5.3. Now given $n \geq N$, $m = 2n + k$, there is an equivariant graph homomorphism $SG_{n,k} \to_{D_{2m}} B_\varepsilon(W_{n,k})$ by Proposition 5.3. Now if $\overline{w}_r(\xi_{n,k}) = 0$ then by Proposition 6.1 and Proposition 6.2 there is a D_{2m}-invariant triangulation Y of $E_r D_{2m}$ such that $\chi(B_\varepsilon(W_{n,k}) \times_{D_{2m}} Y^1) < k + 2 + r$ and therefore $\chi(SG_{n,k} \times_{D_{2m}} Y^1) < k + r + 2$. Since stable Kneser graphs are vertex critical with respect to the chromatic number by a theorem of Schrijver (1978) and $\mathrm{Aut}(SG_{n,k}) = D_{2m}$ for $n > 1$ by a theorem of Braun (2010), we can invoke Theorem 3.3 to conclude the proof. $\square$

Calculations using Proposition 4.3 now show that Proposition 6.3 is applicable in the cases needed for Theorem 1.12.

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Generalized triangulations, pipe dreams, and simplicial spheres

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Abstract. We exhibit a canonical connection between maximal $(0,1)$-fillings of a moon polyomino avoiding north-east chains of a given length and reduced pipe dreams of a certain permutation. Following this approach we show that the simplicial complex of such maximal fillings is a vertex-decomposable and thus a shellable sphere. In particular, this implies a positivity result for Schubert polynomials. For Ferrers shapes, we moreover construct a bijection to maximal fillings avoiding south-east chains of the same length which specializes to a bijection between k-triangulations of the n-gon and k-fans of Dyck paths. Using this, we translate a conjectured cyclic sieving phenomenon for k-triangulations with rotation to k-flagged tableaux with promotion.

Résumé. Nous décrivons un lien canonique entre les $(0,1)$-remplissages maximaux d'un polyomino-lune évitant les chaînes Nord-Est d'une longueur donnée, et les "pipe dreams" réduits d'une certaine permutation. En suivant cette approche nous montrons que le complexe simplicial de tels remplissages maximaux est une sphère "vertex-decomposable" et donc "shellable". En particulier, cela entraîne un résultat de positivité sur les polynômes de Schubert. De plus, nous construisons, dans le cas des diagrammes de Ferrers, une bijection vers les remplissages maximaux évitant les chaînes Sud-Est de même longueur, qui se spécialise en une bijection entre les k-triangulations d'un n-gone et les k-faisceaux de chemins de Dyck. A l'aide de celle-ci, nous traduisons une instance conjecturale du phénomène de tamis cyclique pour les k-triangulations avec rotation dans le cadre des tableaux k-marqués avec promotion.

Keywords: k-triangulation, enumerative combinatorics, pipe dream, fans of Dyck paths, flagged Schur function, Schubert polynomial, Edelman-Greene insertion

1 Introduction

Fix positive integers n and k such that $2k < n$. A k**-triangulation** of a convex n-gon is a maximal collection of diagonals in the n-gon such that no $k+1$ diagonals mutually cross. A k**-fan of Dyck paths** of length 2ℓ is a collection of k Dyck paths from $(0,0)$ to (ℓ,ℓ) which do not cross (although they may share edges).

The following theorem is the first main result in this article. It extends results by S. Elizalde [Eli07] and C. Nicolás [Nic09].

Theorem 1.1 *There is an explicit bijection between k-triangulations of a convex n-gon and k-fans of Dyck paths of length $2(n-2k)$.*

A **north-east chain** of length ℓ in a Ferrers shape λ is a sequence of ℓ boxes in λ such that every box in the sequence is strictly north and strictly east of the preceding one, and for which the smallest rectangle containing all boxes in the sequence is also contained in λ. A k**-north-east filling** of λ is a $(0,1)$-filling which does not contain any north-east chain of 1's of length $k+1$, and in which the number of 1's is maximal. As usual, we identify a $(0,1)$-filling with its set of boxes filled with 1's and draw them by marking its set of boxes by $+$'s. See Figure 1(a) for an example. The set of all k-north-east fillings of λ is denoted by $\mathcal{F}_{NE}(\lambda,k)$. **South-east chains**, k**-south-east fillings** and $\mathcal{F}_{SE}(\lambda,k)$ are defined similarly.

It is well known that k-triangulations of the n-gon can be seen as k-north-east fillings of the staircase shape $(n-1,\ldots,2,1)$, and furthermore, k-fans of Dyck paths of length $2(n-2k)$ can be seen as k-south-east fillings of the same staircase (see e.g. [Kra06, Rub06]). Thus, the second main theorem is a clear extension of the first. It answers a questions raised by C. Krattenthaler in [Kra06].

Theorem 1.2 *Let λ be a Ferrers shape and let k be a positive integer. There is an explicit bijection between k-north-east and k-south-east fillings of λ.*

The constructed bijection goes through two intermediate objects, namely through pipe dreams and flagged tableaux, both arising in the theory of Schubert polynomials. The third main theorem is a central step in the proof of Theorem 1.2 and it concerns the connection between north-east chains and reduced pipe dreams.

Theorem 1.3 *Let λ be a Ferrers shape and let k be a positive integer. There exists a canonical bijection between k-north-east fillings of λ and reduced pipe dreams of a permutation depending on λ and k.*

This bijection will be described in Section 2. A variation of the argument gives the following generalization to moon polyominoes as defined in Section 2.2.

Theorem 1.4 *Let M be a moon polyomino and let k be a positive integer. Then there exists a canonical bijection between k-north-east fillings of M and reduced pipe dreams (of a given permutation) living inside M.*

We will use the construction to obtain new properties and simple proofs for known properties of k-north-east fillings and of k-triangulations. In particular, we obtain the following corollaries.

Corollary 1.5 *The simplicial complex with facets being k-north-east fillings of a moon polyomino M is the join of a vertex-decomposable, triangulated sphere with a full simplex. In particular, it is shellable and Cohen-Macaulay.*

Corollary 1.6 *Let S be a stack polyomino and λ the Ferrers shape obtained from S be properly rearranging its columns. Let σ and τ be the associated permutations. Then the difference*

$$\mathfrak{S}_\sigma(x_1,x_2,\ldots)-\mathfrak{S}_\tau(x_1,x_2,\ldots)$$

of Schubert polynomials is monomial positive.

The bijection for k-triangulations has the additional property that the cyclic action given by rotation of the n-gon corresponds to a promotion-like operation on flagged tableaux and thus transforms a conjectured cyclic sieving phenomenon (CSP) into the context of k-flagged tableaux.

Conjecture 1.7 *Let $\mathcal{FT}(\lambda,k)$ be the set of k-flagged tableaux and let ρ be the promotion-like cyclic action on $\mathcal{FT}(\lambda,k)$. The triple*

$$\Big(\mathcal{FT}(\lambda,k),\langle\rho\rangle,F(q)\Big),$$

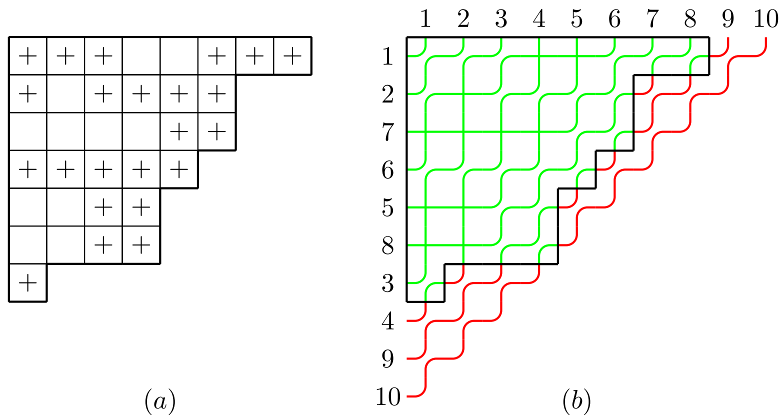

Fig. 1: A 2-north-east filling of $\lambda = (8, 6, 6, 5, 4, 4, 1)$ and its associated reduced pipe dream.

exhibits the CSP, where

$$F(q) := \prod_{1 \leq i \leq j < n-2k} \frac{[i+j+2k]_q}{[i+j]_q}$$

is a natural q-analogue of the cardinality of $\mathcal{F}_{NE}(\lambda, k)$.

2 From north-east fillings to pipe dreams

In this section we exhibit a connection between k-north-east fillings of Ferrers shapes as well as of stack and moon polyominoes on the one hand and reduced pipe dreams on the other. This generalizes a construction by the second author for k-triangulations [Stu10].

Reduced pipe dreams (or rc-graphs) were introduced by S. Fomin and A. Kirillov in [FK96] (see also [BB93] and [KM05, Section 1.4]). They play a central role in the combinatorics of Schubert polynomials of A. Lascoux and M.-P. Schützenberger. A **pipe dream** of size n is a filling of the staircase shape $(n-1, \ldots, 2, 1)$ where each box contains two crossing pipes ┼ or two turning pipes ╯╭. See Figure 1(b) for an example. A pipe dream is identified with its set of boxes containing two crossing pipes ┼. The permutation $\pi(D)$ of a pipe dream D is obtained by following the pipes starting from the top and going all the way to the left, and then reading $\pi(D)$ on the left from top to bottom in one line notation. For example, the permutation of the pipe dream in Figure 1(b) is $[1, 2, 7, 6, 5, 8, 3, 4, 9, 10]$. A pipe dream is **reduced** if two pipes cross at most once. We say that a pipe dream lives inside a set M of boxes in the staircase shape if all its crossings are contained in M. For a given permutation π and a set M of boxes, denote the set of reduced pipe dreams for π by $\mathcal{RP}(\pi)$ and the set of reduced pipe dreams for π which live inside M by $\mathcal{RP}(\pi, M)$.

2.1 A bijection between north-east fillings and reduced pipe dreams

Starting with a k-north-east filling of λ, one obtains a pipe dream by replacing every 1 by two turning pipes and every 0 by two crossing pipes. Afterwards, λ is embedded into the smallest staircase containing it, and all boxes in the staircase outside of λ are replaced by turning pipes. In other words, a k-north-east

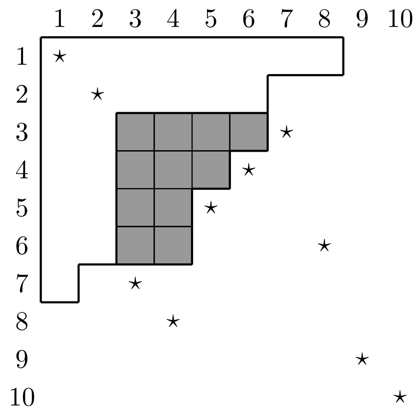

Fig. 2: $\sigma_2(\lambda) = [1, 2, 7, 6, 5, 8, 3, 4, 9, 10]$ for $\lambda = (8, 6, 6, 5, 4, 4, 1)$.

filling of λ and its associated pipe dream are complementary $(0, 1)$-fillings of λ when both are identified with their sets of boxes. For example, the $+$'s in the pipe dream in Figure 1(b) and the marked boxes in (a) are complementary $(0, 1)$-fillings of λ. The pieces in boxes outside of λ are drawn in the pipe dream in red whereas pieces within λ are drawn in green. We call this identification between k-north-east fillings of λ and reduced pipe dreams **complementary map**.

For a permutation $\sigma \in \mathcal{S}_n$, define its (**Rothe**) **diagram** (see [Man01, Section 2.1]) to be the set of boxes in the staircase shape given by

$$\mathrm{D}(\sigma) := \big\{(i, \sigma_j) : i < j, \sigma_i > \sigma_j\big\}.$$

For example, the diagram of $\sigma_2(\lambda)$ in Figure 2 is given by the shaded area. Clearly, the number of boxes in $\mathrm{D}(\sigma)$ equals the **length** of σ, i.e., the minimal number of simple transpositions needed to write σ. A permutation is called **dominant** if its diagram is a Ferrers shape containing the box $(1, 1)$. By construction, different permutations in $\mathcal{S}_n$ have different shapes and one can obtain every Ferrers shape in this way for some n. Thus, starting with a Ferrers shape λ, let $\sigma(\lambda)$ be the unique dominant permutation $\sigma \in \mathcal{S}_n$ for which $\mathrm{D}(\sigma) = \lambda$, where n is given by the smallest staircase shape containing λ. Moreover, define $\sigma_k(\lambda)$ to be

$$\mathbf{1}_k \times \tau := [1, 2, \ldots, k, \tau_1 + k, \ldots, \tau_n + k] \in \mathcal{S}_{n+k}$$

where $\tau = \sigma(\mu)$ and μ is obtained from λ by removing its first k rows and columns. Graphically, this means that $\sigma_k(\lambda)$ is obtained by removing the first k columns and rows from $\sigma(\lambda)$. Note that the north-west corner of $\sigma_k(\lambda)$ remains in box $(k + 1, k + 1)$. See Figure 2 for $\sigma_2(\lambda)$ with λ as in Figure 1. The following theorem is a more precise reformulation of Theorem 1.3.

Theorem 2.1 *Let λ be a Ferrers shape and let $\sigma = \sigma_k(\lambda)$. The complementary map from k-north-east fillings to pipe dreams is a bijection between $\mathcal{F}_{NE}(\lambda, k)$ and $\mathcal{RP}(\sigma)$.*

2.2 Generalizations to moon polyominoes

The results in the previous section can be partially generalized to moon polyominoes which were studied by J. Jonsson in [Jon05]. A polyomino M (i.e., a set of boxes in the positive integer quadrant) is called

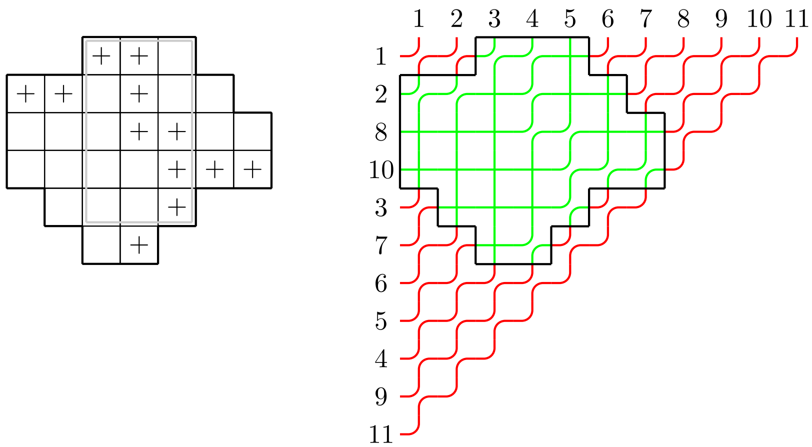

Fig. 3: A 1-north-east filling of a moon polyomino and its associated pipe dream.

convex if for any two boxes in M lying in the same row or column, all boxes in between are also contained in M. Moreover, M is called **intersection-free** if for any two columns (or equivalently, rows) of M, one is contained in the other. A polyomino is called a **moon polyomino** if it is convex and intersection-free.

Without loss of generality we consider always moon polyominoes which are north-west justified, namely, they contain boxes both in the first row and in the first column. Observe that Ferrers shapes are special types of moon polyominoes. A k-filling of a moon polyomino is defined exactly in the same way as for a Ferrers shape. See Figure 3 for an example.

