A Chromatic Partition Polynomial

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Abstract

A polynomial in two variables is defined by $C_n(x,t) = \sum_{\pi \in \Pi_n} \chi(G_\pi, x) \cdot t^{|\pi|}$*, where* Π_n *is the lattice of partitions of the set* $\{1,2,\ldots,n\}$*,* G_π *is a certain interval graph defined in terms of the partition* π*,* $\chi(G_\pi, x)$ *is the chromatic polynomial of* G_π *and* $|\pi|$ *is the number of blocks in* π*. It is shown that* $C_n(x,t) = \sum_{k=0}^{n} t^k \sum_{i=0}^{k} \binom{n-i}{n-k} S(n,i)(x)_i$*, where* $S(n,i)$ *is the Stirling number of the second kind and* $(x)_i = x(x-1)\cdots(x-i+1)$*. As a special case,* $C_n(-1,-t) = A_n(t)$*, where* $A_n(t)$ *is the* n*-th Eulerian polynomial. Moreover,* $A_n(t) = \sum_{\pi \in \Pi_n} a_\pi \cdot t^{|\pi|}$*, where* a_π *is the number of acyclic orientations of* G_π*.*

On définit un polynôme en deux variables par $C_n(x,t) = \sum_{\pi \in \Pi_n} \chi(G_\pi, x) \cdot t^{|\pi|}$*, où* Π_n *est le treillis des partitions de l'ensemble* $\{1,2,\ldots,n\}$*,* G_π *est un certain graphe défini en termes de la partition* π*,* $\chi(G_\pi, x)$ *est le polynôme chromatique de* G_π *et* $|\pi|$ *est le nombre de blocs de* π*. On montre que* $C_n(x,t) = \sum_{k=0}^{n} t^k \sum_{i=0}^{k} \binom{n-i}{n-k} S(n,i)(x)_i$ *où* $S(n,i)$ *est le nombre de Stirling de deuxième espèce et* $(x)_i = x(x-1)\cdots(x-i+1)$*. En particulier,* $C_n(-1,-t) = A_n(t)$*, où* $A_n(t)$ *est le* n*-ième polynôme eulérien. De plus,* $A_n(t) = \sum_{\pi \in \Pi_n} a_\pi \cdot t^{|\pi|}$*, où* a_π *est le nombre d'orientations acycliques de* G_π*.*

1 Introduction

The Eulerian polynomials $A_n(t)$ (for $n = 0,1,2,\ldots$), which can be defined by

$$\sum_{k \geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}},$$

are ubiquitous in enumerative combinatorics and make frequent appearances in other branches of mathematics as well. The best known interpretation of the coefficients of $A_n(t)$ is perhaps the one which says that the i-th coefficient counts the number of permutations of $[n] := \{1,2,\ldots,n\}$ with $i-1$ *descents*, i.e. the number of permutations $a_1 a_2 \cdots a_n$ such that $a_j > a_{j+1}$ for exactly $i-1$ values of j.

Another much studied statistic is the Stirling number of the second kind, $S(n,k)$, which counts the number of partitions of an n–element set into k blocks.

In this paper we construct a link between these two statistics by establishing a bijection between the set of permutations with k descents and the set of pairs $(\pi, \mathcal{A}_\pi)$ where π is a partition of $[n]$ into $n-k$ blocks and $\mathcal{A}_\pi$ is an acyclic orientation of a

*Partially supported by grants from The Icelandic Council of Science and The Royal Swedish Academy of Sciences, respectively.

certain graph G_π determined by π. We thus get a polynomial $\sum_{\pi\in\Pi_n} a_\pi \cdot t^{|\pi|}$, where a_π is the number of acyclic orientations of G_π, $|\pi|$ the number of blocks of π and Π_n is the lattice of partitions of $[n]$, and this polynomial equals $A_n(t)$.

We then generalize this polynomial by replacing a_π by $\chi(G_\pi, x)$, the chromatic polynomial of G_π. The resulting polynomial, which we call $C_n(x,t)$, satisfies $C_n(-1,-t) = A_n(t)$, which is shown using a theorem of Stanley [11] on the number of acyclic orientations of graphs. $C_n(x,t)$ can be expressed in terms of Stirling numbers of the second kind, namely $C_n(x,t) = \sum_{k=0}^{n} t^k \sum_{i=0}^{k} \binom{n-i}{n-k} S(n,i)(x)_i$, where $(x)_i$ is the falling factorial defined by $(x)_i = x(x-1)(x-2)\cdots(x-i+1)$.

