Yet another triangle for the Genocchi numbers

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Abstract

We give a new refinement of the Genocchi numbers, counting permutations with alternating excedances according to first letter. These numbers are related to the Seidel triangle for the Genocchi numbers and to a recent refinement by Kreweras of the Genocchi numbers.

1 Introduction

The study of Genocchi numbers, it is claimed, goes back to Euler. In what sense that is true is somewhat unclear, but during the last two or three decades the Genocchi numbers have been studied by Dumont and some collaborators [2, 3, 4, 5, 6, 7]. In recent years there has been a flurry of activity in this field, viz. [1, 9, 10, 11, 12, 13].

The Genocchi numbers are cousins to the Euler numbers which count the *alternating* permutations, that is, permutations $a_1 a_2 \cdots a_d$ such that

$$a_1 > a_2 < a_3 > \cdots.$$

Whereas the Euler numbers thus count permutations with alternating *descents*, the Genocchi numbers count, among other things, permutations with alternating *excedances*, that is, permutations $a_1 a_2 \cdots a_d$ such that $a_i > i$ if and only if i is odd (and $i < d$).

The Genocchi numbers can be defined in many other ways, although most definitions so far have been related to permutations with alternating excedances or variations thereof.

Several generalizations and refinements of the Genocchi numbers are known, see [1, 4, 7, 11]. The purpose of this paper is to present yet another generalization of these numbers and to elicit the relation of that generalization to previous such, notably to the Seidel triangle for the Genocchi numbers, see [4].

Our generalization consists of counting permutations with alternating excedances according to the first letter of each permutation. This is, of course, a refinement of the Genocchi numbers, but it turns out also to include the Genocchi numbers among its constituents.

We give a recursive formula (with a bijective proof) for the number of excedance-alternating permutations with a given first letter. We then use this formula to establish the relation of these numbers to the Seidel triangle for the Genocchi

$n\backslash k$	2	3	4	5	6	7
1	1					
2	1					
3	1	1				
4	2	1				
5	2	3	3			
6	8	6	3			
7	8	14	17	17		
8	56	48	34	17		
9	56	104	138	155	155	
10	608	552	448	310	155	
11	608	1160	1608	1918	2073	2073

Table 1: Seidel triangle for the Genocchi numbers.

numbers. We also show that counting these permutations according to last letter gives a different statistic which turns out to be equal to a statistic studied by Kreweras [11], namely the number of so-called *Dumont permutations of the first kind* with a given first (or last) letter.

The number of excedance-alternating permutations with a given first letter and the number of those with a given last letter satisfy the same recurrence, but have slightly different initial conditions.

2 Preliminaries

The beginning of the Seidel triangle for the Genocchi numbers is given in Table 1. This triangle can be generated from the corresponding *Seidel matrix*, see [4]. We shall, however, take as our definition the recurrence defined by summing the entries in even (odd) numbered rows from right to left (left to right) to obtain the successive new entries in each row. More precisely, the entry in row n and column k, denoted S_n^k, is given by the following:

$$S_1^1 = 1, \qquad S_n^k = 0 \text{ if } k < 2 \text{ or } k > (n+3)/2,$$

$$S_{2n}^k = \sum_{i \geq k} S_{2n-1}^i, \qquad S_{2n+1}^k = \sum_{i \leq k} S_{2n}^i.$$

The edges of this Seidel triangle consist of the *Genocchi numbers* G_{2n} (on the rightmost diagonal) and the *median Genocchi numbers* H_{2n+1} (in the leftmost column). The Genocchi number G_{2n} is the number of permutations $\pi = a_1 a_2 \cdots a_{2n+1}$ in the symmetric group $\mathcal{S}_{2n+1}$ such that

$$\begin{array}{ll} a_i < a_{i+1} & \text{if } a_i \text{ is odd,} \\ a_i > a_{i+1} & \text{if } a_i \text{ is even.} \end{array} \qquad (1)$$

For example, there are exactly $G_4 = S_5^4 = 3$ permutations in $\mathcal{S}_5$ satisfying these conditions, namely

$$21435 \qquad 34215 \qquad 42135. \tag{2}$$

The median Genocchi number H_{2n+1} is the number of permutations in $\mathcal{S}_{2n+1}$ such that

$$\begin{array}{ll} a_i > i & \text{if } i \text{ is odd,} \\ a_i < i & \text{if } i \text{ is even.} \end{array}$$

According to Kreweras [11, page 53], H_{2n+1} also counts the number of permutations satisfying (1) and beginning with n or $n+1$. For example, $H_5 = S_5^2 = 2$, corresponding to the fact that exactly two of the permutations in (2) begin with 2 or 3.