To connect k-north-east fillings of a moon polyomino M and pipe dreams of a certain permutation $\sigma = \sigma_k(M)$, we must relate maximal fillings of M which do not contain a $(k+1)$-north-east chain in one of its maximal rectangles and maximal fillings of the staircase $(n-1,\dots,2,1)$ which do not contain a north-east chain of length $r_\sigma(p,q)+1$ in any rectangle $[p]\times[q]$. Define $\sigma_k(M)$ as follows: for a maximal rectangle R in M of width and height both strictly larger than k, let $(a+1,b+1)$ and (i,j) be its north-west and south-east corner boxes. Mark the box $(i+b, j+a)$ with $a+b+k$. $\sigma_k(M)$ is the permutation with this collection as its essential set. This means that the diagram $\mathrm{D}(\sigma)$ of σ has $(i+b,j+a)$ as a south-east corner with labels $r_\sigma(i+b,j+a) = a+b+k$. Using [Man01, 2.2.8], it is easy to see that this construction is well defined. Note that maximal rectangles of width or height less than or equal to k cannot contain any north-east chain of length larger than k and thus do not contribute to the essential set of the corresponding permutation. For example, the moon polyomino M in Figure 3(a) has maximal rectangles $(a+1,b+1)-(i,j)$ given by

$$\begin{gathered}(1,3)-(5,5),\quad (1,3)-(6,4),\\ (3,1)-(4,7),\quad (2,2)-(5,5),\\ (2,1)-(4,6),\end{gathered}$$

where the first maximal rectangle is highlighted. Thus, for $k=1$, the resulting essential set and the associated diagram can be seen in Figure 4 and the associated permutation is $\sigma_1(M) = [1,2,8,10,3,7,6,5,4,9]$. As all maximal rectangles in a Ferrers shape are of the form $[p]\times[q]$, the definition of $\sigma_k(\lambda)$ reduces in

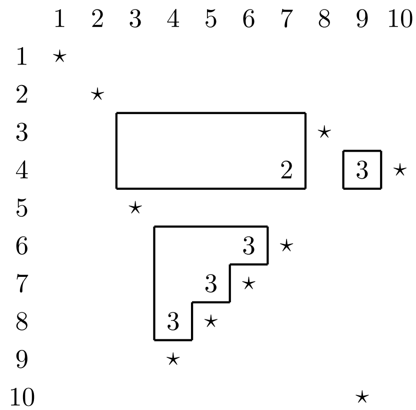

Fig. 4: The essential set and the diagram $\mathrm{D}(\sigma)$ for $\sigma = [1, 2, 8, 10, 3, 7, 6, 5, 4, 9]$.

this case to the definition given in the previous section.

The following theorem is a more precise reformulation of Theorem 1.4.

Theorem 2.2 *The complementary map from k-north-east fillings of a moon polyomino M to pipe dreams of $\sigma = \sigma_k(M)$ is a bijection between $\mathcal{F}_{NE}(M, k)$ and $\mathcal{RP}(\sigma, M)$.*

We now use this theorem together with the main theorem in [Jon05] to get new insights on pipe dreams. A **stack polyomino** is a moon polyomino where every column starts in the first row. Let S be a stack polyomino and let λ be the Ferrers shape obtained from S by properly rearranging the columns. J. Jonsson proved in [Jon05, Theorem 14] that the number of k-north-east fillings of S with a given number of $+$'s in every row equals the number of k-north-east fillings in λ with the same number of $+$'s in every row. Moreover, he conjectured that this property still holds if the stack polyomino S is replaced by a moon polyomino. Therefore, we obtain the following corollary and the conjecture for the analogous statement for moon polyominoes.

Corollary 2.3 *Let S be a stack polyomino and let λ be the associated Ferrers shape. The number of pipe dreams in $\mathcal{RP}(\sigma_k(S), S)$ with a given number of crossings in every row is equal to the number of pipe dreams in $\mathcal{RP}(\sigma_k(\lambda))$ with the same number of crossings in every row.*

2.3 The simplicial complex of north-east-fillings

We are now in position to obtain Corollary 1.5. The canonical connection between k-north-east fillings and reduced pipe dreams can be used in the same way as described in the proof of [Stu10, Corollary 1.3] for k-triangulations in this more general setting. For the necessary background on simplicial complexes and in particular on subword complexes, we refer to [KM04]. A box in a moon polyomino M is called **passive** if it is not contained in any north-east chain in M of length $k + 1$. Let $\Delta(M, k)$ be the simplicial complex with vertices being the collection of boxes in M, and with facets being k-north-east fillings of M.

Corollary 2.4 $\Delta(M, k)$ *is the join of a vertex-decomposable, triangulated sphere and a full simplex of dimension $i - 1$, where i equals the number of passive boxes in M. In particular, it is shellable and Cohen-Macauley.*

$$\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
 & & & 4 & 5 & & & & \\
\hline
 & 3 & & & & & \\
\cline{1-6}
3 & 4 & 5 & 6 & & \\
\cline{1-6}
 & & & & \\
\cline{1-5}
5 & 6 & & \\
\cline{1-4}
6 & 7 & & \\
\cline{1-4}
 \\
\cline{1-1}
\end{array}
\qquad
\begin{pmatrix}
1 & 1 & 2 & 3 & 3 & 3 & 3 & 5 & 5 & 6 & 6 \\
5 & 4 & 3 & 6 & 5 & 4 & 3 & 6 & 5 & 7 & 6
\end{pmatrix}$$

$$\begin{array}{|c|c|c|c|}
\hline
3 & 4 & 5 & 6 \\
\hline
4 & 5 & 6 \\
\cline{1-3}
5 & 6 \\
\cline{1-2}
6 & 7 \\
\cline{1-2}
\end{array}
\qquad
\begin{array}{|c|c|c|c|}
\hline
1 & 1 & 2 & 3 \\
\hline
3 & 3 & 3 \\
\cline{1-3}
5 & 5 \\
\cline{1-2}
6 & 6 \\
\cline{1-2}
\end{array}$$

Fig. 5: Labelling of the crossing boxes in the pipe dream in Figure 1(b), the corresponding compatible sequence, and its insertion and recording tableau.

2.4 A mutation-like operation on pipe dreams

Generalizing the notion in the previous section, one can define a pure simplicial complex $\Delta(\sigma)$ for any $\sigma \in \mathcal{S}_n$ by defining the facets as the complements in the staircase of reduced pipe dreams in $\mathcal{RP}(\sigma)$ (see [KM04]). Using the property that two pipes in a reduced pipe dream D cross at most once, one can define a mutation-like operation on facets of $\Delta(\sigma)$ as follows. One can mutate the facet $F(D)$ of $\Delta(\sigma)$ associated to D at a vertex b if the two pipes in D which touch in b cross somewhere else. In other words, one can mutate $F(D)$ at a vertex b if the starting points $i < j$ of the two pipes in D which touch in b form an inversion of σ. The mutation of $F(D)$ at such a vertex b is then defined to be the facet $F(D')$ for the reduced pipe dream D' such that

(i) the two turning pipes in b are replaced in D' by two crossing pipes,

(ii) the unique crossing b' of those two pipes is replaced in D' by two turning pipes.

By construction, the pipe dream $D' = (D \cup b) \setminus b'$ is again in $\mathcal{RP}(\sigma)$ and thus its complement $F(D') = (F(D) \setminus b) \cup b'$ forms another facet of $\Delta(\sigma)$.

3 From pipe dreams to south-east fillings

In this section we describe a bijection between pipe dreams for $\sigma_k(\lambda)$ and k-south east fillings of λ, for a Ferrers shape λ. For the sake of readability, we do this construction in several steps.

3.1 From pipe dreams to flagged tableaux

Define a k**-flagged tableau** as a semistandard tableau in which the entries in the i-th row are smaller than or equal to $i+k$, and denote the set of k-flagged tableaux of shape λ by $\mathcal{FT}(\lambda, k)$. These were introduced by M. Wachs [Wac85] in the study of **flagged Schur functions**, thus the choice of terminology. We now present a bijection between the set $\mathcal{RP}(\sigma)$ of reduced pipe dreams of $\sigma = \sigma_k(\lambda)$ and the set $\mathcal{FT}(\mu, k)$ of k-flagged tableaux of shape $\mu = \mathrm{D}(\sigma)$.

For a reduced pipe dream $D \in \mathcal{RP}(\sigma)$ with σ being of length ℓ, define the **reading biword** to be the $2 \times \ell$ array by reading $\binom{i}{i+j-1}$ for every crossing box (i, j) in D row by row from east to west and from north to south. See Figure 5 for an example. It is known (and easy to check) that this gives a bijection between $\mathcal{RP}(\sigma)$ and the set of **compatible sequences** $\mathcal{CS}(\sigma)$, defined by S. Billey, W. Jockush and R. Stanley in [BJS93] as the set of all $2 \times \ell$ arrays of the form $\binom{a_1,\ldots,a_\ell}{b_1,\ldots,b_\ell}$ satisfying the following properties:

1. $a_1 \leq a_2 \leq \cdots \leq a_\ell$,
2. if $a_i = a_{i+1}$, then $b_i > b_{i+1}$,
3. $b_1 b_2 \cdots b_\ell$ is a reduced word for σ, where i denotes the simple transposition $s_i = (i, i+1)$, and
4. $a_i \leq b_i$.

One can see from the definition that a compatible sequence t for σ can be written as the concatenation $t = t_1 \cdots t_m$, where $t_i = \binom{i^{|w_i|}}{w_i}$, and w_i is decreasing. Observe that σ fixes all $j \leq k$ and thus, every letter in w_i is larger than or equal to $\max(i, k)$.

Define a map $\mathcal{CS}(\sigma) \to \mathcal{FT}(\mu, k)$ as follows. Let $t \in \mathcal{CS}(\sigma)$ be a compatible sequence for σ. Insert the letters of the word formed by the bottom row of t using column Edelman–Greene insertion [EG87] into a tableau, while recording the corresponding letters from the first row. This produces an insertion tableau $P(t)$ and a recording tableau $Q(t)$. The image in $\mathcal{FT}(\mu, k)$ is now defined to be $Q(t)$.

Theorem 3.1 *Let $\sigma = \sigma_k(\lambda)$ for a Ferrers shape λ, and let $\mu = \mathrm{D}(\sigma)$. The map sending D in $\mathcal{RP}(\sigma)$ to the recording tableau of the reading biword of D is a bijection between $\mathcal{RP}(\sigma)$ and $\mathcal{FT}(\mu, k)$.*

An example of the bijection can be seen in Figure 5.

3.2 A cyclic action on flagged tableaux

In this subsection we define a cyclic action on k-flagged tableaux. The **flagged promotion** $\rho(Q)$ of a k-flagged tableau Q is defined as follows.

(i) Delete all the instances of the letter 1,
(ii) apply jeu de taquin to the remaining entries,
(iii) subtract 1 from all the entries,
(iv) label each empty box on row i with $i + k$.

One can easily see that $\rho(Q)$ is indeed a k-flagged tableau, since the empty boxes after step (iii) must form a horizontal strip, which means there is at most one empty box per column. Furthermore, as every box gets moved at most up by one row, and at the end one subtracts 1 from all the entries, the tableau obtained after step (iii) is k-flagged as well. The argument is finalized with the observation that if one adds a horizontal strip in which every box gets added its maximum possible value, the tableau is still k-flagged, since the row-weakness is assured by the maximality of the value of the entries on each row, and the column-strictness is assured by the fact that the entries in row $i-1$ are all strictly less than the maximal value on row i.

3.3 From flagged tableaux to fans of paths and south-east fillings

We proceed as in [FK97] to obtain a reverse plane partition of height k from a k-flagged tableau. Let λ be a Ferrers shape and let $\mu = \mathrm{D}(\sigma_k(\lambda))$. Since every entry in row i of a k-flagged tableau of shape μ is less than or equal to $i + k$ and greater than or equal to i (as the tableau is semistandard), one can subtract i from all the entries in row i, for all rows, and obtain a reverse plane partition of height k and shape μ, or equivalently, a k-fan of noncrossing north-east paths inside μ. To obtain a bijection between k-flagged

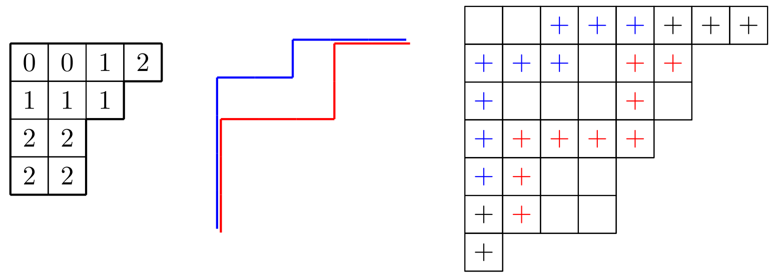

Fig. 6: Reverse plane partition corresponding to Figure 5 and its corresponding 2-fan of paths.

tableaux of shape μ and the set $\mathcal{F}_{SE}(\lambda, k)$ of k-south-east fillings of the shape λ, one lifts the i-th path from the bottom by $i-1$ and turns it into a path of $+$'s inside λ. See Figure 6 for an example; the red marks come from the red path, the blue from the blue path, and the additional black marks are contained in any 2-south-east filling.

Putting the described bijections together, we obtain Theorem 1.2.

Theorem 3.2 *Let λ be a Ferrers shape. The composition of the described maps is a bijection between $\mathcal{F}_{NE}(\lambda)$ and $\mathcal{F}_{SE}(\lambda)$.*

As mentioned in the introduction, k-triangulations of the n-gon can be seen as k-north-east fillings of the staircase shape $(n-1, \ldots, 2, 1)$, and k-fans of Dyck paths of length $2(n-2k)$ can be seen as k-south-east fillings of the same staircase (see e.g. [Kra06, Rub06]). Thus, we obtain Theorem 1.1. See Figure 7 for an example.

Corollary 3.3 *In the case where λ is the staircase shape $(n-1, \ldots, 2, 1)$, the described map is a bijection between k-triangulations of the n-gon and k-fans of noncrossing Dyck paths of length $2(n-2k)$.*

4 Properties of north-east fillings and k-triangulations

Using Theorem 2.2, we obtain several properties of k-north-east fillings of moon polyominoes and of Ferrers shapes and k-triangulations in particular. Some of them where already known while others where only conjectured.

The first property was proved in the case of stack polyominoes by J. Jonsson in [Jon05, Theorem 10]. It follows immediately from Theorem 2.1.

Corollary 4.1 *Every k-north-east filling of a moon polyomino M contains i many boxes where i equals the total number of boxes in M minus the length of $\sigma_k(M)$. In particular i equals the number of boxes in the first k rows and columns in the case of Ferrers shapes.*

The second property is part of the main theorem in [PS09, Theorem 1.4(i)] and concerns the star property as described as well in [Stu10]; for the notion used here, we refer as well to the latter.

Corollary 4.2 *Every k-triangulation of the n-gon consists of exactly $n-2k$ k-stars.*

Using the description of mutations for k-triangulations in Section 2.4, one can also describe the mutation of a facet in the simplicial complex

$$\Delta_{n,k} := \Delta(\mathbf{1}_k \times [n-2k, \ldots, 1]).$$

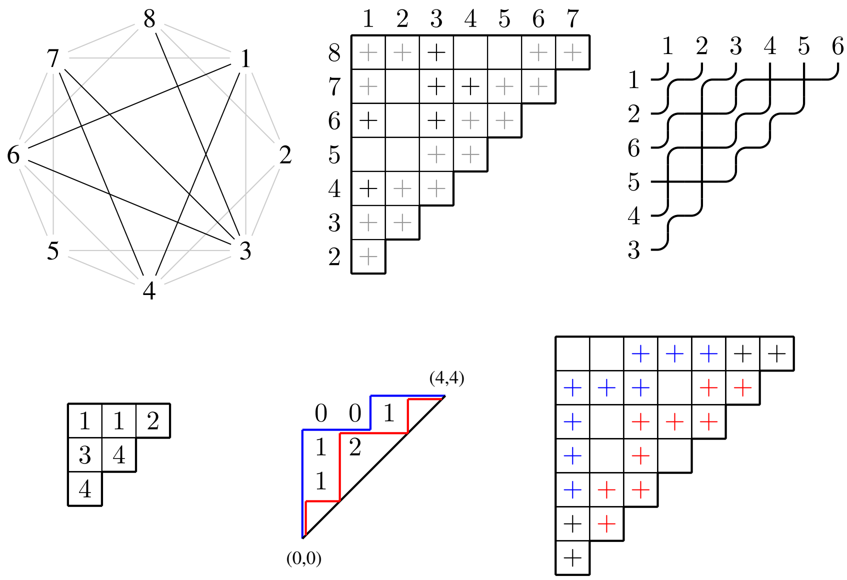

Fig. 7: An example of all the steps in the bijection: a 2-triangulation of the 8-gon, the 2-north-east filling of $(7, \ldots, 1)$, the pipe dream for $[1, 2, 6, 5, 4, 3]$, the 2-flagged tableau of shape $(3, 2, 1)$, the 2-fan of Dyck paths of length 8, and finally the 2-south-east filling of $(7, \ldots, 1)$.

