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Allowed patterns of β-shifts

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Abstract. For a real number $\beta > 1$, we say that a permutation π of length n is allowed (or realized) by the β-shift if there is some $x \in [0, 1]$ such that the relative order of the sequence $x, f(x), \dots, f^{n-1}(x)$, where $f(x)$ is the fractional part of βx, is the same as that of the entries of π. Widely studied from such diverse fields as number theory and automata theory, β-shifts are prototypical examples of one-dimensional chaotic dynamical systems. When β is an integer, permutations realized by shifts have been recently characterized. In this paper we generalize some of the results to arbitrary β-shifts. We describe a method to compute, for any given permutation π, the smallest β such that π is realized by the β-shift.

Résumé. Pour un nombre réel $\beta > 1$, on dit qu'une permutation π de longueur n est permise (ou réalisée) par β-*shift* s'il existe $x \in [0, 1]$ tel que l'ordre relatif de la séquence $x, f(x), \dots, f^{n-1}(x)$, où $f(x)$ est la partie fractionnaire de βx, soit le même que celui des entrées de π. Largement étudiés dans des domaines aussi divers que la théorie des nombres et la théorie des automates, les β-*shifts* sont des prototypes de systèmes dynamiques chaotiques unidimensionnels. Quand β est un nombre entier, les permutations réalisées par décalages ont été récemment caractérisées. Dans cet article, nous généralisons certains des résultats au cas de β-*shifts* arbitraires. Nous décrivons une méthode pour calculer, pour toute permutation donnée π, le plus petit β tel que π soit réalisée par β-*shift*.

Keywords: beta-shift, forbidden pattern, consecutive pattern, shift map, real base expansion, dynamical system

1 Introduction

Forbidden order patterns in piecewise monotone maps on one-dimensional intervals are a powerful tool to distinguish random from deterministic time series. This contrasts with the fact that, from the viewpoint of symbolic dynamics, chaotic maps are able to produce any symbol pattern, and for this reason they are used in practice to generate pseudo-random sequences. However, this is no longer true when one considers order patterns instead, as shown in (1; 2). From now on, we will use the term patterns to refer to order patterns.

The allowed patterns of a map on a one-dimensional interval are the permutations given by the relative order of the entries in the finite sequences (usually called orbits) obtained by successively iterating the map, starting from any point in the interval. For any fixed piecewise monotone map, there are some permutations that do not appear in any orbit. These are called the forbidden patterns of the map. Understanding the forbidden patterns of chaotic maps is important because the absence of these patterns is what distinguishes sequences generated by chaotic maps from random sequences.

Determining the allowed and forbidden patterns of a given map is a difficult problem in general. The only non-trivial family of maps for which the sets of allowed patterns have been characterized are shift maps. The first results in this direction are found in (1), and a characterization and enumeration of the allowed patterns of shift maps appears in (6). For another family, the so-called logistic map, a few basic properties of their set of forbidden patterns have been studied in (7).

The focus of this paper are the allowed and forbidden patterns of β-shifts, which are a natural generalization of shifts. The combinatorial description of β-shifts is more elaborate than that of shifts, yet still simple enough for β-shifts to be amenable to the study of their allowed patterns. At the same time, β-shifts are good prototypes of chaotic maps because they exhibit important properties of low-dimensional chaotic dynamical systems, such as sensitivity to initial conditions, strong mixing, and a dense set of periodic points. The origin of β-shifts lies in the study of expansions of real numbers in an arbitrary real base $\beta > 1$, which were introduced by Rényi (10). Measure-theoretic properties of β-shifts and their connection to these expansions have been extensively studied in the literature (see for example (3; 8; 9; 11)). For instance, it is known that the base-β expansion of β itself determines the symbolic dynamics of the corresponding β-shift. Finally, β-shifts have also been considered in computability theory (12).

Related to the study of the allowed patterns of β-shifts, we are interested in the problem of determining, for a given permutation π, what is the largest β such that π is a forbidden pattern of the β-shift. We call this parameter the shift-complexity of the permutation. Putting technical details aside, this problem is equivalent to finding the smallest β such that π is realized by (i.e., is an allowed pattern of) the β-shift.

In Section 2 we formally define allowed and forbidden patterns of maps, and we describe shifts and β-shifts from a combinatorial perspective. In Section 3 we study some properties of the domain of β-shifts, we define shift-complexity, and we introduce two relevant real-valued statistics on words. Sections 4 and 5 describe how to determine the shift-complexity of a given permutation π, by expressing this parameter as a root of a certain polynomial whose coefficients depend on π in a non-trivial way. In Section 6 we give examples of the usage of our method for particular permutations.

2 Background and notation

Let $[n] = \{1, 2, \dots, n\}$, and let $\mathcal{S}_n$ be the set of permutations of $[n]$. In the rest of the paper, the term permutation will always refer to an element of $\mathcal{S}_n$ for some n. For a real number x, we use $\lfloor x \rfloor$, $\lceil x \rceil$, and $\{x\}$ to denote the floor, ceiling, and fractional part of x, respectively. Most of the words considered in this paper will be infinite words over the alphabet $\{0, 1, 2, \dots\}$ that use only finitely many different letters.