In [7], Dumont and Viennot gave a combinatorial interpretation of all the numbers in the Seidel triangle for the Genocchi numbers, in terms of functions $h : [n] \longrightarrow [n]$ satisfying $h(k) \leq (k+1)/2$ for all $k \in [n]$, where $[n] = \{1, 2, \ldots, n\}$.

The purpose of the present paper is to study the number of *excedance-alternating* permutations in $\mathcal{S}_n$.

Definition 2.1 *A permutation $\pi = a_1 a_2 \cdots a_n$ in $\mathcal{S}_n$ is* excedance-alternating *if it satisfies the following conditions:*

$$\begin{array}{ll} a_i > i & \textit{if } i \textit{ is odd and } i < n, \\ a_i \leq i & \textit{if } i \textit{ is even.} \end{array} \tag{3}$$

We consider the distribution of these permutations according to their first letter and the relation of that statistic to the Seidel triangle for the Genocchi numbers and to other related permutation statistics. However, it turns out to be convenient to also consider permutations beginning with 1 but otherwise satisfying (3). In other words, permutations for which the first condition in (3) is replaced by: $a_i > i$ *if i is odd and* $3 \leq i < n$.

Definition 2.2 *For k and n such that $2 \leq k \leq n$, the number of excedance-alternating permutations in $\mathcal{S}_n$ beginning with k is denoted E_n^k. When $k = 1$ we let E_n^1 denote the number of permutations in $\mathcal{S}_n$ beginning with 1 but satisfying (3) for all $i > 1$. Moreover, we set $E_n^k = 0$ for $k < 1$ and $k > n$.*

Lemma 2.3 *For all n, $E_n^1 = E_n^2$.*

Proof: An excedance-alternating permutation beginning with 2 must start with 21. A permutation counted by E_n^1 must start with 12. There is an obvious bijection between these two sets of permutations. □

$n\backslash k$	1	2	3	4	5	6	7	8	9
2	1	1							
4	1	1	2						
6	3	3	6	4	4				
8	17	17	34	28	36	20	20		
10	155	155	310	276	380	284	324	172	172

Table 2: The numbers E_{2n}^k of excedance-alternating permutations in $\mathcal{S}_{2n}$ whose first letter is k (with an added column for $k = 1$).

Consider now the triangle in Table 2, consisting of the numbers E_{2n}^k. As an example, $E_6^3 = 6$, because the excedance-alternating permutations on six letters beginning with 3 are precisely the following six:

314265 315264 324165 325164 325461 315462.

The reason for omitting the rows for odd n is that they are identical to the respective even numbered rows as we now show.

Lemma 2.4 *For all $n \geq 2$ and for all k we have $E_{2n}^k = E_{2n-1}^k$.*

Proof: Let A_n^k be the set of excedance-alternating permutations in $\mathcal{S}_n$ that begin with k. A permutation $\pi = a_1 a_2 \cdots a_{2n}$ in A_{2n}^k must have $a_{2n-1} = 2n$ since $a_{2n-1} > 2n - 1$. There is a one-to-one correspondence between such permutations and permutations in A_{2n-1}^k, obtained by removing the letter $a_{2n-1} = 2n$ from a permutation in A_{2n}^k and, conversely, inserting $2n$ before the last letter in a permutation in A_{2n-1}^k. □

Remark 2.5 It is easy to prove that the terms in each row alternate in size. Namely, given an excedance-alternating permutation beginning with $2k$, interchanging $2k$ and $2k+1$ gives an excedance-alternating permutation (beginning with $2k+1$), but the converse is not true in general, since $2k$ may be a fixed point.

Studying the triangle in Table 2 and the Seidel triangle in Table 1, one sees a curious relationship. Namely, each entry in an even numbered row in the Seidel triangle is a sum of two entries from the corresponding row in Table 2. For example, we have

$$S_{10}^3 = 552 = 380 + 172 = E_{10}^5 + E_{10}^8.$$

We shall prove that this is a general pattern.

A consequence of this, Corollary 3.5, is that each of the entries in odd numbered rows in the Seidel triangle can be written as a sum (of varying length) of numbers from the corresponding row in Table 2. As an example,

$$S_{11}^3 = 1160 = 380 + 284 + 324 + 172 = E_{10}^5 + E_{10}^6 + E_{10}^7 + E_{10}^8.$$

First, we show that the triangle for excedance-alternating permutations can be constructed recursively. Namely, each entry E_n^k in this triangle equals its left neighbor, E_n^{k-1}, plus/minus the sum of certain entries on the preceding row. For example, $28 = 34 - (3+3)$ and $380 = 276 + (28 + 36 + 20 + 20)$.

