FPSAC 2011, Reykjavík, Iceland DMTCS proc. **AO**, 2011, 411–422

On the evaluation of the Tutte polynomial at the points $(1,-1)$ and $(2,-1)$

Andrew Goodall[1] and Criel Merino[2] and Anna de Mier[3†] and Marc Noy[3]

[1] *Department of Applied Mathematics and Institute of Theoretical Computer Science, Charles University, Praha, Czech Republic*
[2] *Universidad Nacional Autónoma de México, Coyoacán, México D.F., México*
[3] *Universitat Politècnica de Catalunya, Barcelona, Spain.*

Abstract. C. Merino [Electron. J. Combin. 15 (2008)] showed that the Tutte polynomial of a complete graph satisfies $t(K_{n+2}; 2, -1) = t(K_n; 1, -1)$. We first give a bijective proof of this identity based on the relationship between the Tutte polynomial and the inversion polynomial for trees. Next we move to our main result, a sufficient condition for a graph G to have two vertices u and v such that $t(G; 2, -1) = t(G - \{u, v\}; 1, -1)$; the condition is satisfied in particular by the class of threshold graphs. Finally, we give a formula for the evaluation of $t(K_{n,m}; 2, -1)$ involving up-down permutations.

Résumé. C. Merino [Electron. J. Combin. 15 (2008)] a montré que le polynôme de Tutte du graphe complet satisfait $t(K_{n+2}; 2, -1) = t(K_n; 1, -1)$. Le rapport entre le polynôme de Tutte et le polynôme d'inversions d'un arbre nous permet de donner une preuve bijective de cette identité. Le résultat principal du travail est une condition suffisante pour qu'un graphe ait deux sommets u et v tels que $t(G; 2, -1) = t(G - \{u, v\}; 1, -1)$; en particulier, les graphes "threshold" satisfont cette condition. Finalement, nous donnons une formule pour $t(K_{n,m}; 2, -1)$ qui fait intervenir les permutations alternées.

Keywords: Tutte polynomial, increasing tree, threshold graph, generating function, up-down permutation

1 Introduction

The Tutte polynomial is one of the most studied polynomial graph invariants. For a graph $G = (V, E)$, it is given by

$$t(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(G)-r(A)} (y-1)^{|A|-r(A)},$$

where $r(A)$ is the *rank* of A, defined as $|V| - c(G_A)$, where $c(G_A)$ is the number of connected components of the spanning subgraph $G_A = (V, A)$ induced by A. (Although the definition of the Tutte polynomial allows multiple edges and loops, all graphs in this paper are simple.)

We refer to (7) for details about the many combinatorial interpretations of evaluations of the Tutte polynomial at various points of the plane and also along several algebraic curves. For example, $t(G; 1, 1)$

†The third and fouth authors are partially supported by Projects MTM2008-03020 and Gen. Cat. DGR 2009SGR1040.

is the number of spanning trees of G when G is connected and $t(G;2,1)$ is the number of spanning forests of G. A pair of interpretations especially relevant to the context of this paper are that $t(G;2,0)$ is the number of acyclic orientations of G and that $t(G;1,0)$ is the number of acyclic orientations of G with a unique fixed source. With this in mind, it follows that $t(K_{n+1};1,0)=t(K_n;2,0)$ (in fact, the same is true of any graph G with a universal vertex). As for curves, the hyperbolae $H_q=\{(x,y):(x-1)(y-1)=q\}$ play a significant role in the theory of the Tutte polynomial. In particular, for $q\in\mathbb{N}$ the Tutte polynomial specializes on H_q to the partition function of the q-state Potts model.

In this paper we shall be concerned with evaluations of the Tutte polynomial at the points $(1,-1)$ and $(2,-1)$. Merino (6) proved the following identity, which is the starting point for our work:

$$t(K_{n+2};1,-1)=t(K_n;2,-1).$$

Non-trivial relationships between evaluations of the Tutte polynomial at points on different hyperbolae are uncommon. Here, the point $(2,-1)$ lies on the hyperbola H_{-2} and $(1,-1)$ on the hyperbola H_0.

We are interested in whether there are other graphs G with the property that $t(G;1,-1)=t(G-\{u,v\};2,-1)$ for some pair of vertices u and v. Merino's proof when G is a complete graph uses generating functions. It is not very difficult to adapt his proof to show the property holds for complete bipartite graphs and for graphs that are the join of a clique and a coclique. (By a clique we mean a complete graph, K_n, and by a coclique a graph with no edges, $\overline{K}_n$; the join $G+H$ of two graphs G and H is formed by taking their disjoint union and adding an edge between each vertex of G and each vertex of H.) Our main result (Theorem 1 below) generalizes these examples by giving sufficient conditions for a graph to have this property; moreover, it describes graphs for which the property still holds when each vertex is replaced by an arbitrary clique or coclique of twin vertices.

The rest of the paper is organized as follows. Section 2 is devoted to giving a bijective proof of Merino's theorem by using an interpretation of the Tutte polynomial given by Gessel and Sagan (2). After introducing the necessary notation, in Section 3 we state the main theorem and discuss its consequences. Section 4 is devoted to its proof, including some intermediate results. Finally, in Section 5 we give a formula for $t(G;2,-1)$ when G is a complete bipartite graph, $K_{n,m}$, or a graph of the form $K_n+\overline{K}_m$.