2.1 Allowed patterns of a map

Given a finite sequence $x_1, x_2, \dots, x_n$ of different elements of a totally ordered set X, define its standardization $\mathrm{st}(x_1, x_2, \dots, x_n)$ to be the permutation of $[n]$ that is obtained by replacing the smallest element in the sequence with 1, the second smallest with 2, and so on. For example, $\mathrm{st}(4, 7, 1, 6.2, \sqrt{2}) = 35142$.

Fix a map $f : X \to X$. For each $x \in X$ and $n \geq 1$, we define

$$\mathrm{Pat}(x, f, n) = \mathrm{st}(x, f(x), f^2(x), \dots, f^{n-1}(x)) \in \mathcal{S}_n,$$

provided that there is no pair $1 \leq i < j \leq n$ such that $f^{i-1}(x) = f^{j-1}(x)$.

Given $\pi \in \mathcal{S}_n$, we say that π is *realized* by f, or that π is an *allowed pattern* of f, if there is some $x \in X$ such that $\mathrm{Pat}(x, f, n) = \pi$. The set of all permutations realized by f is denoted by $\mathrm{Allow}(f) = \bigcup_{n \geq 1} \mathrm{Allow}_n(f)$, where $\mathrm{Allow}_n(f) = \{\mathrm{Pat}(x, f, n) : x \in X\} \subseteq \mathcal{S}_n$. The remaining permutations are called *forbidden patterns* of f.

2.2 Shift maps

Special cases of dynamical systems are shift systems. Shifts are interesting from a combinatorial perspective due to their simple definition, and at the same time they are important dynamical systems because they exhibit some key features of low-dimensional chaos.

For each $N \geq 2$, let $\mathcal{W}_N$ be the set of infinite words on the alphabet $\{0, 1, \ldots, N-1\}$, equipped with the lexicographic order. The *shift on N symbols* is defined to be the map

$$\begin{array}{cccc} \Sigma_N : & \mathcal{W}_N & \longrightarrow & \mathcal{W}_N \\ & w_1 w_2 w_3 \ldots & \mapsto & w_2 w_3 w_4 \ldots . \end{array}$$

For a detailed description of the associated dynamical system, see (1). According to the above definitions, we have for example that $\mathrm{Pat}(2102212210\ldots, \Sigma_3, 7) = 4217536$.

Let $\Upsilon_N \subset \mathcal{W}_N$ be the set of all words of the form $u(N-1)^\infty$, where u is a finite word, and we use the notation $x^\infty = xxx\ldots$. Then $\mathcal{W}_N \setminus \Upsilon_N$ is closed under shifts, and the map

$$\begin{array}{cccc} \varphi : & \mathcal{W}_N \setminus \Upsilon_N & \longrightarrow & [0,1) \\ & w_1 w_2 w_3 \ldots & \mapsto & \sum_{i \geq 1} w_i N^{-i} \end{array}$$

is an order-preserving bijection, also called an *order-isomorphism*. The map $M_N = \varphi \circ \Sigma_N \circ \varphi^{-1}$ from $[0, 1)$ to itself is the so-called *sawtooth map*

$$M_N(x) = \{Nx\}.$$

We say in this case that Σ_N and M_N are order-isomorphic. As a consequence, Σ_N and M_N have the same allowed and forbidden patterns.

Allowed and forbidden patterns of shifts (equivalently, sawtooth maps) were first studied in (1), where the authors prove the following result.

Theorem 2.1 ((1)) *For $N \geq 2$, the shortest forbidden patterns of the shift Σ_N have length $N + 2$.*

For example, the shortest forbidden patterns of Σ_4 are $162534, 435261, 615243, 342516, 453621, 324156$. In fact, it was later shown in (6) that there are exactly six forbidden patterns of Σ_N of minimum length.

Proposition 2.2 ((6)) *For every $N \geq 2$, the shortest forbidden patterns of Σ_N, which have length $n = N + 2$, are $\{\rho, \rho^R, \rho^C, \rho^{RC}, \tau, \tau^C\}$, where*

$$\rho = 1\,n\,2\,(n-1)\,3\,(n-2)\,\ldots, \qquad \tau = \ldots\,4\,(n-1)\,3\,n\,2\,1,$$

and ^{R} and ^{C} denote the reversal (obtained by reading the entries from right to left) and complementation (obtained by replacing each entry i with $n+1-i$) operations, respectively.

A formula is given in (6) to compute, for any given permutation π, the minimum number of symbols needed in an alphabet in order for π to be realized by a shift, that is,

$$N(\pi) := \min\{N : \pi \in \mathrm{Allow}(\Sigma_N)\}. \tag{1}$$

The formula given to compute for $N(\pi)$ relies on a bijection between $\mathcal{S}_n$ and the set $\mathcal{T}_n$ of cyclic permutations of $[n]$ with a distinguished entry. For example, underlining the distinguished entry, we have

$$\mathcal{T}_3 = \{\underline{2}31, 2\underline{3}1, 23\underline{1}, \underline{3}12, 3\underline{1}2, 31\underline{2}\}.$$

Given $\pi = \pi(1)\pi(2)\dots\pi(n) \in \mathcal{S}_n$, let $\hat{\pi} \in \mathcal{T}_n$ be the permutation whose cycle decomposition is $(\pi(1), \pi(2), \dots, \pi(n))$, with the entry $\pi(1)$ distinguished. For example, if $\pi = 892364157$, then

$$\hat{\pi} = (\underline{8}, 9, 2, 3, 6, 4, 1, 5, 7) = 536174\underline{8}92.$$

For $\hat{\pi} \in \mathcal{T}_n$, let $\operatorname{des}(\hat{\pi})$ denote the number of descents of the sequence that we get by deleting the distinguished entry from the one-line notation of $\hat{\pi}$. For example, $\operatorname{des}(536174\underline{8}92) = 4$. With these definitions, we can now state the aforementioned formula for $N(\pi)$.