Lastly, we refine $C_n(x,t)$ by restricting it to partitions of a given *type*. The type of a partition π of $[n]$ is the partition of the *integer* n whose parts are the sizes of the blocks of π. This polynomial, called $C_\lambda(x)$, when evaluated at $x=-1$, gives a refinement of the Eulerian numbers, but is itself refined by the previously known statistic counting permutations by *descent set*. The descent set of a permutation $p = a_1a_2\cdots a_n$ is $D(p) = \{i \mid a_i > a_{i+1}\}$, i.e. the set of indices at which the descents of p occur. We show that $C_\lambda(x)$ can be expressed in terms of those partitions of n which are refined by λ, i.e. those partitions which can be obtained from λ by adding some of its parts.

2 The link between partitions and permutations

A *partition* π of $[n]$ (or any set) is a collection $\{B_1, B_2, \ldots, B_m\}$ of nonempty subsets of $[n]$ such that $B_i \cap B_j = \emptyset$ for all $i \neq j$ and such that $\bigcup_i B_i = [n]$. The B_i's are called the *blocks* of π and the *size* of B_i is its number of elements. We call π a k–partition if it has k blocks and write $|\pi|$ for the number of blocks in π. We will frequently represent a partition by writing the elements of each block in decreasing order and separating the blocks by dashes. For example, 531–2–94–876 is a 4-partition of [9].

Given a *permutation* $p = a_1a_2\cdots a_n$ in the symmetric group $\mathcal{S}_n$, we define its *descent blocks* to be the maximal decreasing contiguous subwords of p. For example, the descent blocks of 641573982 are 641, 5, 73 and 982, or 641–5–73–982 in our partition notation.

Each descent block of $p \in \mathcal{S}_n$ of size k has $k-1$ descents and, since there are no descents between two descent blocks, the total number of descents in p, $d(p)$, equals the sum of the block sizes minus the number of descent blocks, i.e. $d(p) = n - \#p$, where $\#p$ is the number of descent blocks in p.

Thus, every permutation p with k descents has $n-k$ descent blocks and hence defines an $(n-k)$–partition of $[d]$, but two different permutations can define the same partition, such as 3241 and 4132, whose descent blocks are 32–41. However, given a partition π of $[d]$, it is easy to determine which permutations have descent blocks corresponding to π. Namely, if we write each block of π in decreasing order, then every ordering of these blocks such that no descent occurs between two blocks gives a permutation whose descent blocks correspond to π. As an example, given the partition $\pi = 52\text{–}4\text{–}31$, we get four different permutations, namely 31452, 31524, 45231 and 52314. The two remaining permutations arising from these blocks, 43152 and 52431, have descents between the original blocks and thus don't have descent blocks corresponding to π.

From now on, we assume that the elements of each block in a partition π of $[n]$ are ordered decreasingly, and when we refer to a permutation in $\mathcal{S}_n$ obtained from an ordering of the blocks of π, we mean the permutation obtained by concatenating the blocks of π in the prescribed order. Also, call an ordering of the blocks *descent–free* if no descent occurs between blocks. For example, 21,43 is a descent–free ordering of the blocks 21–43, whereas 43,21 is not.

Let $r(\pi)$ denote the number of descent–free orderings of the blocks of a k–partition π of $[n]$ (e.g. $r(52\text{–}4\text{–}31) = 4$). Then, since each of the $r(\pi)$ permutations generated in this way has $n - k = n - |\pi|$ descents, and since every permutation in $\mathcal{S}_n$ with $n - k$ descents is uniquely generated in this way, we see that

$$A_n(t) = \sum_{\pi \in \Pi_n} r(\pi) \cdot t^{n-|\pi|+1}, \tag{1}$$

where Π_n is the set of all partitions of $[n]$. By symmetry of $A_n(t)$, we also have the more appealing formula

$$A_n(t) = \sum_{\pi \in \Pi_n} r(\pi) \cdot t^{|\pi|}. \tag{2}$$

These two identies yield the following:

Corollary 1 *Let Π_n^k be the rank $(n-k)$–subset of Π_n, i.e. the set of π in Π_n such that $|\pi| = k$. Then*

$$\sum_{\pi \in \Pi_n^k} r(\pi) = \sum_{\pi \in \Pi_n^{n+1-k}} r(\pi).$$

■

As we mentioned before, it is easy to prove the symmetry of the Eulerian polynomials when their coefficients are interpreted as counting permutations by number of descents. One bijection which accomplishes this is the "reversing" map $R : a_1 a_2 \cdots a_n \mapsto a_n a_{n-1} \cdots a_1$. In the present context, however, why Corollary 1 is true is not at all clear, because the lattice Π_n is *not* self–dual (and even that would not suffice). It is possible, of course, to construct a bijection between the pairs $\{(\pi, \mathcal{O}_\pi) \mid \mathcal{O}_\pi$ a descent-free ordering of the blocks in $\pi\}$ for $\pi \in \Pi_n^{n+1-k}$ respectively $\pi \in \Pi_n^k$ by "translating" the above mentioned bijection of permutations into the partition setup. This, however, will not result in a bijection which is "natural" with

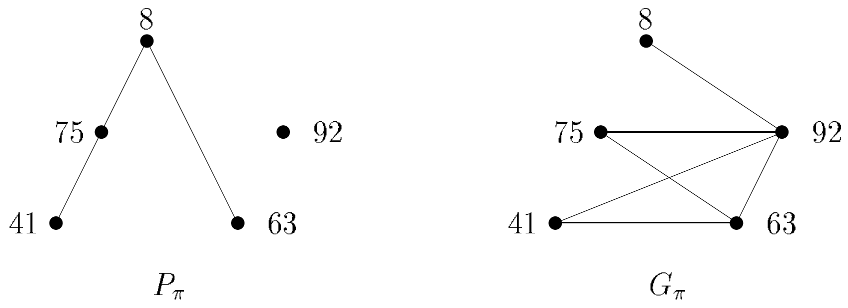

Figure 1: *The poset determined by the partition* $\pi = 41–63–75–8–92$ *and the corresponding incomparability graph.*

respect to the structure of Π_n. That is, two permutations arising from the same partition $\pi \in \Pi_n^k$ will not necessarily be sent to permutations arising from the same partition in Π_n^{n+1-k}.

If we order the blocks of a k-partition π by their respective least elements, it is not hard to show that $r(\pi) = \prod_{i=1}^{k}(p(i)+1)$, where $p(i)$ is the number of blocks which precede the i-th block and which contain a number that is larger than the least number in the i-th block. However, we will derive this in a different way.

Let $\pi \in \Pi_n$ be a k–partition with blocks $B_1, B_2, \ldots, B_k$. Define a partial ordering on the B_i by setting $B_i < B_j$ if the largest element of B_i is smaller than the least element of B_j (equivalently, *every* element of B_i is smaller than each element of B_j). It follows that an arbitrary ordering of the blocks B_i gives rise to a permutation p with $n-k$ descents if and only if the ordering of the blocks is descent-free.

This means, in the terminology of [14], Chapter 4, that $r(\pi)$ is the number of descent–free permutations (self-bijections) of the poset defined by the blocks of π, ordered as above. We will review this briefly now.

Let P be a poset on elements $x_1, x_2, \ldots, x_n$ and $\phi : P \to P$ a bijection. We refer to ϕ as a *permutation of* P and say that ϕ has a *descent* at i if $\phi(x_{i+1}) < \phi(x_i)$, where $<$ is the ordering in P. The *descent polynomial* $D_P(t)$ of P is the polynomial whose k–th coefficient is the number of permutations ϕ with exactly k descents. Let G_P be the *incomparability graph* of P, i.e. the graph whose vertices are the elements of P and with edges (x, y) for each pair of elements $x, y \in P$ such that x and y are incomparable. Fig. 1 shows the poset P_π corresponding to the partition $\pi = 41–63–75–8–92$ and the associated incomparability graph G_π.

It was shown, first in [8] and later, independently, in [2] (see also [3]), and, still later and independently, in [14], that the descent polynomial of a poset P and the chromatic polynomial $\chi(G_P, k)$ of G_P carry the same information. More precisely, if $D_P(t) = d_0 + d_1 t + \cdots + d_{n-1} t^{n-1}$ then

$$\sum_{k \geq 0} \chi(G_P, k) t^k = \frac{d_{n-1} t + d_{n-2} t^2 + \cdots + d_0 t^n}{(1-t)^{n+1}}.$$

It follows from this (see, e.g., Prop. 1.4.2 in [12]) that d_0, the number of permutations $\phi : P \to P$ with no descents, equals $(-1)^{|G_P|} \cdot \chi(G_P, -1)$, where $|G_P|$ is the number of vertices in G_P (and hence the degree of $\chi(G_P, n)$). For any graph G, in turn, $(-1)^{|G|} \cdot \chi(G_P, -1)$ equals the number of acyclic orientations of G. This was shown in [11], Corollary 1.3, and a bijective proof for the special case when G is an incomparability graph was given in [14].