2 Bijective proof

We start by giving a bijective proof of Merino's identity

$$t(K_{n+2};1,-1)=t(K_n;2,-1). \tag{1}$$

To translate this identity into combinatorial terms, we use an interpretation of the Tutte polynomial due to Gessel and Sagan (2). They express $t(G;1+x,y)$ as a generating function of spanning forests of G according to the number of connected components and an "external activity" $\epsilon(F)$ (it is not the usual external activity for trees as defined by Tutte). More concretely, let $\mathcal{T}(G)$ and $\mathcal{F}(G)$ be the set of spanning trees and spanning forests of a graph G, respectively (assume G is connected from now on). The evaluations we are interested in are

$$t(G;1,y)=\sum_{T\in\mathcal{T}(G)}y^{\epsilon(T)},\quad t(G;2,y)=\sum_{F\in\mathcal{F}(G)}y^{\epsilon(F)}. \tag{2}$$

Moreover, we want first to look at these expressions when G is a complete graph. We recall the facts from (2) needed for this. Consider the usual order on $[n]$ and root each tree in a forest in $\mathcal{F}(K_n)$ at its smallest vertex. We say that a vertex u *precedes* a vertex v if u and v are in the same component and u lies on the unique path from the root to v. Then the external activity $\epsilon(F)$ of a forest F is equal to the number of inversions of F, where an *inversion* is a pair (u, v) such that u precedes v in F and v is smaller than u. Therefore, $t(K_n; 1, y)$ is the generating function for inversions in trees with n vertices (rooted at 1) and $t(K_n; 2, y)$ is the generating function for inversions in forests with n vertices. Henceforth we use the notation $\mathrm{inv}(F)$ instead of $\epsilon(F)$ for referring to the external activity of a spanning forest of K_n.

Remark. The fact that the Tutte polynomial of K_n at $x = 1$ is the inversion polynomial is well known. Gessel and Wang (3) prove that the inversion polynomial is the generating function of connected subgraphs of K_n counted by number of edges, and Beissinger (1) gives a bijection between trees counted by numbers of inversions and by (Tutte's) external activity. Kuznetsov, Pak and Postnikov (4) prove that $t(K_n; 1, y)$ is the inversion polynomial by showing they satisfy the same recurrence relation.

Let $\mathcal{T}_n$ denote the set of labelled trees on $[n]$ rooted at 1, and similarly let $\mathcal{F}_n$ denote the set of labelled forests on $[n]$ where each component is rooted at its minimum vertex. Identity (1) can be then rephrased as

$$\sum_{T \in \mathcal{T}_{n+2}} (-1)^{\mathrm{inv}(T)} = \sum_{F \in \mathcal{F}_n} (-1)^{\mathrm{inv}(F)}.$$

To prove this identity, we first cancel out some terms in the sums, so that all remaining terms are positive. A forest $F \in \mathcal{F}_n$ is *increasing* if it has no inversions and it is *even* if all non-root vertices have an even number of children.

Lemma 1 *(i)* $\sum_{T \in \mathcal{T}_n} (-1)^{\mathrm{inv}(T)}$ *equals the number of even increasing trees in* $\mathcal{T}_n$.

(ii) $\sum_{F \in \mathcal{F}_n} (-1)^{\mathrm{inv}(F)}$ *equals the number of even increasing forests of* $\mathcal{F}_n$.

Proof: The second statement follows directly from the first, which appears in (4). The proof given there proceeds by showing that even increasing trees are counted by up-down (or alternating) permutations, which in turn satisfy the same recurrence as the inversion polynomial evaluated at $y = -1$. Alternatively, it is not very difficult to define an involution on trees that fixes even increasing trees and reverses the parity of the number of inversions in the remaining trees (see the end of Section 3.3 of (4)). □

To complete the proof of identity (1), we give a bijection between even increasing trees with $n + 2$ vertices and even increasing forests with n vertices. The core of the bijection is the following lemma, whose easy proof is omitted.

Lemma 2 *Let* T *be an even increasing tree and let* u *be any vertex of* T. *Then the forest* F *obtained from* T *by removing all edges in the unique path from the root to* u *is even and increasing.*

Now, given an even increasing tree T with $n+2$ vertices we construct an even increasing forest F with n vertices. First, remove the edges of the path that goes from 1 to $n + 2$, resulting in an even increasing forest with $n+2$ vertices. Now remove vertices 1 and $n+2$ (the latter being an isolated vertex), obtaining an even increasing forest with n vertices labelled from 2 to $n + 1$. Relabel them from 1 to n to obtain the desired forest. See Figure 1 for an illustration of the process.

Conversely, we show how to recover T if F is given. First, increase all the labels by 1, so that they run from 2 to $n + 1$. Of the components of F, let $T_1, \ldots, T_k$ be those where the root has even degree and let $U_1, \ldots, U_l$ be those with odd root-degree; let the roots of these components be $r_1, \ldots, r_k$ and $s_1, \ldots, s_l$, respectively, and assume also that $s_1 < s_2 < \cdots < s_l$. Construct T by adding vertices 1 and $n + 2$ and edges $\{1, r_1\}, \ldots, \{1, r_k\}, \{1, s_1\}, \{s_1, s_2\}, \ldots, \{s_l, n + 2\}$. It is clear that this procedure recovers T.

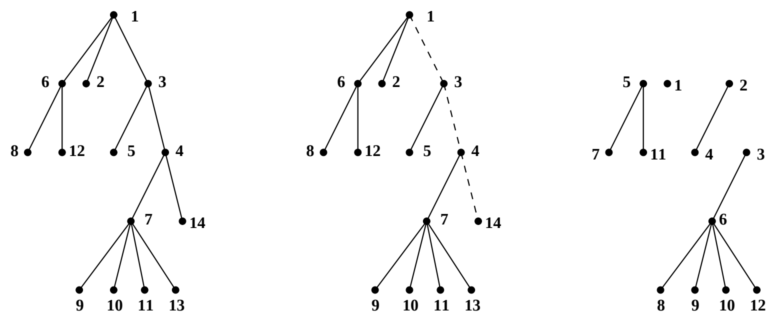

Fig. 1: Obtaining an even increasing forest from an even increasing tree.