Theorem 2.3 ((6)) *Let $\pi \in \mathcal{S}_n$, and let $\hat{\pi}$ be defined as above. Then $N(\pi)$ is given by*

$$N(\pi) = 1 + \operatorname{des}(\hat{\pi}) + \epsilon(\hat{\pi}),$$

where $\epsilon(\hat{\pi}) = 1$ if the one-line notation of $\hat{\pi}$ starts with the distinguished entry followed by 1, or ends with n followed by the distinguished entry, and $\epsilon(\hat{\pi}) = 0$ otherwise.

The distribution of the descent sets of cyclic permutations is studied in (4). The goal of the present paper is to obtain a formula to compute the analogue of $N(\pi)$ for the more general case of β-shifts, which we define next.

2.3 β-shifts

These maps are a natural generalization of shift maps, and have been extensively studied in the literature (11; 8) from a measure-theoretic perspective. Let us begin by defining their order-isomorphic counterparts on the unit interval, which we call β-sawtooth maps. For any real number $\beta > 1$, define the β-sawtooth map

$$\begin{array}{cccc} M_\beta: & [0,1) & \longrightarrow & [0,1) \\ & x & \mapsto & \{\beta x\} \end{array}$$

(see Figure 1). In the rest of the paper we will assume, unless otherwise stated, that β is a real number with $\beta > 1$. Note that when β is an integer we recover the definition of standard sawtooth maps.

To describe the corresponding map on infinite words, called the β-shift, let us first define its domain, $W(\beta)$. As shown by Rényi (10), every nonnegative real number x has a β-expansion

$$x = w_0 + \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots,$$

where $w_0 = \lfloor x \rfloor$, $w_1 = \lfloor \beta\{x\} \rfloor$, $w_2 = \lfloor \beta\{\beta\{x\}\} \rfloor$, ... This expansion has the property that $w_i \in \{1, 2, \dots, \lceil \beta - 1 \rceil\}$ for $i \geq 1$. If $0 \leq x < 1$, then $w_0 = 0$ and $w_i = \lfloor \beta M_\beta^{i-1}(x) \rfloor$ for $i \geq 1$.

Let $W_0(\beta)$ be the set of infinite words $w = w_1 w_2 w_3 \dots$ that are obtained in this way as β-expansions of numbers $x \in [0,1)$. The lexicographic order (which will be denoted by $<$ throughout the paper) makes $W_0(\beta)$ into a totally ordered set. The map $[0,1) \to W_0(\beta)$, $x \mapsto w$ is an order-isomorphism. For any $w = w_1 w_2 w_3 \dots \in W_0(\beta)$, we can recover $x \in [0,1)$ as

$$x = \sum_{i \geq 1} w_i \beta^{-i}.$$

Of particular interest is the β-expansion of β itself, for which we use the notation

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots.$$

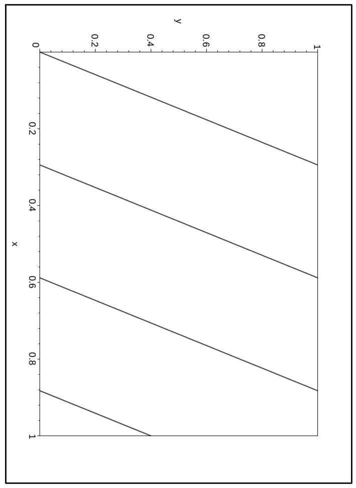

Fig. 1: The β-sawtooth map $M_\beta(x)$ for $\beta = 3.4$.

One can define $A_0 = \beta$, and for $i \geq 0$, $a_i = \lfloor A_i \rfloor$ and $A_{i+1} = \beta(A_i - a_i)$. Then it follows by induction that

$$A_i = \beta^{i+1} - a_0\beta^i - a_1\beta^{i-1} - \cdots - a_{i-1}\beta. \tag{2}$$

If β is such that its β-expansion is finite, i.e., it has only finitely many nonzero terms a_i, we let a_q be the last nonzero term of the expansion, so

$$\beta = a_0 + \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \cdots + \frac{a_q}{\beta^q},$$

and we let $y = (a_0a_1 \ldots a_{q-1}(a_q-1))^\infty$. Define

$$W(\beta) = \{w_1w_2w_3 \ldots : w_kw_{k+1}w_{k+2} \ldots < a_0a_1a_2 \ldots \text{ for all } k \geq 1\}. \tag{3}$$

It follows from Parry (9) that $W_0(\beta)$ is precisely the set of words in $W(\beta)$ that do not end in y. For example, if $\beta = N \in \mathbb{Z}$, then $W(N) = \mathcal{W}_N$, the set of infinite words on the alphabet $\{0, 1, \ldots, N-1\}$, whereas $W_0(N)$ does not include words ending in $(N-1)^\infty$. If $\beta = 1 + \sqrt{2}$, then $a_0a_1a_2 \ldots = 210^\infty$, so $W(\beta)$ is the set of words over $\{0, 1, 2\}$ where every 2 is followed by a 0, but $W_0(\beta)$ does not include words ending in $y = (20)^\infty$. Clearly, if β has an infinite β-expansion, then $W(\beta) = W_0(\beta)$.