If π is a partition of $[n]$, let G_π be the incomparability graph of the poset defined (as above) by π and let a_π be the number of acyclic orientations of G_π. We can then rewrite equations (1) and (2) to get the following:

Theorem 2 $A_n(t) = \sum_{\pi \in \Pi_n} a_\pi \cdot t^{n-|\pi|+1}$. *Equivalently,* $A_n(t) = \sum_{\pi \in \Pi_n} a_\pi \cdot t^{|\pi|}$. ■

Let $B_1, B_2, \ldots B_k$ be the blocks of a partition π and let a_i and b_i be the least and the largest element, respectively, of B_i, for each i. Then it is easy to see that G_π is isomorphic to the *interval graph* defined by the intervals (on the real line, say) $[a_1, b_1], [a_2, b_2], \ldots, [a_k, b_k]$, that is, the graph whose vertices are these intervals and where two vertices are adjacent *iff* their corresponding intervals have a nonempty intersection.

The following lemma, which is easy to prove, holds in a more general context than the one stated here, namely for all chordal (triangulated) graphs (see page 34 in [10]).

Lemma 3 *Let G be an interval graph on k intervals $I_i = [a_i, b_i]$, labeled so that $a_i \le a_j$ if $i < j$. For each i, let $p(i)$ denote the number of intervals I_j with $j < i$ and such that $I_i \cap I_j \neq \emptyset$. Then the chromatic polynomial of G is given by*

$$\chi(G, n) = \prod_{i=1}^{k} (n - p(i)).$$ ■

Corollary 4 *Let π be a k-partition with blocks $B_1, B_2, \ldots, B_k$ (labelled as the corresponding intervals in Lemma 3), let P_π be the poset determined by π and let $p(i)$ be the number of blocks B_j with $j < i$ and such that B_i and B_j are incomparable in P_π. Then $a_\pi = \prod_{i=1}^{k} (p(i) + 1)$.*

Proof: $a_\pi = (-1)^{|G_\pi|} \cdot \chi(G_\pi, -1) = (-1)^k \prod_{i=1}^{k} (-1 - p(i)) = \prod_{i=1}^{k} (p(i) + 1)$. ■

We now define a polynomial $C_n(x, t)$ in two variables, which is an obvious generalization of the polynomial $\sum_{\pi \in \Pi_n} a_\pi \cdot t^{|\pi|}$ in Theorem 2.

Definition 5 *Let* Π_n *and* G_π *be as before. The* n-th chromatic partition polynomial *is* $C_n(x,t) = \sum_{\pi \in \Pi_n} \chi(G_\pi, x) \cdot t^{|\pi|}$*. As a convention,* $C_0(x,t) = 1$*.*

Corollary 6 $C_n(-1,-t) = A_n(t)$*, where* $A_n(t)$ *is the* n*-th Eulerian polynomial.* ■

The polynomial $C_n(x,t)$ can be expressed in a particularly nice way. Let $(x)_i$ denote the falling factorial defined by $(x)_i = x(x-1)\cdots(x-i+1)$, where $(x)_0 = 1$. By definition, the chromatic polynomial of a graph G, when expanded in the basis $\{(x)_i\}_{i\geq 0}$, has as its coefficient to $(x)_i$ the number of ways of partitioning the vertices of G into i *stable* sets. A set of vertices is stable if no two of its vertices are adjacent. Recall that Π_n^k is the set of partitions of $[n]$ with k blocks.

Theorem 7 $C_n(x,t) = \sum_{k=0}^{n} t^k \sum_{i=0}^{k} \binom{n-i}{n-k} S(n,i)(x)_i.$ ■

The theorem can be proved by induction. However, after learning of our conjecture to this effect, Richard Stanley [13] found a bijective proof which we sketch here.