This mutation corresponds to removing a diagonal in a k-triangulation and replacing it by the unique other diagonal which gives a k-triangulation. This operation is called *flip* in [PS09, Theorem 1.4(iii)].

Corollary 4.3 *A facet F in the simplicial complex $\Delta_{n,k}$ can be mutated at any vertex $d = (i, j) \in F$ for which $k < |i - j| < n - k$.*

The next property of the constructed bijection is a refined counting of k-triangulations, as conjectured by C. Nicolas [Nic09].

Theorem 4.4 *The number of k-triangulations of a convex n-gon having degree d in a given vertex is given by the determinantal expression*

$$\det \begin{pmatrix} \mathrm{Cat}_{n-2k} & \cdots & \mathrm{Cat}_{n-k-2} & B^k_{n-k-1}(d) \\ \vdots & \ddots & \vdots & \vdots \\ \mathrm{Cat}_{n-k-1} & \cdots & \mathrm{Cat}_{n-3} & B^k_{n-2}(d) \end{pmatrix},$$

where Cat_ℓ is the usual Catalan number, and where $B^k_\ell(d) = \frac{2k+d-3}{\ell}\binom{2\ell-2k-d+2}{\ell-1}$.

4.1 Rotation of the n-gon and a CSP for flagged tableaux

There is a natural cyclic action ρ on k-triangulations given by rotating the vertex labels in the n-gon counterclockwise. The following conjecture is due to V. Reiner [Rei09].

Conjecture 4.5 (V. Reiner) *Let λ be the staircase shape $(n-1,\ldots,2,1)$ and let k be a positive integer. The triple*

$$\Big(\mathcal{F}_{NE}(\lambda,k),\langle\rho\rangle,F(q)\Big)$$

exhibits the cyclic sieving phenomenon (CSP) as described in [RSW04].

We can describe the cyclic action on k-triangulations induced by rotation in terms of the cyclic action on flagged tableaux as defined in Section 3.2.

Theorem 4.6 *The constructed bijection maps the cyclic action on k-triangulations to the cyclic action given by flagged promotion on flagged tableaux.*

Using this connection, we obtain the following corollary.

Corollary 4.7 *Conjecture 1.7 is equivalent to Conjecture 4.5.*

5 Schubert polynomials and geometry of Schubert varieties

In this section we use the results about moon polyominoes in Section 2.2 to obtain new properties of Schubert polynomials. It was shown in [FK96] that Schubert polynomials are a generating series for pipe dreams, more precisely, for a permutation σ,

$$\mathfrak{S}_\sigma(x_1,\ldots,x_n)=\sum_{D\in\mathcal{RP}(\sigma)}\prod_{(i,j)\in D}x_i.$$

We obtain the following theorem and thus Corollary 1.6.

Theorem 5.1 *Let S be a stack polyomino, let λ be the associated Ferrers shape and let k be a positive integer. Then*

$$\mathfrak{S}_{\sigma_k(S)}(x_1,x_2,\ldots)-\mathfrak{S}_{\sigma_k(\lambda)}(x_1,x_2,\ldots)$$

is monomial positive. In particular, $\mathfrak{S}_{\sigma_k(S)}(1,1,\ldots)$ is greater than or equal to the number of k-flagged tableaux of shape λ.

Let B be the subgroup of $GL(n)$ consisting of upper triangular matrices. For each $\sigma\in\mathcal{S}_n$ there is a subvariety X_σ of $GL(n)/B$ known as the Schubert variety (see, e.g., [Man01, Chapter 3]). A. Knutson and E. Miller showed in [KM05] that the multiplicity of the point X_e at the Schubert variety X_σ is given by the specialization $\mathfrak{S}_\sigma(1,1,\ldots)$. Thus, from the previous theorem, one obtains the following corollary.

Corollary 5.2 *The multiplicity of the point X_e at the Schubert variety $X_{\sigma_k(S)}$ is greater than or equal to the multiplicity of the point X_e at the Schubert variety $S_{\sigma_k(\lambda)}$.*

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A q-analog of Ljunggren's binomial congruence

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Abstract. We prove a q-analog of a classical binomial congruence due to Ljunggren which states that

$$\binom{ap}{bp} \equiv \binom{a}{b}$$

modulo p^3 for primes $p \geqslant 5$. This congruence subsumes and builds on earlier congruences by Babbage, Wolstenholme and Glaisher for which we recall existing q-analogs. Our congruence generalizes an earlier result of Clark.

Résumé. Nous démontrons un q-analogue d'une congruence binomiale classique de Ljunggren qui stipule:

$$\binom{ap}{bp} \equiv \binom{a}{b}$$

modulo p^3 pour p premier tel que $p \geqslant 5$. Cette congruence s'inspire d'une précédente congruence prouvée par Babbage, Wolstenholme et Glaisher pour laquelle nous présentons les q-analogues existantes. Notre congruence généralise un précédent résultat de Clark.

Keywords: q-analogs, binomial coefficients, binomial congruence

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1 Introduction and notation

Recently, q-analogs of classical congruences have been studied by several authors including (Cla95), (And99), (SP07), (Pan07), (CP08), (Dil08). Here, we consider the classical congruence

$$\binom{ap}{bp} \equiv \binom{a}{b} \mod p^3 \tag{1}$$

which holds true for primes $p \geqslant 5$. This also appears as Problem 1.6 (d) in (Sta97). Congruence (1) was proved in 1952 by Ljunggren, see (Gra97), and subsequently generalized by Jacobsthal, see Remark 6.

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Let $[n]_q := 1 + q + \dots q^{n-1}$, $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$ and

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}$$

denote the usual q-analogs of numbers, factorials and binomial coefficients respectively. Observe that $[n]_1 = n$ so that in the case $q = 1$ we recover the usual factorials and binomial coefficients as well. Also, recall that the q-binomial coefficients are polynomials in q with nonnegative integer coefficients. An introduction to these q-analogs can be found in (Sta97).

We establish the following q-analog of (1):

Theorem 1 *For primes $p \geqslant 5$ and nonnegative integers a, b,*

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} - \binom{a}{b+1}\binom{b+1}{2}\frac{p^2-1}{12}(q^p-1)^2 \mod [p]_q^3. \tag{2}$$

The congruence (2) and similar ones to follow are to be understood over the ring of polynomials in q with integer coefficients. We remark that $p^2 - 1$ is divisible by 12 for all primes $p \geqslant 5$.

Observe that (2) is indeed a q-analog of (1): as $q \to 1$ we recover (1).

Example 2 Choosing $p = 13$, $a = 2$, and $b = 1$, we have

$$\binom{26}{13}_q = 1 + q^{169} - 14(q^{13}-1)^2 + (1 + q + \dots + q^{12})^3 f(q)$$

where $f(q) = 14 - 41q + 41q^2 - \dots + q^{132}$ is an irreducible polynomial with integer coefficients. Upon setting $q = 1$, we obtain $\binom{26}{13} \equiv 2$ modulo 13^3.

Since our treatment very much parallels the classical case, we give a brief history of the congruence (1) in the next section before turning to the proof of Theorem 1.

2 A bit of history

A classical result of Wilson states that $(n-1)! + 1$ is divisible by n if and only if n is a prime number. “In attempting to discover some analogous expression which should be divisible by n^2, whenever n is a prime, but not divisible if n is a composite number”, (Bab19), Babbage is led to the congruence

$$\binom{2p-1}{p-1} \equiv 1 \mod p^2 \tag{3}$$

for primes $p \geqslant 3$. In 1862 Wolstenholme, (Wol62), discovered (3) to hold modulo p^3, “for several cases, in testing numerically a result of certain investigations, and after some trouble succeeded in proving it to hold universally” for $p \geqslant 5$. To this end, he proves the fractional congruences

$$\sum_{i=1}^{p-1} \frac{1}{i} \equiv 0 \mod p^2, \tag{4}$$

$$\sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 0 \mod p \tag{5}$$

for primes $p \geqslant 5$. Using (4) and (5) he then extends Babbage's congruence (3) to hold modulo p^3:

$$\binom{2p-1}{p-1} \equiv 1 \mod p^3 \tag{6}$$

for all primes $p \geqslant 5$. Note that (6) can be rewritten as $\binom{2p}{p} \equiv 2$ modulo p^3. The further generalization of (6) to (1), according to (Gra97), was found by Ljunggren in 1952. The case $b = 1$ of (1) was obtained by Glaisher, (Gla00), in 1900.

In fact, Wolstenholme's congruence (6) is central to the further generalization (1). This is just as true when considering the q-analogs of these congruences as we will see here in Lemma 5.

A q-analog of the congruence of Babbage has been found by Clark (Cla95) who proved that

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} \mod [p]_q^2. \tag{7}$$

We generalize this congruence to obtain the q-analog (2) of Ljunggren's congruence (1). A result similar to (7) has also been given by Andrews in (And99).

Our proof of the q-analog proceeds very closely to the history just outlined. Besides the q-analog (7) of Babbage's congruence (3) we will employ q-analogs of Wolstenholme's harmonic congruences (4) and (5) which were recently supplied by Shi and Pan, (SP07):

Theorem 3 *For primes $p \geqslant 5$,*

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q} \equiv -\frac{p-1}{2}(q-1) + \frac{p^2-1}{24}(q-1)^2[p]_q \mod [p]_q^2 \tag{8}$$

as well as

$$\sum_{i=1}^{p-1} \frac{1}{[i]_q^2} \equiv -\frac{(p-1)(p-5)}{12}(q-1)^2 \mod [p]_q. \tag{9}$$

This generalizes an earlier result (And99) of Andrews.

3 A q-analog of Ljunggren's congruence

In the classical case, the typical proof of Ljunggren's congruence (1) starts with the Chu-Vandermonde identity which has the following well-known q-analog:

Theorem 4

$$\binom{m+n}{k}_q = \sum_j \binom{m}{j}_q \binom{n}{k-j}_q q^{j(n-k+j)}.$$

We are now in a position to prove the q-analog of (1).

Proof of Theorem 1: As in (Cla95) we start with the identity

$$\binom{ap}{bp}_q = \sum_{c_1+\ldots+c_a=bp} \binom{p}{c_1}_q \binom{p}{c_2}_q \cdots \binom{p}{c_a}_q q^{p\sum_{1\leqslant i\leqslant a}(i-1)c_i - \sum_{1\leqslant i<j\leqslant a} c_i c_j} \tag{10}$$

which follows inductively from the q-analog of the Chu-Vandermonde identity given in Theorem 4. The summands which are not divisible by $[p]_q^2$ correspond to the c_i taking only the values 0 and p. Since each such summand is determined by the indices $1 \leqslant j_1 < j_2 < \ldots < j_b \leqslant a$ for which $c_i = p$, the total contribution of these terms is

$$\sum_{1 \leqslant j_1 < \ldots < j_b \leqslant a} q^{p^2 \sum_{k=1}^{b} (j_k - 1) - p^2 \binom{b}{2}} = \sum_{0 \leqslant i_1 \leqslant \ldots \leqslant i_b \leqslant a-b} q^{p^2 \sum_{k=1}^{b} i_k} = \binom{a}{b}_{q^{p^2}}.$$

This completes the proof of (7) given in (Cla95).

To obtain (2) we now consider those summands in (10) which are divisible by $[p]_q^2$ but not divisible by $[p]_q^3$. These correspond to all but two of the c_i taking values 0 or p. More precisely, such a summand is determined by indices $1 \leqslant j_1 < j_2 < \ldots < j_b < j_{b+1} \leqslant a$, two subindices $1 \leqslant k < \ell \leqslant b+1$, and $1 \leqslant d \leqslant p-1$ such that

$$c_i = \begin{cases} d \text{ for } i = j_k, \\ p - d \text{ for } i = j_\ell, \\ p \text{ for } i \in \{j_1, \ldots, j_{b+1}\} \backslash \{j_k, j_\ell\}, \\ 0 \text{ for } i \notin \{j_1, \ldots, j_{b+1}\}. \end{cases}$$

For each fixed choice of the j_i and k, ℓ the contribution of the corresponding summands is

$$\sum_{d=1}^{p-1} \binom{p}{d}_q \binom{p}{p-d}_q q^{p \sum_{1 \leqslant i \leqslant a} (i-1) c_i - \sum_{1 \leqslant i < j \leqslant a} c_i c_j}$$

which, using that $q^p \equiv 1$ modulo $[p]_q$, reduces modulo $[p]_q^3$ to

$$\sum_{d=1}^{p-1} \binom{p}{d}_q \binom{p}{p-d}_q q^{d^2} = \binom{2p}{p}_q - [2]_{q^{p^2}}.$$

We conclude that

$$\binom{ap}{bp}_q \equiv \binom{a}{b}_{q^{p^2}} + \binom{a}{b+1} \binom{b+1}{2} \left(\binom{2p}{p}_q - [2]_{q^{p^2}} \right) \mod [p]_q^3. \tag{11}$$

The general result therefore follows from the special case $a = 2$, $b = 1$ which is separately proved next. □

4 A q-analog of Wolstenholme's congruence

We have thus shown that, as in the classical case, the congruence (2) can be reduced, via (11), to the case $a = 2$, $b = 1$. The next result therefore is a q-analog of Wolstenholme's congruence (6).

Lemma 5 *For primes $p \geqslant 5$,*

$$\binom{2p}{p}_q \equiv [2]_{q^{p^2}} - \frac{p^2 - 1}{12} (q^p - 1)^2 \mod [p]_q^3.$$

Proof: Using that $[an]_q = [a]_{q^n} [n]_q$ and $[n+m]_q = [n]_q + q^n [m]_q$ we compute

$$\binom{2p}{p}_q = \frac{[2p]_q [2p-1]_q \cdots [p+1]_q}{[p]_q [p-1]_q \cdots [1]_q} = \frac{[2]_{q^p}}{[p-1]_q!} \prod_{k=1}^{p-1} \left([p]_q + q^p [p-k]_q \right)$$

which modulo $[p]_q^3$ reduces to (note that $[p-1]_q!$ is relatively prime to $[p]_q^3$)

$$[2]_{q^p} \left(q^{(p-1)p} + q^{(p-2)p} \sum_{1 \leqslant i \leqslant p-1} \frac{[p]_q}{[i]_q} + q^{(p-3)p} \sum_{1 \leqslant i < j \leqslant p-1} \frac{[p]_q [p]_q}{[i]_q [j]_q} \right). \tag{12}$$

Combining the results (8) and (9) of Shi and Pan, (SP07), given in Theorem 3, we deduce that for primes $p \geqslant 5$,

$$\sum_{1 \leqslant i < j \leqslant p-1} \frac{1}{[i]_q [j]_q} \equiv \frac{(p-1)(p-2)}{6} (q-1)^2 \mod [p]_q. \tag{13}$$

Together with (8) this allows us to rewrite (12) modulo $[p]_q^3$ as

$$\begin{aligned} [2]_{q^p} \Bigg(q^{(p-1)p} + q^{(p-2)p} \left(-\frac{p-1}{2}(q^p - 1) + \frac{p^2-1}{24}(q^p-1)^2 \right) + \\ + q^{(p-3)p} \frac{(p-1)(p-2)}{6} (q^p-1)^2 \Bigg). \end{aligned}$$

Using the binomial expansion

$$q^{mp} = ((q^p - 1) + 1)^m = \sum_k \binom{m}{k} (q^p - 1)^k$$

to reduce the terms q^{mp} as well as $[2]_{q^p} = 1 + q^p$ modulo the appropriate power of $[p]_q$ we obtain

$$\binom{2p}{p}_q \equiv 2 + p(q^p - 1) + \frac{(p-1)(5p-1)}{12}(q^p-1)^2 \mod [p]_q^3.$$

Since

$$[2]_{q^{p^2}} \equiv 2 + p(q^p - 1) + \frac{(p-1)p}{2}(q^p - 1)^2 \mod [p]_q^3$$

the result follows. □

Remark 6 Jacobsthal, see (Gra97), generalized the congruence (1) to hold modulo p^{3+r} where r is the p-adic valuation of

$$ab(a-b)\binom{a}{b} = 2a\binom{a}{b+1}\binom{b+1}{2}.$$

It would be interesting to see if this generalization has a nice analog in the q-world.