3 Statement of the main result

As mentioned before, it is not difficult to adapt Merino's generating function proof to show that complete bipartite graphs $K_{n,m}$ and the graphs $K_n + \overline{K}_m$ satisfy an identity analogous to (1). Since both these graphs can be seen as a graph K_2 where each vertex has been substituted by a clique or a coclique, it seems natural to consider graphs constructed from a fixed graph by replacing vertices by cliques or cocliques.

Let $\mathbb{N}$ denote the set of non-negative integers. Given a connected graph $G = (V, E)$, $\mathbf{n} \in \mathbb{N}^V$ and $\mathbf{c} \in \{0,1\}^V$, define $G(\mathbf{c}; \mathbf{n})$ to be the graph obtained from G by replacing each vertex $k \in V$ by K_{n_k} if $c_k = 1$ or by $\overline{K}_{n_k}$ if $c_k = 0$; then, for each edge $kl \in E$ join the (co)clique on n_k vertices to the (co)clique on n_l vertices by adding an edge for each of the $n_k n_l$ pairs of vertices. For example, $K_1(1; n) = K_n$, $K_1(0; n) = \overline{K}_n$ and $K_2((0,0); (m,n)) = K_{m,n}$. Note that $K_r((1,1,\dots,1); (n_1,\dots,n_r)) = K_1(1; n_1 + \cdots + n_r) = K_{n_1+n_2+\cdots+n_r}$.

We are looking for parameters $G, \mathbf{c}$ with the property that for all $\mathbf{n} \in \mathbb{N}^V$ there exist vertices u, v of $G(\mathbf{c}; \mathbf{n})$ such that $t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}, \mathbf{n}) - \{u, v\}; 2, -1)$, where $n_i, n_j \geq 1$ if u, v belong to the (co)cliques at vertices i, j of G. In fact, we shall find $i, j \in V$ such that for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$ we have

$$t(G(\mathbf{c}; \mathbf{n}); 1, -1) = t(G(\mathbf{c}; \mathbf{n}'); 2, -1) \tag{3}$$

where $\mathbf{n}'$ is obtained from $\mathbf{n}$ by subtracting 1 from the ith and jth components. In other words, the vertices u, v of $G(\mathbf{c}; \mathbf{n})$ are taken from the fixed (co)cliques that replace the vertices i and j of G in making the graph $G(\mathbf{c}; \mathbf{n})$.

Theorem 2 in Section 4 characterizes pairs $(G, \mathbf{c})$ for which this holds. The following theorem rewrites this characterization in terms of induced subgraphs. (See Figure 2 for an illustration of the statement.) For a subset of vertices $U \subseteq V$, $G[U]$ denotes the subgraph of G induced by the vertices in U.

Theorem 1 *Let $G = (V, E)$ be a simple graph and i and j distinct vertices of G such that $\{i, j\}$ is a vertex cover of G. Let $A = \{v \in V \setminus \{i,j\} : vi \in E, vj \notin E\}$, $B = \{v \in V \setminus \{i,j\} : vi \notin E, vj \in E\}$ and $C = \{v \in V \setminus \{i,j\} : vi \in E, vj \in E\}$.*

Then $t(G; 1, -1) = t(G - \{i, j\}; 2, -1)$ if the following conditions hold:

(i) $G[A]$ and $G[B]$ are cocliques, and $G[C \cup \{i, j\}]$ is a clique (in particular, $ij \in E$);

(ii) there is no induced pair of disjoint edges $2P_2$ with endpoints in $A \cup B$, nor an induced path of length three P_4 with both endpoints in A or both endpoints in B;

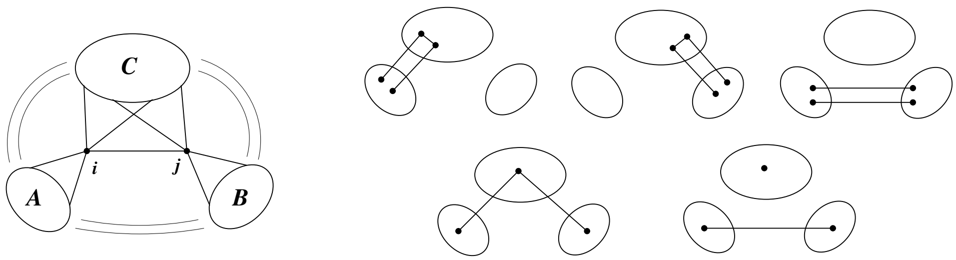

Fig. 2: On the left, structure of the graph described in Theorem 1; A and B induce cocliques, and $C \cup \{i,j\}$ induces a clique. On the right, the five forbidden induced subgraphs.

(iii) there is no induced path of length two P_3 with one endpoint in A and the other in B, nor the complement of such a path.

Furthermore, if G satisfies these conditions then so does any graph obtained from G by replacing a vertex of $A \cup B \cup \{i,j\}$ by a coclique of twin vertices, or a vertex of $C \cup \{i,j\}$ by a clique of twin vertices.

Since K_2 satisfies the conditions of the theorem (it is the simplest case $A = B = C = \emptyset$), we recover complete graphs, complete bipartite graphs, and the join of a clique and a coclique. If we take $G = K_3$, we have $A = B = \emptyset$ and $|C| = 1$. This means that we cannot replace the three vertices of a K_3 by cocliques, but all the other possibilities are fine.

The case $B = \emptyset$ gives a much richer class of graphs, threshold graphs. These are the graphs for which the vertices can be ordered so that each one is adjacent to either all or none of the previous ones. They are also the graphs with no induced P_4, C_4 or $2P_2$. (See (5) for a wealth of characterizations and applications.)