Let G_π be as usual. To avoid confusion, we call a partition of the vertices of G_π (i.e. of the blocks of the partition π) into i stable sets an *i–separation of* π. We need to show that each partition $\tau \in \Pi_n^i$, for $0 \leq i \leq k$, gives rise to $\binom{n-i}{n-k}$ distinct i–separations of partitions in Π_n^k, and that each such i–separation of each $\pi \in \Pi_n^k$ arises uniquely in this way.

Given $\tau \in \Pi_n^i$, write the elements of each block of τ in ascending order. There are $n-i$ places between adjacent elements in blocks of τ. Pick $k-i$ of these places. This can be done in $\binom{n-i}{k-i} = \binom{n-i}{n-k}$ ways and gives a partition $\pi \in \Pi_n^k$ if we break up each block of τ at the places picked. The desired i–separation of π is obtained by letting two blocks of π belong to the same (stable) set *iff* they were contained in the same block of τ. As an example, let $n = 9$, $k = 7$, $i = 3$, $\tau = 8431$–72–965 and suppose we pick the following four places within blocks of τ, indicated by bars: 8|4|31–7|2–9|65. Then $\pi = 8$–4–31–7–2–9–65 and the desired 3–separation of π is $\{13,4,8\}$, $\{2,7\}$, $\{56,9\}$.

Corollary 8 $A_n(t) = \sum_{k=0}^{n} t^k \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} \cdot i! \cdot S(n,i).$
Thus, the Eulerian number $A(n,k)$*, which is the* k*-th coefficient of* $A_n(t)$ *satisfies*

$$A(n,k) = \sum_{i=0}^{k} (-1)^{k-i} \binom{n-i}{n-k} \cdot i! S(n,i).$$ ■

Corollary 8 is equivalent to Theorem E in section 6.5 in [4]. In fact, Corollary 6 could be proved directly from Theorem 7, using this relationship between Stirling numbers and Eulerian numbers. It seems, however, that such a proof would raise the question answered by the bijective proof presented here. A more reasonable desire would be to see a *direct* bijective proof of Theorem 2, using a bijection between the set of permutations of $[n]$ on one hand and the set of pairs $(\pi, \mathcal{A}_\pi)$, where $\mathcal{A}_\pi$ is an acyclic orientation of G_π, on the other. We now sketch such a bijection.

A *source* in a directed graph is a vertex v none of whose incident edges points into v. In particular, an isolated vertex is a source. It is easy to see that in any acyclic orientation of a finite graph there must be at least one source. Let π be a partition of $[n]$ with blocks $B_1, B_2, \ldots, B_k$ and suppose we are given an acyclic orientation $\mathcal{A}_\pi$ of G_π. Observe that two sources in a directed graph cannot be adjacent. Thus, if π is a partition, and B_i and B_j are two sources in an acyclic orientation of G_π, then every element of B_i must be smaller than each element of B_j (i.e. $B_i < B_j$ in P_π), or vice versa. We now construct a permutation p of $[n]$, with descent blocks $B_1, B_2, \ldots, B_k$, from $\mathcal{A}_\pi$ as follows: Let B_i be that source of G_π whose elements are smallest. Then the permutation p begins with the elements of B_i, ordered decreasingly. Now remove B_i and all its incident edges from G_π. Let B_j be the source with the least elements in the resulting graph. Append the elements of B_j, in decreasing order, to those of B_i already placed. Continue in this way until there is nothing left of the graph. This gives a descent–free ordering of the blocks $B_1, B_2, \ldots, B_k$. Conversely, given a permutation p with descent blocks $B_1, B_2, \ldots, B_k$, in this order, let π be the partition with blocks $B_1, B_2, \ldots, B_k$ and construct an acyclic orientation of G_π by orienting edges from B_i to B_j if $i < j$. For an example, see Figure 2.

Remark 9 The above bijection can be modified to apply to an arbitrary poset P, thus giving a new bijective proof (simpler than that in [14], mentioned above) of the fact that the number of descent-free permutations of P equals the number of acyclic orientations of G_P. Namely, label the elements of P with $[n]$ in a natural way, i.e. so that $i < j$ in P implies $i < j$ as integers. Given an acyclic orientation of G_P, let the first letter of the corresponding permutation be the least label among all sources in G_P, remove that source and repeat the process as above. Conversely, given a descent-free permutation p of P, orient the edges of G_P from i to j if i precedes j in p.