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Representations on Hessenberg Varieties and Young's Rule

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Abstract. We combinatorially construct the complex cohomology (equivariant and ordinary) of a family of algebraic varieties called regular semisimple Hessenberg varieties. This construction is purely in terms of the Bruhat order on the symmetric group. From this a representation of the symmetric group on the cohomology is defined. This representation generalizes work of Procesi, Stembridge and Tymoczko. Here a partial answer to an open question of Tymoczko is provided in our two main result. The first states, when the variety has multiple connected components, this representation is made up by inducing through a parabolic subgroup of the symmetric group. Using this, our second result obtains, for a special family of varieties, an explicit formula for this representation via Young's rule, giving the multiplicity of the irreducible representations in terms of the classical Kostka numbers.

Résumé. Nous construisons la cohomologie complexe (équivariante et ordinaire) d'une famille de variétés algébriques appelées variétés régulières semisimples de Hessenberg. Cette construction utilise exclusivement l'ordre de Bruhat sur le groupe symétrique, et on en déduit une représentation du groupe symétrique sur la cohomologie. Cette représentation généralise des résultats de Procesi, Stembridge et Tymoczko. Nous offrons ici une réponse partielle à une question de Tymoczko grâce à nos deux résultats principaux. Le premier déclare que lorsque la variété a plusieurs composantes connexes, cette représentation s'obtient par induction à travers un sous-groupe parabolique du groupe symétrique. Nous en déduisons notre deuxième résultat qui fournit, pour une famille spéciale de variétés, une formule explicite pour cette représentation par la règle de Young, et donne ainsi la multiplicité des représentations irréductibles en termes des nombres classiques de Kostka.

Keywords: Bruhat order, combinatorial representation theory, flag varieties, Young's rule

1 Introduction

In this paper we study a representation of the symmetric group on the complex cohomology (ordinary and equivariant) of a family of algebraic varieties called *regular semisimple Hessenberg varieties*. This representation exposes connections between the combinatorics of the symmetric group, the geometry of the varieties, and representation theory. Also, this representation generalizes representations in work of Procesi [P], Stembridge [St], and Tymoczko [T3]. Procesi and Stembridge studied this same representation in the case when the variety is the toric variety associated to the Coxeter complex in type A_n using ordinary cohomology. Tymoczko studied it when the variety is the flag variety using equivariant cohomology.

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These varieties are examples of regular semisimple Hessenberg varieties. In each case, a decomposition of the representation into irreducible representations are known. Here we partially answer an open question of Tymoczko [T3]. The question Tymoczko asks is; "can one obtain similar decompositions of this representation for the cohomology of all regular semisimple Hessenberg varieties?"

We answer this question in two cases. The first states, when the Hessenberg variety is disconnected, the representation is a particular induced representation through a parabolic (*i.e. Young*) subgroup [Theorem 4.10]. The second result provides an explicit irreducible decomposition of the representation for parabolic Hessenberg varieties [Definition 4.11] via Young's rule (see [JK, Chapter 2]). We give this decomposition in terms of classical Kostka numbers [Theorem 4.15].

Our approach is combinatorial. We study these varieties via a combinatorial graph called the moment graph. These graphs are subgraphs of the Bruhat graph for S_n [BB, Chapter 2]. This allows us to use tools from Coxeter groups, for example the Bruhat order, parabolic subgroups, and minimal coset representatives. In fact, the results of this abstract can be extended to other Coxeter groups in other Lie types. We chose to remain in type A to make the connection to combinatorics (*e.g.* partitions and Kostka numbers) more evident.

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2 Hessenberg varieties.

Fix $G = GL_n(\mathbb{C})$ and let B be the subgroup of upper-triangular matrices. Let the respective Lie algebras be $\mathfrak{g}$ and $\mathfrak{b}$. The *flag variety* is the homogenous space G/B. It is known to be a smooth complex projective variety [H, Section 21]. *Hessenberg varieties* are a family of subvarieties of the flag variety parametrized by an element $\mathsf{X} \in \mathfrak{g}$ and a function $h : \{1, 2, \cdots, n\} \longmapsto \{1, 2, \cdots, n\}$ [dMPS],[T2].

Hessenberg varieties are the space of ordered bases which represent X in a form (*i.e. Hessenberg form*) under which numerical algorithms can be efficiently performed [dMPS]. Hessenberg varieties have also found applications in other fields, including combinatorics, geometry, and representation theory. Well-known examples include the flag variety, the toric variety associated to the Coxeter complex and the Springer variety [P],[Sp],[St],[T3].

When X is semisimple with distinct eigenvalues (*i.e. regular semisimple*) the Hessenberg varieties are smooth [dMPS, Theorem 6]. In this case, we construct the equivariant cohomology combinatorially using GKM theory [GKM],[KT],[T1]. This approach presents the equivariant cohomology using a combinatorial graph. These graphs are subgraphs of the Bruhat graph (see Figure 3 for examples).

Definition 2.1 *An h-*function *is a non-decreasing function $h : \{1, 2, \cdots, n\} \longmapsto \{1, 2, \cdots, n\}$ such that $h(i) \geq i$ for each i. Let $E_{i,j}$ be the $n \times n$ matrix which is one in entry $\{i, j\}$ and zero elsewhere. A* Hessenberg space *is the complex vector space spanned by the $E_{i,j}$ such that $h(j) \geq i$ for each pair i, j. Hessenberg spaces will be denoted H_h and h-functions $h = h(1)h(2)\cdots h(n)$.*

Example 2.2 *We write H_h as the set of matrices with $*$'s in positions $\{i, j\}$ such that $h(j) \geq i$ and 0 in the other positions. For example, let $h(i) = i + 1$ for $i = 1, 2, \cdots, n - 1$. Then H_h is the complex span of the matrices $E_{i,j}$ where $i \leq j + 1$.*

$$H_{23455} = \left\{ \begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix} \right\}$$

Fig. 1: A Hessenberg space.

Definition 2.3 *Fix* $\mathsf{X} \in \mathfrak{g}$ *and an* h*-function. A* Hessenberg variety *is the subvariety of* G/B *given by*

$$\mathsf{X}_h := \{gB \in G/B \mid g^{-1}\mathsf{X}g \in H_h\}.$$

This is a closed set in G/B and hence a projective variety.

Example 2.4 *Examples of Hessenberg varieties.*

(1.) When $h(i) = n$ *for all* i*, then* $H_h = \mathfrak{g}$*. For any* $\mathsf{X} \in \mathfrak{g}$ *the Hessenberg variety* X_h *is the flag variety* G/B*.*

(2.) When $h(i) = i$ *for all* i*, then* $H_h = \mathfrak{b}$*. When* $\mathsf{X} \in \mathfrak{g}$ *is nilpotent the Hessenberg variety is the Springer variety [Sp].*

(3.) Consider the given by h*-function,* $h(i) = i + 1$ *for* $i = 1, 2, \cdots, n-1$*. When* $\mathsf{X} \in \mathfrak{g}$ *is regular semisimple (see Definition 3.5) the Hessenberg variety is the toric variety associated with the Coxeter complex of* S_n *[dMPS, Theorem 11]. There is a symmetric group representation the cohomology of this variety induced by the action of* S_n *on the root system. Procesi and Stembridge give two approaches to decomposing the cohomology into irreducible representations[P],[St]. In Section 4 we study a generalization of this representation on other regular semisimple Hessenberg varieties.*

3 GKM Theory for regular semisimple Hessenberg varieties.

Our goal is to study Hessenberg varieties via a representation of the symmetric group on the (ordinary and equivariant) cohomology with complex coefficients. Among the results we prove is that when the Hessenberg variety has multiple connected components the representation is a permutation representation.

We follow a combinatorial approach. We construct the equivariant cohomology using GKM theory [GKM]. This gives a presentation of both the equivariant and ordinary cohomology of regular semisimple Hessenberg varieties in terms of the Bruhat order of the symmetric group.

This combinatorial viewpoint is a primary advantage of using GKM theory. The representation we will study uses equivariant cohomology in an essential way. In fact, it is not obvious that the ordinary homology carries an S_n-representation without the GKM approach (see Remark 4.6).

Here we introduce GKM theory as needed for our purposes. More thorough background can be found in either the source [GKM] or the expository article [T1]. For more examples, GKM theory has been used to calculate equivariant cohomology for the Grassmannians, Schubert varieties and the flag variety [KT],[T3],[T4].

Let X be a smooth complex projective variety which carries an action of a complex algebraic torus T. GKM theory allows us to view $H_T^*(X, \mathbb{C})$, the equivariant cohomology of X, as a free module over $\mathbb{C}[t_1, t_2, \cdots t_n]$, the polynomial algebra over the Lie algebra of T.

In order to apply GKM theory X must have finitely many of both the T-fixed points and one-dimensional T-orbits. Denote the fixed points by X^T and the one-dimensional orbits by $\{\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k\}$. If X satisfies these finiteness conditions, we call it a *GKM space*. In fact, the category of GKM spaces is larger than smooth category. All a GKM space must satisfy is a technical condition called *equivariant formality* [GKM, Section 1.2].

The fixed points and one dimensional orbits form a one-skeleton in X relative to the T-action. We construct a combinatorial graph, called the *moment graph*, from this one-skeleton. The vertices are the fixed points and there is an edge between fixed points if they are the two fixed points in the closure of an orbit, $\mathcal{O}_i$. Each edge is labeled by α_i, the $\mathbb{C}[t_1, \cdots t_n]$-annihilator of $\mathfrak{t}_i$, the Lie algebra of the point-wise stabilizer of $\mathcal{O}_i$. Further, we direct the edge from $x \xmapsto{\alpha_i} y$ if and only if the torus acts on the tangent space $T_x(\overline{\mathcal{O}_i})$ with weight α_i and on $T_y(\overline{O_i})$ with weight $-\alpha_i$. [GKM, Section 7.1].

Definition 3.1 *Let X be a GKM space. The* moment graph of X *is the graph* $\Gamma(X) = (\mathcal{V}, \mathcal{E})$ *where the vertices are* $\mathcal{V} = X^T$ *and the labeled edges are*

$$\mathcal{E} = \{x \xmapsto{\alpha_i} y \mid x, y \in \overline{\mathcal{O}_i} \cap X^T \text{ and } \alpha_i \text{ is the annihilator of } \mathfrak{t}_i.\}.$$

All GKM spaces have a localization map $H_T^*(X, \mathbb{C}) \longrightarrow \mathbb{C}[t_1, \cdots, t_n]^{\oplus X^T}$ that is in fact injective. This is what permits the GKM presentation of the equivariant cohomology.

Theorem 3.2 (GKM presentation [GKM]) *Let X be a GKM space with moment graph $\Gamma(X)$. Then the equivariant cohomology of X is given by*

$$H_T^*(X, \mathbb{C}) := \left\{ p : X^T \longmapsto \mathbb{C}[t_1, \cdots, t_n] \;\middle|\; \text{for } x \xmapsto{\alpha_i} y, \text{ the difference } p_x - p_y \in \alpha_i \right\}.$$

The forgetful map which sets each $t_i = 0$ relates the equivariant cohomology to the ordinary cohomology $H^*(X, \mathbb{C})$.

Proposition 3.3 *There is a ring isomorphism*

$$H^*(X, \mathbb{C}) \cong \frac{H_T^*(X, \mathbb{C})}{\langle t_1, \cdots, t_n \rangle H_T^*(X, \mathbb{C})}.$$

One consequence is that any free $\mathbb{C}[t_1, \cdots, t_n]$-module basis for equivariant cohomology can be viewed as $\mathbb{C}$-vector space basis for ordinary cohomology by scalar restriction. Finally, the next result is helpful later (see Section 3.2).

Lemma 3.4 *Let X be GKM space with moment graph $\Gamma(X)$. Then the connected components of X are GKM spaces whose moment graph the connected graph components of* $\Gamma(X)$.

3.1 Regular semisimple Hessenberg varieties.

Here we give the GKM presentation for a family of Hessenberg varieties. Recall that S_n embeds into G as the subgroup of permutation matrices. This identification is key to exposing the connection between the geometry of the Hessenberg varieties and the combinatorics of the symmetric group.

Definition 3.5 *A semisimple element* $\mathsf{X} \in \mathfrak{g}$ *is* regular *when its eigenvalues are all distinct.*

Fix a regular semisimple $\mathsf{X} \in \mathfrak{g}$. In fact, because conjugation (change of basis) is an isomorphism of varieties we may assume X is diagonal with distinct diagonal entries. De Mari-Procesi-Shayman proved that the Hessenberg varieties of regular semisimple X are smooth [dMPS, Theorem 6]. Therefore, for X_h to be a GKM space we only need an appropriate torus action.

Let T be the subgroup of diagonal matrices in G. The action of T on G/B given by $t \cdot gB = tgB$ restricts to X_h because $T = C_G(\mathsf{X})$. With respect to this torus action G/B is a GKM space, and so is X_h.

Definition 3.6 *Let* $w \in S_n$. *The* inversions of *w is the set* $\mathrm{inv}(v) := \left\{ i < j \mid w^{-1}(i) > w^{-1}(j) \right\}$.

Proposition 3.7 *Every regular semisimple Hessenberg variety is a GKM space, with moment graph* $\Gamma(\mathsf{X}_h) = (\mathcal{V}, \mathcal{E})$ *given by:*

$$\begin{array}{rcl} \mathcal{V} & := & \{wB \mid w \in S_n\} \\ \mathcal{E} & := & \left\{ w'B \overset{t_i - t_j}{\longmapsto} wB \mid w' = (ij)w,\ i < j \in \mathrm{inv}(w) \text{ and } w^{-1}(i) \leq h(w^{-1}(j)) \right\} \end{array}$$

Furthermore, the equivariant cohomology is

$$H^*_T(\mathsf{X}_h, \mathbb{C}) := \left\{ p : S_n \longmapsto \mathbb{C}[t_1, \cdots, t_n] \mid \text{for } w'B \overset{t_i - t_j}{\longmapsto} wB \text{ the difference } p_w - p_{w'} \in \langle t_i - t_j \rangle \right\}.$$

Proof outline: J. Carrell proved the moment graph for G/B is as above for the function $h(i) = n$ for all i [C]. Any regular semisimple Hessenberg variety X_h carries the same torus action as G/B. Therefore the moment graph of X_h is a subgraph of that of G/B. It is then a direct calculation to show which orbits from G/B are contained in X_h. □

Example 3.8 *To determine whether a tuple of polynomials is a class one must check that the difference of polynomials at adjacent vertices are multiples of the* $t_i - t_j$. *Figure 2 provides an example in* $H^*_T(\mathsf{X}_{223}, \mathbb{C})$), *where one is a class and the other not.*

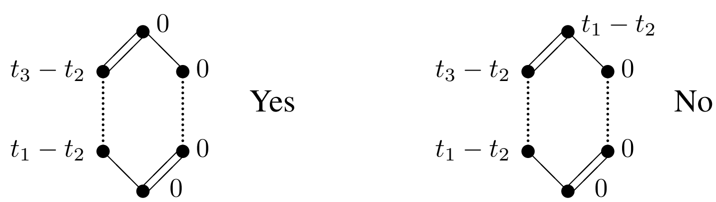

Fig. 2: Examples of the equivariant condition.

The relation $w'B \overset{t_i - t_j}{\longmapsto} wB$ if $i < j \in \mathrm{inv}(w)$ is exactly the condition defining the cover relation for the Bruhat order on the symmetric group [BB, Section 2.1]. When the Hessenberg variety is the flag variety the moment graph is a labeled Bruhat graph. Hence, the moment graph of any regular semisimple Hessenberg variety is a labeled subgraph of the Bruhat graph. This fact is the underpinning of the combinatorics of this paper. Because of it we are able to use facts about the Bruhat order, parabolic subgroups, and minimal coset representatives as we study these varieties.

Frequently, we will suppress the coset notation and write edges as $w' \overset{t_i - t_j}{\longmapsto} w$. The conditions (Proposition 3.7) defining an edge are cumbersome. This can remedied by calculating a right-hand version of the edge condition.