Corollary 1 *Let G be a threshold graph and let u and v be the first and the last vertex in an ordering as above. Then $t(G;1,-1) = t(G - \{u,v\};2,-1)$.*

It is by no means the case that all graphs G for which there exist two vertices $\{u,v\}$ such that $t(G;1,-1) = t(G - \{u,v\};2,-1)$ arise from Theorem 1. For instance, taking G to be a cycle of length 6 and u,v two vertices at distance two in the cycle yields such a graph.

4 Proof of the main result

This section is devoted to proving Theorem 1. We begin by finding the generating function for the Tutte polynomials of the family $G(\mathbf{c},\mathbf{n})$ and then we express the relationship between the evaluations at $(1,-1)$ and $(2,-1)$ as a differential equation. The statement of Theorem 2 is read from the solutions of this equation, and finally Theorem 1 is deduced.

Let us fix a connected graph G with two distinguished vertices i,j and a $\{0,1\}$-labelling of the vertices, that is, $\mathbf{c} \in \{0,1\}^V$. We look for conditions so that $G(\mathbf{c};\mathbf{n})$ satisfies (3) for all $\mathbf{n} \in \mathbb{N}^V$ with $n_i, n_j \geq 1$.

The following are well-known facts: $t(K_2;x,y) = x$, $t(K_3;x,y) = x^2 + x + y$, $t(\overline{K}_n;x,y) = 1$, and if G has blocks $G_1,\ldots,G_k$, then $t(G;x,y) = t(G_1;x,y)\cdots t(G_k;x,y)$. From this it follows that every vertex $k \in V \backslash \{i,j\}$ is adjacent to either i or j. Indeed, suppose k is not adjacent to either i or j, and choose a neighbour l of k. Then it is easy to check that equation (3) does not hold if we take $\mathbf{n}$ to be zero everywhere except $n_i = n_j = n_k = n_l = 1$. So from now on we assume that i and j together cover V.

The proof relies on the use of generating functions. Let $\mathbf{u} = (u_k : k \in V)$ and define

$$T(x,y;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} t(G(\mathbf{c};\mathbf{n});x,y)\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}, \qquad \mathbf{u}^\mathbf{n} = \prod_k u_k^{n_k}, \quad \mathbf{n}! = \prod_k n_k!,$$

taking $t(G(\mathbf{c},\mathbf{0});x,y) = t(\emptyset;x,y) = 1$. Equation (3) holds if and only if

$$\frac{\partial^2 T(1,-1;\mathbf{u})}{\partial u_i \partial u_j} = T(2,-1;\mathbf{u}). \tag{4}$$

The next lemma follows by a change of variables.

Lemma 3 *Let $G = (V,E)$ be a connected graph containing vertices i and j such that $ki \in E$ or $kj \in E$ for every $k \in V \setminus \{i,j\}$. Define*

$$S(z,w;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} \frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!} \sum_{A\subseteq E(G(\mathbf{c};\mathbf{n}))} z^{|A|}w^{c(A)}.$$

Then

$$\frac{\partial^2 T(x,y;\mathbf{u})}{\partial u_i \partial u_j} = \frac{1}{x-1}\frac{\partial^2 S(y-1,(x-1)(y-1);\frac{\mathbf{u}}{y-1})}{\partial u_i \partial u_j} \tag{5}$$

and

$$T(2,y;\mathbf{u}) = S(y-1,y-1;\frac{\mathbf{u}}{y-1}). \tag{6}$$

As an induced subgraph of $G(\mathbf{c};\mathbf{n})$ is of the form $G(\mathbf{c};\mathbf{m})$ for some $\mathbf{m}$, we deduce that $S(z,w;\mathbf{u}) = e^{C(z;\mathbf{u})w}$, where $C(z;\mathbf{u})$ is the exponential generating function (EGF) for connected spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ (counted by number of edges). The term $F(z;\mathbf{u}) = e^{C(z;\mathbf{u})}$ is the EGF for spanning subgraphs of $\{G(\mathbf{c};\mathbf{n}) : \mathbf{n} \in \mathbb{N}^V\}$ and it is given by

$$F(z;\mathbf{u}) = \sum_{\mathbf{n}\in\mathbb{N}^V} (1+z)^{q(\mathbf{n})}\frac{\mathbf{u}^\mathbf{n}}{\mathbf{n}!}, \quad \text{with} \quad q(\mathbf{n}) = \sum_{kl\in E} n_k n_l + \sum_{k\in V} c_k \binom{n_k}{2}. \tag{7}$$

Let $f(\mathbf{u}) = F(-2;\mathbf{u})$. By combining Lemma 3 and Equation (7), Equation (4) becomes

$$\frac{\partial f(\mathbf{u})}{\partial u_i}\frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u})\frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = 2. \tag{8}$$

Solving the differential equation (8) will put conditions on the quadratic form $q(\mathbf{n})$ that translate to structural conditions on the graph G and the clique/coclique parameter $\mathbf{c}$ that together specify the graph $G(\mathbf{c};\mathbf{n})$. This is Theorem 2 below.

We use $\mathbb{I}(P)$ to denote the indicator function, equal to 1 when the statement P is true and 0 otherwise.