We define the β-shift Σ_β to be the map

$$\begin{array}{cccc} \Sigma_\beta : & W(\beta) & \longrightarrow & W(\beta) \\ & w_1w_2w_3 \ldots & \mapsto & w_2w_3w_4 \ldots . \end{array}$$

For $x \in [0, 1)$, if $w \in W_0(\beta)$ is the word given by the β-expansion of x, then $\Sigma_\beta(w)$ is the word given by the β-expansion of $M_\beta(x)$. In particular, M_β and the restriction of Σ_β to $W_0(\beta)$ are order-isomorphic.

Besides, this restriction of the domain does not change the set of allowed patterns of Σ_β, and therefore $\mathrm{Allow}(\Sigma_\beta) = \mathrm{Allow}(M_\beta)$.

A well-studied problem is the connection between β-expansions and the ergodic properties of the corresponding β-shift (see (11) and references therein). In this paper, rather than the measure-theoretic properties of β-shifts, we are concerned with their allowed and forbidden patterns.

3 The shift-complexity of a permutation

In this section we establish some properties of the domain $W(\beta)$ of the β-shift, and we define a real-valued statistic on permutations, which we call the shift-complexity.

Proposition 3.1 *Let* $1 < \beta \le \beta'$. *Then* $W(\beta) \subseteq W(\beta')$ *and* $\mathrm{Allow}(\Sigma_\beta) \subseteq \mathrm{Allow}(\Sigma_{\beta'})$.

Proof: Let $a = a_0a_1\dots$ be the β-expansion of β, and let $a' = a'_0a'_1\dots$ be the β'-expansion of β'. To prove that $W(\beta) \subseteq W(\beta')$, using the definition in equation (3) it is enough to show that $a \le a'$. Let A_i be as defined in equation (2), and let A'_i be defined analogously for β'. Suppose that the first entry where a and a' differ is $a_i \ne a'_i$. We claim that $A_j \le A'_j$ for $0 \le j \le i$. This follows by induction since $A_0 = \beta \le \beta' = A'_0$, and if $A_j \le A'_j$ for some $j < i$, then $\lfloor A_j \rfloor = a_j = a'_j = \lfloor A'_j \rfloor$ implies that $A_j - a_j \le A'_j - a'_j$, so $A_{j+1} = \beta(A_j - a_j) \le \beta'(A'_j - a'_j) = A'_{j+1}$. But then $a_i = \lfloor A_i \rfloor \le \lfloor A'_i \rfloor = a'_i$, so $a_i < a'_i$ and we are done.

To prove that $\mathrm{Allow}(\Sigma_\beta) \subseteq \mathrm{Allow}(\Sigma_{\beta'})$, let $\pi \in \mathrm{Allow}(\Sigma_\beta)$. By definition, there exists a word $w \in W(\beta)$ such that $\mathrm{Pat}(w, \Sigma_\beta, n) = \pi$, where n is the length of π. But then $w \in W(\beta')$, and since $\mathrm{Pat}(w, \Sigma_{\beta'}, n) = \pi$, we see that $\pi \in \mathrm{Allow}(\Sigma_{\beta'})$. □

Because of Proposition 3.1, it is clear from the definition of β-shifts that for $\beta < \beta'$, the restriction of $\Sigma_{\beta'}$ to $W(\beta)$ is equal to Σ_β. In the rest of the paper, we will write Σ instead of Σ_β when it creates no confusion. Now we can give the key definition of this section. We call $B(\pi)$ the *shift-complexity* of π.

Definition 3.2 *For any permutation* π*, let*

$$B(\pi) = \inf\{\beta : \pi \in \mathrm{Allow}(\Sigma_\beta)\}.$$

Equivalently, $B(\pi)$ is the supremum of the set of values β such that π is a forbidden pattern of Σ_β. If we think of the β-shifts Σ_β as a family of functions parameterized by β, then the values of β for which there is a permutation π with $B(\pi) = \beta$ correspond to *phase transitions* where the set of allowed patterns of Σ_β changes.

To compute the value of $B(\pi)$ for given π, we will define two real-valued statistics on words. For an infinite word $w = w_1w_2\dots$, we use the notation $w_{i\to} = w_iw_{i+1}\dots$ for $i \ge 1$.

In the rest of this section, v and w denote words in $\mathcal{W}_N$ for some arbitrary positive integer N. We define the series

$$f_w(\beta) = \frac{w_1}{\beta} + \frac{w_2}{\beta^2} + \cdots + \frac{w_n}{\beta^n} + \cdots.$$

This series is convergent for $\beta > 1$, and in this interval,

$$f'_w(\beta) = -\frac{w_1}{\beta^2} - \frac{2w_2}{\beta^3} - \cdots < 0,$$

assuming that $w \neq 0^\infty$. Since $\lim_{\beta\to\infty} f_w(\beta) = 0$, it follows that there is a unique solution to $f_w(\beta) = 1$ satisfying $\beta \geq 1$. Such value of β will be denoted by $\hat{b}(w)$. We define $\hat{b}(0^\infty) = 0$ by convention. Additionally, let

$$b(w) = \sup_{i\geq 1} \hat{b}(w_{i\to}).$$

Note that $\hat{b}(w) \leq b(w) \leq N$.