Corollary 3.9 *There is an edge between w and w' if and only if $w' = w(i'j')$ for $i' < j'$ and $h(i') \geq j'$.*

Proof: If $w' \overset{t_i - t_j}{\longmapsto} w$ is an edge then $(ij)w = ww^{-1}(ij)w = w(w^{-1}(j)w^{-1}(i))$. Direct calculation shows $i' = w^{-1}(j) < w^{-1}(i) = j'$ satisfies the condition. □

This right-hand condition makes it easier to construct the moment graph, but we must still use the left-hand version to calculate the classes $p \in H_T^*(\mathsf{X}_h, \mathbb{C})$. The transpositions $(i'j')$ satisfying the conditions of this corollary will be called *right-transpositions*.

Example 3.10 *For $GL_3(\mathbb{C})$, up to homeomorphism, there are four regular semisimple Hessenberg varieties. In this case, we can identify these varieties. When $h = 123$, the variety is the fixed point set of the torus. For $h = 223$, the variety is three disjoint copies of $\mathbb{CP}^1$. For $h = 233$, the variety is the toric variety associated with the decomposition of the Coxeter complex (see also Example 2.4). Lastly, $h = 333$ is the flag variety $GL_3(\mathbb{C})/B$ (see also Example 2.4).*

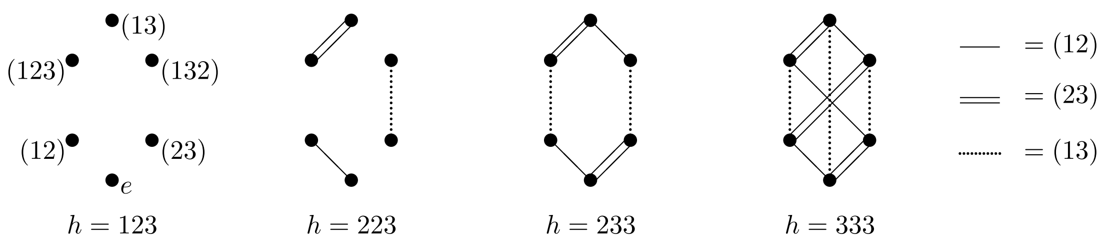

Fig. 3: Moment graphs of regular semisimple Hessenberg varieties.

3.2 Disconnected regular semisimple Hessenberg varieties.

In this section we give a criterion for when X_h is disconnected and give an explicit decomposition of X_h into homeomorphic connected components. Fix a regular semisimple element X and an h-function h. For the rest of the paper, *all* Hessenberg varieties will be regular semisimple.

Definition 3.11 *The* parabolic subgroup *of* X_h *is* $W_h := \langle (ij) \in S_n \mid h(i) \geq j \rangle$.

The name *parabolic subgroup* comes from the theory of Coxeter groups. In fact, W_h is generated by the *simple transpositions* $(i\, i+1)$ such that $h(i) \geq i+1$, and so is a parabolic subgroup of the Coxeter group S_n. These subgroups also arise in the representation theory of the symmetric group, where they are called *Young subgroups*. It will be important to know that up to isomorphism, these subgroups have the form $S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$ for $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ a partition of n,

Since the parabolic subgroup is generated by the *simple* transposition satisfying the right-hand condition they do not uniquely determine the Hessenberg variety. For example, the Hessenberg varieties X_{2334} and X_{3334} both have parabolic subgroup isomorphic to $S_{(3,1)}$. Despite this parabolic subgroups are useful when describing the moment graph.

Every permutation $u \in W_h$ can be written as a product of right-transpositions. In terms of the moment graph, this product corresponds to a path between the identity and u. Hence, this subgroup generates the graph component containing the identity. By Lemma 3.4, this graph component corresponds to another GKM space. We will call this space the *identity component* and denote it X_h°.

Lemma 3.12 *Fix an h-function h. There are $[S_n : W_h]$ connected components of regular semisimple Hessenberg variety.*

Proof: From Lemma 3.4 we know it is sufficient to count the graph components of the moment graph. Now by Corollary 3.9, the permutations $u, v \in S_n$ are in the same graph component if and only if there is a $w \in W_h$ such that $u = vw$. This is equivalent to $uW_h = vW_h$. Hence, there are $[S_n : W_h]$ connected components of X_h. □

This lemma shows that the components of the moment graph respect the right multiplication structure of W_h. Hence, the moment graph is composed of isomorphic graph components indexed by the left cosets of W_h. This combinatorial property hints that when the Hessenberg variety is disconnected then connected components are homeomorphic. This is true.

Proposition 3.13 *For a disconnected Hessenberg variety, $\mathsf{X}_h^\circ \cong \prod_{i=1}^k \mathsf{X}_h^{\lambda_i}$, where the $\mathsf{X}_h^{\lambda_i}$ are regular semisimple Hessenberg varieties in $GL_{\lambda_i}(\mathbb{C})$.*

Proof outline: Suppose $W_h \cong S_\lambda$ for $\lambda = (\lambda_1, \cdots, \lambda_k)$ a partition of n. For $gB \in \mathsf{X}_h^\circ$ the product $g^{-1}\mathsf{X}g$ is mapped to the subspace of H_h consisting of block diagonal matrices with dimensions given by λ. This gives k independent conditions each of which describes a Hessenberg variety in $GL_{\lambda_i}(\mathbb{C})$. □

Corollary 3.14 *Let X_h be a disconnected regular semisimple Hessenberg variety. Then the connected components of X_h are all homeomorphic.*

Proof outline: Let $\mathcal{J}$ be a connected component of X_h and pick $u \in \mathcal{J}^T$. Consider the map given by left translation by u^{-1}. This maps $\mathcal{J}$ homeomorphically onto $(u^{-1}\mathsf{X}u)_h^\circ$ *i.e.* the identity component of the Hessenberg variety corresponding to the regular semisimple element $u^{-1}\mathsf{X}u$ and the same h-function h. By Proposition 3.13, $\mathcal{J}$ is homeomorphic to X_h°. □

4 A representation of the symmetric group.

In this section we define a representation of the symmetric group on the equivariant cohomology of regular semisimple Hessenberg varieties. Geometrically this representation is defined from an action of S_n on the the moment graph. Here we review necessary background on the representation theory of the symmetric group. A classic source for these results is [JK].

The representation ring of S_n has two free $\mathbb{Z}$-bases, both parameterized by partitions of n. The first basis is the collection of irreducible representations V^λ with characters χ^λ. The second basis consists of *permutation representations* P^λ with character ψ^λ. These are obtained from the left multiplication action of S_n on the cosets of S_λ, *i.e.* the cosets of *Young subgroups*. Equivalently, each P^λ is constructed by inducing the trivial representation of S_λ to S_n. We will be interested in decomposing the P^μ in terms of the V^λ.

Definition 4.1 *The* lexicographic order *on partitions of n is given by*

$$\lambda > \mu \text{ if the first non-vanishing } \lambda_i - \mu_i \text{ is positive.}$$

Definition 4.2 *The* Kostka numbers $K_{\mu\lambda}$ *are the number of semistandard Young tableaux of shape μ and weight λ.*

Consider the matrix with Kostka numbers as entries. If we order the rows and columns (*i.e.* partitions) in lexicographic order we obtain a transition matrix between permutation representations and irreducible representations. This is classically known as *Young's Rule*.

Proposition 4.3 (Young's Rule [JK]) *Let τ^λ denote the character of the trivial representation for the Young subgroup S_λ. Then the induced character* $\mathrm{Ind}_{S_\lambda}^{S_n}\tau^\lambda$ *is given by*

$$\mathrm{Ind}_{S_\lambda}^{S_n}\tau^\lambda := \psi^\lambda = \chi^\lambda + \sum_{\mu > \lambda} K_{\mu\lambda}\chi^\mu.$$

4.1 The representation on the cohomology.

The symmetric group acts on $\mathbb{C}[t_1, \cdots, t_n]$ by permuting variables. That is for $w \in S_n$ and a polynomial $f(t_1, \cdots, t_n)$, the action of w on $f(t_1, \cdots, t_n)$ is given by

$$w * f(t_1, t_2, \cdots, t_n) = f(t_{w(1)}, t_{w(2)}, \cdots, t_{w(n)}).$$

This action is a ring automorphism of $\mathbb{C}[t_1, \cdots, t_n]$. We can extend this to a representation of S_n on $H_T^*(\mathsf{X}_h)$.

Proposition 4.4 *Let X_h be a regular semisimple Hessenberg variety. There is a representation of S_n on $H_T^*(\mathsf{X}_h, \mathbb{C})$ given by*

$$(w \cdot p)_u = w * p_{w^{-1}u}.$$

Further, using the isomorphism of Proposition 3.3 this is a representation on $H^*(\mathsf{X}_h, \mathbb{C})$.

We defer the proof until after the next example. This action is easiest understood when $w = (ij)$ is a transposition. In this case, the action of (ij) interchanges the polynomials across edges in the moment graph for G/B labeled $t_i - t_j$, and permutes the variables. For example Figure 4 shows the action of (12) on a class in $H_T^*(\mathsf{X}_{233}, \mathbb{C})$.

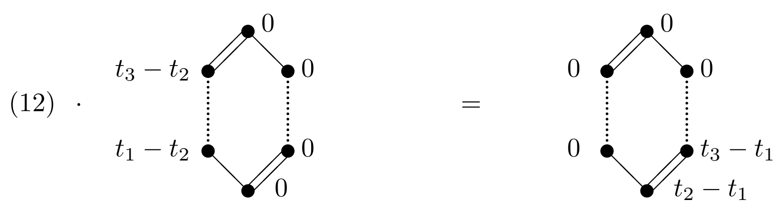

Fig. 4: The action on an equivariant class.

The next Lemma shows that S_n acts on the moment graph of regular semisimple Hessenberg varieties. It is key to proving Proposition 4.4.

Lemma 4.5 *Let $v \in S_n$ and $w' \stackrel{t_i - t_j}{\longmapsto} w$ be an edge in the moment graph. The map $\varphi_v : S_n \to S_n$ defined by $\varphi_v(w) = v^{-1}w$ sends the edge $w' \stackrel{t_i - t_j}{\longmapsto} w$ to*

- *$v^{-1}w' \longmapsto v^{-1}w$ with label $t_{v^{-1}(i)} - t_{v^{-1}(j)}$ if $i < j \notin \mathrm{inv}(v)$.*
- *$v^{-1}w \longmapsto v^{-1}w'$ with label $t_{v^{-1}(j)} - t_{v^{-1}(j)}$ if $i < j \in \mathrm{inv}(v)$.*

Proof: The proof in both cases is similar. We prove it when $i < j \in \mathrm{inv}(v)$. We have

$$v^{-1}w = v^{-1}(ij)w' = (v^{-1}(i)v^{-1}(j))v^{-1}w'$$

and $v^{-1}(i) > v^{-1}(j)$. Therefore, we must check that $v^{-1}(j) < v^{-1}(i) \in \mathrm{inv}(v^{-1}w')$ and that

$$(v^{-1}w')^{-1}(v^{-1}(j)) \leq h((v^{-1}w')^{-1}(v^{-1}(i))).$$

This follows directly from the relation $(v^{-1}w')^{-1}(v^{-1}(i)) = w^{-1}(j)$ and $(v^{-1}w')^{-1}(v^{-1}(j)) = w^{-1}(i)$. $\square$

Proof of Proposition 4.4: Let $u' \stackrel{t_i - t_j}{\longmapsto} u$ be an edge and $p \in H_T^*(\mathsf{X}_h, \mathbb{C})$. We must show that $w \cdot p$ satisfies the equivariant condition, *i.e.* $(w \cdot p)_u - (w \cdot p)_{u'} \in \langle t_i - t_j \rangle$. This follows from the action on the moment graph

$$(w \cdot p)_u - (w \cdot p)_v = w * (p_{w^{-1}(u)} - p_{w^{-1}(v)}) \in w * \langle t_{w^{-1}(i)} - t_{w^{-1}(j)} \rangle = \langle t_i - t_j \rangle.$$

The second claim is immediate. $\square$

Remark 4.6 *In the case of G/B, the group S_n acts on all of G/B by left multiplication. Therefore, the representation on the cohomology is defined geometrically by this action. This is not the case for general Hessenberg varieties. For example, if $h = 233$ consider the matrices*

$$\mathsf{X} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \quad g = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad w_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Direct calculation using Definition 2.3 gives

$$g^{-1}\mathsf{X}g = \begin{pmatrix} 3 & 0 & 0 \\ -2 & 2 & 0 \\ 0 & -1 & 1 \end{pmatrix} \quad (w_{(12)}g)^{-1}\mathsf{X}(w_{(12)}g) = \begin{pmatrix} 3 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 1 & 2 \end{pmatrix}.$$

This means $gB \in \mathsf{X}_{233}$ while $w_{(12)} \cdot gB \notin \mathsf{X}_{233}$. In other words, this Hessenberg variety is not invariant under the left multiplication action of S_n, only its moment graph is. For this reason, the representation will vary as the moment graph varies, so the combinatorial approach GKM theory provides is valuable when studying this representation.

Tymoczko studied the representation on G/B using the same GKM approach we use here. She obtained a combinatorial proof that the representation on ordinary cohomology is trivial [T4]. This result is known in the literature, but the proofs rely on geometric arguments.

Theorem 4.7 (Tymoczko [T4]) *The representation on $H^*(G/B, \mathbb{C})$ decomposes into $|S_n|$ copies of the trivial representation.*

4.2 The representation on disconnected Hessenberg varieties.

Let $w_i, \cdots, w_k$ be the system of coset representatives of W_h minimal length [BB, Section 2.4]. Proposition 3.14 allows us to write X_h as the disjoint union of the translates $w_i\mathsf{X}_h^\circ$. Hence the equivariant cohomology is:

$$H_T^*(\mathsf{X}_h, \mathbb{C}) = \bigoplus_{w_i} H_T^*(w_i\mathsf{X}_h^\circ, \mathbb{C}). \tag{1}$$

Next we determine an explicit isomorphism between $H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$ and $H_T^*(w_i\mathsf{X}_h^\circ, \mathbb{C})$. This will be key to showing $H_T^*(\mathsf{X}_h, \mathbb{C})$ is the induced representation of $H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$ through W_h.

Proposition 4.8 *There is an isomorphism given by $\varphi_{w_i} : H_T^*(\mathsf{X}_h^\circ, \mathbb{C}) \to H_T^*(w_i\mathsf{X}_h^\circ, \mathbb{C})$ defined by*

$$p_u \longmapsto p_{w_i u} := w_i * p_u.$$

Proof outline: This is a direct computation using the same argument as Proposition 4.4. $\square$

With this we have descriptions of the variety X_h, the moment graph $\Gamma(\mathsf{X}_h)$, and the equivariant cohomology $H_T^*(\mathsf{X}_h, \mathbb{C})$ in terms of the analogs for the identity component X_h°. Further, from Proposition 3.13 and the the Künneth formula we have

$$H_T^*(\mathsf{X}_h^\circ, \mathbb{C}) \;\cong\; H_T^*\left(\prod_{i=1}^{k} \mathsf{X}_h^{\lambda_i}, \mathbb{C}\right) \;\cong\; \bigotimes_{i=1}^{k} H_T^*(\mathsf{X}_h^{\lambda_i}, \mathbb{C}). \tag{2}$$

Lemma 4.9 *The equivariant cohomology of X_h° is a representation of W_h.*

Proof: Let $W_h \cong S_\lambda = S_{\lambda_1} \times \cdots \times S_{\lambda_k}$. From Proposition 3.13 and Equation 2 we define the representation on $H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$ component-wise. $\square$

This leads to the first main theorem.

Theorem 4.10 *Let X_h be a disconnected Hessenberg variety with parabolic subgroup W_h. Then as representations $H_T^*(\mathsf{X}_h) = \mathrm{Ind}_{W_h}^{S_n} H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$.*

Proof: Proposition 3.14 gives that

$$H_T^*(\mathsf{X}_h, \mathbb{C}) = \bigoplus_{w_i \text{ coset reps}} H_T^*\left(w_i\mathsf{X}_h^\circ, \mathbb{C}\right),$$

and by Lemma 4.9 $H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$ is W_h-stable. It follows from Proposition 4.8 and Equation 1 that each $p \in H_T^*(\mathsf{X}_h, \mathbb{C})$ is uniquely expressed as $p = \sum_{w_i} w_i * p^i$ for some $p^i \in H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$. By definition $H_T^*(\mathsf{X}_h, \mathbb{C})$ is the induced representation $\mathrm{Ind}_{W_h}^{S_n} H_T^*(\mathsf{X}_h^\circ, \mathbb{C})$. $\square$

This result permits us to decompose the ordinary cohomology into irreducible representations when the Hessenberg variety is *parabolic*.