Using Theorem 7, we get a combinatorial interpretation of another specialization of $C_n(x,t)$. First, we need to compute the exponential generating function of $C_n(x,t)$, but before that, observe that $C_n(x,t) = \sum_i S(n,i)(x)_i \sum_k t^k \binom{n-i}{n-k} = (1+t)^n \sum_i S(n,i)(x)_i \left(\frac{t}{1+t}\right)^i$. Also, let $S_n(x,t) = \sum_i S(n,i)(x)_i \left(\frac{t}{1+t}\right)^i$, so $C_n(x,t) = (1+t)^n S_n(x,t)$.

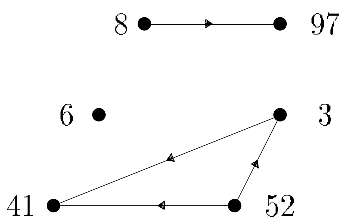

Figure 2: *An acyclic orientation of* G_π*, for* $\pi = 41\text{–}52\text{–}3\text{–}6\text{–}97\text{–}8$*, which corresponds to the permutation* 523416897.

Theorem 10 $\sum_{n\geq 0} C_n(x,t)\frac{z^n}{n!} = \left(\frac{1+te^{(1+t)z}}{1+t}\right)^x.$

Proof: As both sides of the equation have constant term equal to 1 (when viewed as power series in z), it suffices to prove the above identity for the logarithmic derivative (with respect to z) of each side. This reduces the problem to proving the identity

$$(1+te^{(1+t)z})\sum_{n\geq 0}\frac{[(1+t)z]^n}{n!}S_{n+1}(x,t) = xt(1+t)e^{(1+t)z}\sum_{n\geq 0}\frac{[(1+t)z]^n}{n!}S_n(x,t).$$

Setting $v = (1+t)z$, the above identity is equivalent to

$$\sum_{n\geq 0}\frac{v^n}{n!}\left[S_{n+1}(x,t) + t\sum_i \binom{n}{i} S_{i+1}(x,t)\right] = xt\sum_{n\geq 0}\frac{v^n}{n!}\sum_i\binom{n}{i}S_i(x,t),$$

which, in turn, is equivalent to

$$S_{n+1}(x,t) = t\sum_i \binom{n}{i}\,[xS_i(x,t) - S_{i+1}(x,t)].$$

This last identity follows from the basic recurrence for the Stirling numbers of the second kind and the identity $\sum_{i=k}^{n}\binom{n}{i}S(i,k) = S(n+1,k+1)$. ■

Corollary 11 $\sum_{n\geq 0} C_n(-x,-t)\frac{z^n}{n!} = \left(\sum_{n\geq 0} A_n(t)\frac{z^n}{n!}\right)^x.$

Proof: Follows from the well-known identity $\sum_{n\geq 0} A_n(t)\frac{z^n}{n!} = \frac{1-t}{1-te^{(1-t)z}}$. ■

Thus, the coefficient to $t^i x^k$ in $C_n(-x,-t)$ is the number of permutations of $[n]$ with exactly $i-1$ descents and exactly k *left-to-right minima*, i.e. exactly k values of m such that $a_m < a_\ell$ for all $\ell < m$ (see, e.g., [5], or [6]).

3 A refinement of $C_n(x,t)$

We will now refine the polynomial $C_n(x,t)$ by restricting it to partitions of a given *type*. The type of a partition π with blocks $B_1, B_2, \ldots, B_k$ is type$(\pi) = \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, where $\lambda_i = \#B_i$. By convention, we label the B_i so that $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k$. The *length* of λ, denoted $\ell(\lambda)$, is the number of parts of λ, i.e. $\ell(\lambda) = k$. Thus, type(π) is a partition of the integer n. As an example, if $\pi = 531\text{–}762\text{–}4\text{–}8\text{–}9$ then $\lambda = (1,1,1,3,3)$.

To minimize the confusion in what follows, we will always let π and τ denote partitions of the *set* $[n]$ but λ and μ will denote partitions of the *integer* n.

Let I_n be the poset of partitions of n, ordered by refinement, i.e. if $\lambda, \mu \in I_n$ then $\lambda \leq \mu$ if μ can be obtained from λ by adding some of the parts of λ together. For example, $(1,1,2,3,3) \leq (1,2,3,4) \leq (1,4,5) \not\leq (3,7)$.