The following result describes the relationship between the permutation statistic B and the word statistic b. Its proof is omitted in this extended abstract, but it appears in (5).

Proposition 3.3 *For any* $\pi \in \mathcal{S}_n$, $B(\pi) = \inf\{b(w) : \mathrm{Pat}(w, \Sigma, n) = \pi\}$.

4 Computation of $B(\pi)$: from permutations to words

Suppose we are given $\pi \in \mathcal{S}_n$ with $n \geq 2$. The goal of this section and the next one is to describe a method to compute the shift-complexity of π. In the rest of the paper, we refer to the condition $\mathrm{Pat}(w, \Sigma, n) = \pi$ by saying that w *induces* π. In some cases we will be able to find a word w inducing π such that $b(w)$ is smallest for all such words; when this happens, $B(\pi) = b(w)$ and the infimum in Proposition 3.3 is a minimum. In other cases we will find a sequence of words $w^{(m)}$ inducing π where $b(w^{(m)})$ approaches $B(\pi)$ as m grows. This section is devoted to finding a word w or a sequence $w^{(m)}$ with the above properties. In Section 5 we show how to compute the values of the statistic b on these words in order to obtain $B(\pi)$.

Let $N = N(\pi)$ for the rest of this section. From the definitions, it is clear that $B(\pi) \leq N$, and that there is some word $z \in W(N) = \mathcal{W}_N$ that induces π. The explicit construction of such words z is given in (6). It is important to notice that to find words w and $w^{(m)}$ as described above, it is enough to consider only words in $\mathcal{W}_N$. Indeed, if $B(\pi) = N$ (we will later see in equation (5) that this case never happens, but cannot rule it out just yet), then any $z \in \mathcal{W}_N$ inducing π satisfies $b(z) = B(\pi)$, and we can just take $w = z$. On the other hand, to deal with the case $B(\pi) < N$, note that any word z with $b(z) < N$ must be in $\mathcal{W}_N$. This applies to $z = w$ for any word w satisfying $b(w) = B(\pi)$, and also to $z = w^{(m)}$ for words in the above sequence, provided that $b(w^{(m)})$ is close enough to $B(\pi)$. For convenience, a word w inducing π and satisfying

$$b(w) = B(\pi) \ \text{ or } \ B(\pi) \leq b(w) < N \tag{4}$$

will be called a *small* word. For words in $\mathcal{W}_N$ inducing π we can apply Corollary 2.13 from (6), which we restate here.

Proposition 4.1 ((6)) *Let $N = N(\pi)$ as above, and suppose that $z \in \mathcal{W}_N$ induces π. Then the entries $z_1 z_2 \dots z_{n-1}$ are uniquely determined by π.*

In the rest of this section, we let $\zeta = \zeta(\pi) = z_1 z_2 \dots z_{n-1}$ be the word defined in Proposition 4.1. It follows from (6) that the entries of ζ can be computed as follows:

- Write the sequence of (unassigned) variables $z_{\pi^{-1}(1)} z_{\pi^{-1}(2)} \cdots z_{\pi^{-1}(n)}$ in this order and remove z_n from it.
- For each pair $z_i z_j$ of adjacent entries in the sequence with z_i to the left of z_j, insert a vertical bar between them if and only if $\pi(i+1) > \pi(j+1)$.

- In the case that $\pi(n) = 1$ and $\pi(n-1) = 2$, insert a vertical bar before the first entry in the sequence (which is $z_{\pi^{-1}(2)}$ in this case).
- Set each z_i in the sequence to equal the number of vertical bars to its left.

For example, if $\pi = 892364157 \in \mathcal{S}_9$, the sequence with z_n removed is $z_7z_3z_4z_6z_8z_5z_1z_2$, which becomes $z_7|z_3z_4|z_6z_8|z_5z_1|z_2$ after inserting the bars, so $\zeta(\pi) = z_1z_2\dots z_8 = 34113202$.

It is shown in (6, Lemma 2.8) that if $1 \leq i, j < n$ are such that $\pi(i) < \pi(j)$ and $\pi(i+1) > \pi(j+1)$, then the corresponding entries in $\zeta(\pi)$ satisfy $z_i < z_j$. This statement is logically equivalent to the following.

Lemma 4.2 ((6)) *If $1 \leq i, j < n$ are such that $z_j \leq z_i$ and $\pi(j) > \pi(i)$, then $\pi(j+1) > \pi(i+1)$ and $z_j = z_i$.*

The conclusion $z_j = z_i$ is clear from the fact that if $z_j < z_i$, then $z_{j\to} < z_{i\to}$, contradicting $\pi(j) < \pi(i)$. From Proposition 4.1 we see that $0 \leq z_1, \dots, z_{n-1} \leq N-1$. It is shown in (6) that if $\pi(n-1) = n-1$ and $\pi(n) = n$, then $0 \leq z_1, \dots, z_{n-1} \leq N-2$.