Definition 4.11 *Whenever the Hessenberg space H_h is a parabolic subalgebra of $\mathfrak{g}$ we call the Hessenberg variety* parabolic.

$$H_{3334} = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\} \quad H_{2334} = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \right\}$$

Fig. 5: A parabolic Hessenberg space and a non-parabolic Hessenberg space.

In other words, a Hessenberg variety is parabolic whenever H_h "forms a block-staircase" in $\mathfrak{g}$. The size of the blocks correspond to the parts of λ in $W_h \cong S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

Example 4.12 *Compare H_{3334} which is a parabolic Hessenberg, and H_{2334} which is not. They have both have parabolic subgroup isomorphic to $S_{(3,1)}$, but $E_{3,1} \notin H_{2334}$ (see Figure 5).*

Proposition 4.13 *The identity component of a parabolic Hessenberg is homeomorphic to the product $GL_{\lambda_1}(\mathbb{C})/B_{\lambda_1} \times \cdots \times GL_{\lambda_k}(\mathbb{C})/B_{\lambda_k}$, where $\lambda = (\lambda_1, \cdots, \lambda_k)$ is the partition corresponding to the group $W_h \cong S_\lambda$.*

Proof: Use Proposition 3.13 and check that each factor in the product is isomorphic to a flag variety. □

Let $\chi^{(n)}$ be the character of the trivial representation of S_n so $\tau^\lambda = \chi^{(\lambda_1)} \times \cdots \times \chi^{(\lambda_k)}$ is the trivial character of S_λ. As a corollary we obtain the following.

Corollary 4.14 *Let X_h° be the identity component of a parabolic Hessenberg. Then the W_h-representation on $H^*(\mathsf{X}_h^\circ, \mathbb{C})$ is trivial and has $|W_h|\tau^\lambda$ as its character.*

Proof: The proposition gives $\mathsf{X}_h^\circ \cong GL_{\lambda_1}(\mathbb{C})/B_{\lambda_1} \times \cdots \times GL_{\lambda_k}(\mathbb{C})/B_{\lambda_k}$. From Tymoczko's result (Theorem 4.7) and Lemma 4.9 the character is

$$|S_{\lambda_1}|\chi^{(\lambda_1)} \times \cdots \times |S_{\lambda_k}|\chi^{(\lambda_k)} = \left(\prod_{i=1}^{k} |S_{\lambda_i}| \right) \chi^{(\lambda_1)} \times \cdots \times \chi^{(\lambda_k)} = |W_h|\tau^\lambda.$$

□

Finally, we obtain our main result. From Theorem 4.10 together with Corollary 4.14 we have that $H^*(\mathsf{X}_h, \mathbb{C}) = |W_h|P^\lambda$, the permutation representation associated to $W_h \cong S_\lambda$. Using Young's rule we obtain the irreducible decomposition of the ordinary cohomology for all parabolic regular semisimple Hessenberg varieties.

Theorem 4.15 *Let X_h be a parabolic regular semisimple Hessenberg variety, with parabolic subgroup $W_h \cong S_\lambda$. The character of the representation χ^h decomposes in ordinary cohomology as*

$$\chi^h = |W_h|\chi^\lambda + \sum_{\mu > \lambda} |W_h| K_{\mu\lambda} \chi^\mu.$$

Proof: We know $H^*(\mathsf{X}_h, \mathbb{C}) = \mathrm{Ind}_{W_h}^{S_n} H^*(\mathsf{X}_h^\circ, \mathbb{C})$. For parabolic X_h the character on the identity component is $|W_h|\tau^\lambda$ (see Corollary 4.14). Young's Rule gives the result. □

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Noncommutative Symmetric Hall-Littlewood Polynomials

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Abstract. Noncommutative symmetric functions have many properties analogous to those of classical (commutative) symmetric functions. For instance, ribbon Schur functions (analogs of the classical Schur basis) expand positively in noncommutative monomial basis. More of the classical properties extend to noncommutative setting as I will demonstrate introducing a new family of noncommutative symmetric functions, depending on one parameter. It seems to be an appropriate noncommutative analog of the Hall-Littlewood polynomials.

Résumé. Les fonctions symétriques non commutatives ont de nombreuses propriétés analogues à celles des fonctions symétriques classiques (commutatives). Par exemple, les fonctions de Schur en rubans (analogues de la base de Schur classique) admettent des développements à coefficients positifs dans la base des monômes non commutatifs. La plupart des propriétés classiques s'étendent au cas non commutatif, comme je le montrerai en introduisant une nouvelle famille de fonctions symétriques non commutatives, dépendant d'un paramètre. Cette famille semble être un analogue non commutatif approprié de la famille des polynômes de Hall-Littlewood.

Keywords: symmetric functions

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1 Introduction and Results.

Classical (commutative) Hall-Littlewood polynomials are well studied objects with applications ranging from representation theory Desarmenien et al. (1994) to Bethe ansatz in math physics Kirillov and Reshetikhin (1988). While there have been several attempts to find an object in the algebra on noncommutative symmetric function that would mimic some of the properties of the classical Hall-Littlewood polynomials Hivert (1998); Bergeron and Zabrocki (2005); Novelli et al. (2010) each with its own merit, I think it is worthwhile to continue the search. In particular, the new noncommutative symmetric functions I will introduce in 5 enjoy the following properties

- they reduce to noncommutative ribbon Schur functions at $t = 0$
- they reduce to noncommutative monomial functions at $t = 1$
- coefficients of the expansion of ribbon Schur functions in the new basis are polynomials in t with positive integer coefficients;

2 Operations on Compositions

To set the stage, I need some definitions, most of which are standard. Bases of the algebra of noncommutative symmetric functions are labeled by compositions, which will be denoted by capital Latin letters. Let $I = (i_1, \ldots, i_n)$ be a composition, i.e. an ordered set of positive integers $(i_1, \ldots, i_n)$, called parts of the composition I. The sum of all parts of a composition, its **weight**, is denoted by $|I|$ and the number of parts in the composition, its **length** – by $\ell(I)$. Alternatively, given the weight of a composition I , it can be specified by its **descent set**, $D(I)$. If $D(I) = \{d_1, \ldots, d_k\}$ with $d_1 < d_2 < \ldots < d_r < |I| - 1$, then $I = (d_1, d_2 - d_1, \ldots, |I| - d_k)$. A descent set can be defined for any word from an ordered alphabet as the set of positions of letters, which are greater than their right neighbor.
Major index of a composition with n parts is $maj(I) = \sum_k (n-k) i_k \equiv \sum_r d_r$.

Two types of multiplication for compositions were defined in Gelfand et al. (1995). For two compositions $I = (i_1, \ldots, i_{r-1}, i_r)$ and $J = (j_1, j_2, \ldots, j_s)$ define

$$I \triangleright J = (i_1, \ldots, i_{r-1}, i_r + j_1, j_2, \ldots, j_s),$$
$$I \cdot J = (i_1, \ldots, i_r, j_1, \ldots, j_s),$$

Reverse refinement order for compositions is defined as follows. Let $I = (i_1, \ldots, i_n)$, $J = (j_1, \ldots, j_s)$, $|J| = |I|$. Then $J \preceq I$ if every part of J can be obtained from consecutive parts of I:

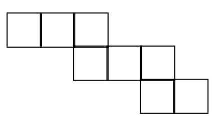
$$J = (i_1 + \ldots + i_{p_1}, i_{p_1+1} + \ldots + i_{p_2}, \ldots, i_{p_{k-1}+1} + \ldots + i_{p_k}, \ldots, i_{p_s} + \ldots + i_n)$$

for some nonnegative $p_1, \ldots, p_s$. (The convention $p_0 = 0$ will be implied below.) The composition $(p_1, \ldots, p_s)$ will be denoted by I_J.

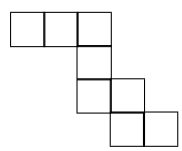
Example 1 *For instance, consider* $I = (3, 3, 2)$*:*

$$|(3,3,2)| = 8 \quad \ell(3,3,2) = 3 \quad D(3,3,2) = \{3,6\}$$

Also $(3,3,2) = (3, 1+2, 2) \preceq (3,1,2,2)$.



$\preceq$



Here $p_1 = 1,\ p_2 = 3,\ p_3 = 4$*, i.e.* $I_J = (1,3,4)$

3 Standard Bases for Noncommutative Symmetric Functions

The algebra **NSym**, introduced originally in Gelfand et al. (1995), is a noncommutative associative graded algebra generated by noncommutative power sums (of the first kind) Ψ_k of degree k. Products of power sums form a multiplicative basis:

$$\Psi^I = \Psi_{i_1} \cdot \ldots \cdot \Psi_{i_k}$$

As in the classical (commutative) theory of symmetric functions, there is a number of useful bases of **NSym**. Noncommutative **monomial symmetric functions** Tevlin (2007) can be defined as

Definition 1

$$M^I = \sum_{J \preceq I} \frac{(-1)^{\ell(I)-\ell(J)}}{\prod_{k=0}^{s-1}(\ell(I) - p_k)} \Psi^J, \tag{1}$$

where $s = \ell(J)$.

Example 2 *Suppose the task is to obtain* M^{312}. *There are four compositions* J, *such that* $J \preceq 312$*:*

$$\begin{aligned}
&312 \quad \textit{with} \quad p_1 = 1 \quad p_2 = 2 \quad p_3 = 3 \\
&3^2 \quad \textit{with} \quad p_1 = 1 \quad p_2 = 3 \\
&42 \quad \textit{with} \quad p_1 = 2 \quad p_2 = 3 \\
&6 \quad \textit{with} \quad p_1 = 3
\end{aligned}$$

Therefore

$$M^{312} = \frac{1}{3}\left(\Psi_6 - \Psi_4\Psi_2 - \frac{1}{2}\Psi_3^2 + \frac{1}{2}\Psi_3\Psi_1\Psi_2\right)$$

Noncommutative **complete (homogeneous) symmetric** function S_n can be defined as

Definition 2

$$S_n = \sum_{|I|=n} M^I$$

with a corresponding multiplicative basis functions

$$S^I = \prod_{k=1}^{\ell(I)} S_{i_k}$$

Finally, noncommutative **ribbon Schur** basis has been defined by Gelfand et al. (1995) as:

Definition 3

$$R^I = \sum_{J \preceq I} (-1)^{\ell(I)-\ell(J)} S^J \tag{2}$$

4 Transition between Different Bases.

4.1 Ribbon and Monomial.

Analogously to the situation on the commutative theory, where Schur functions expand positively in the monomial basis, noncommutative ribbon Schur functions expand in the noncommutative monomial basis with positive coefficients. This was proved in Hivert et al. (2009), who also provided the combinatorial meaning of coefficients.

Proposition 1 (Hivert et al. (2009)) *The coefficient* K_{IJ} *in the expansion*

$$R^I = \sum_J K_{IJ} M^J$$

is equal to the number of packed words such that the descent set of u, $D(u) = I$ *and the word count* $WC(u) = J$.

In the following section I recall definitions of objects in this statement as the main results of the present paper generalize this result.

4.2 Compositions Associated to Packed Words.

Packed words are defined the following way:

Definition 4 *The packed word $u = pack(w)$ associated with a word $w \in A$ (A is an alphabet) is obtained by the following process. If $b_1 < b_2 < ... < b_r$ are the letters occurring in w, u is the image of w by the homomorphism $b_i \mapsto a_i$. A word u is said to be packed if $pack(u) = u$.*

Example 3 *Restricting A to the first five integers, the following is the list of words w_k, such that $pack(w_k) =$ 13132:*

$$w_1 = 13132,\ w_2 = 14142,\ w_3 = 14143,\ w_4 = 24243,\ w_5 = 15152,\ w_6 = 15153,$$
$$w_7 = 25253,\ w_8 = 15154,\ w_9 = 25254,\ w_{10} = 35354$$

Put differently, these are words in which if each letter is replaced by a smaller one (respecting the order $1 < 2 < ...$), one gets the word made out of smallest possible letters, $u = 13132$. A packed word is such that cannot be simplified this way.
There are two ways to associate a composition to each packed word u:

1. First one considers a descent set of a word, $D(u)$, and builds a corresponding composition. For instance, if $u = 2113$, its the descent set $\{1\}$. Therefore the corresponding composition is (13); for $u = 3221$, $D(3221) = \{1, 3\}$, which corresponds to (121).

2. Secondly, the word composition $WC(u)$ of u is the composition whose descent set is given by the positions of the *last* occurrences of each letter in u. For example,

$$WC(1543421323) = (23221)$$

   Indeed, the descent set is $\{2, 5, 7, 9, 10\}$ since the last 5 is in position 2, the last 4 is in position 5, the last 1 is in position 7, the last 2 is in position 9, the last 3 is in position 10.

Finally, with each word one can associate a statistics $sinv(u)$, called **special inversion**, that counts the number of times the last occurrence (reading from left to right) of a smaller letter is to right of a larger one Novelli et al. (2010). For example,

$$sinv(3221) = 4$$

since the second 2 is to the right of 3, 1 is to right of both 2's and 3.

4.3 Monomial and Complete.

In this section I will find the explicit formula for the coefficients $\varrho_{J|K}$ in the following expansion

$$M^J = \sum_K \varrho_{J|I} S^I$$

Consider two compositions I and J and take the first (largest) K such that $K \preceq I$ and $K \preceq J$. Put differently, $D(K) = D(I) \cap D(J)$. More precisely, let

$$
\begin{aligned}
&I = (i_1, \ldots, i_n) \\
&J = (j_1, \ldots, j_l) \\
&K = (k_1, \ldots, k_m) \quad \text{with} \quad m \leq n, \quad m \leq s \\
&K = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{m-1}+1} + \ldots + i_n) \\
&K = (j_1 + \ldots + j_{r_1}, \ldots, j_{r_{m-1}+1} + \ldots + j_l)
\end{aligned}
$$

Proposition 2

$$
\varrho_{J|I} = (-1)^{\ell(I)-\ell(J)} \frac{\prod_{s \in \{1, \ell(J)\}/\{r_k\}} (\ell(J) - s) \prod_{k=1}^{m} (\ell(J) - r_k + i_{p_k})}{\ell(J)!} \tag{3}
$$

Proof (*sketch*): The expansion of power sums in complete is known Gelfand et al. (1995). Inserting this expansion in the definition of monomials above one gets a fairly ugly sum, which can be recognized as an expansion of a certain determinant. Calculation of the determinant results in (3). □

Example 4 *Consider* $\varsigma_{21^2|1^22}$

$$
\begin{aligned}
&K = 2^2 \quad \ell(K) \equiv m = 2 \\
&I = 21^2 \Rightarrow \begin{cases} p_1 = 1 & i_{p_1} \equiv i_1 = 2 \\ p_2 = 3 & i_{p_2} \equiv i_3 = 1 \end{cases} \\
&J = 1^22 \Rightarrow \begin{cases} r_1 = 2 \\ r_2 = 3 \end{cases} \\
&\{1, 2, 3\}/\{2, 3\} = 1 \\
&\varsigma_{21^2|1^22} = \frac{(3-1) \cdot (3-2+2) \cdot (3-3+1)}{3!} = \frac{2 \cdot 3 \cdot 1}{3!} = 1
\end{aligned}
$$

5 Noncommutative Hall-Littlewood Polynomials.

This section contains one the most important results of this paper. Below I will define what seems to be an appropriate noncommutative analog of Hall-Littlewood Polynomials. Classical Hall-Littlewood polynomials $P_{\boldsymbol{\lambda}}(t)$ enjoy the following properties Macdonald (1995) (please refer to this book for notations):

1. $$P_{\boldsymbol{\lambda}}(0) = s_{\boldsymbol{\lambda}}$$

2. $$P_{\boldsymbol{\lambda}}(1) = m_{\boldsymbol{\lambda}}$$

3. $$s_{\boldsymbol{\lambda}} = \sum_{\boldsymbol{\kappa}} K_{\boldsymbol{\lambda}\boldsymbol{\kappa}}(t) P_{\boldsymbol{\kappa}}(t),$$

   where Kostka-Foulkes polynomials $K_{\boldsymbol{\lambda}\boldsymbol{\kappa}}(t)$ are polynomials in t with positive integer coefficients, whose combinatorial meaning is well-understood Lascoux and Schutzenberger (1978), Kirillov (1998).