Theorem 2 *A pair G and $\mathbf{c}$ satisfies equation (3) for all $\mathbf{n}$ if and only if the following conditions hold:*

(i) $ij \in E$*;*

(ii) for each $k \in V \setminus \{i, j\}$*,* $\mathbb{I}(ki \in E) + \mathbb{I}(kj \in E) = c_k + 1$*;*

(iii) for all $U \subseteq V \setminus \{j\}$*, either* j *has odd degree in* $G[U \cup \{j\}]$ *or there is a vertex* $k \in U$ *whose degree in the induced subgraph* $G[U]$ *has the same parity as* c_k*.*

Proof: We have already observed that each $k \in V \setminus \{i, j\}$ must be adjacent to at least one of i and j. We now wish to find all f that solve equation (8). We differentiate the expression for $f(\mathbf{u})$ in terms of $q(\mathbf{n})$, obtaining

$$\frac{\partial f(\mathbf{u})}{\partial u_i} = \sum_{\mathbf{n} \in \mathbb{N}^V} (-1)^{q(\mathbf{n}) + \Delta_i q(\mathbf{n})} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!}, \qquad \frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} = \sum_{\mathbf{n} \in \mathbb{N}^V} (-1)^{q(\mathbf{n}) + \Delta_{i,j} q(\mathbf{n})} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!},$$

where $\Delta_i q(\mathbf{n}) = q(\ldots, n_i + 1, \ldots) - q(\ldots, n_i, \ldots)$ and $\Delta_{i,j} q(\mathbf{n}) = q(\ldots, n_i + 1, \ldots, n_j + 1, \ldots) - q(\ldots, n_i, \ldots, n_j, \ldots)$.

Multiplying power series we find that

$$\frac{\partial f(\mathbf{u})}{\partial u_i} \frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u}) \frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} =$$

$$\sum_{\mathbf{n} \in \mathbb{N}^V} \frac{\mathbf{u}^{\mathbf{n}}}{\mathbf{n}!} \sum_{\mathbf{m} \leq \mathbf{n}} (-1)^{q(\mathbf{m}) + q(\mathbf{n} - \mathbf{m})} \left((-1)^{\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n} - \mathbf{m})} - (-1)^{\Delta_{i,j} q(\mathbf{m})} \right) \prod_k \binom{n_k}{m_k}. \tag{9}$$

(Here we write $\mathbf{m} \leq \mathbf{n}$ to mean $m_k \leq n_k$ for each $k \in V$.)

After some manipulation, we find that the relative parity of $\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n} - \mathbf{m})$ and $\Delta_{i,j} q(\mathbf{m})$ is given by

$$\Delta_i q(\mathbf{m}) + \Delta_j q(\mathbf{n} - \mathbf{m}) + \Delta_{i,j} q(\mathbf{m}) \equiv \sum_{k \sim j} n_k + \mathbb{I}(i \sim j) \pmod 2, \tag{10}$$

where two vertices a, b satisfy $a \sim b$ either if $ab \in E$ or if $a = b$ and $c_a = 1$. If the right-hand side of equation (10) is zero then the coefficient of $\mathbf{u}^{\mathbf{n}}$ in equation (9) is equal to zero. Since the constant term ($\mathbf{n} = \mathbf{0}$) should be equal to 2 we must have $i \sim j$. Therefore, we need to focus only on the coefficients of $\mathbf{u}^{\mathbf{n}}$ where $\sum_{k \sim j} n_k \equiv 0 \pmod 2$, which are the ones we still do not know are equal to zero. For them we find the expression

$$\frac{1}{\mathbf{n}!} [\mathbf{u}^{\mathbf{n}}] \left(\frac{\partial f(\mathbf{u})}{\partial u_i} \frac{\partial f(\mathbf{u})}{\partial u_j} - f(\mathbf{u}) \frac{\partial^2 f(\mathbf{u})}{\partial u_i \partial u_j} \right) = 2 \sum_{\mathbf{m} \leq \mathbf{n}} (-1)^{q(\mathbf{m}) + q(\mathbf{n} - \mathbf{m}) + \Delta_{i,j} q(\mathbf{m})} \prod_k \binom{n_k}{m_k}.$$

So we wish to find necessary and sufficient conditions for this coefficient of $\frac{1}{\mathbf{n}!} \mathbf{u}^{\mathbf{n}}$ to equal zero for all $\mathbf{n} \neq \mathbf{0}$ subject to $\sum_{k \sim j} n_j \equiv 0 \pmod 2$ and $i \sim j$. After some easy manipulation, we find that the coefficient we are interested in can be rewritten as:

$$\begin{aligned}
0 &= \sum_{\mathbf{m} \leq \mathbf{n}} (-1)^{\sum_k m_k \sum_{l \sim k} [n_l + \mathbb{I}(l = i) + \mathbb{I}(l = j) + \mathbb{I}(l = k)]} \prod_k \binom{n_k}{m_k} \\
&= \prod_k \sum_{m_k \leq n_k} (-1)^{\left[\sum_{l \sim k} n_l + \mathbb{I}(i \sim k) + \mathbb{I}(j \sim k) + \mathbb{I}(k \sim k)\right] m_k} \binom{n_k}{m_k} \\
&= \prod_k \left[1 + (-1)^{\sum_{l \sim k} n_l + \mathbb{I}(i \sim k) + \mathbb{I}(j \sim k) + \mathbb{I}(k \sim k)} \right]^{n_k}.
\end{aligned} \tag{11}$$

By taking each n_k to be even, for the expression (11) to be zero it is necessary that, for each $k \in V$,

$$\mathbb{I}(i \sim k) + \mathbb{I}(j \sim k) + \mathbb{I}(k \sim k) \equiv 1 \pmod 2. \tag{12}$$

Thus if $c_k = 1$ in $G(\mathbf{c}; \mathbf{n})$ (a clique) the vertex k must be adjacent to both i and j, whereas if $c_k = 0$ (a coclique) then the vertex k must be adjacent to exactly one of i, j. Since by assumption $i \sim j$ and $\sum_{l \sim j} n_l \equiv 0 \pmod 2$ we can assume $n_j = 0$, otherwise we have a zero factor and we are done.