Proposition 4.1 and the paragraph preceding it imply that for any small word w (as defined by condition (4)) inducing π, the first $n-1$ entries of w are given by $\zeta = z_1z_2\dots z_{n-1}$. The next theorem shows how to find the remaining entries $w_{n+1}w_{n+2}\dots$. Its proof, which is quite involved, is omitted in this extended abstract, but it can be found in (5).

Theorem 4.3 *Let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, and if $c \neq 1$, let $k = \pi^{-1}(c-1)$.*

(a) If $c = 1$, let $w = \zeta 0^\infty$. Then w induces π, and for any other word v that induces π, we have $b(v) > b(w)$. In particular,

$$B(\pi) = b(w) = \hat{b}(w_{\ell\to}).$$

(b) If $c \neq 1$ and $\ell > k$, let

$$w = z_1z_2\cdots z_{n-1}\, z_kz_{k+1}\cdots z_{\ell-2}(z_{\ell-1}{+}1)0^\infty.$$

Then w induces π, and for any other word v that induces π, we have $b(v) \geq b(w)$. In particular,

$$B(\pi) = b(w) = \hat{b}(w_{\ell\to}).$$

(c) If $c \neq 1$ and $\ell < k$, let h be such that $\pi(h)$ is the maximum of $\pi(k+1), \pi(k+2), \dots, \pi(n)$. For each $m \geq 0$, let

$$w^{(m)} = z_1z_2\cdots z_{n-1}\,(z_kz_{k+1}\cdots z_{n-1})^m z_kz_{k+1}\cdots z_{h-2}(z_{h-1}{+}1)0^\infty.$$

Then $w^{(m)}$ induces π for $m \geq \frac{n-2}{n-k}$, and for any word v that induces π, there exists an m_0 such that $b(v) > b(w^{(m)})$ for $m \geq m_0$. In particular,

$$B(\pi) = \lim_{m\to\infty} b(w^{(m)}).$$

Additionally, $b(w^{(m)}) = \hat{b}(w^{(m)}_{\ell\to})$.

Some examples of applications of the above results are given in Section 6. Section 5 deals with the problem of computing $b(w)$ and $\lim_{m\to\infty} \hat{b}(w^{(m)})$, where w and $w^{(m)}$ are the above words. We end this section looking in more detail at the phase transitions where new patterns become allowed for β-shifts, and discussing the relationship between $B(\pi)$ and $N(\pi)$.

Proposition 4.4 *For every $\pi \in \mathcal{S}_n$, $\pi \notin \mathrm{Allow}(\Sigma_{B(\pi)})$. In particular, the infimum in Definition 3.2 is never a minimum, and the shift-complexity of π is the maximum β such that π is a forbidden pattern of Σ_β.*

One can rephrase Proposition 4.4 by stating that $\pi \in \mathrm{Allow}(\Sigma_\beta)$ if and only if $\beta > B(\pi)$. It follows from this observation and the definition of $N(\pi)$ (see equation (1)) that

$$N(\pi) = \lfloor B(\pi) \rfloor + 1. \tag{5}$$

5 Computation of $B(\pi)$: the equations

In this section we find the shift-complexity of an arbitrary permutation π by expressing it as the unique real root greater than 1 of a certain polynomial $P_\pi(\beta)$. Given a finite word $u_1u_2\dots u_r$, define the polynomial

$$p_{u_1u_2\dots u_r}(\beta) = \beta^r - u_1\beta^{r-1} - u_2\beta^{r-2} - \dots - u_r.$$

Theorem 5.1 *For any $\pi \in \mathcal{S}_n$ with $n \geq 2$, let $\zeta = \zeta(\pi) = z_1z_2\dots z_{n-1}$ as defined in Section 4. Let $c = \pi(n)$, $\ell = \pi^{-1}(n)$, and if $c \neq 1$, let $k = \pi^{-1}(c-1)$. Define a polynomial $P_\pi(\beta)$ as follows. If $c = 1$, let*

$$P_\pi(\beta) = p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta);$$

if $c \neq 1$ and $\ell > k$, let

$$P_\pi(\beta) = p_{z_\ell z_{\ell+1}\dots z_{n-1} z_k z_{k+1}\dots z_{\ell-1}}(\beta) - 1;$$

if $c \neq 1$ and $\ell < k$, let

$$P_\pi(\beta) = \begin{cases} p_{z_\ell z_{\ell+1}\dots z_{n-c}}(\beta) & \text{if } \pi \text{ ends in } 12\dots c, \\ p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta) - p_{z_\ell z_{\ell+1}\dots z_{k-1}}(\beta) & \text{otherwise.} \end{cases}$$

Then $B(\pi)$ is the unique real root with $\beta \geq 1$ of $P_\pi(\beta)$.

Note that $P_\pi(\beta)$ is always a monic polynomial with integer coefficients. For $\pi \in \mathcal{S}_n$, its degree is never greater than the maximum of $n-\ell$ and $n-k$, and in particular never greater than $n-1$.

Proof: In the case $c = 1$, letting $w = \zeta 0^\infty$, we know by Theorem 4.3(a) that

$$B(\pi) = b(w) = \hat{b}(w_{\ell\to}) = \hat{b}(z_\ell z_{\ell+1}\dots z_{n-1}0^\infty).$$

Thus, $B(\pi)$ is the unique solution with $\beta \geq 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \dots + \frac{z_{n-1}}{\beta^{n-\ell}} = 1,$$

which multiplying by $\beta^{n-\ell}$ is equivalent to $p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta) = 0$.