We wish to compute $C_\lambda(x) := \sum \chi(G_\pi, x)$, where the sum is over all π of type λ. Evaluating this polynomial at $x = -1$ will yield a refinement of the Eulerian numbers $A(n,k)$, because we have $(-1)^k \sum_{\ell(\lambda)=k} C_\lambda(-1) = \sum_{\pi \in \Pi_n^k} a_\pi = A(n,k)$. On the other hand, $(-1)^k C_\lambda(-1)$ is refined by the known statistic recording the distribution of permutations in $\mathcal{S}_n$ by *descent set.* The descent set of a permutation $p = a_1 a_2 \cdots a_n$ is $D(p) = \{i \mid a_i > a_{i+1}\}$, i.e. the set of indices at which the descents of p occur. This statistic, first discovered by MacMahon [9], and then rediscovered several times, involves *binomial determinants* (see [7]). To see that $(-1)^k C_\lambda(-1)$ is refined by the descent set statistic, observe that it equals the number of permutations whose descent blocks constitute a partition of $[n]$ of type λ, whereas all permutations with a given descent set give rise to partitions of the same type, since the descent set determines the sizes of the descent blocks.

The lattice of partitions of $[n]$, which we denote by Π_n, is ordered by setting $\pi \leq \tau$ if τ is *refined* by π, that is, if each block of π is contained in some block of τ. As an example, $1\text{–}52\text{–}63\text{–}4 \leq 1\text{–}542\text{–}63 \not\leq 541\text{–}632$. Let π be a partition with blocks $B_1, B_2, \ldots, B_k$ and type$(\pi) = \lambda$. An i–separation of π defines a unique partition $\tau \geq \pi$ by letting $B_{i_1} \cup B_{i_2} \cup \cdots \cup B_{i_m}$ be a single block in τ if $\{B_{i_1}, B_{i_2}, \ldots, B_{i_m}\}$ is one of the stable sets of the i–separation in question (see the proof of Theorem 7). In order to give a nice expression for $C_\lambda(x)$ we need to understand how many i–separations of partitions of a given type arise from τ.

Define $f(\lambda, \mu)$ to be the number of ways of obtaining a partition π of type λ from a partition τ of type μ in the way described in the proof of Theorem 7, i.e. by picking some of the places between elements of blocks of τ and breaking up each block at the places picked (recall that the elements of each block are always written in decreasing order). As an example, the block 76421 can be split into 76–4–21 and 7–642–1 (to name a few), but not 74–621.

Then the same proof as for Theorem 7 yields the following result, where $\#(\mu)$ is the number of partitions of type μ (which has a well known expression).

Theorem 12 $C_\lambda(x) = \sum_{\mu \geq \lambda} f(\lambda,\mu) \cdot \#(\mu) \cdot (x)_{\ell(\mu)}$. ■

Setting $\lambda = (1,1,\ldots,1)$ yields the following well-known identity:

Corollary 13 $\sum_{k=0}^{n} S(n,k)(x)_k = x^n$.

Proof: Let π be the unique partition of $[n]$ into n blocks and $\lambda =$ type$(\pi) = (1,1,\ldots,1)$. Thus, for any μ, we have $\mu \geq \lambda$ and $f(\lambda,\mu) = 1$, so we get

$$x^n = \chi(G_\pi, x) = \sum_{\mu \in I_n} \#(\mu) \cdot (x)_{\ell(\mu)} = \sum_{k=0}^{n} S(n,k)(x)_k. \quad ■$$

Theorem 12 can be used to express the *Euler numbers* (not to be confused with the Eulerian numbers $A(n,k)$) in terms of the number of partitions of certain types. The Euler number E_n is defined as the number of *alternating* permutations in $\mathcal{S}_n$, i.e. permutations $a_1a_2\cdots a_n$ such that $a_1 > a_2 < a_3 > \cdots$. A result of André [1] states that the exponential generating function of the Euler numbers is given by $\sum_{n\geq 0} E_n x^n/n! = \tan x + \sec x$. For n odd, they satisfy $E_n = (-1)^{(n+1)/2} A_n(-1)$. For even n, $A_n(-1) = 0$, explaining why the formula only holds for odd n.