Since expression (11) depends only on the parity of each n_k, it is enough to look at $n_k \in \{0, 1\}$. In terms of the graph G, this is to say we may assume each vertex k is either deleted or is present as a single vertex; if this graph satisfies the required conditions then so does $G(\mathbf{c}; \mathbf{n})$ for all $\mathbf{n} \in \mathbb{N}^V$.

Define $U \subseteq V \setminus \{j\}$ by $U = \{k \in V : n_k \neq 0\}$. Since we assume $\sum_{k \sim j} n_k \equiv 0 \pmod 2$ we restrict attention to U such that the induced subgraph $G[U]$ of G has the property that the number of vertices $k \in U$ such that $kj \in E$ is even. A necessary and sufficient condition that expression (11) is zero (under the assumption that $i \sim j$, $n_j = 0$ and $\sum_{k \sim j} n_k \equiv 0 \pmod 2$) is that for any such choice of U there is a vertex k of $G[U]$ of odd degree if $k \sim k$ or of even degree if $k \not\sim k$ (i.e., there is a vertex k of degree the same parity as c_k in the induced subgraph on U). □

From this theorem we wish now to deduce the induced subgraph characterization of Theorem 1. First we need to give some properties of the pairs $(G, \mathbf{c})$ that satisfy the conditions of Theorem 2. Condition (ii) implies that the following sets partition $V \setminus \{i, j\}$ (recall Figure 2):

$$\begin{aligned} A &= \{k \in V \setminus \{i, j\} : ki \in E, c_k = 0\}, \\ B &= \{k \in V \setminus \{i, j\} : kj \in E, c_k = 0\}, \\ C &= \{k \in V \setminus \{i, j\} : ki, kj \in E, c_k = 1\}. \end{aligned}$$

The next lemma is equivalent to saying that the values of c_i and c_j can be chosen freely.

Lemma 4 *If $G = (V, E)$, $i, j \in V$ and $\mathbf{c} \in \{0, 1\}^V$ satisfy the conditions of Theorem 2, then so do G and $\mathbf{c}'$ where $\mathbf{c}'$ is $\mathbf{c}$ with c_i replaced by $1 - c_i$ or with c_j replaced by $1 - c_j$ (or both).*

Proof: The conditions of Theorem 2 are clearly independent of the value of c_j. To see they do not depend on the value of c_i either, suppose on the contrary that there is an induced subgraph $G[U]$ with $i \in U \subseteq V \setminus \{j\}$ such that j has even degree in $G[U \cup \{j\}]$ and where vertex i is the only one in $G[U]$ with degree congruent to $c_i \pmod 2$, as required by condition (iii) of Theorem 2.

Suppose first that $c_i = 0$ and set $A' = A \cap U$, $B' = B \cap U$ and $C' = C \cap U$. The degree of i in $G[U]$ is $|A'| + |C'|$ and the degree of j in $G[U \cup \{j\}]$ is $1 + |B'| + |C'|$; since both degrees are even, we conclude that $|A'| + |B'|$ is odd. Since the vertices in $A' \cup B'$ are by assumption the ones that have odd degree in $G[U]$, we reach a contradiction because no graph has an odd number of vertices of odd degree.

The case $c_i = 1$ is treated by an analogous parity argument. □

Corollary 2 *The induced subgraphs $G[A]$ and $G[B]$ are cocliques and the induced subgraph $G[C \cup \{i, j\}]$ is a clique.*

Lemma 4 and Corollary 2 imply that condition (iii) of Theorem 2 is satisfied if and only if:

($\star$) for all $U \subseteq V \setminus \{i, j\}$ such that $|U \cap (B \cup C)|$ is even, the induced subgraph $G[U]$ contains either a vertex in $A \cup B$ of even degree or a vertex in C of odd degree.

Proof of Theorem 1. We prove that if G contains none of the five induced subgraphs described in the statement of Theorem 1 (and depicted in Figure 2), then condition $(\star)$ holds.

Suppose for a contradiction that there is $U \subseteq V \setminus \{i,j\}$ that contains none of the five induced subgraphs and for which condition $(\star)$ fails, that is, $|U \cap (B \cup C)|$ is even, all vertices in $U \cap (A \cup B)$ have odd degree and all vertices in $U \cap C$ have even degree. We lose no generality by assuming that $U \cap A = A, U \cap B = B, U \cap C = C$. For any vertex $x \in U$, let A_x (respectively, B_x, C_x) be its set of neighbours in A (resp., in B, C). The following two claims hold because otherwise we could find one of the forbidden induced subgraphs.

Claim 1. Let D be one of A, B, or C and let E be one of $\{A,B,C\} \setminus \{D\}$. If $x, y \in D$, then E_x and E_y are comparable sets.

Claim 2. If x is a vertex in C, then $A_x \cup B_x$ induces a complete bipartite graph.

Claim 1 with $D = C$ and $E = A$ implies that there is a vertex $a_0 \in A$ adjacent to all vertices of C that have at least one neighbour in A.

Now let $B' \subseteq B$ be the set of those vertices that are not adjacent to any vertex of C. If B' is non-empty, each of its vertices must be adjacent to at least one vertex in A, because vertices in B have odd degree. Suppose $b \in B'$ is adjacent to $a \in A$. If a is not adjacent to every vertex in C, then we find the fifth graph in Figure 2 as an induced subgraph, therefore a must be adjacent to all vertices in C. Now, since the neighbourhoods of vertices of B in A are nested (Claim 1), we conclude that there is some vertex in A adjacent to all vertices in B and hence to all vertices in C. But that makes this vertex have degree equal to $|B \cup C|$, which is even and hence contradicts our assumption. Therefore, B' must be empty.