3 Main results

Theorem 3.1 *If k is odd and $k \geq 3$ then*

$$E_{2n}^k = E_{2n}^{k-1} + \sum_{i \geq k-1} E_{2n-2}^i.$$

If k is even then

$$E_{2n}^k = E_{2n}^{k-1} - \sum_{i \leq k-2} E_{2n-2}^i.$$

Equivalently, if k is even then

$$E_{2n}^k = E_{2n}^{k-1} + \sum_{i \geq k-1} E_{2n-2}^i - \sum_i E_{2n-2}^i.$$

Proof: Let A_n be the set of excedance-alternating permutations in $\mathcal{S}_n$ and let A_n^k be the set of those that begin with k. Let $\pi = a_1 a_2 \cdots a_{2n} \in A_{2n}^k$ (so $a_1 = k$).

Assume first that k is odd. We partition A_{2n}^k into two sets, according to the following two cases:

(i) If $a_{k-1} \neq k-1$, then we can interchange k and $k-1$ in π to obtain a permutation in A_{2n}^{k-1}. Conversely, given a permutation π in A_{2n}^{k-1}, interchanging k and $k-1$ in π yields a permutation in A_{2n}^k. As an example, 315264 and 215364 belong to A_5^3 and A_5^2, respectively.

This gives a one-to-one correspondence between permutations in A_{2n}^k with $a_{k-1} \neq k-1$ on one hand, and all the permutations in A_{2n}^{k-1} on the other.

(ii) Suppose then that $a_{k-1} = k-1$. Then

$$\pi = k a_2 a_3 \cdots a_{k-2}(k-1) a_k \cdots a_{2n}.$$

Consider the transformation from A_{2n}^k to $\mathcal{S}_{2n-2}$ defined as follows, when $k \geq 5$:

$$\pi = k a_2 a_3 \cdots a_{k-2}(k-1) a_k \cdots a_{2n} \quad \longmapsto \quad a'_{k-2} a'_2 a'_3 \cdots a'_{k-3} a'_k \cdots a'_{2n} = \pi',$$

where

$$a'_i = \begin{cases} a_i & \text{if } a_i < k-1, \\ a_i - 2 & \text{if } a_i > k. \end{cases}$$

In the case when $k = 3$, and thus $a_2 = 2$, we instead set $\pi' = a'_3 a'_4 \cdots a'_{2n}$ and the following reasoning goes through with a_3 replacing a_{k-2} (and some minor modifications).

We claim that π' is excedance-alternating and that $a'_{k-2} \geq k-1$, so that π' belongs to A^i_{2n-2} for some $i \geq k-1$. To see that $a'_{k-2} \geq k-1$, observe that $a_{k-2} > k-2$, so, since $a_{k-2} \neq k, (k-1)$, we have $a_{k-2} \geq k+1$, and thus $a'_{k-2} \geq k-1$.

For the places $i = 2, 3, \ldots, k-3$, we have that $a_i > i$ if and only if $a'_i > i$ because if $a_i \leq i$ then $a'_i = a_i \leq i$ and otherwise, if $a_i > i$, then either $a'_i = a_i \geq i$ or else $a_i > k$, so $a'_i > k-2 > i$.

It remains to be shown that for $i \geq k$ we have $a'_i > i-2$ if and only if i is odd, that is, if and only if $a_i > i$. Clearly, if $a_i > i \geq k$ then $a'_i = a_i - 2 > i - 2$. Conversely, if $a_i \leq i$ then either $a_i < k-1$, in which case $a'_i = a_i < k-1 \leq i-1$, so $a'_i = a_i \leq i-2$, or else $a_i > k$, so $a'_i = a_i - 2 \leq i - 2$.

If k is even, consider the effect of switching k and $k-1$ in a permutation in A^k_{2n}. For example, for $k=4$, we have $415362 \mapsto 315462$. This always yields a permutation in A^{k-1}_{2n}. Conversely, switching $k-1$ and k in a permutation in A^{k-1}_{2n} yields a permutation in A^k_{2n} *except* when $a_{k-1} = k$, because then $k-1$ becomes a fixed point rather than an excedance. To establish the second identity in the theorem we thus need to show that the number of permutations in A^{k-1}_{2n} with $a_{k-1} = k$ equals $\sum_{i \leq k-2} E^i_{2n-2}$, that is, equals the number of permutations in A_{2n-2} with first letter smaller than or equal to $k-2$. To this end we consider the following transformation $\pi \longmapsto \pi'$ of such permutations,