Corollary 14 *Let c_n^k be the number of partitions of $[2n]$ into k blocks of even sizes. Then*

$$E_{2n} = \sum_{k=1}^{n} (-1)^{n-k} \cdot k! \cdot c_n^k.$$

Proof: All the descent blocks of an alternating permutation in $\mathcal{S}_{2n}$ have size 2, so such a permutation arises from a partition π of type $\lambda = (2,2,\ldots,2)$. Conversely, any permutation arising from a partition of type $(2,2,\ldots,2)$ is alternating. If $\tau \geq \pi$, where type$(\pi) = (2,2,\ldots,2)$, then every block of τ has even size. Also, given a block in such a τ, it can be split into linearly ordered blocks of size 2 in only one way, i.e. $f(\lambda,\mu) = 1$, where $\mu =$ type(τ). The number of permutations arising from π is given by $(-1)^n\chi(G_\pi,-1)$, so, letting $\lambda = (2,2,\ldots,2)$, we get

$$E_{2n} = (-1)^n \sum_{\text{type}(\pi)=\lambda} \chi(G_\pi,-1) = (-1)^n \sum_{\mu\geq\lambda} f(\lambda,\mu) \cdot \#(\mu) \cdot (-1)_{\ell(\mu)} =$$

$$(-1)^n \sum_{k=1}^{n} 1 \cdot c_n^k \cdot (-1)^k \cdot k! = \sum_{k=1}^{n} (-1)^{n-k} \cdot k! \cdot c_n^k. \quad ■$$

It has been pointed out to the author by Ira Gessel how this result can be obtained from generating functions. At the end of this paper, we give a combinatorial proof of Corollary 14.

Obviously, one could generalize Corollary 14 by replacing c_n^k with the corresponding number of partitions of $[dn]$ into k blocks of sizes divisible by d, in which case the Euler number E_{2n} would be replaced by a generalized Euler number E_{dn}^d counting the number of permutations of $[dn]$ with descents at positions $d, 2d, 3d, \ldots, (n-1)d$.

The following formula for the Euler numbers E_{2n-1}, similar to Corollary 14, has been found by Sheila Sundaram [16]. Her result stems from homological properties of the lattice Π_n, studied in [15], and is likewise generalized to partitions of $[dn]$ vs. permutations of $[dn-1]$.

Proposition 15 (Sundaram [16]) $E_{2n-1} = \sum_{k=1}^{n} (-1)^{n-k} \cdot (k-1)! \cdot c_n^k.$ ■

We conclude the paper with a combinatorial proof of Corollary 14 and Proposition 15. The proof easily generalizes to partitions of $[dn]$ into blocks of sizes divisible by d.

Note first that $k! \cdot c_n^k$ counts the number of *ordered* partitions of $[2n]$ that have k blocks, all of even size. Each such ordered partition gives rise to a permutation of $[2n]$ if we concatenate the blocks in the prescribed order, writing the elements of each block in decreasing order. Some permutations arising in this way will arise from several different ordered partitions, e.g. 31876542, which arises from 31–876542, 31–8765–42, 31–87–6542, and 31–87–65–42.

Clearly, each permutation p arising in this way has descent blocks of even sizes. Moreover, p arises from precisely those ordered partitions of $[2n]$ that can be obtained from p by splitting p into its descent blocks and then, possibly, further splitting some of the descent blocks. Each descent block of size 2ℓ can be split at any of the $(\ell - 1)$ places between consecutive pairs of its elements. In the above example, the descent block 876542 could be split precisely at the places indicated by bars in 87|65|42.

Given a permutation p of $[2n]$, let $S(p) = \{i_1, i_2, \ldots, i_m\}$ be the set of all such places in p. Thus, the set of ordered partitions of $[2n]$ that give rise to the permutation p is in one-to-one correspondence with the set of subsets of $S(p)$. Moreover, for each p, the cardinality of the subset is either always of the same parity as the number of blocks in the corresponding partition or else always of the opposite parity. But the parity of the number of blocks determines the sign with which a partition appears in the sum in Corollary 14, so the total contribution of a permutation to the sum is $\sum_{i=0}^{m} (-1)^i \binom{m}{i}$, which equals 0 *except* when $S(p) = \emptyset$. When $S(p) = \emptyset$, p arises from just one ordered partition with n blocks of size 2, in which case p is alternating.

To prove Proposition 15, we proceed as above, except that each ordered partition is required to begin with that block which contains $2n$ (as its first element). This results in $k!$ being replaced by $(k-1)!$. Deleting $2n$ from the resulting alternating permutations gives all *reverse* alternating permutations of $[2n-1]$, which are easily seen to be equinumerous with the alternating ones.

ACKNOWLEDGEMENT

My thanks to two referees. One made numerous useful suggestions and pointed out the generalization mentioned in Remark 9. The other one pointed out Theorem 10 and Corollary 11.

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