Polynomials $P^I(t)$ to be introduced below have the following properties

1. $$P^I(0) = R^I$$

2. $$P^I(1) = M^I$$

3. $$R^I = \sum_J K_{IJ}(t) P^J(t),$$

   where $K_{IJ}(t)$ are polynomials in t with positive integer coefficients, whose meaning will be explaned below.

4. Moreover in the commutative limit noncommutative Hall-Littlewood polynomials for hook compositions $I = n1^k$ reduce to classical Hall-Littlewood polynomials $Q_{\boldsymbol{\lambda}}(t)$ where $\boldsymbol{\lambda} = n1^k$.

I will define noncommutative Hall-Littlwood polynomials through their expansion in complete basis. Once again for two compositions I and J select the first (largest) $K \preceq I, K \preceq J$ (i.e. $D(K) = D(I) \cap D(J)$) and let

$$\begin{aligned}
&I = (i_1, \ldots, i_n) \\
&J = (j_1, \ldots, j_s) \\
&K = (k_1, \ldots, k_m) \quad \text{with} \quad m \le n, \quad m \le s \\
&K = (i_1 + \ldots + i_{p_1}, \ldots, i_{p_{m-1}+1} + \ldots + i_n) \\
&K = (j_1 + \ldots + j_{r_1}, \ldots, j_{r_{m-1}+1} + \ldots + j_s)
\end{aligned}$$

Definition 5

$$P^J(t) = \sum_I \varsigma_{I|J}(t) S_I, \quad \textit{where}$$

$$\varsigma_{I|J}(t) = (-1)^{\ell(I)-\ell(J)} t^{|I|-\sum_{k=1}^m i_{p_k}} \frac{\prod_{s\in\{1,\ell(J)\}/\{r_k\}}(1-t^{\ell(J)-s}) \prod_{k=1}^m \left(1-t^{\ell(J)-r_k+i_{p_k}}\right)}{\left[\ell(J)\right]_t!}, \quad (4)$$

Notice that for a given J each polynomial $\varsigma_{JI}(t)$ has the same top degree

$$\frac{\ell(J)(\ell(J)-1)}{2} + |J|$$

The first two properties of noncommutative Hall-Littlewood functions follow directly from the definition. The last part of the formula involving products is a direct t-generalization of (3) and insures that

$$P^J(1) = M^J,$$

The first part of the formula with the power of t ensures that at $t = 0$ only the terms corresponding to $I \preceq J$ survive since in that and only that case $K = I$ and $\sum_{k=1}^m i_{p_k} = |I|$. This ensures that

$$P^J(0) = R^J$$

Further, if one denotes

$$Q^J(t) = \left[\ell(J)\right]_t! P^J(t)$$

then the statement is that in the commutative limit it is these function that for hook compositions reduce to classical Hall-Littlewood polynomials.
Consider an example of (4) for $J = 1^2 2$

Example 5

$$Q^{1^22}(t) = -\varsigma_{1^4|1^22}S_1^4 + \varsigma_{21^2|1^22}S_{21^2} + \varsigma_{121|1^22}S_{121} + \varsigma_{1^22|1^22}S_{1^22} - \varsigma_{2^2|1^22}S_{2^2} - \varsigma_{31|1^22}S_{31}$$
$$- \varsigma_{13|1^22}S_{13} + \varsigma_{4|1^22}S_4$$
$$\varsigma_{1^4|1^22} = (1-t^{3-1+1})(1-t^{3-2+1})(1-t^{3-3+1}) = (1-t)(1-t^2)(1-t^3)$$
$$K = 1^22; \quad i_{p_1} = 1, i_{p_2} = 1, i_{p_3} = 1; \quad r_k = k; \quad s = \{\emptyset\}$$
$$\varsigma_{21^2|1^22} = (1-t^{3-1})(1-t^{3-2+2})(1-t^{3-3+1}) = (1-t)(1-t^2)(1-t^3)$$
$$K = 2^2; \quad i_{p_1} = 2, i_{p_2} = 1; \quad r_1 = 2, r_2 = 3; \quad s = \{1\}$$
$$\varsigma_{121|1^22} = (1-t^{3-2})(1-t^{3-1+1})(1-t^{3-3+1}) = (1-t)^2(1-t^3)$$
$$K = 13; \quad i_{p_1} = 1, i_{p_2} = 1; \quad r_1 = 1, r_2 = 3; \quad s = \{2\}$$
$$\varsigma_{1^22|1^22} = (1-t^{3-1+1})(1-t^{3-2+1})(1-t^{3-3+2}) = (1-t^2)^2(1-t^3)$$
$$K = 1^22; \quad i_{p_k} = i_k; \quad r_k = k; \quad s = \{\emptyset\}$$
$$\varsigma_{2^2|1^22} = (1-t^{3-1})(1-t^{3-2+2})(1-t^{3-3+2}) = (1-t^2)^2(1-t^3)$$
$$K = 2^2; \quad i_{p_1} = 2, i_{p_2} = 2; \quad r_1 = 2, r_2 = 3; \quad s = \{1\}$$
$$\varsigma_{31|1^22} = (1-t^{3-1})(1-t^{3-2})(1-t^{3-3+1}) = (1-t)^2(1-t^2)$$
$$K = 4; \quad i_{p_1} = 1; \quad r_1 = 3; \quad s = \{1,2\}$$
$$\varsigma_{13|1^22} = (1-t^{3-2})(1-t^{3-1+1})(1-t^{3-3+3}) = (1-t)(1-t^3)^2$$
$$K = 13; \quad i_{p_1} = 1, i_{p_2} = 3; \quad r_1 = 1, r_2 = 3; \quad s = \{2\}$$
$$\varsigma_{4|1^22} = (1-t^{3-1})(1-t^{3-2})(1-t^{3-3+4}) = (1-t)(1-t^2)(1-t^4)$$
$$K = 4; \quad i_{p_1} = 4; \quad r_1 = 3; \quad s = \{1,2\}$$
$$Q^{1^22}(t) = -t(1-t)(1-t^2)(1-t^3)S_1^4 + t(1-t)(1-t^2)(1-t^3)S_{21^2} + t^2(1-t)^2(1-t^3)S_{121} +$$
$$+ (1-t^2)^2(1-t^3)S_{1^22} - (1-t^2)^2(1-t^3)S_{2^2} - t^3(1-t)^2(1-t^2)S_{31} - (1-t)(1-t^3)^2S_{13} +$$
$$+ (1-t)(1-t^2)(1-t^4)S_4$$

I now turn to the multiplication rule for noncommutative Hall-Littlewoods.

Proposition 3

$$P^I(t) \cdot P^J(t) = \sum_{K \preceq I} t^{maj(I_K) + \binom{|K|+1}{2}} \left\{ \begin{bmatrix} \ell(I) + \ell(J) \\ \ell(K) \end{bmatrix}_t P^{K \cdot J}(t) + \begin{bmatrix} \ell(I) + \ell(J) - 1 \\ \ell(K) \end{bmatrix}_t P^{K \triangleright J}(t) \right\}, \tag{5}$$

Proof: The proof is by induction. □

It is worth noticing that this formula interpolates between multiplication formulas for noncommutative ribbon Schur functions (at $t = 0$)

$$R^I R^J = R^{I \cdot J} + R^{I \triangleright J}$$

and monomial (at $t = 1$)

$$M^I M^J = \sum_{K \preceq I} \binom{\ell(I) + \ell(J)}{\ell(K)} M^{K \cdot J}(t) + \binom{\ell(I) + \ell(J) - 1}{\ell(K)} M^{K \triangleright J}(t)$$

6 Expansion of Ribbon Schur in the Hall-Littlewood Basis.

Finally I would like to address the last property of noncommutative Hall-Littlewood symmetric functions, namely the expansion of ribbon Schur in this basis.

Proposition 4

$$R^I = \sum_J K_{IJ}(t) P^J(t), \tag{6}$$

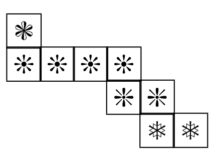
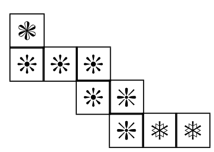
where $K_{IJ}(t)$ are polynomials in t with positive coefficients. More precisely,

$$K_{IJ}(t) = \sum_{u = pack(u),\ D(u) = I,\ WC(u) = J} t^{maj(J) - sinv(u)} \tag{7}$$

The proof of this statement requires two steps. First – the expansion for the expansion of homogeneous basis in terms of Hall-Littlewood.; second – explicit expression for (7).

6.1 Relative Decomposition of Two Compositions.

Consider a decomposition of composition J relative to composition I, i.e. a filling of composition J with sub-compositions of weight $(i_1, i_2, \ldots, i_n)$. Denote by h_k the row of J that contains the last cell of i_k. Rows will be labeled from bottom up starting from zero. Call the sequence (which is actually a partition with some zero parts) of $(h_1, \ldots, h_n)$ relative decomposition of J with respect to I. (Compare with a related notion in Gelfand et al. (1995)). For instance, consider a decomposition of $J = 1323$ with respect to $I = 1422$:

$I =$  $J =$ 

where I have marked all rows of I with different symbols so that one can see where they end up within J.

Thus the decomposition of $J = 1323$ with respect to $I = 1422$ is $h(J, I) = (3, 1, 0, 0)$ Given such a decomposition $h(J, I)$, form a product:

$$\xi_{I|J} = \prod_{k=1}^{\ell(I)-1} \begin{bmatrix} i_k + h_k \\ i_k \end{bmatrix}_q , \tag{8}$$

So if $J = 1323$ and $I = 1422$ as in the example above, then

$$\xi_{1422|1323} = \begin{bmatrix} 1+3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 4+1 \\ 4 \end{bmatrix}_q \begin{bmatrix} 2+0 \\ 2 \end{bmatrix}_q$$

Proposition 5

$$\sum_{I \preceq K} (-1)^{\ell(K)-\ell(I)} \prod_{k=1}^{\ell(I)-1} \begin{bmatrix} i_k + h_k \\ i_k \end{bmatrix}_q = \sum_{u=pack(u),\ D(u)=K,\ WC(u)=J} q^{sinv(u)}, \tag{9}$$

Example 6 *Let $K = 121$ and $J = 21^2$, then $I_1 = 121, I_2 = 31, I_3 = 13, I_4 = 4$*

$$\xi_{121|21^2} - \xi_{31|21^2} - \xi_{13|21^2} + \xi_{4|21^2} = \underbrace{\begin{bmatrix} 1+2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 2+1 \\ 2 \end{bmatrix}_q}_{h(21^2|121)=(2,1)} - \underbrace{\begin{bmatrix} 1+2 \\ 1 \end{bmatrix}_q}_{h(21^2|13)=(2,0)} - \underbrace{\begin{bmatrix} 3+1 \\ 3 \end{bmatrix}_q}_{h(21^2|31)=(1,0)} +1$$
$$= q^2(1+q+q^2)$$

The corresponding packed words are: $2132, 3132, 3231$ *and the number of inversions of these words:*

$$inv(2132) = 2,\ inv(3132) = 3,\ inv(3231) = 4$$

i.e. the right-hand side is $q^2 + q^3 + q^4$

The proof of (9) *will be published separately Tishbi and Tevlin (2010).*

6.2 Special Inversion Statistics and the Expansion of Ribbon Schur Functions.

By induction one can establish the following expansion of noncommutative complete symmetric function in noncommutative Hall-Littlewood basis:

Proposition 6

$$S^I = \sum_J \rho_{IJ} H^J(t),$$

where

$$\rho_{IJ} = t^{maj(J)} \prod_{i=1}^{\ell(I)-1} \begin{bmatrix} i_k + h_k \\ i_k \end{bmatrix}_{\frac{1}{t}} \tag{10}$$

Note, that ρ_{IJ} may be thought of as noncommutative t-supernomial coefficents, compare with Schilling (2002).

Using the definition of ribbon Schur functions (2) and (9) the statement (6) is immediate.

7 Comments.

Obviously one would like to know if there is a representation-theoretic and/or geometric interpretation of the above results. In that context it would be interesting to see if there is a noncommutative analog of the plethystic substitution and, therefore, an appropriate analog of modified Hall-Littlewood functions.
Also, the question of specialization of noncommutative Hall-Littlewood functions and q-series identities implied by, for instance, multiplication rule, has not been explored.

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On the monotone hook hafnian conjecture

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Abstract. We investigate a conjecture of Haglund that asserts that certain graph polynomials have only real roots. We prove a multivariate generalization of this conjecture for the special case of threshold graphs.

Résumé. Nous étudions une conjecture de Haglund qui affirme que certaines polynômes des graphes ont uniquement des racines réelles. Nous prouvons une généralisation multivariée de cette conjecture pour le cas particulier des graphes à seuil.

Keywords: hafnian, threshold graphs, monotone hook, only real roots, stable polynomial

1 Introduction

Caianiello (1973) defined the *hafnian* for (upper) triangular arrays as the "signless" pfaffian. There are slight variations in the literature on how to extend the original definition to matrices; we will use the following one. Let $C = (c_{ij})$ be a $2n \times 2n$ symmetric matrix, the hafnian of C is defined as

$$\mathrm{haf}(C) = \frac{1}{n!2^n} \sum_{\sigma \in \mathfrak{S}_{2n}} \prod_{k=1}^{n} c_{\sigma(2k-1),\sigma(2k)}, \tag{1}$$

where $\mathfrak{S}_{2n}$ denotes the symmetric group on $2n$ elements.

The mth hook of a triangular array (or shifted Ferrers board) $A = (a_{ij})_{1 \leq i < j \leq n}$ is the set of cells

$$\mathrm{hook}_m = \{(i, m) \mid i = 1, \ldots, m-1\} \cup \{(m, j) \mid j = m+1, \ldots, n\}. \tag{2}$$

The direction along the mth hook is the one in which the quantity $i+j$ is increasing where $(i, j) \in \mathrm{hook}_m$. A *monotone hook triangular array* has real entries decreasing along at least $n-1$ of its hooks, or possibly along all n of them. Analogously, a monotone hook matrix is a real symmetric matrix whose entries above the diagonal form a monotone hook triangular array.

In this paper we discuss results on the following conjecture of Haglund (2000):

Conjecture 1.1 (Monotone Hook Hafnian (MHH)) *Let A be a $2n \times 2n$ monotone hook matrix. Let J_{2n} denote the $2n \times 2n$ matrix of all ones. Then the polynomial $\mathrm{haf}(zJ_{2n} + A) \in \mathbb{R}[z]$ has only real roots.*

In Haglund (2000) the MHH conjecture was proven for adjacency matrices of a class of graphs called threshold graphs, and as a corollary for all monotone hook $\{0, 1\}$ matrices. And it was also verified for all $2n \times 2n$ monotone hook matrices for $n \leq 2$.

The main result of this paper is a (multivariate) generalization of the MHH conjecture for the special case of adjacency matrices of threshold graphs. We begin by discussing a closely related problem involving permanents and monotone column matrices. In Section 2, we introduce the machinery needed for our proofs from the theory of stable polynomials. In Section 3, we prove the multivariate generalization of the MHH conjecture for the special case of threshold graphs. We also mention some negative results and some open problems. We conclude in Section 4 by giving an alternative proof of the MHH conjecture for some threshold graphs whose adjacency matrices can be written in a special form.