In the case $c \neq 1$ and $\ell > k$, Theorem 4.3(b) states that if we now let

$$w = z_1 z_2 \cdots z_{n-1}\, z_k z_{k+1} \cdots z_{\ell-2}(z_{\ell-1}{+}1)0^\infty,$$

then $B(\pi) = b(w) = \hat{b}(w_{\ell\rightarrow})$. Thus, $B(\pi)$ is the unique solution with $\beta \geq 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}} + \frac{z_k}{\beta^{n-\ell+1}} + \cdots + \frac{z_{\ell-2}}{\beta^{n-k-1}} + \frac{z_{\ell-1}+1}{\beta^{n-k}} = 1,$$

which multiplying by β^{n-k} is equivalent to $p_{z_\ell z_{\ell+1}\dots z_{n-1} z_k z_{k+1}\dots z_{\ell-1}}(\beta) - 1 = 0$.

Finally, if $c \neq 1$ and $\ell < k$, it follows from Theorem 4.3(b) that letting

$$w^{(m)} = z_1 z_2 \cdots z_{n-1}\,(z_k z_{k+1} \cdots z_{n-1})^m z_k z_{k+1} \cdots z_{h-2}(z_{h-1}{+}1)0^\infty,$$

where $\pi(h) = \max\{\pi(k+1), \pi(k+2), \dots, \pi(n)\}$, we have $B(\pi) = \lim_{m\to\infty} b(w^{(m)})$ and $b(w^{(m)}) = \hat{b}(w^{(m)}_{\ell\rightarrow})$. Here $\hat{b}(w^{(m)}_{\ell\rightarrow})$ is the unique solution with $\beta \geq 1$ of

$$\begin{aligned}\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{k-1}}{\beta^{k-\ell}} + \left(\frac{z_k}{\beta^{k-\ell+1}} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}}\right)\left(1 + \frac{1}{\beta^{n-k}} + \frac{1}{\beta^{2(n-k)}} + \cdots + \frac{1}{\beta^{m(n-k)}}\right)\\ + \frac{z_k}{\beta^{n-\ell+m(n-k)+1}} + \cdots + \frac{z_{h-2}}{\beta^{n-\ell+m(n-k)+h-k-1}} + \frac{z_{h-1}+1}{\beta^{n-\ell+m(n-k)+h-k}} = 1. \quad (6)\end{aligned}$$

For fixed m, it is clear that $\hat{b}(w^{(m)}_{\ell\rightarrow}) > 1$, because $w^{(m)}_{\ell\rightarrow}$ has at least two nonzero entries, since $z_\ell \geq 1$. Suppose first that not all of the entries $z_k, \dots, z_{n-1}$ are zero. In this case, making m go to infinity in equation (6) and using that $B(\pi) = \lim_{m\to\infty} \hat{b}(w^{(m)}_{\ell\rightarrow})$, we see that $B(\pi)$ is the solution with $\beta > 1$ of

$$\frac{z_\ell}{\beta} + \frac{z_{\ell+1}}{\beta^2} + \cdots + \frac{z_{k-1}}{\beta^{k-\ell}} + \left(\frac{z_k}{\beta^{k-\ell+1}} + \cdots + \frac{z_{n-1}}{\beta^{n-\ell}}\right)\frac{1}{1-\frac{1}{\beta^{n-k}}} = 1.$$

Multiplying by $\beta^{k-\ell}(\beta^{n-k}-1)$ we get

$$(\beta^{n-k}-1)(z_\ell\beta^{k-\ell-1} + z_{\ell+1}\beta^{k-\ell-2} + \cdots + z_{k-1}) + z_k\beta^{n-k-1} + \cdots + z_{n-1} = \beta^{n-\ell} - \beta^{k-\ell},$$

which can be rearranged as $p_{z_\ell z_{\ell+1}\dots z_{n-1}}(\beta) = p_{z_\ell z_{\ell+1}\dots z_{k-1}}(\beta)$.

In the case where $z_k = \cdots = z_{n-1} = 0$, $B(\pi)$ is the solution with $\beta \geq 1$ of

$$z_\ell\beta^{k-\ell-1} + z_{\ell+1}\beta^{k-\ell-2} + \cdots + z_{k-1} = \beta^{k-\ell},$$

or equivalently $p_{z_\ell z_{\ell+1}\dots z_{k-1}}(\beta) = 0$. This situation only happens when π ends in $123\dots c$. Indeed, one can use Lemma 4.2 to show that the condition $z_k = \cdots = z_{n-1}$ forces the sequence $\pi(k), \pi(k+1), \dots, \pi(n)$ to be monotonic, which can only happen if $k = n-1$. Now, Lemma 4.2 again and the fact that $z_k = 0$ imply that if d_i is the entry following i in π, then $1 \neq d_1 < d_2 < \cdots < d_{c-1} = c$, which forces the ending of π to be $123\dots c$. We remark that since $z_{n-c+1} = \cdots = z_{n-1} = 0$ in this case, we have that $p_{z_\ell z_{\ell+1}\dots z_{k-1}}(\beta) = \beta^{c-2} p_{z_\ell z_{\ell+1}\dots z_{n-c}}(\beta)$. □

6 Examples

In this section we give examples where Theorems 4.3 and 5.1 are used to construct words inducing a given permutation and to determine its shift-complexity.