Hence, every vertex in B (if any) must be adjacent to at least one vertex in C. By Claim 1 again, there is a vertex in C adjacent to all vertices in B. Any vertex with this property must be adjacent to some vertex in A, and hence to a_0 as well (otherwise it would have degree $|B| + |C| - 1$, which is odd). Also, a_0 is adjacent to all vertices of B by Claim 2. Now, let C' be the vertices in C that are not adjacent to a_0. Since a_0 has odd degree and $|B \cup C|$ is even, $|C'|$ is odd. Now, any $c' \in C'$ cannot be adjacent to all of B, because we just showed that in this case it would be adjacent to a_0 as well. But then, if c' is not adjacent to, say, $b' \in B$, then the edge $a_0 b'$ and vertex c' form one of the forbidden induced subgraphs.

Therefore, we are forced to have $B = \emptyset$. Then either there is a vertex in C not adjacent to a_0, and hence to no vertex in A, or a_0 is adjacent to every vertex in C. But in the former case there is a vertex in C of odd degree and in the latter case a_0 has even degree. □

5 Evaluating $t(K_{n,m};2,-1)$ and $t(K_n + \overline{K}_m;2,-1)$

As mentioned in the proof of Lemma 1, it is known that $t(K_n;2,-1)$ is the number of up-down permutations of $[n+1]$. The corresponding exponential generating function is $\sec(t)(\tan(t)+\sec(t))$ (recall that the EGF for up-down permutations is $\tan(t)+\sec(t)$). In this section we focus on the evaluations $t(K_{n,m};2,-1)$ and $t(K_n + \overline{K}_m;2,-1)$.

Let $B(u,v)$ be the bivariate EGF for $t(K_{m,n};2,-1)$. Here are the first few values of $t(K_{m,n};2,-1)$ for $1 \le m \le n$. (That the first column is given by 2^n and the second by $(3^{n+1}-1)/2$ is easy to prove from the definition and properties of the Tutte polynomial.)

$n\backslash m$	1	2	3	4	5	6
1	2					
2	4	13				
3	8	40	176			
4	16	121	736	4081		
5	32	364	3008	21616	144512	
6	64	1093	12160	111721	927424	7256173

By Lemma 3 and Equations (6) and (7)

$$B(u,v) = F(-2;-u/2,-v/2)^{-2} = \left(\sum_{m,n\geq 0} (-1)^{mn} \frac{u^m}{(-2)^m m!} \frac{v^n}{(-2)^n n!} \right)^{-2}.$$

Since the EGF for $(-1)^{nm}$ is $e^u \cosh(v) + e^{-u} \sinh(v)$, using some hyperbolic function identities, we obtain $B(u,v) = (\cosh(u)\cosh(v) - \sinh(u) - \sinh(v))^{-1}$.

We would like to extract from $B(u,v)$ the coefficient of $u^m v^n$. Let D^m denote the operation of taking the derivative m times with respect to u. Then

$$D^m(B(u,v))\big|_{u=0} = \sum_{n\geq 0} t(K_{m,n};2,-1)\frac{v^n}{n!}.$$

Let $g = \cosh(u)\cosh(v) - \sinh(u) - \sinh(v)$. Applying the rule for the derivative of a product to the equality $D^m(g \cdot g^{-1}) = 0$ we obtain the following recursion

$$gD^m(g^{-1}) = -\sum_{k=0}^{m-1} \binom{m}{k} D^{m-k}(g)D^k(g^{-1}).$$

It is easy to show by induction that, for $i \geq 1$, $D^{2i}(g) = \cosh(u)\cosh(v) - \sinh(u)$ and $D^{2i-1}(g) = \sinh(u)\cosh(v) - \cosh(u)$. Evaluating at $u = 0$ and using the above recurrence, we arrive at

$$D^m(g^{-1})\big|_{u=0} = -e^v \left(\sum_{k=0}^{m-1} \binom{m}{k} D^k(g^{-1})\big|_{u=0} \left(\delta^0_{k,m}\cosh(v) - \delta^1_{k,m}\right) \right),$$

where $\delta^0_{k,m}$ (respectively, $\delta^1_{k,m}$) is equal to 1 if m and k have the same parity (resp., different parity), and zero otherwise.

Writing b_m for $D^m(g^{-1})\big|_{u=0}$, we have

$$b_m = \sum_{k=0}^{m-1} \binom{m}{k} b_k \left(e^v \delta^1_{k,m} - \frac{1}{2}(1+e^{2v})\delta^0_{k,m} \right).$$

Since $b_0 = e^v$, it follows that b_k is a linear combination of exponentials e^{lv}, the first ones being

$$e^v, e^{2v}, \frac{1}{2}(3e^{3v} - e^v), \frac{1}{2}(6e^{4v} - 4e^{2v}), \frac{1}{2}(2e^v - 15e^{3v} + 15e^{5v}).$$

Let $b_{m,j}$ be the coefficient of e^{jv} in b_m, so that $t(K_{m,n}; 2, -1) = \sum_{j=1}^{m+1} b_{m,j} j^n$. The $b_{m,j}$ satisfy the recurrence

$$b_{m,j} = \sum_{k=0}^{m-1} \binom{m}{k} \left(b_{k,j-1}\delta^1_{k,m} - \frac{1}{2}(b_{k,j} + b_{k,j-2})\delta^0_{k,m} \right). \tag{13}$$

The first values of $b_{m,j}$ are given in the table below. From them one guesses the Pascal-like recurrence stated in Theorem 3, which can be proved by induction.