1.1 A closely related problem: the MCP theorem

Recall the definition of the permanent of an $n \times n$ matrix B with entries b_{ij}:

$$\mathrm{per}(B) = \sum_{\sigma \in \mathfrak{S}_n} \prod_{i=1}^{n} b_{i,\sigma(i)}. \tag{3}$$

A *monotone column matrix* $A \in \mathbb{R}^{n \times n}$ has real entries a_{ij} weakly decreasing down columns: $a_{ij} \geq a_{i+1,j}$ for all $1 \leq i < n$ and $1 \leq j \leq n$. Building on recent developments in the theory of stable polynomials by Borcea and Brändén (see, for instance, Borcea and Brändén (2010) or Wagner (2011) for a survey) the following theorem was proved in Brändén et al. (2010):

Theorem 1.2 (Monotone Column Permanent (MCP)) *Let A be an $n \times n$ monotone column matrix, and J_n the $n \times n$ matrix of all ones. Then the polynomial* $\mathrm{per}(zJ_n + A) \in \mathbb{R}[z]$ *has only real roots.*

As pointed out in Haglund (2000) the MHH conjecture can be viewed as the analog of the MCP theorem from the complete bipartite graph, $K_{n,n}$ to the complete graph, K_{2n}, in the following sense. Given an (undirected) weighted graph G with edge weights w_{ij} define the following graph polynomial

$$\phi(G; z) = \sum_{\mathcal{M}} \prod_{ij \in \mathcal{M}} (w_{ij} + z), \tag{4}$$

where $\mathcal{M}$ runs over all perfect matchings of G. If $A_G = \begin{pmatrix} 0 & B_G \\ B_G^\top & 0 \end{pmatrix}$ is the adjacency matrix of a weighted $K_{n,n}$ graph G, then $\phi(G; z) = \mathrm{per}(zJ_n + B_G)$. Similarly, if A_H is the adjacency matrix of a weighted K_{2n} graph H, then $\phi(H; z) = \mathrm{haf}(zJ_{2n} + A_H)$.

Another interesting connection comes from the following basic fact. The hafnian of a matrix does not depend on the diagonal entries of the matrix. Let A^* be a monotone hook matrix that is monotone along *all* of its hooks. In this case, for the purpose of studying $\mathrm{haf}(A^*)$ we can assume that A^* is a symmetric *monotone column* matrix (by picking the diagonal entries accordingly). Since A^* is symmetric this will imply that A^* is monotone both down columns and down rows.

In the next section, we describe the results from the theory of stable polynomials that we need for our proof. These results were successfully employed in the proof of the MCP theorem and they turn out to be useful for the MHH conjecture, as well.

2 Stable polynomials and stability preservers

A multivariate polynomial $f \in \mathbb{R}[z_1, \ldots, z_n]$ is *stable* if it does not vanish when the imaginary parts of the z_i's are positive. This notion of stability — sometimes referred to as the upper half-plane property

(Fisk (2008)) — has been the focus of some recent research (Brändén (2007); Borcea and Brändén (2009, 2010)). If $f \in \mathbb{R}[z_1, \ldots, z_n]$ does not vanish when the real parts of z_i's are positive then we call it *Hurwitz-stable* (Brändén (2007)), sometimes called right half-plane stable.

The technique we use to show that a (univariate) polynomial $f \in \mathbb{R}[z]$ has real roots only is to show that f is stable, since the two notions coincide in $\mathbb{R}[z]$. We show stability of f by finding a stable multivariate generalization of it, namely a suitable stable polynomial $g(z_1, \ldots, z_n) \in \mathbb{R}[z_1, \ldots, z_n]$ that can be reduced to $f(z)$ using certain operations that preserve stability. Operations that preserve stability play a crucial role in this framework, since they allow for manipulation of multivariate polynomials while maintaining their stability. The following lemma gives a list of such operations that we will be using throughout the paper. The results are taken from Brändén (2007); Borcea and Brändén (2009); Wagner (2011).

Lemma 2.1 *The following operations preserve stability of polynomials in* $\mathbb{R}[z_1, \ldots, z_n]$

1. *Permutation: for any permutation* $\sigma \in \mathfrak{S}_n$, $f \mapsto f(z_{\sigma(1)}, \ldots, z_{\sigma(n)})$.

2. *Diagonalization: for* $1 \leq i < j \leq n$, $f \mapsto f(z_1, \ldots, z_n)|_{z_i = z_j}$.

3. *Specialization: for* a *with* $\Im(a) > 0$, $f \mapsto f(a, z_2, \ldots, z_n)$.

4. *Translation:* $f \mapsto f(z_1 + t, z_2, \ldots, z_n) \in \mathbb{R}[z_1, \ldots, z_n, t]$.

5. *Differentiation:* $f \mapsto \partial f / \partial z_1$.

3 Multivariate MHH for threshold graphs

In this section we prove a multivariate generalization of the following theorem of Haglund (see Theorem 2.2 of Haglund (2000)).

Theorem 3.1 (MHH for threshold graphs) *Let* A_G *denote the adjacency matrix of a (non-weighted) threshold graph* G *on* $2n$ *vertices. Then* $\operatorname{haf}(zJ_{2n} + A_G)$ *is stable.*

Threshold graphs have been widely studied and are known to have several equivalent definitions. For our purposes, the following definition will come in handy (see Theorem 1.2.4 in Mahadev and Peled (1995)). A graph G on n vertices is a *threshold graph* if it that can be constructed starting from a one-vertex graph by adding vertices one at a time in the following way. Start at step 1 with a single vertex v_1. At each step i, for $2 \leq i \leq n$, the vertex v_i being added is either isolated (has degree 0) or dominating (has degree $i - 1$ at the time when added).

By definition, $\operatorname{haf}(zJ_{2n} + A_G)$ is invariant under the permutation of the vertices in G. Hence, we can assume that the vertices of a threshold graph are labeled in the order of the above construction. This means that in any column i for $2 \leq i \leq 2n$ the entries above the diagonal entry of $zJ_{2n} + A_G$ are equal to either z (if v_i was added as isolated vertex) or $z + 1$ (if v_i was added as dominating vertex). This suggests the multivariate generalization that we show next. The idea essentially is to use a separate variable z_i for the entries above the diagonal in each column i (see explicit construction of $A_{2n}(\mathbf{z})$ in Proposition 3.2 below).

3.1 A multivariate generalization for hafnians

The following proposition is a direct application of the idea of the proof of Theorem 3.4 of Brändén et al. (2010) to hafnians.

Proposition 3.2 *Let $z_1, \ldots, z_{2n}$ denote commuting indeterminates and let $A_{2n}(\mathbf{z}) = (a_{ij})$ denote the $2n \times 2n$ symmetric matrix with entries $a_{ij} = z_{\max(i,j)}$. Then $\operatorname{haf}(A_{2n}(\mathbf{z}))$ is a stable polynomial in $\mathbb{R}[z_2, \ldots, z_{2n}]$ (z_1 only appears on the diagonal).*

Proof: We use induction. Clearly, $\operatorname{haf}\begin{pmatrix} z_1 & z_2 \\ z_2 & z_2 \end{pmatrix} = z_2$ is stable, which settles the base case. Next we show that for $n \geq 2$, $\operatorname{haf}(A_{2n})$ is stable if $\operatorname{haf}(A_{2n-2})$ is stable. This follows from the differential recursion:

$$A_{2n}(\mathbf{z}) = z_{2n} A_{2n-2}(\mathbf{z}) + 2 z_{2n-1} z_{2n} \sum_{i=2}^{2n-2} \frac{\partial A_{2n-2}(\mathbf{z})}{\partial z_i}. \tag{5}$$

This recursion can be seen from the expansion of the hafnian along the last column. The differential operator in the right-hand side preserves stability (analogously to the one in Theorem 3.4 of Brändén et al. (2010)). □

The proposition gives a multivariate version of the MHH conjecture for certain matrices. Unfortunately, it is not clear how to transition from here to the general MHH conjecture. Nevertheless, Proposition 3.2 does imply Theorem 3.1 in the following way.

Proof of Theorem 3.1: Let G be a threshold graph on $2n$ vertices with an adjacency matrix A_G. Assume that the vertices of G are ordered as in the definition of the vertex-by-vertex construction above. It is easy to see that $A_{2n}(\mathbf{z})$ in Proposition 3.2 specializes to $zJ_{2n} + A_G$ if, for all i, we set

$$z_i = \begin{cases} z, & \text{if } v_i \text{ was added as an isolated vertex,} \\ z+1, & \text{if } v_i \text{ was added as a dominating vertex.} \end{cases} \tag{6}$$

This last operation consists of diagonalizing, translating and specializing the variables and thus preserves stability of $\operatorname{haf}(A_{2n}(\mathbf{z}))$ by Lemma 2.1. Hence, we obtain that $\operatorname{haf}(zJ_{2n} + A_G)$ is a stable polynomial. To complete the proof, note that a stable univariate polynomial with real coefficients has only real roots. □

Remark 3.3 *The same proof goes through if we allow edges to have weights other than zero or one with the additional restriction that when we add a dominating vertex, all edges incident to it must have the same weight (but this weight need not be 1 as before).*

3.2 Negative results and open problems

In Brändén et al. (2010) a more general, multivariate version of the MCP theorem was also obtained:

Theorem 3.4 (Multivariate MCP) *Let A be a monotone column matrix, and let $Z_n = \operatorname{diag}(z_1, \ldots, z_n)$ be the $n \times n$ diagonal matrix of n indeterminates. Then $\operatorname{per}(J_n Z_n + A) \in \mathbb{R}[z_1, \ldots, z_n]$ is stable.*

Similarly to the multivariate MCP theorem, we would like to have a multivariate analog of the MHH conjecture that would imply the univariate case. We consider the following three multivariate generalizations. For a monotone hook matrix $A = (a_{ij})$ of size $2n \times 2n$, the diagonal matrix Z_{2n} of $2n$ indeterminates, and the matrix J_{2n} of all ones of the same size, let

- $h_1(A, \mathbf{z}) = \mathrm{haf}(Z_{2n}J_{2n} + J_{2n}Z_{2n} + A) = \mathrm{haf}(z_i + z_j + a_{ij})$,
- $h_2(A, \mathbf{z}) = \mathrm{haf}(Z_{2n}J_{2n} + A) = \mathrm{haf}(z_i + a_{ij})$.
- $h_3(A, \mathbf{z}) = \mathrm{haf}(Z_{2n}J_{2n}Z_{2n} + A) = \mathrm{haf}(z_iz_j + a_{ij})$,

For convenience, we also provided a shorthand for these matrices that is sometimes easier to work with. The motivation behind the definition of these multivariate graph polynomials is the following.

Proposition 3.5 *Let A be a monotone hook matrix. Assume that either*

1. *$h_1(A, \mathbf{z})$ is stable, or*
2. *$h_2(A, \mathbf{z})$ is stable, or*
3. *$h_3(A, \mathbf{z})$ is Hurwitz-stable.*

Then the conclusion of the MHH conjecture (Conjecture 1.1) holds for the matrix A.

Unfortunately, these generalizations fail to be stable even for $\{0, 1\}$ matrices.

Proposition 3.6 *There is a monotone hook $\{0, 1\}$ matrix A for which $h_1(A, \mathbf{z})$ is not stable, and $h_2(A, \mathbf{z})$ is not stable, and $h_3(A, \mathbf{z})$ is not Hurwitz-stable.*

Since all these polynomials are multi-affine (they have degree at most one in each variable) we can use the following criterion (Theorem 5.6 in Brändén (2007)) to check whether they are stable or not.

Theorem 3.7 *A multi-affine polynomial $f \in \mathbb{R}[z_1, \dots, z_n]$ is stable if and only if*

$$\Delta_{i,j} f := \frac{\partial f}{\partial z_i}(x) \cdot \frac{\partial f}{\partial z_j}(x) - \frac{\partial^2 f}{\partial z_i \partial z_j}(x) \cdot f(x) \geq 0$$

for all $x \in \mathbb{R}^n$ and $1 \leq i, j \leq n$.

Proof of Proposition 3.6: Let

$$A = \begin{pmatrix} * & 0 & 0 & 1 \\ 0 & * & 1 & 1 \\ 0 & 1 & * & 1 \\ 1 & 1 & 1 & * \end{pmatrix},$$

where $*$ denotes a wildcard, since the hafnian does not depend on the diagonal elements.

We have that

$$h_1(A, \mathbf{z}) = (z_1 + z_2)(z_3 + z_4 + 1) + (z_1 + z_3)(z_2 + z_4 + 1) + (z_1 + z_4 + 1)(z_2 + z_3 + 1)$$

and from this we get that $\Delta_{3,4} h_1(A, z) = 4z_1^2 + 4z_2^2 + 4z_1z_2 + 2z_2$ which can take on negative values (e.g., when $z_1 = 0$ and $-1/2 < z_2 < 0$). Similarly, one can easily check that $\Delta_{1,3} h_2(A, \mathbf{z}) = -z_2 - 1$. Checking Hurwitz-stability for $h_3(A, z_1, z_2, z_3, z_4)$ can be reduced to checking stability for $\tilde{h}_3(A, \mathbf{z}) = h_3(A, \frac{z_1}{i}, \frac{z_2}{i}, \frac{z_3}{i}, \frac{z_4}{i})$ because multiplication by the imaginary unit i maps the right half-plane to the upper half-plane. Since $\tilde{h}_3(A, \mathbf{z})$ has only real coefficients we can apply Brändén's criterion again to get that

$$\Delta_{3,4} \tilde{h}_3(A, \mathbf{z}) = z_1(9z_1z_2^2z_3z_4 + 3z_1z_2z_3 + 3z_2^2z_3 + 3z_1z_2z_4 + z_1 - 2z_2)$$

an expression that takes on negative values as well (e.g., when $z_1 = z_2 = 1$ and $z_3 = z_4 = 0$). □

4 An alternative proof for a subclass of threshold graphs

In this section, we make use of the α-permanents and their close connections to hafnians to prove that $\mathrm{haf}(zJ + A)$ has only real roots for certain $\{0, 1\}$ matrices A of a special form.

Recall the definition of α-permanent of an $n \times n$ matrix $B = (b_{ij})$ (Vere-Jones (1998)):

$$\mathrm{per}(B; \alpha) = \sum_{\pi \in \mathfrak{S}_n} \alpha^{\nu(\pi)} \prod_{i=1}^{n} b_{i,\pi(i)}, \tag{7}$$

where $\nu(\pi)$ is the number of disjoint cycles of the permutation π. Special cases of this formula when $\alpha = 1$ and when $\alpha = -1$ yield $\mathrm{per}(B)$ and $(-1)^n \det(B)$, respectively.

It is known that for real symmetric $n \times n$ matrices B:

$$\mathrm{per}(B; 1/2) = \frac{1}{2^n} \mathrm{haf} \begin{pmatrix} B & B \\ B & B \end{pmatrix}. \tag{8}$$

The combinatorial proof of (8) in Frenkel (2010) allows us to easily extend this equality to any symmetric matrix B (i.e., the entries need not be real numbers, they can be indeterminates as well). We combine this result with the cycle-counting extension of the MCP theorem (Proposition 4.4 of Brändén et al. (2010)). In fact, we only need the following bivariate special case of it (when we diagonalize all $x_i = x$ and $y_i = y$):

Proposition 4.1 *Let F be an $n \times n$ Ferrers matrix, a $\{0, 1\}$ matrix that is weakly decreasing down columns and weakly increasing from left to right across rows. Then, for any $\alpha > 0$, the polynomial $\mathrm{per}(xF + y(J_n - F); \alpha) \in \mathbb{R}[x, y]$ is stable.*

Corollary 4.2 *Let B be an $n \times n$ symmetric monotone column $\{0, 1\}$ matrix, and let $A = \begin{pmatrix} B & B \\ B & B \end{pmatrix}$ then the polynomial $\mathrm{haf}(zJ_{2n} + A)$ has only real roots.*

Proof: From (8) we see that it is equivalent to prove that the $1/2$-permanent of $zJ_n + B$ has real roots only. Since B is a symmetric monotone column $\{0, 1\}$ matrix, it is also a Ferrers matrix, and thus we can apply Proposition 4.1 to it with $\alpha = 1/2$. Translation and diagonalization preserve stability (see Lemma 2.1), so by setting $x = z + 1$ and $y = z$ we get that $\mathrm{per}(zJ_n + B; 1/2)$ is stable, i.e., has real roots only. □

Remark 4.3 *The adjacency matrices of the form $A = \begin{pmatrix} B & B \\ B & B \end{pmatrix}$ where B is a symmetric is usually not a monotone hook matrix. However, it can be shown that for any such matrix A there is a threshold with adjacency matrix equal to A. In other words, Corollary 4.2 is a special case of Theorem 3.1.*

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