(1) Let $\pi = 3421$. Using the construction from (6), described also right after Proposition 4.1 above, we get $\zeta(\pi) = 121$. Theorem 4.3(a) states that $w = 1210^\infty$ induces π and $B(\pi) = b(w) = \hat{b}(210^\infty)$. By Theorem 5.1, $B(\pi)$ is the root with $\beta \geq 1$ of $P_\pi(\beta) = p_{21}(\beta) = \beta^2 - 2\beta - 1$, so $B(3421) = 1 + \sqrt{2}$.

(2) Let $\pi = 735491826$. Using the construction from (6), $\zeta(\pi) = 42326051$. Applying Theorem 4.3(b) with $k = 3$ and $\ell = 5$, we get that $w = 42326051330^\infty$ induces π and $B(\pi) = b(w) = \hat{b}(6051330^\infty)$. By Theorem 5.1, $B(\pi)$ is the real root with $\beta \geq 1$ of

$$P_\pi(\beta) = p_{605132}(\beta) - 1 = \beta^6 - 6\beta^5 - 5\beta^3 - \beta^2 - 3\beta - 3,$$

so $B(735491826) \approx 6.139428921$.

(3) For $\pi = 892364157$, we have seen earlier that $\zeta(\pi) = 34113202$. Applying Theorem 4.3(c) with $k = 5$, $\ell = 2$, and $h = 9$, we have that $w^{(m)} = 34113202(3202)^m32030^\infty$ induces π for $m \geq 2$, and

$$B(\pi) = \lim_{m\to\infty} b(w^{(m)}) = \lim_{m\to\infty} \hat{b}(4113202(3202)^m32030^\infty).$$

By Theorem 5.1, $B(\pi)$ is the real root with $\beta \geq 1$ of

$$P_\pi(\beta) = p_{4113202}(\beta) - p_{411}(\beta) = \beta^7 - 4\beta^6 - \beta^5 - \beta^4 - 4\beta^3 + 2\beta^2 + \beta - 1,$$

so $B(892364157) \approx 4.327613926$.

(4) Let $\pi = (c+1)(c+2)\dots n12\dots c$ for any fixed $1 \leq c \leq n$. Here we get $\zeta(\pi) = 0^{n-c-1}10^{c-1}$. If $1 < c < n$, then $k = n-1$, $\ell = n-c$ and $h = n$, so by Theorem 4.3(c), $w^{(m)} = 0^{n-c-1}10^{c-1}0^m10^\infty$ induces π for $m \geq n-2$, and

$$B(\pi) = \lim_{m\to\infty} \hat{b}(10^{c-1}0^m10^\infty).$$

By Theorem 5.1, $B(\pi) = 1$ is the root of $P_\pi(\beta) = p_1(\beta) = \beta - 1$. If $c = n$, Theorem 4.3(b) gives $w = 0^{n-1}10^\infty$, and if $c = 1$, Theorem 4.3(a) yields $w = 0^{n-2}10^\infty$. In both cases, w induces π and $B(\pi) = \hat{b}(10^\infty) = 1$ as well. It is not hard to see that these are the only permutations with $B(\pi) = 1$.

The values of $B(\pi)$ for all permutations of length 2, 3, and 4 are given in Table 1. They have been computed using the implementation in *Maple* of Theorem 5.1 and the algorithm described in Section 4 to find $\zeta(\pi)$.

Using Theorem 5.1 one can show that for $n \geq 4$, the permutation $\rho = 1\,n\,2\,(n{-}1)\,3\,(n{-}2)\,\dots$ has the property that $B(\pi) < B(\rho)$ for all $\pi \in \mathcal{S}_n \setminus \{\rho\}$. For the proof of this result, as well as a method to determine the length of the shortest forbidden pattern of Σ_β for given $\beta > 1$, we refer the reader to the full version of this extended abstract (5).

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$\pi \in \mathcal{S}_2$	$\pi \in \mathcal{S}_3$	$\pi \in \mathcal{S}_4$	$B(\pi)$	$B(\pi)$ is a root of
$12, 21$	$123, 231, 312$	$1234, 2341, 3412, 4123$	1	$\beta - 1$
		$1342, 2413, 3124, 4231$	1.465571232	$\beta^3 - \beta^2 - 1$
	$132, 213, 321$	$1243, 1324, 2431, 3142, 4312$	$\frac{1+\sqrt{5}}{2} \approx 1.618033989$	$\beta^2 - \beta - 1$
		4213	1.801937736	$\beta^3 - \beta^2 - 2\beta + 1$
		$1432, 2143, 3214, 4321$	1.839286755	$\beta^3 - \beta^2 - \beta - 1$
		$2134, 3241$	2	$\beta - 2$
		4132	2.246979604	$\beta^3 - 2\beta^2 - \beta + 1$
		$2314, 3421$	$1 + \sqrt{2} \approx 2.414213562$	$\beta^2 - 2\beta - 1$
		1423	$\frac{3+\sqrt{5}}{2} \approx 2.618033989$	$\beta^2 - 3\beta + 1$

Tab. 1: The shift-complexity of all permutations of length up to 4.

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