$$(k-1)a_2 a_3 \cdots a_{k-2} k a_k \cdots a_{2n} \quad \longmapsto \quad a'_{k-2} a'_2 a'_3 \cdots a'_{k-3} a'_k \cdots a'_{2n},$$

where a'_i is defined as above. Since k, and thus also $k-2$, is even, we have $a_{k-2} \leq k-2$, so π' begins with a letter smaller than or equal to $k-2$. Define i' analogously to a'_i, that is, by $i' = i$ if $i < k-1$ and $i' = i-2$ if $i > k$. We need to show that $a'_i > i'$ if and only if $a_i > i$. This can be done in a way similar to the case when k is odd, and is omitted.

That the procedure described defines a bijection is straightforward.

The last identity in the theorem follows directly from the second one. □

We have already noted in Lemma 2.3 that the first two entries in each row in the triangle of the E^k_n's are equal. It is also true for $n \geq 6$ that the last two entries are equal. This follows immediately from Theorem 3.1. Moreover, the last entries equal the sum of all numbers in the preceding row and the first entries equal the sum of all but the first entry in the preceding row.

Corollary 3.2 *For all $n \geq 2$ we have $E_{2n}^2 = \sum_{i \geq 2} E_{2n-2}^i$ and $E_{2n}^{2n-1} = \sum_i E_{2n-2}^i$. Thus, E_{2n}^2 equals the number of excedance-alternating permutations in $\mathcal{S}_{2n-2}$.*

Proof: Any permutation $\pi \in A_{2n}^2$ must begin with 21. Removing 21 from π and decreasing each remaining letter by 2 defines a bijection to the (disjoint) union $A_{2n-2}^2 \cup A_{2n-2}^3 \cup \cdots \cup A_{2n-2}^{2n-2}$ which establishes the first identity.

The second identity follows by setting $k = 2n$ in the second identity in Theorem 3.1 and observing that $E_{2n}^{2n} = 0$. □

Remark 3.3 It is proved in [8] that E_{2n}^2, being equal to the number of excedance-alternating permutations in $\mathcal{S}_{2n-2}$, is the Genocchi number G_{2n-2}. It is also shown there that E_n^2 is odd for all n. In fact, E_{2n}^2 is congruent to $(-1)^n$ modulo 4. The numbers E_{2n}^3 are congruent to $(-1)^n \cdot 2$ modulo 8. The numbers E_{2n}^k, for $n > k \geq 4$, are congruent to 4 modulo 8. This can be proved by induction, using Theorem 3.1. Combinatorially it is clear why E_n^k is even for $k \geq 3$. Namely, in an excedance-alternating permutation beginning with a letter ≥ 3, the letters 1 and 2 are interchangeable.

Theorem 3.4 *For all n and k with $2 \leq k \leq n+1$, we have $S_{2n}^k = E_{2n}^{n+3-k} + E_{2n}^{n+k}$.*

Proof: The proof is by induction on n. The base case is $n = 2$ which is easily checked. Assuming the statement true for $n-1$, we have

$$\begin{aligned}
S_{2n}^k &= \sum_{i \geq k} S_{2n-1}^i = \sum_{i \geq k} \sum_{j=2}^{i} S_{2(n-1)}^j = \sum_{i \geq k} \sum_{j=2}^{i} \left(E_{2n-2}^{n+2-j} + E_{2n-2}^{n-1+j} \right) \\
&= \sum_{i \geq k} \left(\sum_{j=2}^{i} E_{2n-2}^{n+2-j} + \sum_{j=2}^{i} E_{2n-2}^{n-1+j} \right) \\
&= \sum_{i \geq k} \left(\sum_{j=n+2-i}^{n} E_{2n-2}^{j} + \sum_{j=n+1}^{n-1+i} E_{2n-2}^{j} \right) \\
&= \sum_{i \geq k} \left(\sum_{j \leq n} E_{2n-2}^{j} - \sum_{j \leq n+1-i} E_{2n-2}^{j} + \sum_{j \geq n+1} E_{2n-2}^{j} - \sum_{j \geq n+i} E_{2n-2}^{j} \right) \\
&= \sum_{i \geq k} \left(\sum_{j} E_{2n-2}^{j} - \left(\sum_{j \leq n+1-i} E_{2n-2}^{j} + \sum_{j \geq n+i} E_{2n-2}^{j} \right) \right) \\
&= \sum_{i \geq k} \left(\sum_{j \geq n+2-i} E_{2n-2}^{j} - \sum_{j \geq n+i} E_{2n-2}^{j} \right).
\end{aligned}$$