$m\backslash j$	1	2	3	4	5	6	7	8
0	1							
1	0	1						
2	-$\frac{1}{2}$	0	$\frac{3}{2}$					
3	0	-2	0	3				
4	1	0	-$\frac{15}{2}$	0	$\frac{15}{2}$			
5	0	$\frac{17}{2}$	0	-30	0	$\frac{45}{2}$		
6	-$\frac{17}{4}$	0	$\frac{231}{4}$	0	-$\frac{525}{4}$	0	$\frac{315}{4}$	
7	0	-62	0	378	0	-630	0	315

Theorem 3 *For* $m \geq 0$, $t(K_{m,n}; 2, -1) = \sum_{j=1}^{m+1} b_{m,j} j^n$, *where* $b_{m,j}$ *is given by*

$$b_{0,1} = 1,\ b_{m,0} = 0,\ b_{m,m+1} = 0,\ b_{m,j} = \frac{j}{2}(b_{m-1,j-1} - b_{m-1,j+1}) \ \textit{for}\ 1 \leq j \leq m.$$

In particular, $b_{m,j} = 0$ *if* m *and* j *are of the same parity.*

The generating function for the numbers $b_{m,j}$ is related to that of up-down permutations, as we now explain. Let $B_j(u) = \sum_{m \geq 0} b_{m,j} \frac{u^m}{m!}$. From the expression we have for $B(u, v)$, we obtain

$$B_1(u) = \frac{2}{1 + \cosh(u)}.$$

The recurrence for the $b_{m,j}$ in Theorem 3 gives

$$B_{j+1}(u) = B_{j-1}(u) - \frac{2}{j} B'_j(u).$$

Finally, solving this equation yields

$$B_j(u) = 2(\tanh^j(\frac{u}{2}))'.$$

Recall that $\tan(x)$ is the EGF of up-down permutations of $[n]$ for odd n (odd up-down permutations). Then $\tanh(x)$ is the EGF for *signed* odd up-down permutations, where the sign depends only on n and is given by $(-1)^{(n-1)/2}$. So $\tanh(x)^j$ is, up to signs, the EGF for permutations that can be decomposed as a sequence of j odd up-down permutations.

For instance, consider $b_{3,2}$. There is one odd up-down permutation of $[1]$ and two odd up-down permutations of [3]. There are thus 16 permutations of $[4]$ that can be split as a sequence of two odd up-down permutations. Then the coefficient of u^3 in $2(\tanh^2(u/2))'$ is $2 \cdot 16/(3! \cdot 2^4) = 2/3!$, which agrees with $|b_{3,2}| = 2$.

We can take similar steps to evaluate $t(K_m + \overline{K}_n; 2, -1)$ and obtain the following unexpected relationship.

Theorem 4 *For $m \geq 0$, $t(K_m + \overline{K}_n; 2, -1) = \sum_{j=1}^{m+1} c_{m,j} j^n$, where $c_{m,j} = b_{m,j}(-1)^{(m-j-1)/2}$.*

The EGF for the sequence $\{c_{m,j}\}_m$ follows immediately from the one for $\{b_{m,j}\}_m$:

$$\sum_{m \geq 0} c_{m,j} \frac{u^m}{m!} = (\tan(u)^j)'.$$

Let us conclude with the open problem of proving the identity $t(K_{n+1,m+1}; 1, -1) = t(K_{n,m}; 2, -1)$ bijectively. The interpretation of Gessel and Sagan (2) of the Tutte polynomial allows us again to recast this identity into combinatorial terms. We take $[n] \cup [m]' = [n] \cup \{1', 2', \ldots, m'\}$ as the vertex set of $K_{n,m}$. We call the vertices in $[n]$ *black* and the ones in $[m]'$ *white*. Black vertices among themselves are ordered by the usual order; the same applies to white vertices. A black vertex is smaller than a white one. A *white inversion* (respectively, *black inversion*) is an inversion where the two vertices involved are white (resp., black). Their union is the set of *monochromatic inversions of F* and its cardinality is $\mathrm{binv}(F)$.

Let $\mathcal{T}_{n,m}$ be the set of spanning trees of $K_{n,m}$ and let $\mathcal{F}_{n,m}$ be the set of spanning forests of $K_{n,m}$, with all trees rooted at its smallest element. A forest in $\mathcal{F}_{n,m}$ is *χ-increasing* if it has no monochromatic inversions, and it is *bi-even* if each non-root vertex has an even number of grandchildren (descendants at distance two). Then the identity $t(K_{n+1,m+1}; 1, -1) = t(K_{n,m}; 2, -1)$ is equivalent to the equality of the numbers of bi-even χ-increasing trees of $\mathcal{T}_{n+1,m+1}$ and of bi-even χ-increasing forests of $\mathcal{F}_{n,m}$.

Problem 1 *Find a bijection between bi-even χ-increasing trees of $\mathcal{T}_{n+1,m+1}$ and bi-even χ-increasing forests of $\mathcal{F}_{n,m}$.*

References

[1] J. S. Beissinger, On external activity and inversions in trees, *J. Combin. Theory Ser. B* **33** (1982), 87–92.

[2] I. M. Gessel and B. E. Sagan, The Tutte polynomial of a graph, depth-first search, and simplicial complex partitions, *Electron. J. Combin* **3** (1996), #R9.

[3] I. M. Gessel and D. L. Wang, Depth-first search as a combinatorial correspondence, *J. Combin. Theory Ser. A* **26** (1979), 308–313.

[4] A. G. Kuznetsov, I. M. Pak and A.E. Postnikov, Increasing trees and alternating permutations, *Russian Math. Surveys* **49** (1994), 79–114.

[5] N. V. R. Mahadev and U. N. Peled, *Threshold graphs and related topics.* (Annals of Discrete Mathematics, 56. North-Holland Publishing Co., 1995).

[6] C. Merino, The number of 0–1–2 increasing trees as two different evaluations of the Tutte polynomial of a complete graph, *Electron. J. Combin.* **15** (2008), #N28.

[7] D. J. A. Welsh, *Complexity: knots, colourings and counting*, (Cambridge Univ. Press, 1993).