We can now rewrite each of the inner sums on the last line, using the first or the last identity in Theorem 3.1. Since $n+2-i$ and $n+i$ have the same parity, either both or neither of the rewritten expressions will contain the term $\sum_i E_{2n-2}^i$, so these terms will cancel each other, if present, and we get

$$\sum_{i\geq k}\left[\left(E_{2n}^{n+3-i}-E_{2n}^{n+2-i}\right)-\left(E_{2n}^{n+1+i}-E_{2n}^{n+i}\right)\right] \quad = \quad E_{2n}^{n+3-k}+E_{2n}^{n+k},$$

as desired. □

Corollary 3.5 *For all n and k with $2\leq k\leq n+1$, $S_{2n+1}^k=\sum_{i=n+3-k}^{n+k} E_{2n}^i$.*

Proof: We have

$$\begin{aligned} S_{2n+1}^k=\sum_{i=2}^k S_{2n}^i &= \sum_{i=2}^k\left(E_{2n}^{n+3-i}+E_{2n}^{n+i}\right)=\sum_{i=2}^k E_{2n}^{n+3-i}+\sum_{i=2}^k E_{2n}^{n+i} \\ &= \sum_{i=n+3-k}^{n+1} E_{2n}^i+\sum_{i=n+2}^{n+k} E_{2n}^i=\sum_{i=n+3-k}^{n+k} E_{2n}^i. \end{aligned}$$ □

Together with Theorem 3.4 the preceding corollary says that summing E_{2n}^{n+3-k} and E_{2n}^{n+k} gives S_{2n}^k whereas summing *all* the E_{2n}^i for i between $n+3-k$ and $n+k$ yields S_{2n+1}^k.

4 Other statistics and further identities

Interestingly, counting excedance-alternating permutations by *last* letter yields a statistic equal to that for the permutations satisfying (1), that is, permutations $a_1a_2\cdots a_n$ such that $a_i<a_{i+1}$ if and only if a_i is odd. The triangle for the number of such permutations, counted by first letter, is given in Table 3. In fact, counting these permutations by *last* letter yields the same statistic. This statistic was studied by Kreweras in [11].

The recursive definition of the triangle in Table 3 is similar to the recursion for the triangle in Table 2 which was established in Theorem 3.1. That is, the recursion for these two triangles is the same except that they have different initial conditions: We have $E_2^2=1$, whereas $K_1^2=0$. The proof that the numbers of excedance-alternating permutations with a given last letter satisfy this recurrence is analogous to the proof of Theorem 3.1 and is omitted.

A consequence of this is that Tables 2 and 3 agree for $k=1,2$ whereas for $k\geq 3$ the table for E_{2n}^k can be obtained by adding each row of the table for K_{2n+1}^k to the next row of the same table, after shifting the first of these two rows two steps to the right. In other words, we have the following theorem.

$n\backslash k$	1	2	3	4	5	6	7	8	9
1	1								
3	1	1	1						
5	3	3	5	3	3				
7	17	17	31	25	31	17	17		
9	155	155	293	259	349	259	293	155	155

Table 3: The numbers K_{2n+1}^k counting excedance-alternating permutations in $\mathcal{S}_{2n+1}$ whose last letter is k.

Theorem 4.1 *For all k and all $n \geq 2$ we have $E_{2n}^k = K_{2n-1}^k + K_{2n-3}^{k-2}$.*

Corollary 4.2 *For all n and k we have:*

$$\begin{aligned} S_{2n}^k &= K_{2n-1}^{n+3-k} + K_{2n-3}^{n+1-k} + K_{2n-1}^{n+k} + K_{2n-3}^{n+k-2}, \\ S_{2n+1}^k &= \sum_{i=n+3-k}^{n+k} \left(K_{2n-1}^i + K_{2n-3}^{i-2}\right). \end{aligned}$$

Proof: Follows from Theorem 3.4, Corollary 3.5 and Theorem 4.1. □

Finally, we list, without proof, a few further results, all of which can be proved easily from the recurrence for the numbers E_{2n}^k and their relationship to the Seidel triangle for the Genocchi numbers.

Theorem 4.3

1. $E_{2n}^k + E_{2n-2}^k = E_{2n}^{2n-k} + E_{2n-2}^{2n-k}$.
2. $E_{2n}^{2n-k} - E_{2n}^k = S_{2n-3}^{n+1-k}$.
3. $E_{2n}^{n+1} + E_{2n}^{n+2} = H_{2n+1}$ *where* $H_{2n+1} = S_{2n}^2$ *is the* median Genocchi number.

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